On Extensions of Intermediate Logics by Strong Negation

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Abstract

In this paper we will study the properties of the least extension $n(\Lambda)$ of a given intermediate logic Λ by a strong negation. It is shown that the mapping from Λ to $n(\Lambda)$ is a homomorphism of complete lattices, preserving and reflecting finite model property, frame–completeness, interpolation and decidability. A general characterization of those constructive logics is given which are of the form $n(\Lambda)$. This summarizes results that can be found already in [13, 14] and [4]. Furthermore, we determine the structure of the lattice of extensions of $n(\boldsymbol{LC})$.

Introduction

Constructive logic is an extension of intuitionistic logic by another connective, the **strong negation**. [1] Basically, this additional connective is motivated by the fact that we can not only *verify* a simple proposition such as *This door is locked.* by direct inspection, but also *falsify* it. An intuitionist is forced to say that the falsity of this sentence is seen only indirectly, namely by seeing that it is impossible for this sentence be true. While accepting that there is a way to deny a sentence weakly — in the sense of the intuitionist —, for a constructivist there also is a way to deny a sentence strongly — namely by stating its falsity. If one assumes that the direct access to the falsity of a sentence is limited to basic propositions, one must assume as well that the strong negation of a complex sentence can be verified by an intuitionist if he was only informed about the truth or falsity of the simple propositions. This leads to Nelsons constructive logic.

[1] It has been brought to my attention by the referee that in the context of logic the term *constructive* is used for logics which have the disjunction property. This might be a reason to avoid the use of the term *constructive* here since we are talking of extensions of Nelson–logic. And in them the disjunction property might fail. It is for this reason that a different term is used below, except for the introduction, where we allow ourselves some relaxed use of terminology.

Constructive logic has attracted attention in logic programming recently, see [9, 10] and [5]. Its main advantage is in admitting the possibility of a direct statement of the falsity of a proposition while retaining the 'negation as failure', which is so characteristic in logic programming. The present essay has been sparked off by a question raised by David Pearce concerning the so–called answer set semantics of Gelfond and Lifschitz, see [2, 3]. In [8], Pearce shows that the constructive logic of the two–element chain is a deductive base for the nonmonotonic logic derived from stable models à la Gelfond & Lifschitz.

While the knowledge about intermediate logics is immense, little is commonly known about constructive logics ([12] puts it as an exercise to clarify the structure of the lattice of extensions of Nelsons's Logic). The main results in this area are contained in the papers by Valentin Goranko [4] and by Andrzej Sendlewski [13] and [14]. The state of the art is nicely summarized in [15]. The quoted papers establish facts about constructive logic using the theory of intuitionistic logic. It is shown that the least extension of an intermediate logic by a strong negation shares many basic properties with its intuitionistic reduct, for example tabularity, decidability and interpolation. We will add some properties to this list. Moreover, we will show in addition that the three–valued frame–semantics of [12] is the geometrical analogue of the construction of so–called *twist algebras*. Using this correspondence, the structure of the lattice of logics based on linear frames will be determined.

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Intermediate and Constructive Logics

Recall that intuitionistic logic is a logic weaker than classical logic, consisting of the following axioms for the connectives $\wedge$, $\vee$, $\neg$ and $\to$ in addition to the rules of substitution and modus ponens. (A different set of basic connectives might be chosen; the results do not depend on the actual choice.)

$(a1)$	$p \to (q \to p)$
$(a2)$	$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$
$(a3)$	$p \wedge q \to p$
$(a4)$	$p \wedge q \to q$
$(a5)$	$p \to (q \to p \wedge q)$
$(a6)$	$p \to p \vee q$
$(a7)$	$p \to q \vee p$
$(a8)$	$(p \to r) \to ((q \to r) \to (p \vee q \to r))$
$(a9)$	$(p \to q) \to ((p \to \neg q) \to \neg p)$
$(a10)$	$\neg p \to (p \to q)$

We define $\bot := p \wedge \neg p$, $\top := p \to p$. We are working over a fixed set V_ω of denumerably many sentence letters, denoted by lower case Roman letters. Lower case Greek letters are reserved for formulae. We denote by **Int** the set of tautologies of intuitionistic logic in these connectives. Classical logic is obtained by adding the axiom $((p \to q) \to p) \to p$. Any logic containing **Int** and contained in classical logic is called an **intermediate logic**. (We do not require that an intermediate logic be different from **Int** or from classical logic, i. e. we do not require it to be strictly intermediate between the two.) We denote the set $\{\wedge, \vee, \neg, \to\}$ of intuitionistic connectives by I; I also denotes the language defined by means of these connectives. There will be no harm in the ambiguity. An **I–algebra** is an algebra of a signature appropriate for I and an **I–homomorphism** is a homomorphism of I–algebras. The operations of $\mathfrak{J}$ corresponding to the operation symbols of I are $\cap$, $\cup$, $-$ and $\to$. If θ is a congruence relation on $\mathfrak{J}$ then $[x]\theta := \{y | x\,\theta\, y\}$ denotes the coset of x, and $p_\theta : x \mapsto [x]\theta$ the natural homomorphism from $\mathfrak{J}$ onto $\mathfrak{J}/\theta$. If $\mathfrak{J} = \langle J, \cap, \cup, \to, - \rangle$ is an I–algebra and $\beta : V \to H$ a function then there exists a unique I–homomorphism from the algebra of I–formulae into $\mathfrak{J}$, denoted by $\overline{\beta}$. Given a pair ϕ, ψ of formulae, we write $\mathfrak{J} \models \phi = \psi$ if $\overline{\beta}(\phi) = \overline{\beta}(\psi)$ for all $\beta : V_\omega \to J$. In addition, we write $\mathfrak{J} \models \phi$ if $\mathfrak{J} \models \phi = 1$, for $1 := x \to x$ for some (and hence all) x. (Thus $1 = \overline{\beta}(\top)$ for all β.) We write $\mathsf{Th}\,\mathfrak{J}$ for the set of all ϕ such that $\mathfrak{J} \models \phi$, and call it the **theory** of $\mathfrak{J}$. Also we write $\mathsf{Eq}\,\mathfrak{J}$ for the set of all equations $\phi = \psi$ such that $\mathfrak{J} \models \phi = \psi$. We call this the **equational theory** of $\mathfrak{J}$. For classes $\mathcal{K}$ of algebras, $\mathsf{Th}\,\mathcal{K}$ and $\mathsf{Eq}\,\mathcal{K}$ denote the intersection of the theory (of the equational theory) of all members of $\mathcal{K}$. A **Heyting algebra** is an I–algebra $\mathfrak{H}$ whose theory contains **Int**, and in which $x = y$ iff $x \leftrightarrow y = 1$, where $x \leftrightarrow y$ abbreviates $(x \to y) \cap (y \to x)$. It holds by definition that if $\mathfrak{H}$ is a Heyting algebra then $\mathfrak{H} \models \phi \leftrightarrow \psi$ iff $\mathfrak{H} \models \phi = \psi$. It follows that for a class $\mathcal{K}$ of Heyting algebras $\mathsf{Eq}\,\mathcal{K} = \{\phi = \psi \mid \phi \leftrightarrow \psi \in \mathsf{Th}\,\mathcal{K}\}$. If $\mathfrak{J}$ is an I–algebra such that $\mathsf{Th}\,\mathfrak{J}$ contains **Int**, then the relation θ defined by $x\,\theta\,y$ iff $x \leftrightarrow y = 1$ is a congruence relation, and the algebra $\mathfrak{J}/\theta$ is a Heyting algebra. We call θ the **natural congruence relation** on $\mathfrak{J}$. Given an intermediate logic Λ we let $\mathsf{Alg}\,\Lambda$ denote the class of Heyting algebras whose theory contains Λ. $\mathsf{Alg}\,\Lambda$ is a variety, since it is determined by the equations $\phi = 1$ for each axiom of Λ. Recall here that a **variety** is a class of algebras closed under the operations P of forming products, H of taking homomorphic images, S of taking subalgebras and I of taking isomorphic copies. In turn, each variety corresponds to an intermediate logic. For if it is determined by a set $\{\phi_i = \psi_i | i \in J\}$ of equations, it is likewise determined by the set of axioms $\{\phi_i \leftrightarrow \psi_i | i \in J\}$. By BIRKHOFFs Theorems, intermediate logics are complete with respect to Heyting algebras in the sense that every intermediate logic is determined by a class of Heyting algebras.

A Heyting algebra can be represented by an algebra of sets in the following way. Take a poset $\mathfrak{f} = \langle f, \leq \rangle$. A set $S \subseteq f$ is a **cone** if for all $x \in S$ and $x \leq y$ we have $y \in S$. Let $Co(\mathfrak{f})$ be the set of cones of $\mathfrak{f}$; $Co(\mathfrak{f})$ is closed under intersection and union, contains f and $\emptyset$. Define the following operations

$$\begin{array}{lcl} -S & = & \bigcup \langle U | U \in Co(\mathfrak{P}), U \cap S = \emptyset \rangle \\ S \to T & = & \bigcup \langle U | U \in Co(\mathfrak{P}), U \cap S \subseteq T \rangle \end{array}$$

Now let $\mathbb{P}$ be any subset of $Co(\mathfrak{f})$ closed under intersection, union, $-$ and $\to$, and which contains at least $\emptyset$ and P. In that case $\langle f, \leq, \mathbb{P} \rangle$ is called a (generalized) frame. (In what is to follow, we will omit the qualification 'generalized'.) It is easy to see by direct verification that $\mathbb{P}$ forms a Heyting algebra under the operations just defined, putting $0 := \emptyset$ and $1 := P$. Any Heyting algebra is isomorphic to the algebra of cones of a frame.

Intermediate logics are therefore also complete with respect to (generalized) frames.

The language of constructive logic, C, is obtained by adding to I a new unary connective $\sim$. The logic **N**, also called **Nelson logic**, is defined to be the least set closed under substitution and modus ponens, containing **Int** and the postulates

$$\begin{array}{lllll} (n1) & \sim(x \to y) & \leftrightarrow & x \wedge \sim y \\ (n2) & \sim(x \wedge y) & \leftrightarrow & \sim x \vee \sim y \\ (n3) & \sim(x \vee y) & \leftrightarrow & \sim x \wedge \sim y \\ (n4) & \sim -x & \leftrightarrow & x \\ (n5) & \sim\sim x & \leftrightarrow & x \\ (n6) & \sim x \vee -x & \leftrightarrow & -x \end{array}$$

The notions of **C–algebra**, **C–homomorphism** etc. are defined as in the I–case. As before, we define satisfaction relations $\mathfrak{C} \models \phi = \psi$ and $\mathfrak{C} \models \phi$ for algebras with appropriate signature, and extend the definition of $\mathsf{Th}\,\mathfrak{C}$, $\mathsf{Eq}\,\mathfrak{C}$ to C–algebras. A **Nelson algebra** is a C–algebra $\mathfrak{N} = \langle N, \cap, \cup, \to, -, \sim\rangle$ whose theory contains **N** and whose reduct to $\{\cap, \cup\}$ is a distributive lattice. (This has given rise to the name *N–lattice* for what we call Nelson algebras.) By virtue of (n5), (n2) and (n3), the map $\sim$ is an antiisomorphism of that lattice.

It is not necessarily the case that the I–reduct of a Nelson algebra is a Heyting algebra, because even though all axioms receive the value 1 we might have $x \leftrightarrow y = 1$ but $x \neq y$. An alternative way of stating this is to say that the biimplication $\leftrightarrow$ is not congruential. (This means that $\{\langle x, y\rangle : x \leftrightarrow y = 1\}$ is not a congruence.) From the truth of $\phi \leftrightarrow \psi$ we can only conclude that ϕ and ψ are truth equivalent, but not that they are equal (see [12]). However, we do have $x = y$ iff both $x \leftrightarrow y = 1$ and $\sim x \leftrightarrow \sim y = 1$. Hence, any equational theory of Nelson algebras can be turned into an axiomatic theory by replacing the equation $\phi = \psi$ by the axioms $\psi \leftrightarrow \phi$ and $\sim\phi \leftrightarrow \sim\psi$. We can summarize this as follows. For a class $\mathcal{L}$ of Nelson–algebras, $\mathsf{Eq}\,\mathcal{L} = \{\phi = \psi \mid \phi \leftrightarrow \psi, \sim\phi \leftrightarrow \sim\psi \in \mathsf{Th}\,\mathcal{L}\}$. Conversely, if $\mathcal{L}$ is the class of algebras satisfying $\{\phi_i = \psi_i \mid i \in J\}$, then $\mathcal{L}$ is also the algebra satisfying $\{\phi_1 \leftrightarrow \psi_i \mid i \in J\} \cup \{\sim\phi_i \leftrightarrow \sim\psi_i \mid i \in J\}$. Therefore, for any N–logic $\mathsf{Alg}\,\Theta$ is a variety and every variety is of this form. Using BIRKHOFFs Theorems again we establish that every logic extending **N** is determined by a class of Nelson algebras which is a variety. Following [4], axiomatic extensions of **N** are called **N–logics**.

The Twist Construction

In analogy to the representation of Heyting algebras as algebras of sets over a frame, one would like to have a representation of Nelson algebras. Such a representation can be given either directly by using three valued interpretations on frames (on the basis of [12]) or by an algebraic construction of Nelson algebras from Heyting algebras, which is due to [16]. (VAKARELOV calls the algebras obtained by this construction *special* while we refer to them as *twist algebras*.) We will perform these constructions and show that they are actually the same construction, based on algebras and frames respectively. Given an

I–algebra $\mathfrak{J} = \langle J, \cap, \cup, \rightarrow, - \rangle$ put

$$J^{\bowtie} = \{\langle x, x' \rangle | x \cap x' = 0\} = \{\langle x, x' \rangle | x' \leq -x\}.$$

Define the following operations.

$$\begin{array}{llll}
(t\cap) & \langle x, x' \rangle \cap \langle y, y' \rangle & := & \langle x \cap y, x' \cup y' \rangle \\
(t\cup) & \langle x, x' \rangle \cup \langle y, y' \rangle & := & \langle x \cup y, x' \cap y' \rangle \\
(t \rightarrow) & \langle x, x' \rangle \rightarrow \langle y, y' \rangle & := & \langle x \rightarrow y, x \cap y' \rangle \\
(t-) & -\langle x, x' \rangle & := & \langle -x, x \rangle \\
(t \sim) & \sim \langle x, x' \rangle & := & \langle x', x \rangle
\end{array}$$

Define $\mathfrak{J}^{\bowtie} = \langle J^{\bowtie}, \cap, \cup, \rightarrow, \sim, - \rangle$. We call the pair $\langle 1, 0 \rangle$ the **unit** of $\mathfrak{J}^{\bowtie}$. It is denoted by 1. It is the maximal element in the lattice $\langle J^{\bowtie}, \cap, \cup \rangle$. The minimal element is $\langle 0, 1 \rangle$.

Proposition 1 *Let $\mathfrak{J}$ be an I–algebra. If $\mathsf{Th}\,\mathfrak{J}$ contains **Int** then $\mathsf{Th}\,\mathfrak{J}^{\bowtie}$ contains **N**.*

Proof. Let $\beta : V_\omega \rightarrow J^{\bowtie}$ be a valuation. It is straightforward to check that for any ϕ, $\overline{\beta}(\phi) \in J^{\bowtie}$. Moreover, there exist $\gamma_1, \gamma_2 : V_\omega \rightarrow J$ such that $\beta(p) = \langle \gamma_1(p), \gamma_2(p) \rangle$. For each ϕ, $\overline{\beta}(\phi) = \langle \overline{\gamma_1}(\phi), y \rangle$ for some y such that $y \cap \overline{\gamma_1}(\phi) = 0$. Hence if ϕ is a theorem of **Int**, then $\overline{\gamma_1}(\phi) = 1$ and so $y = 0$. Thus $\overline{\beta}(\phi) = \langle 1, 0 \rangle$, which is the unit element. Hence, $\mathsf{Th}\,\mathfrak{J}^{\bowtie}$ contains **Int**. Now consider the postulates for $\sim$. Notice that in $\mathfrak{J}^{\bowtie}$, $\langle x, x' \rangle \leftrightarrow \langle y, y' \rangle = \langle x \leftrightarrow y, x \cap y' \cap y \cap x' \rangle = \langle x \leftrightarrow y, 0 \rangle$. Hence the equivalences (n1) – (n6) hold if the first components are equal. This is a matter of direct verification. $\dashv$

If $\mathfrak{H}$ is a Heyting algebra we call $\mathfrak{H}^{\bowtie}$ a **twist–algebra**.

Theorem 2 (Vakarelov) *Let $\mathfrak{H}$ be a Heyting algebra. Then $\mathfrak{H}^{\bowtie}$ is a Nelson algebra.*

Proof. In view of the previous theorem one only has to show that $\langle \mathfrak{J}^{\bowtie}, \cap, \cup \rangle$ is a lattice. Again, this is a question of direct verification. $\dashv$

This observation can be strengthened. Take two Heyting algebras $\mathfrak{H}$ and $\mathfrak{J}$ and an I–homomorphism $h : \mathfrak{H} \rightarrow \mathfrak{J}$. Put $h^{\bowtie} : \mathfrak{H}^{\bowtie} \rightarrow \mathfrak{J}^{\bowtie} : \langle x, y \rangle \mapsto \langle h(x), h(y) \rangle$. This is a C–homomorphism of the corresponding twist algebras. Namely, since h commutes with the basic operations, $h^{\bowtie}$ also commutes with the operations on the twist algebras. For example

$$\begin{array}{rcl}
h^{\bowtie}(\langle x, x' \rangle \rightarrow \langle y, y' \rangle) & = & h^{\bowtie}(\langle x \rightarrow y, x \cap y' \rangle) \\
& = & \langle h(x \rightarrow y), h(x \cap y') \rangle \\
& = & \langle h(x) \rightarrow h(y), h(x) \cap h(y') \rangle \\
& = & h^{\bowtie}(\langle x, x' \rangle) \rightarrow h^{\bowtie}(\langle y, y' \rangle)
\end{array}$$

Theorem 3 (Goranko) *$(-)^{\bowtie}$ is a covariant functor from the category Heyt of Heyting algebras into the category Nels of Nelson algebras.* $\dashv$

Twist algebras can be represented as follows. Let $\mathfrak{H}$ be a Heyting algebra. There exists a general frame $\mathfrak{F} = \langle P, \leq, \mathbb{P}\rangle$ such that the natural algebra over $\mathbb{P}$ is isomorphic to $\mathfrak{H}$. Let $\mathbb{P}^{\bowtie}$ be the set of pairs $\langle S, T\rangle$ such that $S, T \in \mathbb{P}$ and $S \cap T = \emptyset$. The operations on the pairs of sets are as defined by the clauses $(t\cap)$, $(t\cup)$, $(t \to)$, $(t-)$ and $(t\sim)$ above. Suppose then that $\iota : H \to \mathbb{P}$ is an I–isomorphism. Then it is easy to see that $\iota^{\bowtie} : H^{\bowtie} \to \mathbb{P}^{\bowtie}$ is a C–isomorphism.

In [12] an approach was taken via pair valued or — alternatively — via three–valued interpretations on frames. It is as follows. Let $\mathfrak{F} = \langle f, \leq, \mathbb{P}\rangle$ be a frame. A **pair–valuation** into $\mathfrak{F}$ is a pair $\beta = \langle \beta^+, \beta^-\rangle$ of functions from the set of variables into $\mathbb{P}$ such that for every sentence letter p $\beta^+(p) \cap \beta^-(p) = \emptyset$. $\beta^+(p)$ is the set of all worlds at which p is definitely accepted, $\beta^-(p)$ the set of all worlds at which it is definitely rejected. The assumption is that if p is accepted (rejected) at x and $x \leq y$ then p must be accepted (rejected) at y as well. This is extended to all formulae ϕ, using the definitions $(t\cap)$, $(t\cup)$, $(t \to)$, $(t-)$ and $(t\sim)$ as well as the standard clauses for the intuitionistic connectives. A pair valuation can be interpreted as a valuation as follows. $\mathfrak{P} := \langle \mathbb{P}, \cap, \cup, \to, -\rangle$ is a Heyting algebra. A pair valuation into $\mathfrak{F}$ (and likewise into $\mathfrak{P}$) is a (simple) valuation into $\mathfrak{P}^{\bowtie}$, that is, a function $\beta : V_\omega \to \mathbb{P}^{\bowtie}$. We write $\langle \mathfrak{F}, \beta, x\rangle \models \phi$ if $\overline{\beta}(\phi) = \langle S, T\rangle$ where S contains x, and we say that the models **accepts** ϕ at x. If on the other hand $x \in T$ we say that the models **rejects** ϕ at x. Define the **twist frame** $\mathfrak{F}^{\bowtie}$ of $\mathfrak{F}$ by $\langle f, \leq, \mathbb{P}^{\bowtie}\rangle$. A valuation into $\mathfrak{F}^{\bowtie}$ is a function $\beta : V_\omega \to \mathbb{P}^{\bowtie}$. There is a natural correspondence between valuations on $\mathfrak{F}^{\bowtie}$ and pair–valuations on $\mathfrak{F}$. Again, we write $\langle \mathfrak{F}^{\bowtie}, \beta, x\rangle \models \phi$ if $\overline{\beta}(\phi) = \langle S, T\rangle$ for some S containing x. An **N–frame** is a triple $\langle f, \leq, \mathbb{F}\rangle$ where $\mathbb{F}$ is a subset of $Co(\langle f, \leq\rangle)^{\bowtie}$ closed under $\cap$, $\cup$, $\to$, $\sim$ and $-$ as defined above by the clauses $(t\cap)$, $(t\cup)$, $(t \to)$, $(t\sim)$ and $(t-)$. In intuitionistic logic, a frame of the form $\langle f, \leq, \mathbb{P}\rangle$ where $\mathbb{P}$ contains all cones of $\langle f, \leq\rangle$ is called a **Kripke–frame**. Analogously, an N–frame is called a **Kripke–frame** if it is the twist frame of a Kripke–frame. (Notice that it makes no sense to distinguish a special *Kripke–N–frame* from a *Kripke–frame*.) The topological (or geometrical) representation theorems for Heyting algebras can be carried over straightforwardly to Nelson algebras, once the algebraic properties of the twist map are determined. We will return to this issue briefly below.

We can pass from a pair–valuation to a three–valued valuation in the following way. A **3–valuation** is a map $w : P \times \mathit{var} \mapsto \{-1, 0, 1\}$ such that both $w^{-1}(-1)$ and $w^{-1}(1)$ are cones for each individual variable. The interpretation is that p is accepted at x if $w(x, p) = 1$, and that p is rejected at x if $w(x, p) = -1$. The third case, $w(x, p) = 0$, arises when p is not decided at x. Again, this is extended to all formulae, following $(t\cap)$, $(t\cup)$, $(t \to)$, $(t-)$ and $(t \sim)$. We write $\mathfrak{F} \models_3 \phi$ if for all three–valued interpretations w and all worlds x, $w(x, \phi) = 1$. The various acceptance clauses are made to fit in the following way. Given a 3–valuation w put $w_+(p) = \{x | w(x, p) = 1\}$, $w_-(p) = \{x | w(x, p) = -1\}$. Then $1 = w(x, \phi)$ iff $x \in \overline{w_+}(\phi)$. It is clear from the previous considerations that $\overline{w_+}(\phi) \cap \overline{w_-}(\phi) = \emptyset$, so we have a pair–valuation. Conversely, let us be given a pair–valuation $\langle \beta^+, \beta^-\rangle$. Then put $w(x, p) = 1$ iff $x \in \beta^+(p)$ and $w(x, p) = -1$ iff $x \in \beta^-(p)$. This is noncontradictory, and yields a 3–valuation. Thus it does not matter whether we think in terms of pair–valuations or in terms of 3–valuations.

A natural question is whether all Nelson algebras can be obtained as twist algebras from some Heyting algebra. This is generally not so; there is a two element Nelson algebra which does not arise as a twist algebra from a Heyting algebra, because the latter necessarily contain either one or at least three elements. However, the situation is nearly optimal. For a Nelson algebra $\mathfrak{N}$ can at least be embedded into a twist algebra. Moreover, this algebra can be canonically determined. For let $\mathfrak{N}$ be a Nelson algebra. Then $\mathfrak{N} \restriction I$ is not necessarily a Heyting algebra, but $(\mathfrak{N} \restriction I)/\theta$ is a Heyting algebra, where θ is the natural congruence, and $(\mathfrak{N} \restriction I)/\theta$ is also the 'least' Heyting algebra that can be obtained by factorization. We denote it by $\mathfrak{N}_{\bowtie}$, and call it the Heyting algebra **associated with** $\mathfrak{N}$ or the **untwist algebra** of $\mathfrak{N}$. If $h : \mathfrak{N} \to \mathfrak{O}$ is a C–homomorphism of Nelson algebras, then h is also an I–homomorphism. Moreover, if $x \leftrightarrow y = 1$ for $x, y \in N$ then $h(x \leftrightarrow y) = 1$. Hence, if θ_N and θ_O are the natural congruences on $\mathfrak{N}$ and $\mathfrak{O}$, respectively, the map $p_{\theta_o} \circ h$ factors through p_{θ_N}. Thus, there exists a map $h_{\bowtie}$ such that $p_{\theta_O} \circ h = h_{\bowtie} \circ p_{\theta_N}$. Hence we have $h_{\bowtie} : \mathfrak{N}_{\bowtie} \to \mathfrak{O}_{\bowtie}$.

$$\begin{tikzcd}
\mathfrak{N} \restriction I \arrow[r, "h"] \arrow[d, "p_{\theta_N}"'] & \mathfrak{O} \restriction I \arrow[d, "p_{\theta_O}"] \\
\mathfrak{N}_{\bowtie} \arrow[r, "h_{\bowtie}"] & \mathfrak{O}_{\bowtie}
\end{tikzcd}$$

Thus, $(-)_{\bowtie}$ also is a functor. But more can be shown. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is **left adjoined** to a functor $G : \mathcal{D} \to \mathcal{C}$ iff for every $A \in \mathcal{C}$ and $B \in \mathcal{D}$ there exists a bijection $\pi_{AB} : Hom_{\mathcal{C}}(F(B), A) \to Hom_{\mathcal{D}}(B, G(A))$. (This requires that the categories involved are small, i. e. that $Hom(C, D)$ is a set for any given objects C and D. This is the case in our setting.)

Theorem 4 (Sendlewski) *$(-)_{\bowtie}$ is a covariant functor from the category* Nels *into the category* Heyt. *$(-)_{\bowtie}$ is left–adjoined to $(-)^{\bowtie}$. Moreover, $((\mathfrak{H})^{\bowtie})_{\bowtie} \cong \mathfrak{H}$ for all Heyting algebras $\mathfrak{H}$.*

Proof. Fix a Heyting algebra $\mathfrak{H}$ and a Nelson algebra $\mathfrak{N}$. We define a map $\pi : Hom(\mathfrak{N}, \mathfrak{H}^{\bowtie}) \to Hom(\mathfrak{N}_{\bowtie}, \mathfrak{H})$ and a map $\rho : Hom(\mathfrak{N}_{\bowtie}, \mathfrak{H}) \to Hom(\mathfrak{N}, \mathfrak{H}^{\bowtie})$. First π. Let $h : \mathfrak{N} \to \mathfrak{H}^{\bowtie}$. Then there exists an I–homomorphism $j_1 : \mathfrak{N} \restriction I \to \mathfrak{H}$, and a map $j_2 : N \to H$ such that $h(x) = \langle j_1(x), j_2(x)\rangle$. Let θ be the natural congruence on $\mathfrak{N}$. Then put $\pi(h)([x]\theta) := j_1(x)$. To see that this does not depend on the choice of representatives, let $y \in [x]\theta$. Then $x \leftrightarrow y = 1$, and so $h(x \leftrightarrow y) = \langle 1, 0\rangle$. Therefore $j_1(x) = j_1(y)$.

Now let $i : \mathfrak{N}_{\bowtie} \to \mathfrak{H}$. Recall that by the definition of $\mathfrak{N}_{\bowtie}$ its underlying set is the set of cosets $[x]\theta$, where θ is the natural congruence on $\mathfrak{N}$. Then we define $\rho(i) : x \mapsto$

$\langle i([x]\theta), i([\sim x]\theta)\rangle$. This is a C–homomorphism. For example

$$\begin{array}{rcll}
\rho(i)(x\cap y) & = & \langle i([x\cap y]\theta), i([\sim(x\cap y)]\theta)\rangle & = \\
& = & \langle i([x]\theta\cap[y]\theta), i([\sim x]\theta\cup[\sim y]\theta)\rangle & = \\
& = & \langle i([x]\theta)\cap i([y]\theta), i([\sim x]\theta)\cup i([\sim y]\theta)\rangle & = \\
& = & \langle i([x]\theta), i([\sim x]\theta)\rangle\cap\langle i([y]\theta), i([\sim y]\theta)\rangle & = \quad \rho(i)(x)\cap\rho(i)(y)
\end{array}$$

We also have to show that $\rho\circ\pi(h) = h$ and $\pi\circ\rho(i) = i$. Take h and let $h(x) = \langle j_1(x), j_2(x)\rangle$. Then $\pi(h) : [x]\theta \mapsto j_1(x)$ and so $\rho(\pi(h)) : x \mapsto \langle j_1(x), j_1(\sim x)\rangle = \langle j_1(x), j_2(x)\rangle$. Hence $\rho\circ\pi(h) = h$. Next let $i : \mathfrak{N}_{\bowtie} \to \mathfrak{H}$. Then $\rho(i) : x \mapsto \langle i([x]\theta), i([\sim x]\theta)\rangle$. Thus $\pi(\rho(i)) : [x]\theta \mapsto i([x]\theta)$ and so $\pi\circ\rho(i) = i$. Now, finally, consider the identity $1_{\mathfrak{H}^{\bowtie}} : \mathfrak{H}^{\bowtie} \to \mathfrak{H}^{\bowtie}$. Then $\pi(1_{\mathfrak{H}^{\bowtie}}) : (\mathfrak{H}^{\bowtie})_{\bowtie} \to \mathfrak{H}$. This is an isomorphism. For, by definition, $\pi(1_{\mathfrak{H}^{\bowtie}}) : [\langle x, y\rangle]\theta \mapsto x$. Now notice that $[\langle x, y\rangle]\theta = \{\langle x, y'\rangle | x\cap y' = 0\}$. Thus we can choose $\langle x, 0\rangle$ as a representative of the coset of $\langle x, y\rangle$. Hence the map is bijective. $\dashv$

Corollary 5 (Vakarelov) *Let $\mathfrak{N}$ be a Nelson algebra. Then $\mathfrak{N}$ is embeddable into $(\mathfrak{N}_{\bowtie})^{\bowtie}$.*

Proof. Let $1_{\mathfrak{N}_{\bowtie}} : \mathfrak{N}_{\bowtie} \to \mathfrak{N}_{\bowtie}$. Then $\rho(1_{\mathfrak{N}_{\bowtie}}) : [x]\theta \mapsto \langle [x]\theta, [\sim x]\theta\rangle$. Assume that $x \neq y$. Then either $x \leftrightarrow y \neq 1$, that is, $[x]\theta \neq [y]\theta$, or $\sim x \leftrightarrow \sim y \neq 1$, in which case $[\sim x]\theta \neq [\sim y]\theta$. Hence $\rho(1_{\mathfrak{N}_{\bowtie}})$ is injective. $\dashv$

As a consequence, each Nelson algebra can be embedded isomorphically into the algebra of sets of an N–frame. Namely, let $\mathfrak{N}$ be given. Then there exists a frame $\langle f, \leq, \mathbb{P}\rangle$ such that $\mathfrak{N}_{\bowtie} \cong \mathfrak{P}$, where $\mathfrak{P} = \langle \mathbb{P}, \cap, \cup, \to, -\rangle$. Hence there exists an imbedding $i : \mathfrak{N}_{\bowtie} \rightarrowtail \mathfrak{P}$, and so $\rho(i) : \mathfrak{N} \to \mathfrak{P}^{\bowtie}$. Let $\mathbb{F}$ be the image of $\rho(i)$. We now have an N–frame $\langle f, \leq, \mathbb{F}\rangle$ such that $\mathfrak{N}$ is isomorphic to the algebra over the sets of $\mathbb{F}$.

Simulating Twist Algebras

The C–validities of $\mathfrak{H}^{\bowtie}$ can be simulated as intuitionistic validities of $\mathfrak{H}$ by doubling each variable. To make that precise we need the following definition of a *standard form.*

Definition 6 *A C–formula is said to be in* ***standard form*** *if it is built from variables p and strongly negated variables $\sim p$ using only intuitionistic connectives.*

Proposition 7 *For every ψ there exists a $s(\psi)$ which is in standard form, such that in* ***N*** *the equivalence $\psi \leftrightarrow s(\psi)$ holds.*

Proof. Only truth equivalence is required. This follows directly from the laws (n1) – (n6). $\dashv$

Now let ψ be any C–formula and introduce for every variable p of ψ a new variable p^-. To distingish the variable p^- from standard variables we call it a **twistor**. In particular,

p^- is the twistor of p. Let $V := var(\psi)$ and put

$$\psi_{\bowtie} := \bigwedge_{p \in V} p^- \to \neg p. \to .(s(\psi)[p^-/{\sim}p])$$

That means the following. First put ψ into standard form, replace occurrences of $\sim p$ by p^- and then add the condition that $p^- \leq \neg p$. For a finite set of formulae V we denote the formula $\bigwedge\langle p^- \to \neg p | p \in V\rangle$ by γ_V. The set of twistors of ϕ is denoted by $var^-(\phi)$. Furthermore, we write $\widehat{\phi}$ for $s(\phi)[p^-/{\sim}p | p \in var(\phi)]$. Thus $s(\psi)$ can be obtained from $\widehat{\psi}$ by a substitution. We now have $\psi_{\bowtie} = \gamma_V \to \widehat{\psi}$ for some V. The following theorem (in a slightly different form) is due to Valentin Goranko [4].

Theorem 8 (Twist Simulation) *For all Heyting algebras $\mathfrak{H}$ and C–formulae ψ,*

$$\mathfrak{H}^{\bowtie} \models \psi \qquad \Leftrightarrow \qquad \mathfrak{H} \models \psi_{\bowtie}.$$

Proof. Suppose $\mathfrak{H}^{\bowtie} \models \psi$. Then $\mathfrak{H}^{\bowtie} \models s(\psi)$, by truth–equivalence. Thus we may assume that ψ is in standard form. Let $V := var(\phi)$ and β a valuation of $V \cup V^- = V \cup \{p^- | p \in V\}$ in $\mathfrak{H}$. We have to show that $\overline{\beta}(\psi_{\bowtie}) = 1$. Put

$$c := \overline{\beta}(\gamma_V) = \bigcap_{p \in V} \overline{\beta}(p^- \to \neg p)$$

It is enough to prove that $c \leq \overline{\beta}(\widehat{\psi})$. Now let $w^+(p) = c \cap \overline{\beta}(p)$, and let $w^-(p) = c \cap \overline{\beta}(p^-)$. Then we have defined a valuation w into $\mathfrak{H}^{\bowtie}$ by $w : p \mapsto \langle w^+(p), w^-(p)\rangle$. For by construction, $w^+(p) \cap w^-(p) = c \cap \beta(p) \cap \beta(p^-) \leq \beta(p) \cap \overline{\beta}(p^- \to \neg p) \cap \beta(p^-) = 0$. Since $\mathfrak{H}^{\bowtie} \models \psi$ we have $\overline{w^+}(\psi) = 1$. Consequently, we also have $c \leq \overline{w^+}(\psi[p^-/{\sim}p]) = \overline{w_+}(\widehat{\psi})$. On the other hand, $c = \overline{w^+}(\widehat{\psi}) \cap c = \overline{\beta}(\widehat{\psi}) \cap c$ and this shows that $c \leq \overline{\beta}(\widehat{\psi})$. Now assume conversely that $\mathfrak{H} \models \psi_{\bowtie}$. Let v be a valuation on $\mathfrak{H}^{\bowtie}$. There are β^+ and β^- such that $v(p) = \langle \beta^+(p), \beta^-(p)\rangle$. Then $\beta^-(p) \leq -\beta^+(p)$. We expand β^+ to a valuation δ with $\delta(p) = \beta^+(p)$ for $p \in V$ and $\delta(p^-) = \beta^-(p)$. Then since $\overline{\delta}(p^- \to \neg p) = 1$ we get $\overline{\delta}(\psi[p^-/{\sim}p]) = 1$ by construction. It is enough to show $\overline{\beta}^+(\psi) = 1$. But assuming ψ in standard form this follows from the equation $\delta(p^-) = \beta^-(p)$. $\dashv$

For a I–formula ψ put $\psi^{\bowtie} := \psi$. In full analogy to the Twist Simulation we get

Theorem 9 (Untwist Simulation) *For all Nelson algebras $\mathfrak{N}$ and intuitionistic formulae ψ*

$$\mathfrak{N}_{\bowtie} \models \psi \qquad \Leftrightarrow \qquad \mathfrak{N} \models \psi^{\bowtie} \quad (= \psi). \qquad \dashv$$

Corollary 10 *The rule $\psi/\psi_{\bowtie}$ is admissible in all varieties generated by twist algebras. The rule $\psi_{\bowtie}/\psi$ is admissible in all extensions of **N**.*

Proof. Let $\mathcal{J}$ be a class of Heyting algebras. Let $\mathcal{V}$ be the variety generated by the twist algebras $\mathfrak{H}^{\bowtie}$, $\mathfrak{H} \in \mathcal{J}$. Assume $\mathcal{V} \models \psi$. Then $\mathfrak{H}^{\bowtie} \models \psi$ for all $\mathfrak{H} \in \mathcal{J}$. Therefore, by Twist

Simulation, $\mathfrak{H} \models \psi_{\bowtie}$ for all $\mathfrak{H} \in \mathcal{J}$. But then $\mathfrak{H}^{\bowtie} \models \psi_{\bowtie}$ for all $\mathfrak{H} \in \mathcal{J}$. Now let $\mathcal{V}$ be any variety of Nelson algebras. Assume that $\psi_{\bowtie}$ holds in $\mathcal{V}$. Pick $\mathfrak{N} \in \mathcal{V}$. Then $\mathfrak{N} \models \psi_{\bowtie}$ and so $\mathfrak{N}_{\bowtie} \models \psi_{\bowtie}$, by Untwist Simulation. Therefore, by Twist Simulation, $(\mathfrak{N}_{\bowtie})^{\bowtie} \models \psi$. Since $\mathfrak{N}$ is a subalgebra of $(\mathfrak{N}_{\bowtie})^{\bowtie}$, we also have $\mathfrak{N} \models \psi$, as desired. $\dashv$

We extend the notation for twist and untwist to classes of algebras in the following way. If $\mathcal{K}$ ($\mathcal{L}$) is a class of Heyting algebras (Nelson algebras) then $\mathcal{K}^{\bowtie}$ ($\mathcal{L}_{\bowtie}$) is the class of all $\mathfrak{N}^{\bowtie}$, $\mathfrak{N}$ a member of $\mathcal{K}$ (the class of all $\mathfrak{H}^{\bowtie}$, $\mathfrak{H}$ a member of $\mathcal{L}$). Call a variety $\mathcal{V}$ of Nelson algebras a **twist variety** if it is generated by a class of twist algebras.

Theorem 11 *A variety of Nelson algebras is a twist variety iff all rules of the form $\psi/\psi_{\bowtie}$ are admissible.*

Proof. Let $\mathcal{V}$ be a variety of Nelson algebras. Let $\mathcal{W}$ be the twist variety generated by $(\mathcal{V}_{\bowtie})^{\bowtie}$. It is easy to see that $\mathcal{W}$ is the smallest twist variety containing $\mathcal{V}$. By Corollary 10, $\psi/\psi_{\bowtie}$ is admissible. Now assume that $\psi/\psi_{\bowtie}$ is admissible in $\mathcal{V}$. Then $\mathcal{V} \models \psi$ iff $\mathcal{V} \models \psi_{\bowtie}$ iff $\mathcal{V} \models (\psi_{\bowtie})^{\bowtie}$ iff $(\mathcal{V}^{\bowtie})_{\bowtie} \models \psi$ iff $\mathcal{W} \models \psi$. So, $\mathcal{V}$ and $\mathcal{W}$ have the same theorems, and are thus identical. $\dashv$

Properties of the Imbedding

Let $\mathcal{E}\,\mathbf{Int}$ be the lattice of intermediate logics and $\mathcal{E}\,\mathbf{N}$ be the lattice of axiomatic extensions of $\mathbf{N}$. Both are complete lattices, with the meet being the intersection and the join being the least logic generated by the set theoretic union. Define a map from the first into the second by taking $n(\Lambda)$ to be the least N–logic containing (the theorems of) Λ. Define a map $i : \mathcal{E}\,\mathbf{N} \to \mathcal{E}\ \mathbf{Int} : \Theta \mapsto \Theta \cap I$, restricting Θ to its intuitionistic fragment. Given $\Theta \in \mathcal{E}\,\mathbf{N}$ and $\Lambda \in \mathcal{E}\,\mathbf{Int}$ we have $n(\Lambda) \subseteq \Theta$ iff $\Lambda \subseteq i(\Theta)$, as is easily verified. Hence, n is left–adjoint as functor of posets to i. The following is straightforward consequence of Untwist Simulation.

Proposition 12 *(1.) Let Λ be an intermediate logic and $\mathfrak{H}$ a Heyting algebra. If $\mathfrak{H} \in \mathsf{Alg}\,\Lambda$ then $\mathfrak{H}^{\bowtie} \in \mathsf{Alg}\,n(\Lambda)$. (2.) Let Θ be an N–logic and $\mathfrak{N}$ be a Nelson algebra. If $\mathfrak{N} \in \mathsf{Alg}\,\Theta$ then $\mathfrak{N}_{\bowtie} \in \mathsf{Alg}\,i(\Theta)$.*

Theorem 13 *(1.) Let Λ be an intermediate logic and $\Lambda = \mathsf{Th}\,\mathcal{K}$ for some class $\mathcal{K}$ of Heyting algebras. Then $n(\Lambda) = \mathsf{Th}\,\mathcal{K}^{\bowtie}$. (2.) Let Θ be a N–logic and $\Theta = \mathsf{Th}\,\mathcal{L}$ for some class $\mathcal{L}$ of Nelson algebras. Then $i(\Theta) = \mathsf{Th}\,\mathcal{L}_{\bowtie}$.*

Proof. (1.) Suppose $\psi \notin n(\Lambda)$. Then $\psi_{\bowtie} \notin n(\Lambda)$ and so $\psi_{\bowtie} \notin \Lambda$. Hence there exists a $\mathfrak{H} \in \mathcal{K}$ such that $\mathfrak{H} \not\models \psi_{\bowtie}$. By Twist Simulation, $\mathfrak{H}^{\bowtie} \not\models \psi$, and so $\psi \notin \mathsf{Th}\,\mathcal{K}^{\bowtie}$. The reasoning can be played backwards, and so the first claim is proved. (2.) Suppose $\psi \notin i(\Theta)$. Then $\psi \notin \Theta$, by conservativity of the imbedding. Hence there exists a $\mathfrak{N} \in \mathcal{L}$ such that

$\mathfrak{N} \nvDash \psi (= \psi^{\bowtie})$. By Untwist Simulation $\mathfrak{N}_{\bowtie} \nvDash \psi$ and so $\mathcal{L}_{\bowtie} \nvDash \psi$, from which $\psi \notin \mathsf{Th}\,\mathcal{L}_{\bowtie}$, by definition. Again, the reasoning can be played backwards, and this proves the second claim. ⊣

Corollary 14 *If $\mathcal{V}$ is a variety of Nelson algebras, $\mathsf{I}(\mathcal{V}_{\bowtie})$ is a variety of Nelson algebras. If $\mathcal{W}$ is a variety of Heyting algebras then $\mathsf{IS}(\mathcal{W}^{\bowtie})$ is a variety of Nelson algebras.*

Theorem 15 (Sendlewski) *n is a lattice imbedding of $\mathcal{E}\,\boldsymbol{Int}$ into $\mathcal{E}\,\boldsymbol{N}$ commuting with infinite joins and meets. A logic is of the form $n(\Lambda)$ for some intermediate logic Λ iff all rules $\psi/\psi_{\bowtie}$ are admissible. Moreover, $\Lambda = i(n(\Lambda))$; in other words, the minimal N–logic containing a given intermediate logic is a conservative extension.*

Proof. We prove the last assertion first. Let Λ be an intermediate logic, characterized by a variety $\mathcal{V}$ of Heyting algebras. Then $n(\Lambda)$ is characterized by $\mathcal{V}^{\bowtie}$, by Theorem 13, and $i(n(\Lambda)$ is characterized by $(\mathcal{V}^{\bowtie})_{\bowtie}$, again by Theorem 13. But $\mathsf{I}(\mathcal{V}^{\bowtie})_{\bowtie} = \mathsf{I}(\mathcal{V})$ and so $i(n(\Lambda)) = \Lambda$. That n commutes with infinite joins is easy to verify. The case of (infinite) meets is still to be verified. So, let Λ_i, $i \in I$, be a set of intermediate logics. Put $\Theta := \bigcap_{i \in I} \Lambda_i$. Then $n(\Theta) \subseteq \bigcap_{i \in I} n(\Lambda_i)$, since n is isotone. For the converse it is enough to show that in $\bigcap_{i \in I} n(\Lambda_i)$ all rules $\psi/\psi_{\bowtie}$ are admissible. So let $\psi \in \bigcap_{i \in I} n(\Lambda_i)$. Then for each $i \in I$, $\psi \in n(\Lambda_i)$, whence $\psi_{\bowtie} \in n(\Lambda_i)$, and so $\psi_{\bowtie} \in \bigcap_{i \in I} n(\Lambda_i)$. ⊣

Usually, from the fact that an imbedding is a lattice homomorphism not much can be deduced. However, in this special case we can show that a lot of properties of the intermediate logics are preserved under the imbedding, so that the N–extensions are more or less determined by their I–reduct. We say that an extension Λ of $\mathbf{N}$ is **tabular** iff $\Lambda = \mathsf{Th}\,\mathfrak{N}$ for a finite Nelson algebra $\mathfrak{N}$. Λ has the **finite model property** if for every formula $\phi \notin \Lambda$ there is a finite Nelson algebra $\mathfrak{N}$ such that $\mathfrak{N} \vDash \Lambda$ but $\mathfrak{N} \nvDash \phi$. Λ is called **complete** if for every $\phi \notin \Lambda$ there exists a Kripke–N–frame $\mathfrak{F}^{\bowtie}$ such that $\mathfrak{F}^{\bowtie} \vDash \Lambda$ but $\mathfrak{F}^{\bowtie} \nvDash \phi$. Λ is **frame–compact** if for every finitely satisfiable set of formulae is there is a Kripke–N–frame $\mathfrak{F}^{\bowtie}$ for Λ, a valuation β and a point x such that $\langle \mathfrak{F}^{\bowtie}, \beta, x \rangle \vDash \Phi$. Λ is decidable if the problem '$\phi \in \Lambda$' is decidable. These definitions are the natural extensions of the definitions for intermediate logics.

Theorem 16 (Goranko, Sendlewski) *The following properties of intermediate logics are preserved and reflected by passing to the minimal N–extension: finite model property, tabularity, completeness, frame–compactness, and decidability.*

Proof. For tabularity, completeness and finite model property this already follows from Theorem 13. Now assume that Λ is frame–compact. Let Φ be a set of C–formulae and assume that it is $n(\Lambda)$–consistent. Then $\Phi_{\bowtie} := \{\phi_{\bowtie} \mid \phi \in \Phi\}$ is Λ–consistent. For otherwise there exists a finite set $\Delta_{\bowtie}$ of $\Phi_{\bowtie}$ such that $\Delta_{\bowtie} \vdash_{\Lambda} \bot$. We have $\Delta_{\bowtie} := \{\gamma_{v(\phi)} \to \widehat{\phi} \mid \phi \in \Delta\}$, where $v(\phi)$ denotes the set of variables of ϕ. (Recall the definitions of γ_V and $\widehat{\phi}$ from the previous section.) Since Δ is finite, the union of the $v(\phi)$, $\phi \in \Delta$, is also finite.

Let it be W. Then the set $\{\gamma_W \to \widehat{\phi} \mid \phi \in \Delta\}$ is also Λ–inconsistent. Consequently,

$$((\bigwedge \Delta)_{\bowtie} =)\ \gamma_W \to \bigwedge_{\phi \in \Delta} \widehat{\phi} \vdash_\Lambda \bot$$

It follows that Δ is $n(\Lambda)$–inconsistent. But then Φ is $n(\Lambda)$–inconsistent, against our assumption. Thus we have established that $\Phi_{\bowtie}$ is Λ–consistent. It follows that there is a Kripke–frame $\mathfrak{F} = \langle f, \leq, \mathbb{P}\rangle$ and a valuation β and a point $x \in f$ such that $\langle \mathfrak{F}, \beta, x\rangle \models \Phi_{\bowtie}$. Let $\gamma : V_\omega \cup V_\omega^- \to \mathbb{P}^{\bowtie}$ be defined by $\gamma(p) := \langle \beta(p), \beta(p^-)\rangle$. Then $\langle \mathfrak{F}, \gamma, x\rangle \models \Phi$. Hence, $n(\Lambda)$ is frame–compact. Conversely, if $n(\Lambda)$ is frame–compact and Φ is a Λ–consistent set of I–formulae then Φ is also $n(\Lambda)$–consistent and there exists a Kripke–frame $\mathfrak{F}$, a pair valuation $\beta(p) = \langle \gamma_1(p), \gamma_2(p)\rangle$ and a point x such that $\langle \mathfrak{F}, \beta, x\rangle \models \Phi$. Then $\langle \mathfrak{F}, \gamma_1, x\rangle \models \Phi$. Moreover, $\mathfrak{F} \models \Lambda$. Finally, decidability. Suppose that $n(\Lambda)$ is decidable. Let $\psi \in I$ be given. Then, by conservativity, $\psi \in n(\Lambda)$ iff $\psi \in \Lambda$. Thus, Λ is decidable. Now assume conversely that Λ is decidable. Let $\psi \in C$ be given. Then, by Theorem 11, $\psi \in n(\Lambda)$ iff $\psi_{\bowtie} \in n(\Lambda)$. The latter is decidable, since it is equivalent to $\psi_{\bowtie} \in \Lambda$. $\dashv$

An intermediate logic Λ has **interpolation** if whenever $\phi \to \psi \in \Lambda$ there exists a formula χ such that $var(\chi) \subseteq var(\phi) \cap var(\psi)$ and $\phi \to \chi, \chi \to \psi \in \Lambda$. Likewise for extensions of $\mathbf{N}$.

Theorem 17 (Goranko) *Let Λ be an intermediate logic. Then Λ has interpolation iff $n(\Lambda)$ has interpolation.*

Proof. Assume that Λ has interpolation. Let $\phi \to \psi \in n(\Lambda)$. Then $(\phi \to \psi)_{\bowtie} \in n(\Lambda)$. The interpolating formula will be found by close analysis of the formulae simulating the twist. Recall that they were defined by reserving a special, new set of variables of the form p^- called twistors. The set of twistors of ϕ is denoted by $var^-(\phi)$. Furthermore, recall that $\widehat{\phi}$ denotes $s(\phi)[p^-/{\sim}p \mid p \in var(\phi)]$. Let $V := var(\phi)$ and $W := var(\psi) - V$. Then $(\phi \to \psi)_{\bowtie}$ is equivalent in $\mathbf{N}$ to $\gamma_V \wedge \gamma_W. \to .\widehat{\phi} \to \widehat{\psi}$, which in turn is equivalent in $\mathbf{N}$ to

$$\gamma_V \wedge \widehat{\phi}. \to .\gamma_W \to \widehat{\psi}.$$

By assumption on Λ there exists a ρ such that $var(\rho) \subseteq var(\gamma_V \wedge \widehat{\phi}) \cap var(\gamma_W \to \widehat{\psi})$ and

$$\gamma_V \wedge \widehat{\phi} \to \rho, \rho. \to .\gamma_W \to \widehat{\psi} \in n(\Lambda).$$

Now put $\chi := \rho[{\sim}p/p^- \mid p^- \in var^-(\rho)]$. We claim that χ is the desired interpolant. To that end, observe that χ contains no twistors. Furthermore, the variables of χ are common variables of ϕ and ψ, as is easily verified. Next, observe that $\gamma_V \wedge \widehat{\phi} \to \rho$ is equivalent in $\mathbf{N}$ to $(\phi \to \chi)_{\bowtie}$ and that $\rho. \to .\gamma_W \to \widehat{\psi}$ implies $\gamma_V \wedge \gamma_W. \to .(\rho \to \widehat{\psi})$, which is nothing but $(\chi \to \psi)_{\bowtie}$. Hence, by Theorem 15, $\phi \to \chi, \chi \to \psi \in n(\Lambda)$.

Now assume that $n(\Lambda)$ has interpolation. Let $\phi \to \psi \in \Lambda$. Then $\phi \to \psi \in n(\Lambda)$ and there exists a ρ such that $var(\rho) \subseteq var(\phi) \cap var(\psi)$ and $\phi \to \rho, \rho \to \psi \in n(\Lambda)$. We can assume that ρ is in standard form. Then $\rho = \tau[\vec{p}, {\sim}\vec{p}]$ for some I–term $\tau[\vec{p}, \vec{q}]$, where $\vec{p}$ and $\vec{q}$ are sequences of variables of length n. Then let $\chi := \tau[\vec{p}, \neg\vec{p}]$. This is an

I–formula. Furthermore, it is based on the same set of variables as ρ. We claim, finally, that $\phi \to \chi, \chi \to \psi \in \Lambda$. Observe, namely, that by Twist Simulation

$$(\bigwedge_{i<n} q_i \to \neg p_i). \to .\phi \to \tau[\vec{p}, \vec{q}] \in \Lambda \ .$$

A fortiori we have

$$(\bigwedge_{i<n} q_i \leftrightarrow \neg p_i). \to .\phi \to \tau[\vec{p}, \vec{q}] \in \Lambda \ .$$

From this we conclude that $\phi \to \tau[\vec{p}, \neg\vec{p}] = \phi \to \chi \in \Lambda$. Likewise it is shown that $\chi \to \psi \in \Lambda$. ⊣

In [6] it was shown that there are exactly seven intermediate logics with interpolation. (By definition, intermediate logics are consistent. The inconsistent logic has interpolation as well. Also, for the theorem to hold we need to assume that at least one of $\top$ and $\bot$ is a primitive symbol of the language.) Among them are **Int**, **LC**, classical logic and the logic of here–and–there.

Corollary 18 *There exist exactly seven extensions of **N** of the form $n(\Lambda)$ for some logic containing **Int** with interpolation. In particular, **N** and $n(\mathbf{LC})$ have interpolation.* ⊣

A logic Λ has the **disjunction property** if whenever $\phi \vee \psi \in \Lambda$ then either $\phi \in \Lambda$ or $\psi \in \Lambda$. Λ is **Halldén–complete** if for any pair ϕ, ψ of formulae with $var(\phi) \cap var(\psi) = \emptyset$ if $\phi \vee \psi \in \Lambda$ then either $\phi \in \Lambda$ or $\psi \in \Lambda$. Λ is **Maximova–complete** if whenever $\gamma \wedge \delta \to \phi \vee \psi \in \phi \in \Lambda$ and $var(\gamma \to \phi) \cap var(\delta \to \psi) = \emptyset$ then $\gamma \to \phi \in \Lambda$ or $\delta \to \psi \in \Lambda$.

Theorem 19 *Let Λ be an intermediate logic. (1.) If Λ is Maximova–complete then $n(\Lambda)$ is Halldén–complete. (2.) If $n(\Lambda)$ is Halldén–complete, so is Λ. (3.) If $n(\Lambda)$ has the disjunction property, so does Λ.*

Proof. (1.) Let $\phi \vee \psi \in n(\Lambda)$. Then $(\phi \vee \psi)_{\bowtie} \in \Lambda$. Put $V := var(\phi)$ and $W := var(\psi)$. With γ_V, γ_W, $\widehat{\phi}$ and $\widehat{\psi}$ defined as above we have $(\phi \vee \psi)_{\bowtie} = \gamma_V \wedge \gamma_W. \to .\widehat{\phi} \vee \widehat{\psi}$. Moreover, $var(\gamma_V \to \widehat{\phi}) \cap var(\gamma_W \to \widehat{\psi}) = \emptyset$. By Maximova–completeness of Λ we get $(\phi_{\bowtie} =)\gamma_V \to \widehat{\phi} \in \Lambda$ or $(\psi_{\bowtie} =)\gamma_W \to \widehat{\psi} \in \Lambda$. Hence $\phi \in n(\Lambda)$ or $\psi \in n(\Lambda)$. (2.) Let $\phi \vee \psi \in \Lambda$ and let ϕ and ψ have no variables in common. Then $\phi \vee \psi \in n(\Lambda)$ and — by Halldén–completeness of the latter — either $\phi \in n(\Lambda)$ or $\psi \in n(\Lambda)$. Hence $\phi \in \Lambda$ or $\psi \in \Lambda$. (3.) Similar. ⊣

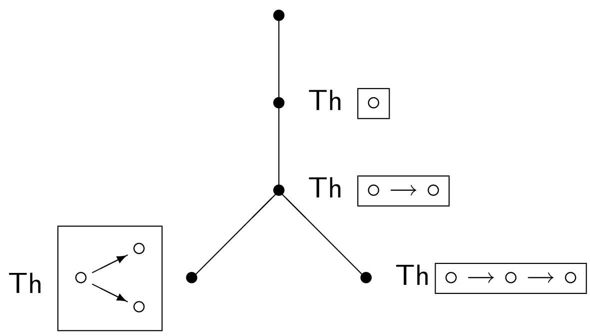
The Upper Part of $\mathcal{E}\,N$

For a class $\mathcal{K}$ the smallest variety containing $\mathcal{K}$ is $\mathsf{HSP}(\mathcal{K})$. A variety is congruence distributive if the lattice of congruences of any algebra from that variety is distributive. Any variety containing at least the two lattice operations $\wedge$ and $\vee$ is congruence distributive (see [1]). Hence the variety of Nelson algebras is congruence distributive. Jónsson has proved that in such varieties the subdirectly irreducible members of $\mathsf{HSP}(\mathcal{K})$ are already in the class $\mathsf{HSUp}(\mathcal{K})$, where $\mathsf{Up}(\mathcal{K})$ is the ultraproduct closure of $\mathcal{K}$. It follows that a finite class of finite algebras generates a finite number of subdirectly irreducible algebras. The relevance of this lies in the fact that the logic of a variety is determined already by the logic of the subdirectly irreducible members. The following theorem has been proved in [13], in essentially the same way.

Theorem 20 (Sendlewski) *A logic in $\mathcal{E}\,\mathbf{N}$ is tabular iff it is of finite codimension.*

Proof. First we show that if $\Lambda \subseteq \Theta$ and Λ is tabular, then so is Θ; in other words the tabular logics form a filter in the lattice of N–logics. Assume that $\Lambda = \mathsf{Th}(\mathfrak{C})$ for a finite Nelson algebra $\mathfrak{C}$. Nelson algebras are congruence distributive, since their reduct to $\cap$ and $\cup$ is a lattice. Θ is determined by its subdirectly irreducible algebras. By Jónsson's Lemma (see [1]), they are contained in $\mathsf{HSUp}\,\mathfrak{C} = \mathsf{HS}\,\mathfrak{C}$, again since $\mathfrak{C}$ is finite. It also follows that if Θ is tabular, then there are finitely many finite subdirectly irreducible algebras in the variety of Θ–algebras. This concludes the proof of the first claim. Now suppose Θ is of finite codimension in $\mathcal{E}\,\mathbf{N}$. Then $i(\Theta)$ is of finite codimension in $\mathcal{E}\,\mathbf{Int}$, and hence tabular (see [12]). Thus $i(\Theta)$ is determined by a finite algebra $\mathfrak{H}$. Then, since $n(i(\Theta)) \subseteq \Theta$, the logic of $\mathfrak{H}^{\bowtie}$ is contained in Θ. The first is tabular. Then the second is tabular as well. Now, assume conversely that Θ is tabular. Then the subdirectly irreducible algebras for Θ are finite and there is a finite number of them. Let them be $\mathfrak{C}_i$, $i < n$, and let k be the sum of their cardinalities. Suppose Λ is a proper extension of Θ. Then every subdirectly irreducible algebra for Λ is a subdirectly irreducible algebra for Θ, and the sum of their cardinalities must be less than k. Hence, Θ is of finite codimension. $\dashv$

These results can be used to gain some understanding of the upper part of $\mathcal{E}\,\mathbf{N}$ up to codimension 4. We start from the well-known picture of the upper part of the lattice of intermediate logics, up to codimension 3. (This picture can be found in [12] on page 294.)



There is exactly one logic of codimension 1, namely classical logic. It is characterized by the axiom $p \vee \neg p$, or alternatively by $((p \to q) \to p) \to p$. Every intermediate logic not containing classical logic is actually contained in it. Classical logic has one lower cover, the logic of tomorrow, or of *here–and–there*. Again, every logic not containing this logic is contained in it. Below, however, the situation is different. What we find is two lower covers, one being the logic of the three–element chain and the other the logic of the fork.

Let us see how this transfers to the N–logics. The least N–extension of classical logic is the logic of a three element algebra. This algebra is due to VAKARELOV. Namely, if we start from a single point poset $\langle\{0\}, \leq\rangle$, then we have three pairs of subsets, $1 := \langle\{0\}, \emptyset\rangle$, $0 := \langle\emptyset, \emptyset\rangle$ and $-1 := \langle\emptyset, \{0\}\rangle$. This algebra satisfies the axioms of classical logic but $\leftrightarrow$ is not a congruence. In fact, the three–element algebra is homomorphically simple and has only one nontrivial subalgebra, namely $\{-1, 1\}$. The logic of the latter two–element algebra extends the logic of the three-element algebra by the axiom $-x \leftrightarrow \sim x$. Therefore, the logic of the three–element algebra is of codimension 2.

$\rightarrow$	-1	0	1
-1	1	1	1
0	1	1	1
1	-1	0	1

	$-$
-1	1
0	1
1	-1

	$\sim$
-1	1
0	0
1	-1

Let us go now to the logic of here–and–there, that is, the logic of the poset $\langle\{0, 1\}, \leq\rangle$. Its minimal N–extension is that of the five elements $-2 := \langle\emptyset, \{0, 1\}\rangle$, $-1 := \langle\emptyset, \{1\}\rangle$, $0 := \langle\emptyset, \emptyset\rangle$; $1 := \langle\{1\}, \emptyset\rangle$ and $2 := \langle\{0, 1\}, \emptyset\rangle$. The operation $\cap$ corresponds to *min* and $\cup$ to *max*. The other connectives act as follows.

$\rightarrow$	-2	-1	0	1	2
-2	2	2	2	2	2
-1	2	2	2	2	2
0	2	2	2	2	2
1	-1	-1	0	2	2
2	-2	-1	0	1	2

	$-$
-2	2
-1	2
0	2
1	-1
2	-2

	$\sim$
-2	2
-1	1
0	0
1	-1
2	-2

The subalgebras must contain at least 2, the unit, and -2. Furthermore, they contain 1 iff they contain -1. This leaves three choices of proper subalgebras, namely $\{2, -2\}$, $\{2, 0, -2\}$, $\{2, 1, -1, -2\}$. It is readily checked that all three sets are closed under the operations. Hence these are all proper subalgebras. However, the first two arose already from the twist construction over the one-point poset. Thus, only the four–element subalgebra based on the set $\{-2, -1, 1, 2\}$ is new.

We will now discuss the structure of the lattice of extensions of $\mathbf{NC} := n(\mathbf{LC})$, where $\mathbf{LC} = \mathbf{Int} + p \rightarrow q. \vee .q \rightarrow p$. (Hence $\mathbf{NC} = \mathbf{N} + p \rightarrow q. \vee .q \rightarrow p$.) In general, given a logic Λ, let $\mathsf{Sfg}(\Lambda)$ denote the set of subdirectly irreducible, finitely generated algebras for Λ. Eevery extension is complete with respect to $\mathsf{Sfg}(\Lambda)$. For each extension Θ, let $S(\Theta)$ be the class of algebras in $\mathsf{Sfg}(\Lambda)$ which validate Θ. The sets $S(\Theta)$ are closed under finite unions and infinite intersections and can be thought of as the closed sets of a topological space over $\mathsf{Sfg}(\Lambda)$. Moreover, S is injective. It is therefore enough to study the space of closed sets in $\mathsf{Sfg}(\Lambda)$. In the case of $\Lambda = \mathbf{NC}$ we have a locally finite variety, so $\mathsf{Sfg}(\mathbf{NC})$ consists of finite algebras only. The induced topology satisfies T_0 (that is, for every set $\{x, y\}$ with two elements there exists an open set O such that $card(O \cap \{x, y\}) = 1$). T_0–spaces induce an ordering on the elements by $x \leq y$ iff $y \in \overline{\{x\}}$. Therefore, put $\mathfrak{A} \leq \mathfrak{B}$ iff $\mathsf{Th}\,\mathfrak{A} \leq \mathsf{Th}\,\mathfrak{B}$. Then $\mathfrak{B} \in \overline{\{\mathfrak{A}\}}$ iff $\mathfrak{A} \leq \mathfrak{B}$. Hence, closed sets in the space are upward closed with respect to $\leq$. The converse need not hold in general, however. If every upward closed set is also closed, the lattice of closed sets is a continuous lattice. In the present case it means that $\bigsqcup$ distributes over $\sqcap$ and $\bigsqcap$ over $\sqcup$. Below we will determine the structure of $\langle \mathsf{Sfg}(\mathbf{NC}), \leq \rangle$. The lattice $\mathcal{E}\,\mathbf{NC}$ is a continuous lattice as we will see, and so the poset actually describes $\mathcal{E}\,\mathbf{NC}$ up to isomorphism.

LC is the logic of linear Kripke–frames. Before we can start, we need some pieces of notation. We denote by $\mathfrak{ch}_n$ the linear Kripke–frame consisting of n points. It is based on the poset $\langle \{x_i \mid i < n\}, \lhd \rangle$ where $x_i \lhd x_j$ iff $j \leq i$. The set of cones of this frames is the collection of sets of the form $\uparrow x_i = \{x_j \mid x_i \lhd x_j\} = \{x_j \mid j \leq i\}$. The algebra of cones of $\mathfrak{ch}_n$ is denoted by $\mathfrak{ch}_n^+$. The following is known about the structure of $\mathcal{E}\,\mathbf{LC}$. (The notation used is as follows. For a lattice $\mathfrak{V}$ we denote by $1 + \mathfrak{V}$ the lattice obtained by adding a new bottom element. Also, $\mathfrak{V}^\perp$ is the dual (i. e. upside down) lattice corresponding to $\mathfrak{V}$.)

Theorem 21 *$\mathcal{E}\,\boldsymbol{LC} \cong 1 + \omega^\perp$. Every proper extension of $\boldsymbol{LC}$ is tabular. Moreover, a proper extension Λ has codimension n iff $\Lambda = \mathsf{Th}\,\mathfrak{ch}_n$.*

This can be shown as follows. Recall that a variety $\mathcal{V}$ is said to be **locally finite** if for every natural number n the algebra freely generated by n elements in $\mathcal{V}$ is finite.

Theorem 22 *The variety of $\boldsymbol{LC}$–algebras is locally finite.*

For a proof of the latter observe that **LC** has the finite model property and is complete with respect to the algebras $\mathfrak{ch}_m$, $m \in \omega$. Now take a finite number n. Let S be any set of $\leq n$ cones in $\mathfrak{ch}_m$, $m \in \omega$. The algebra generated by S in $\mathfrak{ch}_m$ has cardinality $\leq n + 2$. This allows to deduce that the size of the freely n–generated algebra is $\leq n! \times (n + 2)$. Now Theorem 21 follows by observing that first that every extension is locally finite as well and therefore has finite model property. For a proper extension Λ of **LC** there must exist an n such that $\mathfrak{ch}_n$ is not a frame for Λ. Then no $\mathfrak{ch}_m$ for $m \geq n$ is a frame for Λ.

Theorem 23 *Let Λ be an intermediate logic. The variety of Λ–algebras is locally finite iff the variety of $n(\Lambda)$–algebras is locally finite.*

Proof. Let $\mathcal{V} := \mathsf{Alg}\,\Lambda$ be locally finite. Let $\mathcal{W}$ be the variety generated by $\mathcal{V}^{\bowtie}$; then $\mathcal{W} = \mathsf{Alg}\,n(\Lambda)$. Now let n be a natural number and denote by $\mathfrak{F}_{\mathcal{W}}(n)$ ($\mathfrak{F}_{\mathcal{V}}(n)$) the algebra generated freely in $\mathcal{V}$ ($\mathcal{W}$) by the elements a_i, $i < n$. Consider the map $i : \sim a_i \mapsto a_{n+i}$. This map can be extended to an I–homomorphism from $\mathfrak{F}_{\mathcal{W}}(n)$ into $\mathfrak{F}_{\mathcal{V}}(2n)$. This map is injective. Since the latter algebra is finite, so is the former. The other direction is easy. $\dashv$

Corollary 24 *The variety of $\boldsymbol{NC}$–algebras is locally finite.* $\dashv$

Thus every extension of **NC** is the logic of some set of finite algebras. In order to obtain an insight into the structure of $\mathcal{E}\,\mathbf{NC}$ we will look somewhat closer at the model structures for **LC**. Take two chains $\mathfrak{ch}_m$ and $\mathfrak{ch}_n$, and let $m \geq n$. (a.) There exists exactly one imbedding $\iota_{mn} : \mathfrak{ch}_n \rightarrowtail \mathfrak{ch}_m$, sending x_i to x_i, $i < n$. (b.) Each p–morphism $\mathfrak{ch}_m \twoheadrightarrow \mathfrak{ch}_n$ for $m > n$ factors through a p–morphism $\mathfrak{ch}_{n+1} \twoheadrightarrow \mathfrak{ch}_n$. Moreover, for each $i < n$, the following map is a p–morphism

$$\epsilon_n^i : x_j \mapsto \begin{cases} x_j & \text{if } j \leq i \\ x_{j-1} & \text{if } j > i \end{cases}$$

Hence any surjective p–morphism is a composition of maps of the form ϵ_n^i, for some $i, n \in \omega$. Now let $\mathfrak{ch}_n^+$ be the Heyting algebra of cones of $\mathfrak{ch}_n$. Using the duality between finite Heyting algebras and finite Kripke–frames we obtain the following. (a.) Any I–homomorphism $\mathfrak{ch}_m^+ \twoheadrightarrow \mathfrak{ch}_n^+$ is of the form ι_{mn}^+. The congruence δ corresponding to it has the cosets $[\uparrow x_i]\delta = \{\uparrow x_i\}$ if $i < n$ and $[\uparrow x_i]\delta = \{\uparrow x_j \mid j \geq i\}$ for $n \leq i < m$. (b.) Any imbedding $\mathfrak{ch}_m^+ \rightarrowtail \mathfrak{ch}_n^+$ can be factored through imbeddings of the form $(\epsilon_n^i)^+ : \mathfrak{ch}_n^+ \rightarrowtail \mathfrak{ch}_{n+1}^+$. We have

$$(\epsilon_n^i)^+ : \uparrow x_j \mapsto \begin{cases} \uparrow x_j & \text{if } j \leq i \\ \uparrow x_{j+1} & \text{if } j > i \end{cases}$$

Now let us apply this to the constructive case. First, we know that every logic extending **NC** has the finite model property. A finite Nelson algebra for **NC** is a subalgebra of some $\mathfrak{ch}_n^+$, for some n. Its elements are of the following form: (i.) $\langle \uparrow x_i, \emptyset \rangle$; we call such elements **upper elements**, (ii.) $\langle \emptyset, \emptyset \rangle$; we call it the **middle element**, (iii.) $\langle \emptyset, \uparrow x_i \rangle$; we call these elements **lower elements**. The reduct of a Nelson algebra $\mathfrak{N}$ to $\cap$ and $\cup$ is a lattice; the ordering relation is denoted by $\leq$. Moreover, if x, y are elements of $\mathfrak{N}$ then put $[x, y] := \{z \mid x \leq z \leq y\}$. A congruence on an algebra is **trivial** if no coset has more than one element or there exists exactly one coset.

Lemma 25 *Let $\mathfrak{L}$ be a subalgebra of $(\mathfrak{ch}_n^+)^{\bowtie}$ for some n and let θ be a nontrivial congruence on $\mathfrak{L}$. Then θ has exactly two nontrivial cosets, and they are of the form $[x, 1]$ and $[\sim 1, \sim x]$ for some upper element x. All subalgebras of $\mathfrak{ch}_n^+)^{\bowtie}$ are therefore subdirectly irreducible.*

Proof. Let $\mathfrak{L}$ be a Nelson algebra of this type. Consider a congruence θ. Then if θ is not the diagonal, there is an upper element in a nontrivial coset. For let $y \in [x]\theta$, $y \neq x$. If both of them are not upper elements, at least one is lower. Let it be x. Then, as $\sim y \in [\sim x]\theta$, we have an upper element in a nontrivial coset. Furthermore, cosets $[x]\theta$ are convex sets with respect to $\leq$, since the induced homomorphism p_θ is a lattice homomorphism. Let x be an upper element in a nontrivial coset, and let $y \in [x]\theta$ be a nonlower element such that $y \leq x \not\leq y$. (Such an element exists.) Then $1 = y \to x \;\theta\; x \to y = y$. Hence the coset of an upper element is nontrivial iff it contains the unit. Similarly for lower elements. Suppose now that there are at least two cosets. Then $[1]\theta \neq [\sim 1]\theta$. Let x be the least element of $[1]\theta$ (with respect to $\leq$). Then $[1]\theta = [x, 1]$ and $[\sim 1]\theta = [\sim 1, \sim x]$. $\dashv$

It can be checked that if the cardinality of the coset $[1]\theta$ is k then the projection p_θ is the dual of the imbedding $\iota_{n,n+k-1}$. Now let us turn to subalgebras of $(\mathfrak{ch}_n^+)^{\bowtie}$.

Lemma 26 *Let $\mathfrak{L}$ be the twist algebra of $\mathfrak{ch}_n^+$ and let M be a maximal proper subset closed under all operations. Then either (a.) M is the set of non–middle elements, or (b.) M contains all elements except for $\langle \uparrow x_i, \emptyset \rangle$ and $\langle \emptyset, \uparrow x_i \rangle$ for some $i < n$.*

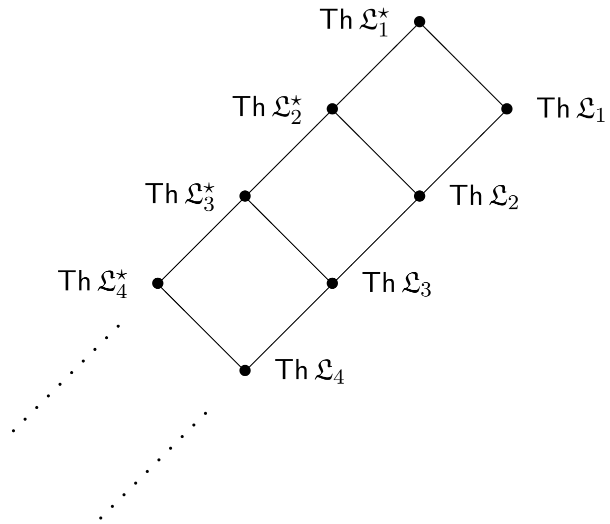
Proof. It is straightforward to check that the sets are closed under all operations. Now let M be a proper subset closed under all operations. Suppose that M contains the middle element. There exists a nonmiddle $x \notin M$. Since also $\sim x \notin M$, we can assume that x is upper, i. e. of the form $\langle \uparrow x_i, \emptyset \rangle$. This shows the theorem. $\dashv$

The subalgebras of the type (b.) are actually isomorphic to $(\mathfrak{ch}_{n-1}^+)^{\bowtie}$. Thus, all Nelson algebras for **NC** fall into either of two classes: (1.) the twist algebra of $\mathfrak{ch}_n^+$, (2.) the algebra of nonmiddle elements of the twist algebra of $\mathfrak{ch}_n^+$. The algebra of the first kind will be denoted by $\mathfrak{L}_n$, the algebra of the second kind by $\mathfrak{L}_n^\star$.

Theorem 27 *The logic $\boldsymbol{NC}^\star := \boldsymbol{NC} + \neg\!\sim\! p \leftrightarrow \neg\neg p$ is the logic of all $\mathfrak{L}_n^\star$. $\boldsymbol{NC}^\star$ is pretabular and the lattice of extensions is isomorphic to $1 + \omega^\perp$.*

Proof. Take $x = \langle \uparrow x_i, \emptyset \rangle$. Then $-\!\sim\! x = \langle 1, \emptyset \rangle$ and $- - x = \langle 1, \emptyset \rangle$. Let now $x = \langle \emptyset, \uparrow x_i \rangle$. Then $-\!\sim\! x = \langle \emptyset, 1 \rangle$ and $- - x = \langle \emptyset, 1 \rangle$. However, $-\!\sim \langle \emptyset, \emptyset \rangle = \langle 1, \emptyset \rangle$, but $- - \langle \emptyset, \emptyset \rangle = \langle \emptyset, 1 \rangle$. The remaining claims are more or less straightforward. $\dashv$

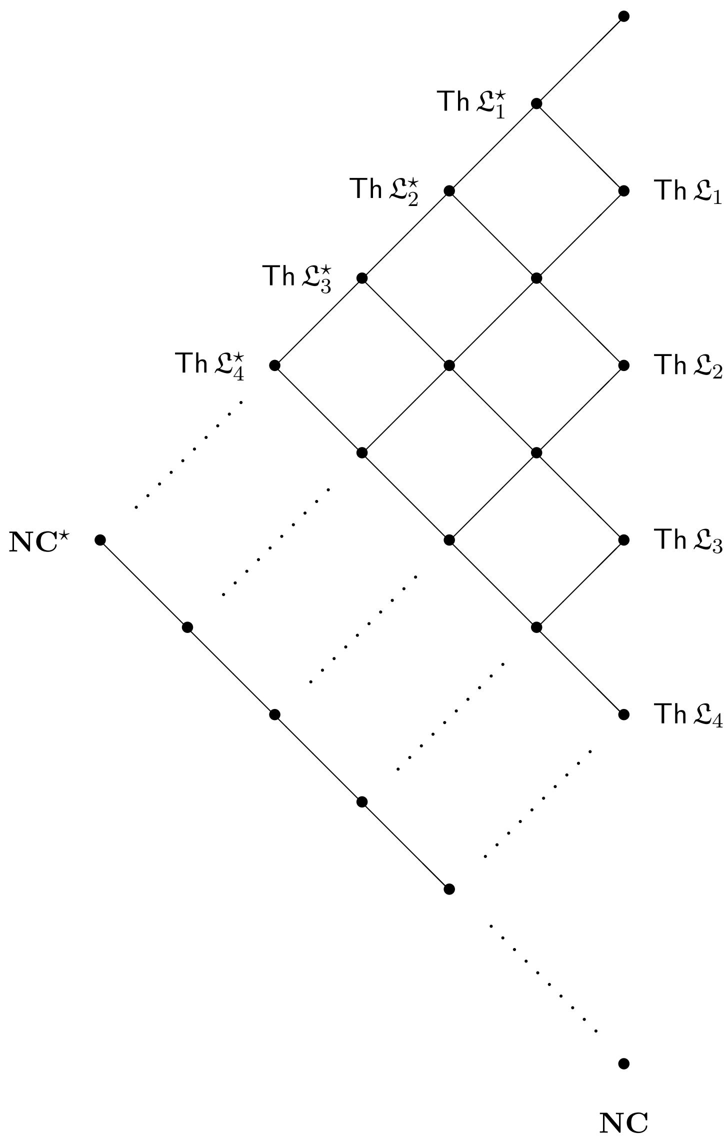
The elements of $\mathfrak{L}_n^\star$ can therefore be characterized by the fact that they satisfy a requirement that eventually a proposition must be decided, either positively or negatively. (So there is a kind of 'last judgement' day.) Finally, the lattice of extensions of **NC** is largely determined by the poset of $\bigsqcap$–irreducible elements in the lattice. These are of finite codimension and correspond to logics of a finite Nelson algebra. This algebra in turn is isomorphic to $\mathfrak{L}_n^\star$ or $\mathfrak{L}_n$ for some n. Since for finite subdirectly irreducible Nelson algebras $\mathsf{Th}\,\mathfrak{M} \subseteq \mathsf{Th}\,\mathfrak{N}$ iff $\mathfrak{N} \in \mathsf{HSUp}\,\mathfrak{M} = \mathsf{HS}\,\mathfrak{M}$ we get the following picture. $\mathsf{Th}\,\mathfrak{L}_m \subseteq \mathsf{Th}\,\mathfrak{L}_n$ iff $n \leq m$, $\mathsf{Th}\,\mathfrak{L}_m \subseteq \mathsf{Th}\,\mathfrak{L}_n^\star$ iff $n \leq m$, $\mathsf{Th}\,\mathfrak{L}_m^\star \subseteq \mathsf{Th}\,\mathfrak{L}_n$ never and $\mathsf{Th}\,\mathfrak{L}_m^\star \subseteq \mathsf{Th}\,\mathfrak{L}_n^\star$ iff $n \leq m$. The picture below shows the poset of $\bigsqcap$–irreducible logics in $\mathcal{E}\,\mathbf{NC}$.



An **index** is a pair $\langle i, j\rangle$ such that $i, j \leq \omega$ are ordinal numbers and $i \leq j$. The set of all indices is denoted by *Ind*. For indices $\mathfrak{i}_1 := \langle i_1, j_1\rangle$ and $\mathfrak{i}_2 := \langle i_2, j_2\rangle$ let $\mathfrak{i}_1 \leq \mathfrak{i}_2$ iff $i_1 \geq i_2$ and $j_1 \geq j_2$. The set of indices forms a lattice with respect to this ordering, denoted by $\mathfrak{Ind}$. Let Θ be an extension of **LC**. The **index** of Θ, $ind(\Theta)$, is obtained as follows. $ind(\Theta) = \langle i, j\rangle$ where i the least upper bound of all m such that $\mathfrak{L}_m \in \mathsf{Alg}\,\Theta$ (if that set is empty $i = 0$ is a least upper bound); and j is the least upper bound of all n such that $\mathfrak{L}_n^\star \in \mathsf{Alg}\,\Theta$. It is clear that $i \leq j$, so $\langle i, j\rangle$ is an index. The assignment of indices can easily be shown to be an isotonic map of posets, and to be bijective. We conclude the following theorem.

Theorem 28 *The map $\Theta \mapsto ind(\Theta)$ is an isomorphism from $\mathcal{E}\,\boldsymbol{NC}$ onto $\mathfrak{Ind}$, the lattice of indices. Hence $\mathcal{E}\,\boldsymbol{NC}$ is continuous.* $\dashv$

THE LATTICE $\mathcal{E}\,\mathbf{NC}$



Given a lattice $\mathfrak{L}$, an element x is called a **splitting element** if there exists a y such that for every element z either $z \leq x$ or $z \geq y$, but not both. If x is a splitting element, y is called the **splitting** of $\mathfrak{L}$ by x and denoted by $\mathfrak{L}/x$. We write $\mathfrak{L}/\{x, y\}$ for $\mathfrak{L}/x \sqcup \mathfrak{L}/y$. With $\mathfrak{L} = \mathcal{E}\,\mathbf{NC}$ and $x = \Theta$ we write $\mathbf{NC}/\Theta$ rather than $\mathcal{E}\,\mathbf{NC}/\Theta$ and $\mathbf{NC}/\mathfrak{N}$ instead of $\mathbf{NC}/\mathsf{Th}\,\mathfrak{N}$. Take an extension Θ of $\mathbf{NC}$. If $ind(\Theta) = \langle \omega, \omega \rangle$ then $\Theta = \mathbf{NC}$. Otherwise $ind(\Theta) = \langle i, \omega \rangle$ for $i \in \omega$ and so $\Theta = \mathbf{NC}/\mathfrak{L}_{i+1}$ or $ind(\Theta) = \langle i, j \rangle$ with $i, j \in \omega$ and then $\Theta = \mathbf{NC}/\{\mathfrak{L}_{i+1}, \mathfrak{L}^*_{j+1}\}$.

Theorem 29 *The splitting elements of $\mathcal{E}\,\mathbf{NC}$ are of the form $\mathsf{Th}\,\mathfrak{L}^\star_n$ and $\mathsf{Th}\,\mathfrak{L}_n$, $n > 0$.*

Every proper extension of $\boldsymbol{NC}$ *is of the form* $\boldsymbol{NC}/P$*, where* P *is a set of splitting logics of size at most 2. Every logic* Θ *such that* $\boldsymbol{NC} \subsetneq \Theta \subseteq \boldsymbol{NC}^\star$ *is of the form* $\boldsymbol{NC}/\mathfrak{L}_n$ *for some* $n > 0$*. Moreover,* $\mathsf{Th}\,\mathfrak{L}_n = \boldsymbol{NC}/\mathfrak{L}^\star_{n+1}$, $n > 0$. ⊣

It is not hard to show that for every splitting logic Θ there exists a formula ϕ such that $\mathcal{E}\,\mathbf{NC}/\Theta = \mathbf{NC} + \phi$. (Suppose $\mathbf{NC}/\Theta = \mathbf{NC} + \Delta_1 + \Delta_2$ and $\mathbf{NC} + \Delta_i \subsetneq \mathbf{NC}/\Theta$, $i = 1, 2$. Then $\mathbf{NC} + \Delta_i \subseteq \Theta$ for $i = 1, 2$, and so $\mathbf{NC} + \Delta_1 + \Delta_2 \subseteq \Theta$. Contradiction.) Hence every extension is finitely axiomatizable, and decidable. Moreover, for every extension Θ of $\mathbf{NC}$ the problem '$\mathbf{NC} + \phi \subseteq \Theta$' is decidable; for it is equivalent to '$\phi \in \Theta$'. Now suppose that Θ is a splitting element of $\mathbf{NC}$. Then '$\mathbf{NC} + \phi \supseteq \mathbf{NC}/\Theta$' is decidable as well, for it is equivalent to '$\mathbf{NC} + \phi \nsubseteq \Theta$'. Likewise, for a finite set P, the problem '$\mathbf{NC} + \phi \supseteq \mathbf{NC}/P$' is decidable; for it is the conjunction of the problems '$\mathbf{NC} + \phi \supseteq \mathbf{NC}/\Theta$' with $\Theta \in P$.

Theorem 30 $\mathcal{E}\,\boldsymbol{NC}$ *is a continuous distributive lattice. The variety of* $\boldsymbol{NC}$*–algebras is locally finite. Every extension of* $\boldsymbol{NC}$ *is finitely axiomatizable, has the finite model property and is decidable. Moreover, for every logic* Λ *containing* $\boldsymbol{NC}$ *the problem '*$\boldsymbol{NC} + \phi = \Lambda$*' is decidable.*

References

[1] Stanley Burris and H. P. Sankappanavar. *A Course in Universal Algebra*. Number 78 in Graduate Texts in Mathematics. Springer, 1981.

[2] Michael Gelfond and Vladimir Lifschitz. The Stable Model Semantics for Logic Programs. In *Proceedings of the 5th International Conference on Logic Programming*, pages 1070 – 1080. MIT–Press, 1988.

[3] Michael Gelfond and Vladimir Lifschitz. Logic Programs with Classical Negation. In *Proceedings of the ICLP-90*, pages 579 – 597. MIT–Press, 1990.

[4] Valentin Goranko. The Craig Interpolation Theorem for Propositional Logics with Strong Negation. *Studia Logica*, 44:291 – 317, 1985.

[5] Heinrich Herre and David Pearce. Disjunctive logic programming, constructivity and strong negation. In *Logics in AI. Proceedings of the European Workshop JELIA 92*, number 633 in Lecture Notes in Artificial Intelligence. Springer, 1992.

[6] Larisa L. Maksimova. Craig's Theorem in superintionistic logics and amalgamable varieties of Pseudo–boolean algebras. *Algebra and Logic*, 16:427 – 455, 1977.

[7] David Nelson. Constructible falsity. *Journal of Symbolic Logic*, 14:16 – 26, 1946.

[8] David Pearce. A New Characterization of Stable Models. unpublished, 1995.

[9] David Pearce and Gerd Wagner. Reasoning with negative information, I: Strong negation in logic programs. In *Language, Knowledge and Intentionality (Acta Philosophica Fennica 49)*, pages 405 – 419. Helsinki, 1990.

[10] David Pearce and Gerd Wagner. Logic programming with strong negation. In Peter Schroeder-Heister, editor, *Extensions of Logic Programming*, number 475 in Lecture Notes in Artificial Intelligence, pages 311 – 326. Springer, 1991.

[11] Helena Rasiowa. N-lattices and constructive logic with strong negation. *Fundamenta Mathematicae*, 46:61 – 80, 1958.

[12] Wolfgang Rautenberg. *Klassische und nichtklassische Aussagenlogik*. Vieweg Verlag, Braunschweig/Wiesbaden, 1979.

[13] Andrzej Sendlewski. Some investigations of varieties of N–lattices. *Studia Logica*, 43:257 – 280, 1984.

[14] Andrzej Sendlewski. Nelson Algebras Through Heyting Ones. *Studia Logica*, 49:106 – 126, 1990.

[15] Andrzej Sendlewski. Axiomatic Extensions of the Constructive Logic with Strong Negation and the Disjunction Property. *Studia Logica*, 55:377 – 388, 1995.

[16] Dimiter Vakarelov. Notes on N–lattices and constructive logic with strong negation. *Studia Logica*, 36:109 – 125, 1977.