GROUPS AND SHEAVES

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Abstract

A group is formed by using the word `and`, as in `John and Mary`. Likewise, a sheaf is formed using the word `or`, as in `John and Mary`. This paper has two goals: to establish arguments in favour of sheaves, and, secondly, to determine the algebraic structure formed by the universe under the operations corresponding to (group forming) `and` and (sheaf forming) `or`.

1 Introduction

The traditional analysis of sentences containing a quantified phrase, for example a quantified subject NP, is to tear apart the quantifier from the NP and use the NP as denoting a restrictor set for the quantification.

(1) `Every student likes Ronald Reagan.`
(2) `No car was sold yesterday.`

The logical form of these sentences is something like the following.

(3) $(\forall x)(\mathsf{student}'(x) \to \mathsf{likes}'(x, \mathsf{r}', \mathsf{now}'))$
(4) $\neg(\exists xy)(\mathsf{car}'(x) \land \mathsf{sell}'(y, x, t) \land t \subseteq \mathsf{yesterday}')$

Montague followed this line of analysis. Moreover, in order to keep the denotational types harmonious, he likened the type of individual constants to that of quantified NPs. `Ronald Reagan`, under this view, would no longer be translated by the constant r' but by the property of properties of individuals $\lambda\mathcal{P}.\mathcal{P}(\mathsf{r}')$. This view has obvious advantages over a view, which we shall endorse here, that `Ronald Reagan` simply denotes the constant r'. For, as Montague pointed out, there is no easy way to interpret (1) unless we give the expression `Ronald Reagan` the same type as that of a quantified NP. Otherwise the type harmony would be broken. Such a conclusion is too hasty, though. For it is likewise possible to turn quantified NPs into something of an object that can be fed as an argument to the verb. Suppose we let `every` denote a function from properties of individuals to sets of objects, namely the function $\lambda\mathcal{P}.\{x : \mathcal{P}(x)\}$. Suppose further that `likes` is true of a set if and only if it is true of every member, then the translation of (1) is now (5), which by stipulation (6) (or, if you wish, meaning postulate (6)) is equivalent to (3).

(5) $\mathsf{likes}'(\{x : \mathsf{student}'(x)\}, \mathsf{r}', \mathsf{now}')$
(6) $(\forall Mxt)(\mathsf{likes}'(M, y, t) \leftrightarrow (\forall y)\mathsf{likes}'(y, x, t))$

The investigation into plurals has brought to light the necessity of introducing something akin to sets. (7) is not the same as (8) since it favours the reading where John marries Mary, which is exactly disfavoured by (8). Moreover, no reduction is even possible for (9), as the unacceptability of (10) shows.

(7) `John and Mary got married.`
(8) `John got married and Mary got married.`
(9) `John and Mary met.`
(10) `*John met and Mary met.`

(9) and (10) offer clear semantic evidence for the existence of groups as opposed to individuals. Given that it is fine to say `Everybody met.` it is suggestive to let `everybody` denote the set of all people.

It is by now perhaps accepted that groups are needed in semantics. Now look at examples with `or`.

(11) `John or Mary got married.`

Unless we return to the Montagovian analysis of individuals as properties of properties of individuals, we shall have to say that `or` can take two individuals and produce what we call a *sheaf*. This is where the present paper takes its beginning. Before we introduce sheaves, however, we take a look once more at groups.

2 Groups

Consider the sentence

(12) `John and Mary met.`

The subject of this sentence is a **group**, the group consisting of John and Mary. There are two concurrent views on what groups are. Link in (Link 1983) takes them to be objects in the free join semilattice generated by the individuals. Let S be a set. Take a binary operation $\oplus$ on S that satisfies the following equations for all $x, y, z \in S$.

$$\begin{aligned} x \oplus (y \oplus z) &= (x \oplus y) \oplus z \\ x \oplus y &= y \oplus x \\ x \oplus x &= x \end{aligned}$$

Then $\langle S, \oplus \rangle$ is called a *semilattice*. Of particular interest are the semilattices freely generated by a certain set E. The free semilattice generated by a finite set E is isomorphic to $\langle \wp(E), \cup \rangle$, so that it can be easily equipped with a meet and a complement operation. For infinite sets we just have to add all complements and we get the finite and cofinite sets, which again form a boolean algebra. Thus, the theory of Link is completely compatible with boolean semantics à la Keenan and Faltz (1985).

However, this view meets certain problems. ((13) is due to Hoeksema (1983).)

(13) `Blücher and Wellington and Napoleon fought against each other.`
(14) `John and Mary and Frank and Susan got married.`
(15) `The students solved the problem groupwise.`

In all of these cases we find that the subject is actually a group of groups. Under the theory by Link, this is the same as a group. Ojeda (1998) studied the plural of Papago, in which we find a kind of plural that suggests that we are dealing with a group of groups. Ojeda analyses this in (Ojeda 1998) in the spirit of Link, using an equivalence relation in order to do away with the need of talking about groups of groups.

Landman has argued in (Landman 1989) that groups are better seen as sets. Thus, we think of the phrase `A and B` or the phrase `A, B, C and D` as forming the set consisting of (the denotation of) `A` and `B` in the first case and `A`, `B`, `C` and `D` in the second. Let us introduce an *n*–ary operation $\circledR_n$ that takes n arguments and forms the set consisting of them. In fact, as is customary in mathematics (see the distinction between $\wedge$ and $\bigwedge$) we may simply postulate a polyadic operation $\circledR$, which takes any finite (or even infinite?) number of arguments. In what is to follow, however, we shall focus on $n = 2$, that is, the binary operation $\circledR_2$, which we nevertheless write $\circledR$. Associativity fails, idempotence fails, but the operation is still commutative: $\circledR(x,y) = \circledR(y,x)$ for all x,y. Given a set E of basic objects, the structure that these operationts create is contained in $\bigcup_{n\in\omega}\wp^n(E)$. (If E is finite, the two are equal.) It is this domain, which will now take over the role of basic objects for what is to come. We call it U, the **universe**.

3 Sheaves

Consider now the sentence

(16) `John or Claver has read the book.`

Proceeding as before we want something to be the subject of (16). For want of a better name, we call it a **sheaf**. We propose that there is a polyadic operation $\vee$ whose binary counterpart $\vee_2$ has exactly the properties of $\oplus$. The reason for this is that sheaves — as opposed to groups — do not allow for meaningful distinction between sheaves of sheaves and sheaves simpliciter. The easiest interpretation of a sheaf is that it is once again a set and $\vee$ the operation of set union. (Notice a slight quirk with the singletons. `John` is not the same as `John or John`, the latter being a sheaf. This can be remedied. I thank Harald Stamm for bringing this to my attention.) Actually, we shall reify sheaves over a set X simply as subsets of X, although this potentially confuses them with groups. Thus we shall call them sheaves nevertheless.

We need to connect the truth of (16) with that of (17) and (18).

(17) `John has read the book.`
(18) `Claver has read the book.`

Viewing the predicate as a function from U to $2 = \{0,1\}$, there is an easy way to implement this. Take a function $f : X \to Y$. Then the function $f^\circ : \wp(X) \to \wp(Y) : S \mapsto f[S] = \{f(x) : x \in S\}$ is the **sheafification** of f. (Sheafification is a covariant functor from Set to Set.) Thus, from the function $\mathsf{has\text{-}read\text{-}the\text{-}book}'$ we can form the function $\mathsf{has\text{-}read\text{-}the\text{-}book}'^\circ$, which sends sheaves of objects to sheaves of truth values. There are four sheaves of truth values. Then (16) is true just in case that one member of the sheaf satisfies it, that is to say, if its sheaf of truth values contains 1. So, define the map $\exists : \wp(2) \to 2$ by $\exists(A) = 1$ iff $1 \in A$.

$$
\begin{tikzcd}
 & \{0,1\} \arrow[dl, dash] \arrow[dr, dash] & & & & 1 \arrow[d, dash] \\
\{0\} \arrow[dr, dash] & & \{1\} \arrow[dl, dash] & {} \arrow[r, "\exists"] & {} & 0 \\
 & \varnothing & & & &
\end{tikzcd}
$$

This is a homomorphism. (16) is thus translated by

$$\exists\mathsf{has\text{-}read\text{-}the\text{-}book}'^\circ(j \vee c)$$

The dual map, $\forall$, is $\forall(A) = 0$ iff $0 \in A$. It corresponds to the distributive reading of a sentence with groups à la Link. We shall return to this below.

4 Sheafification

We may consider a sheaf as something that represents 'uncertainty', or 'lack of knowledge'. (16) is a case in point. Now, it is actually possible to embed `or` in `and` as in (19).

(19) `John and Mary or Susan got married.`
(20) `John got married with Mary of Susan.`

(19) in the meaning of (20) (which is preferrable to (19) because it is less convoluted) seems to express the idea that there was a group consisting of John and the sheaf with Mary and Susan. However, there is no semantical difference with that 'group' and the sheaf consisting of the group of John and Mary and the group of John and Susan. So, sheaf formation commutes with conjunction, in both arguments. Thus, we claim, there is no group of sheaves, there are only sheaves of groups. Algebraically, this comes down to the validity of the following equations (spelled out in the binary case).

$$\begin{array}{rcl} Ⓡ_2(x,y\vee z) & = & Ⓡ_2(x,y)\vee Ⓡ_2(x,z) \\ Ⓡ_2(y\vee z,x) & = & Ⓡ_2(y,x)\vee Ⓡ_2(z,x) \end{array}$$

In general, functions from X to Y can also be seen as sheaves. This will take care of verbs, for example. The basic equation is as follows ('pointwise sheafification').

$$(f\vee g)(x) = f(x)\vee g(x)$$

This accounts for the equivalence between (21) and (22).

(21) `Claver walks or talks.`
(22) `Claver walks or Claver talks.`

Using these laws, we can finally produce structures for them to show that the postulates are consistent and imply no new conditions on group formation. Suppose that $\langle U, Ⓡ\rangle$ is a structure for groups (which is to say that Ⓡ is a commutative polyadic operation, nothing more). Then the following is a structure for sheaves and groups together: $\langle \wp(U), Ⓡ^\circ, \bigcup\rangle$. Here,

$$g\,Ⓡ^\circ\,h := \{x\,Ⓡ\,y : x\in g, y\in h\}$$

So, every object is standardly sheafified, and the operation of forming groups is obligatorily distributed over the members of the sheaf. The postulates above are readily verified to hold in this structure.

5 Basic Sheaves

So far we have considered sheaves that were explicitly constructed from basic objects that were not sheaves themselves. However, we claim that many words denote objects that are based on sheaves. For example, we claim that common nouns denote sheaves. The noun `table` denotes the sheaf of all (individual) objects that are a table. This needs argumentation. (Notice that it automatically makes proper nouns distinct from intransitive verbs, a welcome consequence.) First, notice the problem with groups and plurality. The analysis of plurals by Link is essentially a nominalistic one: groups are conglomerates formed from the individuals. (Conglomerates differ from sets in that conglomerates of conglomerates are nothing but conglomerates. There is no hierarchy of conglomerates as there is for sets. Evidently, conglomerates form a semilattice under union just as sets. Notice that the interpretation of groups as conglomerates is ours, the semilattice interpretation is Link's. However, we cannot make sense of his theory other than assuming that it presupposes a decidedly nominalistic stance on groups.) Taking individuals to be singletons, group formation is technically reduced to set union. In doing so, groups of groups are collapsed into groups. Link thinks this is an advantage, while we think for reasons given above that it is not. Let us grant his point, however. Now, for the denotation of `tables` Link assumes that it is the set (sic!) of all conglomerates of tables, which is needed to make it technically nondistinct from the denotation of `table` simpliciter — which in his view is the conglomerate of all tables. This distinguishes his theory from that of Scha (1981), which makes the denotation of `table` and `tables` come out the same. But this cannot be. Notice for example that `table` triggers singular agreement, while `tables` does not. The argument is based on the observation that number agreement is basically semantically triggered. (See (Landman (2000), following Hoeksema (1983). Evidently, a purely morphological account could be given, saving Scha from this predicament.) Our solution is to assume the converse: groups are sets, while the conglomerates are sheaves. However, in order not to complicate the structure of the universe, we will not assume conglomerates, only sets. This, however, is a technical matter. Important is that sheaves are different from groups not only in semantic respects. We can, for example, formulate the following (semantic) agreement rule.

> An NP triggers singular agreement if it consists entirely of individuals. Otherwise it triggers plural agreement.

For example, the sheaf denoted by `table` consists of all individual tables. So, it triggers singular agreement. Likewise for `John or Mary`. On the other hand, `tables` consists of non–individuals, and so does `John or the children` and therefore it triggers plural agreement. If one subscribes to a morphological theory of plural agreement (for example, to save the theory of Scha) these facts are less elegantly explained. So, we conclude, with Link, against Scha, that `tables` denotes a different object as does `tables`, and moreover, that `table` is distinct from the group of tables. The difference with Link is then twofold: (a) groups are sets (though that is not needed to explain the facts adduced so far), (b) `table` denotes a sheaf, which is an object in its own right.

Another argument comes from the fact that the indefinite determiner is usually very weak and often missing in languages (especially with plural NPs). Bare NPs are typically understood existentially (or else generically). The sheafified denotation of verbs however does take sheaves of objects as arguments, so the indefinite is actually not necessary. It may be checked that this gives the right results with respect to group formation. In a language without determiners (but with number marking) we would say

(23) `Table and chair are in this room.`

to say that there is a table and a chair in this room. Given the rules established above, the subject is the sheaf of groups consisting of a table and a chair each. (23) is true just in case there is such a group in the room. No group denotation of common nouns would establish this for (23). Namely, if the denotation of `table` was the group of tables and the denotation of `chair` the group of chairs, then (23) would say that all the tables and all the chairs are in the room.

Still, it might be objected that this is no reason to suppose that `chair` denotes the sheaf of chairs rather than the group of chairs. For we could say that rather than the indefinite determiner being the identity on sheaves it is simply a function that takes a group and returns the sheaf of its members. However, this view would have problems explaining the contrast between (24) and (25). (I find the German counterpart of (24) not entirely straightforward, it has a taste of a ‘correction’, but the contrast is nevertheless there.)

(24) `A table or chair was in this room.`
(25) `*A table and chair was in this room.`

If the input to the denotation of `a(n)` was a group and not a sheaf, then it is not clear why (25) is ungrammatical. (24) on the other hand *is* acceptable, showing that sheaves can be the input to the function. Thus, we conclude that common nouns denote sheaves of singletons, and `a(n)` denotes a function which takes as input sheaves of single objects and is the identity on them.

Following this, adjectives denote functions from sheaves of objects to sheaves of objects. Plural denotes a function that takes a sheaf of objects into a sheaf over groups of objects. And so on. Some adjectives do not commute with sheaf formation. That is to say, they denote functions over sheaves which are not formed via sheafification. A case in point is `average`. The average X–or–Y is not the same as the average X or the average Y. We give an example. Suppose that the language department teaches two languages, English and French. A language student is either a student of English or a student of French. Suppose that the students of French drink exactly one glass of wine every day, while the students of English are tea totallers. Now look at the following sentences.

(26) `The average student of French drinks one glass of wine per day.`
(27) `The average student of English drinks no glass of wine per day.`
(28) `The average language student drinks` x `glasses of wine per day.`

((27) is pragmatically somewhat odd, but never mind.) The number x is directly propertional to the quotient between the number of students of French and the overall number of students — so, it is not predictable on the basis of the sentences (26) and (27) alone.

6 Quantification and Negation

The four cardinal quantifiers are privileged by the sheaf–semantics: `some`, `every`, `no` and `not every`. They can be interpreted as follows (with ν the denotation of `N`, f the denotation of `V`).

Some N Vs.	No N Vs.	Every N Vs.	Not every N Vs.
$\exists f^{\circ}(\nu)$	$\neg\exists f^{\circ}(\nu)$	$\forall f^{\circ}(\nu)$	$\neg\forall f^{\circ}(\nu)$

The negation sign corresponds with the monotonicity/antimonotonicity with respect to the denotation of `V`, the choice between $\exists$ and $\forall$ with that of `N`. All other quantifiers require a different treatment. First notice that all others invariably take a plural NP. Thus they operate on (sheaves of) groups. Take as an example `most students` (here we look at the unary quantifier). We start with the sheaf of individual students. From it, `most` forms the sheaf of groups that contain most of the students. The reading is then formed regularly. Similarly for `all`. Polyadic quantifiers are treated analogously: first the product sheaf is defined from the basic denotations, and then the quantifier returns a proper sheaf of (tuples of) groups which is fed to the function.

One important question remains to be answered: why did we not introduce some operator to take care of negation? Why is there nothing that corresponds to `not`? There are two reasons. Technically, it is not immediate how we could introduce negation into the structures proposed here. That in itself, however, is not a reason if that would indeed turn out to be necessary. However, that is not the case. Consider the following sentences:

(29) `*Mary or not Susan got married.`
(30) `John and not Bill entered the room.`

The ungrammaticality of (29) shows that we cannot simply combine `or` with `not`. That already points in the right direction. However, (30) shows that `and` and `not` *can* actually be combined.

Fortunately, (30) is not a good counterexample. First, notice that `and` here logical and not group forming. The sentence does not mean that there were two individuals enetering, it says that there was one, and that it was John and not Bill. Second, (30) is only felicitous in certain contexts, namely those in which it has been established that someone entered the room. Evidence for this is the fact that we may not put that part of the sentence into focus.

(31) `*John and not Bill ENTERED the room.`
(32) `*John and not Bill entered the ROOM.`

It seems therefore that negation does not readily fit into the picture. A different story must be told with regard to `except`, but we shall not discuss that here.

7 Conclusion

The present investigation followed a simple logic: if we want something to be the subject of a disjunctive sentence, we must create an object analogous to a group, called *sheaf*. It turns out that the so defined entity behaves essentially like a group in the sense of Link. However, the objects of a sheaf are thought of as being in (nonexclusive) competition. Moreover, common nouns may be taken to denote sheaves, adjectives to denote functions from sheaves to sheaves and so on.

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