

## Judgment and consequence relations

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*ABSTRACT. In this paper I argue that a variety of consequence relations can be subsumed under a common core. The reduction proceeds by taking the unconditional consequence, or judgment, as basic and deriving the conditional consequence via a uniform abstraction scheme. A specific outcome is that it is better not to base such a scheme on the semantic notion of a matrix and valuation but rather on theories and substitutions. I will also briefly look at consequence relations that are not reducible in this way.*

*KEYWORDS: consequence relations, modal logic, judgement.*

DOI:10.3166/JANCL.20.1–13 © 2010 Lavoisier, Paris

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## 1. Introduction

This paper is devoted to the 20th anniversary of the Journal of Applied Non-Classical Logic. My principal aim is to get clear about the diverse notions of consequence in modal logic. Such an enterprise is needed in applications for computational linguistics and natural language (see (Kracht, 2010)). My vision of applied (non)classical logic is that it involves not only hard theorems but also conceptual clarifications. My contribution is of the latter kind. It will show that while in modal logic we use a homogeneous notion of consequence, many applications in computational linguistics and elsewhere suggest to use heterogeneous consequence relations.

In this paper I look into the standard definitions of logical consequence and show that they can be unified under a common scheme. Standardly, consequence relations are defined via the preservation of truth. Here I propose to generalize this as follows: consequence is the preservation of a (truth related) property of propositions, called *mode*. Which one that is depends on the particular notion of truth that we are considering. In modal logic we find several concurrent modes, and this defines various consequence relations for a given logic.

To start, a **rule** is a pair, written  $\Delta/\varphi$ , where  $\Delta$  is a set of formulae and  $\varphi$  a single formula. A **consequence relation** is a set of rules which has certain properties (such as closure under consequence). Typically a consequence relation is a collection of rules that preserve truth. (In general, consequence relations may be derived from any property of formulae.) A rule is **correct** if *whenever  $\Delta$  is true so is  $\varphi$* . If this is so we write

$$\Delta \vdash \varphi \tag{1}$$

There is an important special case to be considered, namely  $\Delta = \emptyset$ . In this case the above reduces to

$$\vdash \varphi \tag{2}$$

(We omit  $\emptyset$  in the notation.) This is read “ $\varphi$  is valid” (or “ $\varphi$  is a tautology”). Notice the superficial discrepancy between the two; where (1) talks about the *truth* of  $\varphi$ , (2) talks about its theoremhood. The explanation lies in a close analysis of the phrase “whenever”. By saying “whenever” we effectively quantify over valuations. We take (1) to say that whatever valuation makes all members of  $\Delta$  true must also make  $\varphi$  true. If  $\Delta = \emptyset$  this is equivalent to saying that  $\varphi$  is a tautology.

As already indicated, we may also think of logical consequence as preserving a different property of formulae, not just truth. We may for example require rules to preserve validity, or theoremhood. Then we get a different interpretation of (1), namely: whenever  $\Delta$  is a theorem, so is  $\varphi$ . Also in this case, (2) reduces to the claim that  $\varphi$  is a theorem. However, again we have to say what we mean by “whenever”. Here, it turns out that the common understanding is that “whenever” means: *for all substitutions*. Thus we read (1) as saying that given a substitution  $s$  such that for all  $\delta$  from  $\Delta$   $s(\delta)$  is a theorem of the logic, so is  $s(\varphi)$ . Under this interpretation (2) says that all substitution instances of  $\varphi$  are theorems. This is equivalent to saying that  $\varphi$  itself is a theorem.

The basic scheme that can be extracted from this is as follows. We consider a particular mode of judgment, say “truth under a valuation”, “theoremhood” and so on. Write “ $\succ$ ” for this mode. Then “ $\succ$ ” determines a consequence relation in the following way.

“ $\Delta \succ \varphi$ ” iff for all circumstances  $c$ : if for all  $\delta \in \Delta$  we have  $\succ_c \delta$  then we also have  $\succ_c \varphi$ .

In what is to follow I shall precisify what I mean by “mode of judgment” and “circumstance” and how  $\succ_c$  and  $\succ$  are related. It will turn out that circumstances are pairs  $\langle s, T \rangle$ , where  $s$  is a substitution and  $T$  a theory of the logic.

## 2. Truth

Let us start with a propositional language  $\mathcal{L}$ , consisting as usual of a set  $\text{Con}$  of constants, a set  $\text{Var}$  variables, and a set  $\text{Op}$  of connectives. The signature is  $\Omega$ , a function that assigns an arity to each symbol of  $\text{Op}$ . This language generates an algebra  $\mathfrak{Fml}_{\mathcal{L}}(\text{Var})$ . A substitution is an endomorphism of  $\mathfrak{Fml}_{\mathcal{L}}(\text{Var})$ , that is, a map  $s : \mathcal{L} \rightarrow \mathcal{L}$  that commutes with the connectives and maps each constant onto itself. The objects of  $\mathcal{L}$  are **formulae**. A **logic** can be identified with a set  $L \subseteq \mathfrak{Fml}_{\mathcal{L}}(\text{Var})$ . We require that a logic be closed under substitutions:  $L \subseteq s[L]$  for all  $s$ . A **consequence relation** by contrast is a particular subset  $\vdash$  of  $\wp(\mathfrak{Fml}_{\mathcal{L}}(\text{Var})) \times \mathfrak{Fml}_{\mathcal{L}}(\text{Var})$  (see (Wójcicki, 1988)). Write  $\Delta \vdash \varphi$  in place of  $\langle \Delta, \varphi \rangle \in \vdash$ . A consequence relation satisfies the following conditions.

- 1)  $\Delta \vdash \varphi$  if  $\varphi \in \Delta$ .
- 2) If  $\Delta \subseteq \Delta'$  and  $\Delta \vdash \varphi$  then  $\Delta' \vdash \varphi$ .
- 3) If  $\Delta \vdash \chi$  for all  $\chi \in \Sigma$  and  $\Sigma \vdash \varphi$  then also  $\Delta \vdash \varphi$ .

$\vdash$  is **structural** if  $\Delta \vdash \varphi$  implies  $s[\Delta] \vdash s(\varphi)$ . (As usual,  $f[X]$  denotes the direct image of the set  $X$  under  $f$ .)

The set of theorems of  $\vdash$  is the set of all  $\varphi$  such that  $\emptyset \vdash \varphi$ . I am interested here in deriving a consequence relation from a logic via a mode, that is, a property of formulae.

The semantics of consequence relations is based on the notion of a matrix. A **matrix** is a pair  $\mathfrak{M} = \langle \mathfrak{A}, D \rangle$ , where  $\mathfrak{A}$  is an  $\Omega$ -algebra and  $D$  a subset of the carrier set of  $\mathfrak{A}$ , called the set of **distinguished values**. Let  $\mathcal{K}$  be a class of matrices. Then  $\vdash_{\mathcal{K}}$  is defined as follows.

**DEFINITION 1.** —  $\Delta \vdash_{\mathcal{K}} \varphi$  iff for all  $\mathfrak{M} = \langle \mathfrak{A}, D \rangle \in \mathcal{K}$ , and every homomorphism  $h : \mathfrak{Fml}_{\mathcal{L}}(\text{Var}) \rightarrow \mathfrak{A}$ : if  $h[\Delta] \subseteq D$  then  $h(\varphi) \in D$ .

The sets  $D$  play the role of deductively closed sets, or **theories**. In fact, rather than taking any algebra we may confine ourselves to the free algebra on a fixed countably infinite set of generators. The deductively closed sets will be the theories. The basic

semantic object to look at is therefore the collection of all matrices  $\langle \mathfrak{M}_{\mathcal{L}}(\text{Var}), T \rangle$ , where  $T$  is an  $L$ -theory. In classical logic, it is enough to study maximally consistent sets, known in modal logic as worlds.

Our standard case is that  $\mathcal{L}$  is the language of monomodal logic. A **(modal) logic** is a subset  $L$  of  $\mathcal{L}$  that contains all theorems of PC, the formula  $\Box(p_0 \rightarrow p_1) \rightarrow (\Box p_0 \rightarrow \Box p_1)$ , and is closed under substitution, and Modus Ponens (MP).  $L$  is **normal** if it is also closed under necessitation, that is, if from  $\varphi \in L$  we may conclude  $\Box\varphi \in L$ . A different formulation (and the one I shall adopt here) is that a logic is a subset of  $\mathcal{L}$  containing  $K$  (the theorems of the minimal system  $K$ ) that is closed under substitution and MP. Given a logic  $L$ , call an  $L$ -**world** any maximal set of formulae that contains  $L$ , is consistent and closed under MP. Given a world  $w$ , a formula  $\varphi$  holds at  $w$ , or is true at  $w$ , if  $\varphi \in w$ .

**DEFINITION 2.** — A **mode** is a subset of  $\mathcal{L}$  or a property of formulae; alternatively, it is a function “ $\succ$ ” from propositions to  $\{0, 1\}$ . A **judgment** on  $\mathcal{L}$  is a pair consisting of a mode and a proposition from  $\mathcal{L}$ . If  $\succ(\varphi) = 1$  we simply say that  $\varphi$  is **accepted** (under  $\succ$ ); and that it is **rejected** otherwise.

With “ $\succ$ ” a mode and “ $\varphi$ ” a formula, the expression “ $\succ(\varphi)$ ” is a judgment. We simply write “ $\succ\varphi$ ” in place of “ $\succ(\varphi) = 1$ ”. I give two examples. Given a world  $w$ , let “ $\succ_w\varphi$ ” iff  $\varphi \in w$ , and let “ $\succ_L\varphi$ ” iff  $\varphi \in L$ . In the first case  $\varphi$  is accepted if it is true in the world, in the second case it is accepted if it is a theorem of the logic.

It seems that we are replicating here classical logic at a higher level. This is true insofar as we allow for only two outcomes. It is false, however, for the reason that the outcomes are of different character. The outcome “accept” means: is accepted under the mode, while “rejected” means: is not accepted under the mode. The latter allows for the possibility that the formula has some other truth value. It need not be false. For example, if we have a matrix of truth values, say  $\mathfrak{M} = \langle \mathfrak{A}, D \rangle$ , and a valuation  $\beta$ , we can define the following mode:  $\vdash_{\langle \mathfrak{M}, \beta \rangle} \varphi$  iff  $\beta(\varphi) \in D$ .

Basically, however, rules contain variables, and the treatment of variables poses problems. Two parameters come into play as soon as we have variables. The first is that a logic may have different theories (or, equivalently, different matrices). The second is that variables must be given values. The consequence is obtained by abstracting over these two parameters. The circumstances are pairs consisting of a matrix from  $\mathcal{K}$  and a valuation. In abbreviated form:

$$\vdash_{\mathcal{K}} = \bigcap_{\mathfrak{M} \in \mathcal{K}} \vdash_{\langle \mathfrak{M}, \beta \rangle} \quad (3)$$

Thus, a formula  $\varphi$  follows from a set  $\Delta$  just in case for any matrix and any valuation, if  $\Delta$  is true under the valuation, so is  $\varphi$ . This is the standard definition. However, there are alternative concepts of truth; one is theoremhood. This concept is already abstracted from the valuation. If we abstract this notion we get a different consequence relation. I shall show how these consequence relations are generated in classical logic, PC, before moving on to the really interesting case, modal logic.

### 3. Classical truth

Let us talk about classical truth. Let  $T$  be a maximally consistent set. Then we say that “ $\varphi$  is true $_T$ ” holds iff  $\varphi \in T$ . However, we can explicate the concept of “true $_T$ ” independently as follows. Write “ $\vdash_T \varphi$ ” to say that  $\varphi$  is true $_T$ . The following now holds.

$$\begin{aligned}
 \vdash_T \top & \quad \text{iff } \textit{true} \\
 \vdash_T \perp & \quad \text{iff } \textit{false} \\
 \vdash_T \neg\varphi & \quad \text{iff } \not\vdash_T \varphi \\
 \vdash_T \varphi \wedge \chi & \quad \text{iff } \vdash_T \varphi \text{ and } \vdash_T \chi \\
 \vdash_T \varphi \vee \chi & \quad \text{iff } \vdash_T \varphi \text{ or } \vdash_T \chi \\
 \vdash_T \varphi \rightarrow \chi & \quad \text{iff } \not\vdash_T \varphi \text{ or } \vdash_T \chi
 \end{aligned} \tag{4}$$

Also, given (4),  $T$  is maximally consistent. In this definition we consider truth in the classical sense. A proposition is either true or false. If it is rejected, that is, if  $\not\vdash_T \varphi$  this is because the proposition is false. So, no subjective element enters here. Truth is independent of whether we know of it. And hence if  $\not\vdash_T \varphi$  then we also have  $\vdash_T \neg\varphi$ .

Notice that this definition of classical truth is fully compositional. This means that it can define truth without recourse to any other mode. Consider by contrast weak Kleene three valued logic. It is a three valued logic with values 0, 1 and  $u$ . Hence a judgment is a boolean combination of “has value 0”, “has value 1” and “has value  $u$ ”. The truth table of  $\neg$  is  $\neg 1 = 0$ ,  $\neg 0 = 1$  and  $\neg u = u$ . Now,  $\neg\varphi$  has value 0 if  $\varphi$  has value 1 and  $\neg\varphi$  has value 1 if  $\varphi$  has value 0. However, if  $\varphi$  has value  $u$ , so does  $\neg\varphi$ . Now, assume that “ $\vdash_T \varphi$ ” stands for “ $\varphi$  has value 1”. Then “ $\vdash_T \neg\varphi$ ” cannot be reduced to a judgment on  $\varphi$  in terms of the mode  $\vdash_T$  alone. For it has to be the judgment that  $\varphi$  has value 0, which cannot be expressed. Hence, in this situation we need a second mode, write it “ $\succ$ ”, where “ $\succ\varphi$ ” means that  $\varphi$  has value 0. (Notice that we could also make it denote “ $\varphi$  has value  $u$ ”. The problem is the same.) Then it is possible to define “ $\vdash_T \neg\varphi$ ” as “ $\succ\varphi$ ” and “ $\succ\varphi$ ” by “ $\vdash_T \varphi$ ”. So, the definition of “ $\varphi$ ” can be established by mutual induction with “ $\succ$ ”. But it cannot be established in its own terms. Notice finally that we also need to stipulate “ $\vdash \varphi$ ” and “ $\succ\varphi$ ” cannot both hold (technically we would have to require this only for the variables and constants). Classical logic is weak Kleene logic in which “ $\succ\varphi$ ” means the same as “ $\not\vdash_T \varphi$ ”.

Using (4) we can reduce the truth of a complex expression to that of simple expressions. Once we reach the ultimate component expressions (propositional variables or constants) no more reduction takes place. The interpretation of constants is fixed, but the interpretation of variables is not. A formula is considered *valid* (under a given concept of truth) if it is true under all changes in the truth of variables. It is in this way that the notion of validity is derived from the notion of truth. So, if  $p$  is a variable it may either happen that  $\vdash_T p$  or that  $\not\vdash_T p$ , but not both. However, there is a theory  $U$  such that  $p \notin U$ . Then  $\vdash_T p$  and  $\not\vdash_U p$ . For complex formulae we use (4). There are now two routes we can take. We can either declare that “ $\vdash_T$ ” and “ $\vdash_U$ ” are different notions of truth, or we declare them one and the same. In the latter case we need to show in what way they are the same. The idea is that they are based on a *valuation*.

For a theory  $T$ , “ $\vdash_T$ ” is based on the valuation  $\beta$  for which  $\beta(p) = 1$  iff  $p \in T$ . (And we write “ $\vdash_\beta$ ” in place of “ $\vdash_T$ ”.) Hence, we consider the meaning of a proposition not to be a truth value but rather a set of valuations. This leads to a canonical interpretation of variables as values  $p^* := \{\beta : \beta(p) = 1\}$ . We can write definitions for the function  $(-)^*$  so that we get

$$\varphi^* = \{\beta : \beta(\varphi) = 1\} \quad (5)$$

Here we may simply assume that a circumstance is a valuation. Write “ $\vdash_\beta$ ” for the truth definition of such a circumstance. Then we have  $\vdash_\beta \varphi$  iff  $\beta \in \varphi^*$  iff  $\beta(\varphi) = 1$ . Alternatively, we can define this using (4) and the base clause stating that  $\vdash_\beta p$  iff  $\beta(p) = 1$ . It is important to realise that valuations are just a matter of separating out the variables from the constants. A constant is a proposition that does not change truth value. It is either true or false; if it is true, it is always true. If it is false, it must always be so. Variables can be either true or false, and — depending on circumstances — they are true here and false there.

It turns out, however, that a different interpretation of variables is to be preferred. Let us look at a different mode, namely validity. Write “ $\Vdash \varphi$ ” to say that  $\varphi$  is a tautology, that is, true under all valuations. Equivalently,  $\varphi$  is a tautology iff it is contained in all theories. If we reread the relational interpretation of “ $\Delta \Vdash \varphi$ ” it says: suppose that all members of  $\Delta$  are valid; then  $\varphi$  is valid, too. But valid means that the formula is true under all valuations. Hence the rule of Modus Ponens turns out to be correct but useless. We even have  $p \Vdash q$ . For  $p$  is not a theorem. Since  $\top \Vdash q$  is a substitution instance, this consequence is not structural. This is however not intended. To see the intended meaning we have to remind ourselves that variables are not placeholders for truth values; they stand in for propositions. They are in actual fact schematic variables. Modus Ponens is better described as  $\varphi; \varphi \rightarrow \chi / \chi$ , where  $\varphi$  and  $\chi$  are schematic letters for formulae.

However, we shall dispense with schematic letters and use substitution instead. Variables stand for propositions, more exactly such propositions that contain no variables (because they are rather metavariables). Call a substitution  $s$  **ground** if for all variables,  $s(p)$  contains no variables. Thus we are aiming for a substitutional view on variables. Under this interpretation Modus Ponens becomes valid again. For to suppose that  $p$  and  $p \rightarrow q$  are tautologies means to suppose that we have a ground substitution  $s$  such that both  $s(p)$  and  $s(p \rightarrow q)$  are tautologies. In this case  $s(q)$  is a tautology as well. In classical logic, “ $\Vdash$ ” coincides with “ $\vdash$ ” (defined below), but in modal logic this is no longer the case as we shall see. On the other hand, once we understand that in connection with validity we assume an understanding of variables as standing in for propositions it is not necessary to restrict ourselves to ground substitutions anymore.

To bring the point home, let us finally compare the different consequence relations. We can totally eliminate talk of valuations by using theories instead. Given a theory

$T, \chi$  is “ $\text{true}_T$ ” iff  $\chi \in T$ .  $\chi$  is “ $\text{true}_\emptyset$ ” or valid if  $\chi$  is  $\text{true}_T$  for all theories  $T$ . We then get

$$\Delta \vdash \varphi \text{ iff for all substitutions } s \text{ and all theories } T: \text{ if } s[\Delta] \subseteq T \text{ then } s(\varphi) \in T. \quad (6)$$

Thus,  $\vdash$  is the consequence relation derived from “ $\text{true}_T$ ”. Here is now the consequence relation derived from “ $\text{true}_\emptyset$ ”.

$$\begin{aligned} \Delta \Vdash \varphi \text{ iff for all substitutions } s: \text{ if for all theories } T \ s[\Delta] \subseteq T \\ \text{then for all theories } T \ s(\varphi) \in T. \end{aligned} \quad (7)$$

Both can be reduced as follows. A circumstance consists in a pair  $\langle s, T \rangle$ , where  $s$  is a substitution and  $T$  a theory. The definition of  $\vdash$  is obviously in the required form. The definition of  $\Vdash$  can be brought into this form as follows:

$$\begin{aligned} \Delta \Vdash \varphi \text{ iff for all substitutions } s \text{ and all theories } T: \text{ if for all theories } T \\ s[\Delta] \subseteq T \text{ then for all theories } T \ s(\varphi) \in T. \end{aligned} \quad (8)$$

In addition to validities we may also study invalidities (see (Skura, 2009) and references therein). A formula is called *invalid* if it is not valid; it is *untrue<sub>T</sub>* if it is not  $\text{true}_T$ . Write “ $\neg_T \varphi$ ” to say that  $\varphi$  is untrue<sub>T</sub>. Then from (4) we get

$$\begin{aligned} \neg_T \top & \text{ iff } \textit{false} \\ \neg_T \perp & \text{ iff } \textit{true} \\ \neg_T \neg \varphi & \text{ iff } \mathcal{A}_T \varphi \\ \neg_T \varphi \wedge \chi & \text{ iff } \neg_T \varphi \text{ or } \neg_T \chi \\ \neg_T \varphi \vee \chi & \text{ iff } \neg_T \varphi \text{ and } \neg_T \chi \\ \neg_T \varphi \rightarrow \chi & \text{ iff } \neg_T \varphi \text{ and } \mathcal{A}_T \chi \end{aligned} \quad (9)$$

Notice that we have not changed the interpretation of the connectives; all that has changed is the mode. We have switched from truth to falsity. If this notion is taken as primitive, the “validities” come out as the formulae that are never true, that is, the contradictions.

(Skura, 2009) discusses calculi of a different sort. Here, “ $\neg \varphi$ ” means  $\varphi$  is *refutable*, that is, there is a truth assignment to the variables of  $\varphi$  that make it false. It can also be seen as satisfiability under “ $\neg$ ”. The consequence relation is however not of the kind discussed here; one indication of this is that it is not closed under substitution but rather inversely closed under substitution.

#### 4. Modal logic

I now discuss in detail the various notions of consequence in modal logic.

Modalities add another wrinkle to the notion of truth. The truth of the formula “ $\Box\varphi$ ” is determined by the valuation and the theory, which we call the “world”. Worlds are maximally consistent sets. Start with a logic  $L$ . Given  $L$ , we construct in the usual way a canonical frame  $\mathfrak{C}_{\text{an}_L}(\text{Var})$ , where  $\text{Var}$  is as before the set of basic propositions. The construction proceeds as follows.  $W_L$  denotes the set of the ultrafilters of  $L$ . These are maximal MP-closed sets  $u \supseteq L$  such that  $\perp \notin u$ . Write  $u \triangleleft v$  if  $\Box\varphi \in u$  implies  $\varphi \in v$ . Further, the admissible sets are the sets of the form  $\{u : \varphi \in u\}$  for some  $\varphi$ . This defines a general frame. The natural valuation is  $\kappa : \varphi \mapsto \{u : \varphi \in u\}$ . The following is important: for any valuation  $\beta$  into the canonical frame there is a substitution  $s$  such that  $\beta = \kappa \circ s$ . For let  $\beta(p)$  be given. This is a set of the form  $\{u : \chi \in u\}$  for some formula  $\chi$  based only on the constants, that is, the members of  $\text{Con}$ . Put  $s(p) := \chi$  and we have  $\beta(p) = \kappa(s(p))$ . This allows us to use the notion of valuation and ground substitution interchangeably. It is to be kept in mind, though, that  $L$  may not be the logic of its canonical frame (namely, if there are not enough constants). For the purpose of exposition, however, I assume that  $L$  is the logic of  $\mathfrak{C}_{\text{an}_L}(\emptyset)$ , though I continue to use  $\mathfrak{C}_{\text{an}_L}(\text{Var})$  for ease of readability.

Let  $\beta$  be a valuation on  $\mathfrak{C}_{\text{an}_L}(\text{Var})$ . Write “ $\text{true}_{\beta,w}$ ” for the concept of being true at  $w$  under  $\beta$ . (As a matter of fact, I should index truth also with the logic but as we assume  $L$  to be fixed, the index  $L$  is omitted.) We now extend the clauses (4) as follows.

$$\Diamond\varphi \text{ is true}_{\beta,w} \text{ iff for some } v \triangleright w: \varphi \text{ is true}_{\beta,v} \quad (10)$$

Again we see that the truth of a formula cannot be determined compositionally. In order to see whether  $\varphi$  is  $\text{true}_{\beta,w}$  you may have to invoke a different notion of truth, namely  $\text{true}_{\beta,v}$  for some other world  $v$ . However, there are now two parameters. By abstracting either  $\beta$  or  $w$  we get new concepts of “truth”. For example, consider the notion “ $\text{true}_w$ ”, obtained by abstracting  $\beta$ . A proposition is “ $\text{true}_w$ ” iff for all valuations  $\beta$  it is “ $\text{true}_{\beta,w}$ ”. We may call this *local validity*. A formula is “ $\text{true}_w$ ”, or locally valid, if it is true at the world independently of the valuation. In other words, from the standpoint of this world the proposition is valid. Similarly, a formula is “ $\text{true}_\beta$ ” if it is true at all worlds; in other words, it is “ $\text{true}_\beta$ ” if it is *globally true*. And, finally, there is “ $\text{true}_\emptyset$ ”, which is *global validity*. The latter notion is nothing but theoremhood. For in the present context the logic is determined by the frame.

Now, as I have emphasized, it is convenient to change from truth relative to a valuation to truth relative to a substitution. Thus, the valuation  $\beta$  is set into correspondence with the substitution  $s$  such that  $\beta(p) = \kappa(s(p))$  for all  $p$ . A circumstance is a pair  $c = \langle s, w \rangle$  consisting of a substitution and a world.

$$\begin{array}{lll} \varphi \text{ is true}_{s,w} & \text{local truth} & \langle \mathfrak{C}_{\text{an}_L}(\text{Var}), \kappa, w \rangle \vDash s(\varphi) \\ \varphi \text{ is true}_s & \text{global truth} & \langle \mathfrak{C}_{\text{an}_L}(\text{Var}), \kappa \rangle \vDash s(\varphi) \\ \varphi \text{ is true}_w & \text{local validity} & \langle \mathfrak{C}_{\text{an}_L}(\text{Var}), w \rangle \vDash \varphi \\ \varphi \text{ is true}_\emptyset & \text{global validity} & \mathfrak{C}_{\text{an}_L}(\text{Var}) \vDash \varphi \end{array} \quad (11)$$

I add that we may also define

$$w_s := \{\chi : s(\chi) \in w\} \quad (12)$$



This too is a maximally consistent set. It is clearly consistent. Also, it is maximal. For if  $\chi \notin w_s$  then  $s(\chi) \notin w$ , whence  $s(\neg\chi) = \neg s(\chi) \in w$ , from which  $\neg\chi \in w_s$ . Hence, we may rewrite the definition of  $\text{true}_{s,w}$  as

$$\varphi \text{ is true}_{s,w} \text{ iff } \langle \mathcal{C}\text{an}, \kappa, w_s \rangle \models \varphi \quad (13)$$

In certain systems (S5 or logics with a universal modality) we can define the global concepts intrinsically. A modality  $\boxtimes$  is called *universal* if its interpretation is the relation  $W \times W$ . Given such a modality,  $\varphi$  is globally true (valid) iff  $\boxtimes\varphi$  is locally true (valid). Validity is however not so easily reduced to truth. We could in principle define some sort of frame with a modality  $[\star]$  such that  $\varphi$  is (locally/globally) valid iff  $[\star]\varphi$  is (locally/globally) true. This would require frames where each world has uncountably many twins, all accessible from each other via the relation  $R(\star)$ . More concretely,  $R(\star)$  is an equivalence relation such that for all  $u$ : (1) if  $u < v$  and  $uR(\star)u'$  then  $u' < v$ , (2) if  $u < v$  and  $vR(\star)v'$  then  $u < v'$ , (3) for any function  $\alpha : \text{Var} \rightarrow \{0, 1\}$  there is a world  $u'$  such that  $uR(\star)u'$  and for all variables  $p \in \text{Var}$ :  $p$  is true at  $u'$  under  $\beta$  iff  $\alpha(p) = 1$ . However, I prefer not to use this structural approach to validity.

## 5. Consequence

We shall now look at the consequence relations defined by these notions. Recall the definition of the consequence relation. Given  $\succ$ , we write  $\Delta \succ \varphi$  if for all circumstances  $c = \langle s, w \rangle$ ,  $\Delta \succ_c \varphi$ . More simply,

$$\succ = \bigcap_c \succ_c \quad (14)$$

Furthermore,  $\Delta \succ_c \varphi$  if whenever  $\succ_c(\delta) = 1$  for all  $\delta \in \Delta$  then also  $\succ_c(\varphi) = 1$ . We still need to clarify the relationship between  $\succ$  and  $\succ_c$ . Effectively, there everything must be reduced to truth, that is,  $\succ_{s,w}$ . This means that circumstances are defined as potential choices for  $s$  and  $w$ .

### 5.1. Local truth

The notion of truth is “ $\text{true}_{s,w}$ ”. We write “ $\Delta \vdash \varphi$ ” in modal logic to say that  $\varphi$  follows from  $\Delta$  in every world under any valuation. Namely, instead of saying that some formula is true at  $w$  under a valuation  $\beta$  we may say that some substitution instance  $s(\varphi)$  is true at  $w$  under the canonical valuation  $\kappa$ . And so we say that  $\Delta \succ_{s,w} \varphi$  if whenever  $\kappa(s(\delta)) \in w$  for all  $\delta \in \Delta$  then  $\kappa(s(\varphi)) \in w$ . Alternatively, if for all  $\delta \in \Delta$  we have that  $\delta$  is  $\text{true}_{s,w}$  then also  $\varphi$  is  $\text{true}_{s,w}$ . This is the standard, local consequence relation of  $L$ .

### 5.2. Local validity

Now let the notion of truth be “true<sub>w</sub>”, that is, look at local validity. “ $\Delta \vdash' \varphi$ ” means that for every substitution  $s$  and every world  $w$ , if  $s(\varphi)$  is true<sub>w</sub>  $s(\Delta)$  is true<sub>w</sub> as well. This reduces to the following.  $\Delta \vdash' \varphi$  iff for all substitutions  $s$  and all worlds  $w$ : if all members of  $s[\Delta]$  are locally valid at  $w$ , then so is  $s(\varphi)$ . I call this the **local admissibility consequence**. It is different from the standard admissible consequence based on global validity (see below). Namely, denecessitation ( $\Box p/p$ ) is admissible in  $K$  (see (Rautenberg, 1979)), but not locally. For consider a world  $w$  in which  $\Box \perp$  is true, that is, a world without successors. Then  $\Box p$  is true<sub>w</sub> but  $p$  is not.

### 5.3. Global truth

Now suppose that truth is “true<sub>s</sub>”, that is, true in all worlds of a frame under a given substitution  $s$ . This yields the standard global consequence relation also written “ $\vDash$ ” (see (Kracht, 1999)). For the definitional scheme says the following.  $\Delta \vDash \varphi$  iff for all substitutions  $s$  and all worlds  $w$ : if all members of  $s[\Delta]$  are true<sub>s</sub> (= globally true) then so is  $s(\varphi)$ . An example is the rule of necessitation ( $p/\Box p$ ). It is a derived rule of  $\vDash_L$  for all logics  $L$ , while the rule of denecessitation ( $\Box p/p$ ) generally is not. For you may have situations where  $\Box \varphi$  is true at all points but  $\varphi$  is not. Namely, think of a frame generated from an irreflexive point  $w$ . Even if  $\Box \varphi$  is true everywhere,  $\varphi$  need not be true at  $w$ .

### 5.4. Global validity

It is based on the notion “true<sub>∅</sub>”. Write  $\vdash^m$  for this consequence. Applying the scheme, we have  $\Delta \vdash^m \varphi$  iff for all substitutions  $s$ : if  $s(\delta)$  is true<sub>∅</sub> for all  $\delta \in \Delta$  then  $\varphi$  is true<sub>∅</sub>. Equivalently,  $\Delta \vdash^m \varphi$  iff for all substitutions  $s$  and worlds  $w$ ,  $s[\Delta] \subseteq L$  then  $s(\varphi) \in L$ . This encodes the standard notion of admissibility of a rule.  $\Delta/\varphi$  is a derived rule of  $\vdash^m$  iff it is admissible for  $L$ .

We have the following relations.

**THEOREM.** — *The following inclusions hold.*

- 1)  $\vdash \subseteq \vDash$ .
- 2)  $\vdash \subseteq \vdash'$ .
- 3)  $\vdash' \subseteq \vdash^m$ .
- 4)  $\vDash \subseteq \vdash^m$ .

**PROOF.** — I show the first inclusion, all the others are similar. Assume that  $\Delta \vdash \varphi$ . We have to show that  $\Delta \vDash \varphi$ . To that effect, assume that  $\Delta$  is true<sub>s</sub>. Then for any world  $w$ ,  $\Delta$  is true<sub>s,w</sub>, and so, by assumption,  $\varphi$  is true<sub>s,w</sub>. Hence  $\varphi$  is true<sub>s</sub>. ■

Equality between these consequence relations may hold for certain logics. In general they are different.

## 6. Generalisations

In this last section I shall discuss two strategies to generalise the notion of a rule.

The first symmetrifies the rules by allowing sets on the right hand side. These are known as **multiple conclusion rules**. They have the form  $\Delta/\Theta$  where  $\Delta$  and  $\Theta$  are both sets of formulae (see (Scott, 1974)).

“ $\Delta \succ \Theta$ ” is accepted if the following holds: if “ $\succ \delta$ ” is accepted for all  $\delta \in \Delta$  then “ $\succ \theta$ ” is accepted for some  $\theta \in \Theta$ .

A famous example is the unmodalised disjunction rule:  $\{p \vee q\}/\{p, q\}$ , or the modalised form  $\{\Box p \vee \Box q\}/\{p, q\}$ . There is also the rule of margins  $\{p \rightarrow \Box p\}/\{p, \neg p\}$ . For these rules we can run the same process of differentiating a local and global variant, and distinguish between truth and validity.

The unmodalised disjunction rule is correct for local truth but not for global truth. It is also not correct for local or global validity. The modalised disjunction rule is correct for local truth if  $L$  is reflexive. It is correct for local validity if  $L$  is reflexive. (For if  $\Box \varphi \vee \Box \chi$  is a constant formula true at  $w$  then  $\Box \varphi$  is true, and so  $\varphi$  is as well; or  $\Box \chi$  is true and therefore  $\chi$ .) It is however also sometimes correct for global validity, for example if  $L = S4$ .

The interpretation of  $\Delta/\Theta$  is that whenever all formulae of  $\Delta$  are true, at least one formula of  $\Theta$  is true. By contraposition, it means that if all formulae of  $\Theta$  are *false*, at least one formula from  $\Delta$  is *false*. So, we have that  $\{\varphi, \chi\}/\varphi \wedge \chi$  is a valid rule of PC; by consequence we have that  $\{\varphi \wedge \chi\}/\{\varphi, \chi\}$  is valid for PC in the refutation sense (taking truth to the “ $\text{false}_T$ ” or “ $\text{untrue}_T$ ”). In this way one may pursue axiomatisation of refutation calculi. It is to be noted, though, that the theorems of this dualised calculus are the contradictions. That is to say,  $\emptyset/\{\varphi\}$  is valid for the mode of “ $\text{untrue}_T$ ” iff  $\varphi$  is a contradiction. For notice that as noted in the discussion of (2), both “true” and “valid” yield the same axioms; so do “is untrue” and “is contradictory”. Thus refutation calculi fall outside of the present discussion for they require a different abstraction scheme; rather than abstracting universally (“for all theories”) we would need to abstract existentially (“there is a theory”).

Another generalisation consists in mixing the modes. So far we have been looking at rules that proceed from a set of formulae with a property P to a formula with the same property P. We may ask, however, whether it may be beneficial to broaden the notion of a rule even further and allow for several properties to be used concurrently. In this case, however, there will again be only one form of turnstile, say  $\vdash$ , while the formulae are each prefixed by the mode. If the mode is “true”, we may write Modus Ponens as

$$\text{true}:p, \text{true}:p \rightarrow q \vdash \text{true}:q \quad (15)$$

Mixing modes is straightforward. Here is an example of a valid mixed rule for K, where gtrue stands for “globally true”.

$$\text{true}:p, \text{gtrue}:p \rightarrow \Box p \vdash \text{true}:\Box p \quad (16)$$

One of the benefits of such an approach is that it allows to use a rather weak language and thus may result in a computationally tractable problem. For the alternative is to encode the properties into the object language, so that P becomes, say, a modal operator, “ $\Box P$ ”. The price to be paid is that embeddings become (at least syntactically) possible. Indeed, (Finger *et al.*, 1993) can be read precisely in this way: if we resist the impulse to embed the temporal operator the logic may become much simpler. We may reinterpret the approach by Finger and Gabbay by introducing a heterogeneous consequence: in addition to “is true” we consider additional properties such as “is true always in the future” etc.

A case where this approach is used is description logic (DL). In DL facts are split into two kinds: terminological and accidental. In DL terms, one separates knowledge into a TBox, containing the terminological facts, and an ABox, containing the accidental facts. While the difference is mainly syntactic (terminological knowledge is typically universal, accidental knowledge is existential) we can also look at the matter in a different way. We can allow in the interpretation of a rule different kinds of premisses: there may be premisses that are locally true, others globally true, others locally valid, and yet other globally valid. A generic example of this kind has been the logic of feature structures in (Gazdar *et al.*, 1988) (see also (Kracht, 1989) for a translation into polymodal logic). The introduction of a modal operator “ $\Box$ ” was motivated mainly to be able to express that certain constraints can be enforced arbitrarily deep, or, more precisely, to say that a constraint holds globally. We can do this without  $\Box$  merely by allowing both for local premisses and for global premisses, in other words, by introducing a split similar to that between ABoxes and TBoxes. Additionally, it may happen that we wish to add a path equation as a premise, to be true either locally or globally. A path is translated simply as a sequence  $\pi$  of modal operators ((Kracht, 2003)), so that the equations correspond to formulae of the form  $\langle \pi \rangle p \leftrightarrow \langle \pi' \rangle p$ . However, it will not do if we add this into the set of our formulae, for the variable is otherwise read existentially. The real import of these formulae is namely universal; we should read it as  $(\forall p)(\langle \pi \rangle p \leftrightarrow \langle \pi' \rangle p)$ , but we lack the syntactic means to say it this way. Instead, we allow to add formulae as being locally or globally *valid*.

I should add that adding validities is tantamount to adding axioms to the logic, so the added power is very high. If we wish to maintain tractability, we must constrain the syntactic shape of the axioms. Path equations, however, are just as dangerous (see (Kracht, 1995)).

## 7. Conclusion

Consequence relations commonly used in modal logic can ultimately all be derived from the single concept of truth in a model. The differences arise due to the abstraction involved in the concept of truth. The abovementioned refutation calculus is obtained by a different mode of abstraction, namely existential rather than universal. The idea of heterogeneous consequence is an interesting notion and deserves further study.

## Acknowledgements

I wish to thank Philippe Balbiani and two anonymous referees for useful comments on this paper.

## 8. References

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