That FINE-Splittings Are Definable

Marcus Kracht *

II. Department of Mathematics

Arnimallee 3

1000 Berlin 33

GERMANY

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Abstract

It is shown that there are finitely many formulas which define the structure of the subframe of maximal points underlying a given finite transitive and refined model. Using this definability result we obtain a constructive proof of a theorem of Fine [85] that all subframe logics have the finite model property.

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1 Preliminaries

If g is a frame and X a set of variables, $\beta : X \longrightarrow 2^g$ is called a **valuation**. Throughout this paper it will be assumed that $X = \{p_1, \ldots, p_\ell\}, \ell \in \omega$. The pair $\langle g, \beta \rangle$ is **refined** for finite g iff $\forall s, t, : s \neq t \Leftrightarrow (\exists P)(\langle g, \beta, s \rangle \models P \ \& \ \langle g, \beta, t \rangle \models \neg P)$. If $\langle g, \beta \rangle$ is not refined there is a map $p_\beta : g \twoheadrightarrow g/\beta$ defined by $p_\beta(s) = p_\beta(t)$ iff $(\forall P)(\langle g, \beta, s \rangle \models P \Leftrightarrow \ \langle g, \beta, t \rangle \models P)$. g/β is called the **refinement** of g with respect to β. β induces a valuation $\gamma : X \longrightarrow 2^{g/\beta}$ by $\gamma(p) = p_\beta[\beta(p)]$. We will write β instead of γ as we will write β for the valuation $\gamma : X \longrightarrow 2^f$ for a subframe $f \subseteq g$ defined by $\gamma(p) = \beta(p) \cap f$. Note that if $\langle g, \beta \rangle$ is refined, this might not be the case for $\langle f, \beta \rangle$. If $\langle g, \beta \rangle$ is a valuation and $q : g \twoheadrightarrow h$ a p-morphism, q is called **admissible** for β iff $(\forall p \in X)(\forall s \in h)(q^{-1}(s) \subseteq \beta(p)$ or $q^{-1}(s) \cap \beta(p) = \emptyset)$. Equivalently, q is admissible for β iff q factors through p_β.

If P is a formula, $dg(P)$ denotes the maximal number of nestings of modal operators, the **degree of P**. We will always assume that $var(P) \subseteq \{p_1, \ldots, p_\ell\}$. We write F^l_n for the set of all such formulas of degree $\leq n$. F^ℓ_n is finite and hence atomic as a boolean algebra with set of atoms A^ℓ_n. $\sharp F^\ell_n = 2^{a(\ell,n)}$, $a(\ell, n) := \sharp A^\ell_n$.

If g is a K4-frame and $s, t \in g$ then t is called a **weak successor** of s, in symbols $s \trianglelefteq t$, if $s \lhd t$ or $s = t$. Call $t \in g$ **maximal** with respect to P in $\langle g, \beta \rangle$ if $(\forall s \in g)(t \lhd s \ \& \ \langle g, \beta, s \rangle \models P \Rightarrow s \lhd t)$. Letting $At(t) \in A^\ell_n$ denote the unique atom with $\langle g, \beta, t \rangle \models At(t)$ we say that t is **n-maximal** or simply **maximal** if t is maximal with respect to $At(t)$. (In Fine [74], s is called *noneliminable* iff s is maximal with respect to some P.) The subframe of all maximal points of $\langle g, \beta \rangle$ is denoted by g^μ. For $t \in g$ we use the symbol t^μ to denote a weak successor of t maximal w.r.t. $At(t)$. Then $t = t^\mu$ iff t is maximal. It is useful to observe that if $s \lhd t^\mu$ and $s \models \Diamond At(t^\mu)$ then there is a weak successor s^μ such that $s^\mu \lhd t^\mu$. We now have the following

Fact 1 *For all $s \in g^\mu, P \in F^\ell_n$: $\langle g, \beta, s\rangle \models P \Leftrightarrow \langle g^\mu, \beta, s\rangle \models P$.*

This is proved by induction on P. The nontrivial case is $P = \Diamond Q$. But observe that if $\langle g, \beta, s\rangle \models \Diamond Q$ then for some $t \rhd s$: $\langle g, \beta, t\rangle \models Q$. Thus, $\langle g, \beta, t^\mu\rangle \models Q$ and, by IH, $\langle g^\mu, \beta, t^\mu\rangle \models Q$ showing $\langle g^\mu, \beta, s\rangle \models \Diamond Q$, since $s \lhd t \unlhd t^\mu$ and so $s \lhd t^\mu$.

This can be stated more generally as follows:

Fact 2 *Let $g^\mu \subseteq h \subseteq g$ and $s \in h$. Then for all $P \in F^\ell_n$: $\langle g, \beta, s\rangle \models P \Leftrightarrow \langle h, \beta, s\rangle \models P$.*

As a consequence, if $t \in h$, then any successor maximal w.r.t. $At(t)$ of t in h is a successor of t maximal w.r.t. $At(t)$ in g, and so t^μ does not depend on h as long as $h \supseteq g^\mu$. If the *depth* of a point t in g, $d^g(t)$, is defined by $d^g(t) = d+1 \Leftrightarrow d^g(t) \not\leq d \& (\forall u)(t \lhd u \rightarrow u \lhd t \vee d^g(u) \leq d)$ and $d^g(t) = 1 \Leftrightarrow (\forall u)(t \lhd u \Rightarrow u \lhd t)$, then with $r := g^\mu/\beta$ and $d^\mu(t) := d^r(t^\mu)$ we have

Fact 3 *For all $t \in g$, $d^\mu(t)$ is the maximum number n such that there is a chain $\langle x_i \mid i \in n\rangle$ with $x_{i+1} \models \neg At(x_i) \wedge \neg\Diamond At(x_i), i \in n-1$. By consequence, $d^\mu(t) \leq a(\ell, n)$.*

This is seen by first noting that if there is a chain $\langle x_i \mid i \in n\rangle$ such that $x_{i+1} \models \neg At(x_i) \wedge \neg\Diamond At(x_i)$, then $x_{i+1} \not\!\lhd\, x_i$, and, starting with x_{n-1}, one can successively replace the x_i by a maximal weak successor x^μ_i so that $x^\mu_i \lhd x^\mu_{i+1} \not\!\lhd\, x^\mu_i$. Reversely, if there is a chain $\langle x_i \mid i \in n\rangle$ of maximal points such that $x_i \lhd x_{i+1} \not\!\lhd\, x_i$ then $x_{i+1} \models \neg At(x_i) \wedge \neg\Diamond At(x_i)$.

So there is a chain of points with $x_{i+1} \models \neg At(x_i) \wedge \neg \Diamond At(x_i)$ iff there is a chain of maximal points of the same length satisfying $x_i \lhd x_{i+1} \not\lhd x_i$ iff $d^\mu(x_0) \geq n$.

For notational purposes we need the frame $f^{<\omega}$ which we get as follows: let f be the frame of ultrafilters of the freely ℓ-generated K4-algebra. Then $f^{<\omega} := \{s \in f \mid d^f(s) < \omega\}$. The canonical valuation $\beta : X \longrightarrow 2^f, p \mapsto \{U \mid p \in U\}$, induces a valuation β on $f^{<\omega}$. $f^{<\omega}$ is a generated subframe of f and for every finite frame g and any valuation γ on g such that $\langle g, \gamma \rangle$ is refined there is a p-morphic embedding $g \rightarrowtail f^{<\omega}$ such that $\gamma(p) = f^{<\omega} \cap \beta(p)$, i.e. $\gamma = \beta$ in our notation.

Let $g = \langle g, \lhd \rangle$ be a frame and $h \subseteq g$. Then $\langle h, \lhd_h \rangle$ is called a **subframe of g** if $\lhd_h = \lhd \cap h^2$. In that case we write $\lhd$ for $\lhd_h$. Say that f **subreduces** to g if there is a subframe $h \subseteq g$ and a surjective p-morphism $h \twoheadrightarrow g$. A logic $\Lambda \supseteq K4$ is a **subframe logic** if its class of frames is closed under the formation of subframes. Furthermore, if $\Lambda \supseteq K4$ is any logic and g a one-generated finite frame, then let Λ_g denote the smallest subframe logic containing Λ such that no Λ_g-frame is subreducible to g, and call Λ_g the **FINE-splitting** of Λ by g. (Actually, this is not quite correct. A strict definition needs a notion of a subframe for a *generalized frame*. Then Λ_g is the smallest extension of Λ such that no subframe of a generalized frame for Λ_g can be mapped p-morphically onto $\langle g, 2^g \rangle$.) It turns out that $\Lambda_g = \Lambda(C_g)$ where $C_g = SF(g) \wedge \Box SF(g). \rightarrow .\neg p_s$ with

$$\begin{aligned} SF(g) = \quad & \bigwedge \langle p_t \rightarrow \neg p_u \mid t \neq u \rangle \\ & \wedge \bigwedge \langle p_t \rightarrow \Diamond p_u \mid t \lhd u \rangle \\ & \wedge \bigwedge \langle p_t \rightarrow \neg \Diamond p_u \mid t \not\lhd u \rangle \end{aligned}$$

Here, t and u range over g and s is a point which generates g. Fine [85] has shown that any subframe logic Λ is a FINE-splitting $K4_G$ with $G = \{g \mid g \notin Fr(\Lambda), g \text{ one-generated}\}$ and that all subframe logics have the finite model property (f.m.p.). His proof consisted in first showing that given any pair $\langle f, \beta\rangle$ one may drop all eliminable points from f to obtain the reduced pair $\langle f^r, \beta\rangle$ and that $\langle f^r, \beta\rangle \nvDash P \Leftrightarrow \langle f, \beta\rangle \nvDash P$. He then proved that in canonical frames based on finitely many sentence letters the property "the reduced subframe does not subreduce to g" is definable by an infinite set of modal formulas; or, equivalently, if the reduced subframe subreduces to g there is a substitution $s : p_t \mapsto Q_t, t \in g$ such that $w \not\models s(C_g)$. Here we prove a result which is different from the original one; namely, we prove that there is a number $\sigma(\ell, n) \in \omega$ and *finitely* many substitutions $s_i, i \in \sigma(\ell, n)$, such that for every finite, ℓ-generated model $\langle f, \beta\rangle$, $\langle f, \beta, w\rangle \not\models s_j(C_g)$ for some $j \in \sigma(\ell, n)$ iff the refinement of the submodel of n-maximal points is subreducible to g.

2 Defining the Maximal Subframe

For $s \in f^{<\omega}$ define $suc^+(s) = \{t \in (f^{<\omega})^\mu \mid s \lhd t \not\lhd s\}$, $cl(s) = \{t \in (f^{<\omega})^\mu \mid s \lhd t \lhd s\}$ as well as $CL(s) = \{At(t) \mid t \in cl(s)\}$. By induction on $d^\mu(s)$ we will now define formulas E_s, L_s. The formulas E_s will encode the structure of the refined submodel of maximal points. The formulas L_s define the layers of that model, that is, the set of all points t in the submodel of maximal points with $d^\mu(t) < d^\mu(s)$. The induction starts with $d^\mu(s) = 0$, where there is nothing to do, except to let $L_0 = \bot$. Now let $d^\mu(s) = d + 1$ with $d \geq 0$:

$$
\begin{aligned}
L_s :=\quad & L_d \\
A_s :=\quad & At(s) \wedge \neg L_s \wedge \Box(L_s \to \neg\Diamond At(s) \wedge \neg At(s)) \\
B_s :=\quad & \Box\neg At(s), \quad \text{if } cl(s) = \emptyset \\
B_s :=\quad & \bigwedge\langle \Diamond A \mid A \in CL(s)\rangle \\
& \wedge \bigwedge\langle \Box(A \to \Diamond B) \mid A, B \in CL(s)\rangle \\
& \wedge \bigwedge\langle \Box(A \to (L_s \vee \Diamond(A \wedge L_s))) \mid A \in A_n^\ell - CL(s)\rangle, \ \text{else} \\
C_s :=\quad & \bigwedge\langle \Diamond E_t \wedge \Box(At(s) \vee \Diamond At(s). \to .\Diamond E_t) \mid t \in suc^+(s)\rangle \\
& \wedge \bigwedge\langle \Box\neg E_t \mid t \notin suc^+(s), d^\mu(t) \leq d\rangle \\
& \wedge \bigwedge\langle \Box(\Box\neg E_t \to (L_s \vee \bigvee\langle A \wedge \Diamond(A \wedge L_s) \mid A \in A_n^\ell\rangle)) \mid t \in suc^+(s)\rangle \\
E_s :=\quad & A_s \wedge B_s \wedge C_s \\
L_{d+1} :=\quad & L_d \vee \bigvee\langle E_u \mid d^\mu(u) = d + 1\rangle
\end{aligned}
$$

Define $SUC^+(s) = \{E_t \mid t \in suc^+(s)\}$. Then if $A \equiv B$ denotes equivalence in K it is clear that $E_s \equiv E_t$ iff $SUC^+(s) = SUC^+(t), CL(s) = CL(t)$ and $At(s) = At(t)$. Define the frame $r = \langle r, \lhd\rangle$ with $r = \{E_s/\equiv \mid s \in f\}$ and $E_s \lhd E_t$ iff either $E_t \in SUC^+(s)$ or $SUC^+(s) = SUC^+(t), CL(s) = CL(t)$ and $At(t) \in CL(s)$. (Henceforth we will not distinguish between E_s and its equivalence class $E_s/\equiv$.) The first lemma shows that the definition of the E_s is *sound* for the maximal points:

Lemma 4 *Let $f^\mu \subseteq g \subseteq f$ and $s \in g^\mu$. Then $s \in f^\mu$ and $\langle g, \beta, s\rangle \models E_s$.*

Proof. If $s \in g^\mu$, then its maximal successor s^μ is in g, since $g \subseteq f^\mu$. Hence, $s = s^\mu$, since s is maximal in g and it follows that s is maximal in f as well. By induction on $d := d^\mu(s)$ we show

(‡) $\langle g, \beta, s\rangle \models E_s$ and if $\langle g, \beta, t\rangle \models L_d$ then $d^\mu(t) \leq d$.

To begin with $d^\mu(s) = 0$, there is nothing to show. Thus let $d^\mu(s) = d + 1$. The proof is broken down into four parts:

(i) $s \models A_s$

This is so because of Fact 2 and $t \models L_s$ implies $t^\mu \models L_s = L_d$ from which $d^\mu(t^\mu) \leq d$. But no maximal successor of s of depth $\leq d$ can satisfy $At(s)$ or $\Diamond At(s)$.

(ii) $s \models B_s$.

The case $CL(s) = \emptyset$ is straightforward. Let therefore $CL(s) \neq \emptyset$. By definition of $CL(s)$ and the fact that $g \supseteq f^\mu$ we get $s \models \Diamond A, \Box(A \to \Diamond B)$ for all $A, B \in CL(s)$. Also $s \models \Box(C. \to .L_s \vee \Diamond(C \wedge L_s))$ for $C \not\in CL(s)$, for if for a successor t: $t \models C$, then $t^\mu \models C$ and by IH and the fact that $d^\mu(t^\mu) \leq d$, $t^\mu \models L_s$. Thus if $t = t^\mu : t \models L_s$ and if $t \lhd t^\mu : t^\mu \models \Diamond(C \wedge L_s)$.

(iii) $s \models C_s$.

$s \models \Diamond E_t \wedge \Box(At(s) \vee \Diamond At(s). \to .\Diamond E_t)$ for all $E_t \in SUC^+(s)$ by the fact that s is $At(s)$-maximal and $s \models \Diamond E_t$ for all $E_t \in SUC^+(s)$. Furthermore, $s \models \neg\Diamond E_t$ for all $t \notin suc^+(s)$ and $d^\mu(t) \leq d$. Finally, suppose for $s \lhd u$ that $u \models \neg\Diamond E_t$ for some $t \in suc^+(s)$. By definition of depth, $d^\mu(u^\mu) \leq d$. Hence, by IH, $u^\mu \models L_s$. If $u = u^\mu : u \models L_s$, if $u \lhd u^\mu$: $u \models A \wedge \Diamond(A \wedge L_s)$ for $A = At(u) \in A_n^\ell$. And so $u \models Z := L_s \vee \bigvee\langle A \wedge \Diamond(A \wedge L_s) \mid A \in A_n^\ell\rangle))$ and so $s \models \Box(\Box\neg E_t \to Z)$ showing (C).

(iv) Now suppose $t \models L_{d+1}$. If also $t \models L_d$ then $d^\mu(t) \leq d$, by IH. Hence let $t \models \neg L_d$. Then $t \models E_u$ for some maximal u with $d^\mu(u) = d+1$ and so $t \models \Diamond E_x$ for some maximal x with $d^\mu(x) = d$. So, $d^\mu(t) > d$. But $t \models B_u$, implying that if $t \lhd v \models \neg At(t) \wedge \Box\neg At(t)$, then $At(v) \notin CL(u)$ and so $v \models L_d \vee \Diamond(At(v) \wedge L_d)$. If $v \models L_d$, $d^\mu(v) \leq d$; but if $v \models \Diamond(At(v) \wedge L_d)$ then $v^\mu \models L_d$ and so $d^\mu(v) = d^\mu(v^\mu) \leq d$ as well. This proves $d^\mu(t) = d + 1$. ◀

Lemma 5 *For all s, t:*

- (a) $E_s \equiv E_t \Leftrightarrow \vdash_{K4} E_s \leftrightarrow E_t$
- (b) $E_s \not\equiv E_t \Leftrightarrow \vdash_{K4} E_s \to \neg E_t$
- (c) $E_s \lhd E_t \Leftrightarrow \vdash_{K4} E_s \to \Diamond E_t$
- (d) $E_s \not\lhd E_t \Leftrightarrow \vdash_{K4} E_s \to \neg\Diamond E_t$

Proof. Since $\nvdash_{K4} \neg E_s$ for all s it is enough to show only ($\Rightarrow$) in each case. (a) is then trivial.

(b) If $SUC^+(s) \neq SUC^+(t)$, for example $E_w \in SUC^+(s) - SUC^+(t)$, then $\vdash_{K4} E_s \to \Diamond E_w, E_t \to \neg\Diamond E_w$, whence $\vdash_{K4} E_s \to \neg E_t$; likewise for $E_w \in SUC^+(t) - SUC^+(s)$.

Let us now suppose $SUC^{+}(s) = SUC^{+}(t)$. If $At(s) \neq At(t)$, the case is clear. Thus, if $At(s) = At(t)$, we must have $CL(s) \neq CL(t)$. Without losing generality we can assume that $A \in CL(s) - CL(t)$. Since $\vdash_{K4} E_s \to \Diamond A, E_s \to \Box(\Diamond At(s) \to \neg L_s)$ (by $\vdash_{K4} A_s \to \Box(At(s) \vee \Diamond At(s). \to .\neg L_s)$ and $\vdash_{K4} E_s \to A_s$) and $\vdash_{K4} E_s \to \Box(A \to \Diamond At(s))$ we get $\vdash_{K4} E_s \to \Diamond(A \wedge \neg L_s)$. But $\vdash_{K4} E_t \to \Box(A \to .L_t \vee \Diamond(A \wedge L_t))$ and since $SUC^{+}(s) = SUC^{+}(t)$ we have $L_s \equiv L_t$ and consequently $\vdash_{K4} E_t \to \Box(A \to .L_s \vee \Diamond(A \wedge L_s))$. If $\vdash_{K4} E_s \to E_t$, then $\vdash_{K4} E_s \to \Diamond(A \wedge \neg L_s) \wedge \Box(A \to .L_s \vee \Diamond(A \wedge L_s)$. But since $\vdash_{K4} E_s \to \Box(A \to \neg L_s)$, we get a contradiction. Thus $\vdash_{K4} E_s \to \neg E_t$.

(c) If $E_t \in SUC^{+}(s)$, the case is trivial. So let us suppose the contrary. Then $SUC^{+}(s) = SUC^{+}(t)$ and $CL(s) = CL(t) \neq \emptyset$. Since E_t is of the form $Q \wedge \Box P$ we succeeded in showing $\vdash_{K4} E_s \to \Diamond E_t$ if only we prove $\vdash_{K4} E_s \to \Diamond Q$. Thus it remains to be shown that

(†) $\vdash_{K4} E_s \to \Diamond(At(t) \wedge \neg L_t \wedge \bigwedge\langle \Diamond B \mid B \in CL(t)\rangle \wedge \bigwedge\langle \Diamond E_u \mid u \in suc^{+}(t)\rangle)$

(i) $\vdash_{K4} E_s \to \Diamond At(t)$.

(ii) $\vdash_{K4} E_s \to \Box(At(t) \to \neg L_t)$ follows from $\vdash_{K4} E_s \to \Box(At(t) \to \Diamond At(s))$, $\vdash_{K4} E_s \to \Box(\Diamond At(s) \to \neg L_s)$ and $L_s \equiv L_t$.

(iii) $\vdash_{K4} E_s \to \Box(At(t) \to \Diamond B)$ for all $B \in CL(t)$, since $CL(s) = CL(t)$.

(iv) $\vdash_{K4} E_s \to \Box(At(t) \to \Diamond E_u)$ for all $E_u \in SUC^{+}(t)$ since $\vdash_{K4} E_s \to \Box(At(t) \to \Diamond At(s))$ and $\vdash_{K4} E_s \to \Box(\Diamond At(s) \to \Diamond E_u)$. Taking (i) together with (ii), (iii) and (iv) yields (†).

(d) Case 1: If $d^\mu(t) < d^\mu(s)$, then $\neg\Diamond E_t$ is a conjunct of E_s.

Case 2: $d^\mu(s) = d^\mu(t)$. Then $L_s \equiv L_t$. Suppose $E_u \in SUC^+(s) - SUC^+(t)$ for some E_u. If $\vdash_{K4} E_s \to \Diamond E_t$ then since we have $\vdash_{K4} E_t \to .(\neg\Diamond E_u) \wedge \neg L_s \wedge \bigwedge\langle A \to \neg\Diamond(A \wedge L_s) \mid A \in A^l_n\rangle$, we get $\vdash_{K4} E_s \to \Diamond(\Box\neg E_u.\wedge.\neg L_s \wedge \bigwedge\langle A \to \Diamond(A \wedge L_s) \mid A \in A^l_n\rangle$, a contradiction to $\vdash_{K4} E_s \to C_s$. Now suppose $E_u \in SUC^+(t) - SUC^+(s)$. Then $\vdash_{K4} E_s \to \Diamond E_t$ yields $\vdash_{K4} E_s \to \Diamond E_u$ in contradiction to $\vdash_{K4} E_s \to \Box\neg E_u$. Thus the case $SUC^+(s) = SUC^+(t)$ is left. Then we must have $CL(s) \neq CL(t)$ or $CL(s) = CL(t) = \emptyset$. The latter case is dealt with as follows. $\vdash_{K4} E_s \to \Diamond E_t$ implies $\vdash_{K4} E_s \to \Diamond At(t)$; and since $E_s \vdash_{K4} \Box(At(t) \to L_s \vee \Diamond(At(t) \wedge L_s)$ (for $At(t) \notin CL(s)$) we have $\vdash_{K4} E_s \to \Diamond(E_t \wedge (L_t \vee \Diamond(At(t) \wedge L_t))$ in contradiction to $E_t \vdash_{K4} \neg L_t \wedge \Box(L_t \to \neg At(t))$. Thus $CL(s) \neq CL(t)$. Now let $B \in CL(t) - CL(s)$. Then $\vdash_{K4} E_s \to \Box(B. \to .L_s \vee \Diamond(B \wedge L_s))$ and $\vdash_{K4} E_t \to \Box(B \to \neg L_s)$ and if $\vdash_{K4} E_s \to \Diamond E_t$ we get $\vdash_{K4} E_s \to \Diamond(B \wedge L_s) \wedge \Box(B \to \neg L_s)$, again a contradiction. Assume finally $C \in CL(s) - CL(t)$. Then $\vdash_{K4} E_s \to \Box(At(t) \to \Diamond C)$ and $\vdash_{K4} E_t \to \Box(C \to .L_s \vee \Diamond(C \wedge L_s)))$ we arrive at a contradiction with $\vdash_{K4} E_s \to \Diamond E_t$ since $\vdash_{K4} E_s \to \Box(C \to \neg L_s)$.

Case 3: $d^\mu(t) > d^\mu(s)$. If there is a x with $d^\mu(x) = d^\mu(s)$ and $E_x \not\lhd E_s \not\lhd E_x$, then $\vdash_{K4} E_s \to \Diamond E_t$ would imply $\vdash_{K4} E_s \to \Diamond E_x$, which is contradiction because of Case 2. But in the other case $E_t \lhd E_s$ and since $\vdash_{K4} E_t \to \Box(L_t \to \Box\neg At(t))$ and $\vdash_{K4} E_s \to L_t$ we get $\vdash_{K4} E_t \to \Box(E_s \to \Box\neg At(t))$ showing $\vdash_{K4} E_s \to \neg\Diamond E_t$. ◀

Since $d^\mu(t) \leq a(\ell, n)$ we immediately have the

Fact 6 $\sharp r \leq e(\ell, n) := a(\ell, n + 2a(\ell, n))$.

This is so because $dg(E_s) = n + 2d^\mu(s)$ and $d^\mu(s) \leq a(\ell, n)$ and the fact that the E_s are incompatible. Another consequence of Lemma 5 is

Proposition 7 *Let g be a finite frame and f^μ be subreducible to g, that is, $f^\mu \supseteq h \overset{p}{\twoheadrightarrow} g$, and let $Q_t := \bigvee \langle E_w \mid w \in h, p(w) = t \rangle$, for $t \in g$. Then*

$\vdash_{K4} SF(g)[Q_t/p_t]$,

where $A[Q_t/p_t]$ is the result of replacing Q_t for all occurrences of p_t **for all** *t in A.*

Proof. It is easily seen that $\vdash_{K4} Q_v \to \neg Q_w$ if $v \neq w$, $\vdash_{K4} Q_v \to \Diamond Q_w$ if $v \lhd w$ and $\vdash_{K4} Q_v \to \neg\Diamond Q_w$ if $v \ntriangleleft w$. ◀

Proposition 8 *The map $\rho : f^\mu \twoheadrightarrow r$ given by $\rho : t \mapsto E_t$ is a p-morphism admissible for β. In addition, $\langle r, \beta \rangle$ is refined.*

Proof. Let $s \lhd t$. Then either $t \lhd s$ or $t \ntriangleleft s$. $t \ntriangleleft s$ implies $E_s \lhd E_t$ by definition, since $\Diamond E_t$ is a conjunct of E_s; if $t \lhd s$ then we have $SUC^+(s) = SUC^+(t), CL(s) = CL(t)$ and $At(t) \in CL(s)$. Thus $E_s \lhd E_t$ as in the proof for Lemma 5(c). Hence $s \lhd t$ implies $E_s \lhd E_t$. Furthermore, if $\rho(s) \lhd E_t$ then since $\langle f^\mu, \beta, s \rangle \models E_s$ and $\vdash_{K4} E_s \to \Diamond E_t$, $\langle f^\mu, \beta, x \rangle \models E_t$ for some $s \lhd x$. Then $E_t \equiv E_x$, that is, $\rho(x) = E_t$. This shows that ρ is a p-morphism.

ρ is clearly admissible and since $\langle f^\mu, \beta, x \rangle \models E_s$, we have $\langle r, \beta, \rho(x) \rangle \models E_s$ but $\langle r, \beta, \rho(x) \rangle \models \neg E_t$ for $E_s \neq E_t$ and consequently $\langle r, \beta \rangle$ is refined. ◀

We have now constructed formulas E_s which completely describe the structure of the refined submodel of maximal points of any given finite model. Using the fact that refined models are *top-heavy* (see Fine [85]) it might be possible to show that these formulas describe the structure of the refined submodel of maximal points of any refined model. But we have not attempted a proof so far. In any case, the established result suffices to prove the subframe theorem in a new way using a technique first applied in Kracht [90] to ordinary splittings. There it was shown that certain frames g preserve f.m.p. beyond K4 in the following sense: if $\Lambda \supseteq K4$ has f.m.p., so does Λ/g, the splitting of Λ by g. Here, we will show that all finite, one-generated frames preserve f.m.p. for FINE-splittings beyond $K4$ *on the condition that* Λ *is a subframe logic.*

3 The Subframe Theorem

Theorem 9 *Let $\Lambda \supseteq K4$ be a subframe logic, g a finite, one-generated frame. If Λ has f.m.p., Λ_g has f.m.p. as well. Moreover, if P is Λ_g-consistent then it has a model of size $\leq e(\sharp var(P), dg(P))$.*

Proof. Suppose P is Λ_g-consistent. Then define

$$P^\sharp := \{C_g \wedge \Box C_g[E_{S(x)}/p_x] \mid S : g \longrightarrow 2^r\},\ E_{S(x)} := \bigvee \langle E_w \mid w \in S(x)\rangle.$$

Then $P; P^\sharp$ is Λ_g-consistent and a fortiori Λ-consistent and thus $\langle z, \zeta, w\rangle \models P; P^\sharp$ for some Λ-frame z. Now we can assume that z is a generated subframe of $f^{<\omega}$ and $\zeta = \beta$ and therefore $\langle f^{<\omega}, \beta, w\rangle \models P; P^\sharp$. Moreover we can have $w = w^\mu$. Let r_w denote the subframe of r generated by E_w. Since $\langle (f^{<\omega})^\mu, \beta, w\rangle \models P$ we also have $\langle r_w, \beta, E_w\rangle \models P$, by Proposition 8. Now suppose $r_w \supseteq h \overset{p}{\twoheadrightarrow} g$. Then let $S : t \mapsto p^{-1}(t)$. Now if $v \in p^{-1}(s)$ with s generating g then $\langle (f^{<\omega})^\mu, \beta, v\rangle \models \neg C_g[E_{S(t)}/p_t]$ since $\langle (f^{<\omega})^\mu, \beta, v\rangle \models E_{S(s)}$ and $\langle (f^{<\omega})^\mu, \beta\rangle \models SF(g)[E_{S(x)}/p_x]$, by Proposition 7. Thus $v \notin z$, that is, $w \ntriangleleft v$. Consequently, r_w is not subreducible to g. $\langle r_w, \beta, w\rangle$ is a Λ_g-model for P and $\sharp r_w \leq e(\ell, n)$ by Fact 6. ◀

By induction one can now show that all FINE-splittings $K4_G$ for finite G have f.m.p. But this is all we need to show the full subframe theorem:

Theorem 10 *All FINE-splittings $K4_G$ of $K4$ have f.m.p. and if P is $K4_G$-consistent it has a model of size $\leq e(\sharp var(P), dg(P))$.*

Proof. Let P be $K4_G$-consistent. Define $G^P := \{g \in G \mid \sharp g \leq e(\ell, n)\}, \ell = \sharp var(P), n = dg(P)$. Then P is $K4_{G^P}$-consistent and has a finite model of size $\leq e(\ell, n)$ by the preceding theorem. But this already is a $K4_G$-model. ◀

The number $e(\ell, n)$ can be computed recursively via $e(\ell, n) = a(\ell, n+2a(\ell, n))$ and $a(\ell, 0) = 2^{2^\ell}$ and $a(\ell, k+1) = 2^{2^\ell} \times 2^{a(\ell,k)}$. It is tempting to conclude that since we can give an a priori bound to the size of a model therefore all subframe logics are decidable. Such a conclusion is not immediate as is shown in Urquhart [81]. On the other hand, the proof given in Fine [85] that there are $2^{\aleph_0}$ subframe logics is also incorrect. It is therefore still an open question whether all subframe logics are decidable. We believe that the answer is positive but have found no means of proving it. We will conclude this paper with some partial solutions to this question. Following Kruskal [60] we say that a relation $\preccurlyeq$ is a **well-partial order (wpo)** if it is a partial order without infinite, strictly descending chains such that every set N of mutually incomparable elements is finite. On the set of finite rooted and transitive Kripke-frames we define $\preccurlyeq$ by $f \preccurlyeq g \Leftrightarrow g$ subreduces to f. Clearly, $\preccurlyeq$ is a partial order without infinite strictly descending chains. If we can prove that $\preccurlyeq$ is a wpo we can show that every FINE-splitting Λ_M can be finitely axiomatized over Λ. It follows that all subframe logics are finitely axiomatizable and thus decidable. For if Λ_M is a FINE-splitting of Λ then letting $N \subseteq M$ to be the set of $\preccurlyeq$-minimal elements of M it is easily seen that $\Lambda_M = \Lambda_N$. Moreover, all frames of N are mutually incomparable. Therefore, N is finite and Λ_M finitely axiomatizable over Λ. Furthermore, if all subframe logics are decidable then $\preccurlyeq$ is a well partial-order. For if not, there is an infinite set N of mutually incomparable frames and therefore $2^{\aleph_0}$ subframe logics.

Theorem 11 *The following are equivalent*

(i) $\preccurlyeq$ *is a well partial order.*

(ii) *All subframe logics are finitely axiomatizable.*

(iii) *All subframe logics are decidable.* ◄

We can show for a restricted class of frames that $\preccurlyeq$ is wpo. With every transitive frame g we can associate a partial order $g^\flat = \langle g^\flat, \leq \rangle$ by letting $g^\flat$ to be the set of clusters of g and $C \leq D$ iff $C = D$ or $(\forall s \in C)(\forall t \in D)(s \lhd t)$. Next we define an indexing function $\iota : g^\flat \to \omega$ by letting $\iota(C) = \sharp\{s|(\exists t \in C)(s \lhd t \lhd s)\}$. Finally, define a partial order $\preccurlyeq$ on the natural numbers by $m \preccurlyeq n \Leftrightarrow m = n = 0$ or $1 \leq m \leq n$ and define $\tau(g) = \langle g^\flat, \iota \rangle$. It is not hard to see that $\epsilon : g \rightarrowtail h$ iff $\tau(g)$ can be embedded in $\tau(h)$ as a partial-order-over-$\langle \omega, \preccurlyeq \rangle$, that is, such that $\iota(C) \leq \iota(\epsilon(C))$. Say that g is a **quasi-tree** if $g^\flat$ is a tree. Denote the set of finite quasi-trees by $\mathcal{Q}$. Qausi-trees are equivalent to trees-over-$\langle \omega, \preccurlyeq \rangle$ in the sense of Kruskal [60]. Now, by the famous result of Kruskal obtained in Kruskal [60], since $\langle \omega, \preccurlyeq \rangle$ is wpo, so is $\langle \mathcal{Q}, \subseteq \rangle$, the space of trees-over-$\langle \omega, \preccurlyeq \rangle$. Therefore any subframe logics axiomatized by a set of finite quasi-trees is finitely axiomatizable. Since linear frames are quasi-trees, we get an interesting corollary, first noted in Fine [71]. If $\Lambda = S4.3/g$ is a splitting of $S4.3$ by g then $\Lambda = S4.3_g$ – and is therefore a subframe logic, since $g \subseteq h$ iff there is a p-morphism $p : h \twoheadrightarrow g$ onto g. So, for any $\Lambda \supseteq S4.3$ and any rooted frame $\Lambda/g = \Lambda_g$. Therefore, all splittings $S4.3/N$ of $S4.3$ by a set N of finite frames have f.m.p. since they are FINE-splittings. Now if $\Lambda \supseteq S4.3$, let $\Lambda_o = S4.3/N$ with $N = \{g \mid g \notin Fr(\Lambda), g \text{ one-generated and finite}\}$. Then since $\Lambda_o \subseteq \Lambda$ and Λ_o has f.m.p. and shares all the finite models with Λ, $\Lambda = \Lambda_o$. Thus, together with Theorem 10 we have the

Theorem 12 *All logics containing $S4.3$ have f.m.p., are finitely axiomatizable and decidable.*

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