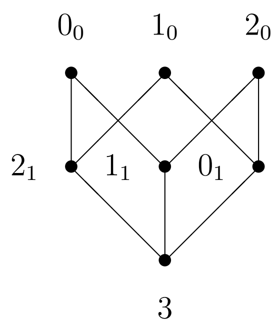
A solution to a problem of Urquhart

In [4], Urquhart gives an example of a modal logic which has the finite model property, is recursively axiomatizable but not decidable. He poses the problem of finding such a logic containing **S4**. We will solve this problem by constructing a logic which in addition to the required properties also defines a locally finite variety of modal algebras. (Recall that a variety is said to be locally finite if all finitely generated algebras in that variety are finite.) This also gives an example of a locally finite variety $\mathcal{V}$ such that $Eq(\mathcal{V})$ is recursively enumerable but not recursive.

The key to the solution is provided in [1]. Let $\mathbf{S4.J}_3$ be the logic of **S4**-frames of depth ≤ 3. Define the frames $b_n = \langle b_n, \lhd \rangle$ by letting

$$\begin{aligned} b_n = &\quad \{i_0 : i \in n\} \cup \{i_1 : i \in n\} \cup \{n\} \\ \lhd = &\quad \{\langle n, s\rangle : s \in b_n\} \cup \{\langle s, s\rangle : s \in b_n\} \cup \{\langle i_1, j_0\rangle : i \neq j\} \end{aligned}$$



The picture shows b_3. The diagram or *frame formula* $D(b_n)$ of b_n is

$$\begin{aligned} D(b_n) = &\quad \Box \bigwedge \langle p_s \to \neg p_t | s \neq t\rangle \\ \wedge &\quad \Box \bigwedge \langle p_s \to \Diamond p_t | s \lhd t\rangle \\ \wedge &\quad \Box \bigwedge \langle p_s \to \neg \Diamond p_t | s \not\lhd t\rangle \\ \wedge &\quad \Box \bigvee \langle p_s | s \in b_n\rangle \\ \wedge &\quad p_n \end{aligned}$$

where s and t range over all points of b_n. For a set $X \subseteq \omega$ we define $\mathbf{S4.J_3/X} = \mathbf{S4.J_3}(\{\neg D(b_n) | n \in X\})$. $\mathbf{S4.J_3/X}$ is the (iterated) splitting of $\mathbf{S4.J_3}$ by all b_n for which $n \in \omega$, to use the terminology and notation of [2]. As is shown in [1], b_n is a frame for $\mathbf{S4.J_3/X}$ iff $n \notin X$. Hence, let X be a recursively enumerable subset of ω which is not recursive. Then $\mathbf{S4.J_3/X}$ is not decidable; for $D(b_n)$ is consistent iff $n \in \omega - X$, the latter being not recursively enumerable. However, by a result of [3], the variety of $\mathbf{S4.J_3}$-algebras is locally finite and so is the variety of $\mathbf{S4.J_3/X}$-algebras. It follows that $\mathbf{S4.J_3/X}$ has the finite model property; it is recursively axiomatizable by construction and contains **S4**.

References:

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