Using Each Other's Words

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Abstract. Natural languages are polyphonic: typically, no two speakers associate identical meanings with all the words they are using. Also, the meanings of words may change over time. Yet we are still missing a formal framework with which to handle this variety. This paper is making a first step by introducing the logic of so-called *deflectors*. These are devices to borrow someone else's language.

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1. Varieties of Meaning

There are essentially two ways in which a sentence can cease be true. One is that it describes a state of affairs that no longer obtains. The other is that the state of affairs is no longer described by that same sentence. The first one is the usual scenario. The sentence "The door is open." is true on one occasion because the door actually is open; and it ceases to be true because the door gets closed. So far so good. Now consider this: nothing changes and yet the sentence ceases to be true. A spectacular case of this sort is the sentence

`There are nine planets.` (1)

(Here, as further down, we are talking about planets of the sun rather than what is also referred to as "moons", like Ceres. Also, in informal discussion I will continue to use the English word "planet" as if it still denotes all the nine customary planets.) It was true in 1980, but it no longer is true. What has happened? Astronomers have decided in 2006 on a new definition of the word "planet", and that new definition excludes Pluto. Hence what is true in astronomical sciences now, in 2014, is rather

`There are eight planets.` (2)

I wish to thank Daniel Milne and Udo Klein for useful discussions.

So, (1) ceased to be true not because the world has changed, but because (1) no longer describes a state of affairs that obtains.

This situation is pervasive, actually. As have argued Kracht and Klein (2014), there is no guarantee that the meaning of words is the same for two different speakers. Any learning algorithm will lead to slight differences between individuals. Thus, whether or not a certain object will be called "green" or "yellow" will differ when the colour is in the middle between the clear cut cases. Or, with respect to the above case it may very well be that plenty of people have not heard about the scientific reform and so continue to use the word "planet" in the old sense. Hence for them (1) is still true. To say that they are wrong is to maintain that there can be only one meaning to a word and that it is in this case up to the astronomers what is the correct use of "planet". But if there is only one correct meaning, how is one to interpret the change that happened in 2006? Can we say that the astronomers talking in the 1980ies have been wrong in saying there are nine planets? Is "planet" one of these infamous non natural kinds that Goodman defined? I think, this is clearly not the case. And we can diagnose that by looking at ways in which we refer to the alternative way of talking. Not even astronomers would say, for example,

`I was wrong in 1986 in calling Pluto a planet.` (3)

Obviously, they think, as we do, that they had every right in 1986 to call Pluto a planet. And that is because the meaning of that term at the time of utterance was such that it included Pluto.

To describe the situation, people would say, for example,

`Pluto is no longer called a planet.` (4)

Contrast this with

`Pluto no longer is a planet.` (5)

Consider an event in 2000 (before the change in definition) whereby Pluto collides with a big asteroid and is being kicked out of its orbit so that it leaves the solar system. As a consequence, astronomers have decided to stick with the original definition of "planet". Then, while (5) would be appropriate today in 2014, (4) definitely would not. The reason is that (4) asserts that a change in the way we call things has happened. (5) however only states that the sentence "Pluto is a planet." has become false.

This much has always been known, of course. A sentence is true because of the way the world is and what the words mean together with the composition algorithm for meanings. Change one of the ingredients and your sentence may change truth value. What has been missing so far is a logical treatment of this phenomenon. Or, to say it more modestly, I am not aware of any such treatment. I wish to provide one here. [1]

[1]There is a resemblance with what is known as two dimensional semantics, see for example Stalnaker (1978). However, the approach taken here is different in that I use two distinct sets

2. Deflectors

If the various meanings would simply coexist side by side, there would be no point in developing a theory. However, language also provides means to quote the meaning of a word from different sources. In addition to talking our own talk we can also talk other people's talk. We can say, for example, "in your words" or "in Finnish" and the like and thereby change the way in which our words are to be interpreted. I call the devices to achieve this *deflectors*. We shall provide a formal theory of such deflectors.

We assume a fixed language, L, that has constants and a fixed mode of interpreting complex expressions. However, the so-called constants will lack a uniform interpretation. Instead, their interpretation may vary according to some *index*. This index can be a time point, a speaker, a dialect, a combination of the two etc. Given an index i and a sentence φ, the expression "$\langle\!\langle i \rangle\!\rangle\varphi$" may be translated as "in the words of i, φ" when i is a person, or as "speaking i, φ" or "in the terminology of i, φ", when it is a mode of expression.

In addition to indices, we also have worlds and a modality $\Diamond$ that ranges over possible worlds. So, "$\Diamond\varphi$" says that φ is possible. Although we can think of several more modal operators, one will suffice to demonstrate the interaction between modality and deflection.

The basic symbols of the language are the following.

1. A set C of constants.
2. A set I of indices.
3. $\langle\!\langle \cdots \rangle\!\rangle$, $\Diamond$, $\neg$, $\wedge$.

There are no variables. A *proposition* is formed as follows.

- Any constant is an proposition.
- If φ, χ are propositions, so are $\neg\varphi$, $\Diamond\varphi$ and $\varphi \wedge \chi$.
- If φ is a proposition and i an index, $\langle\!\langle i \rangle\!\rangle\varphi$ is a proposition.

The brackets "$\langle\!\langle \cdots \rangle\!\rangle$" are called *deflectors*. They allow to quote a proposition in the words of someone else. A *preframe* is a triple $\mathcal{D} = \langle W, R, \{J_i : i \in I\}\rangle$, where $R \subseteq W^2$ is a binary relation and for every index i, $J_i : C \to 2^W$ is an interpretation of the constants.

$$\begin{array}{ll}
\langle \mathcal{D}, (w,i)\rangle \vDash c & :\Leftrightarrow w \in J_i(c) \\
\langle \mathcal{D}, (w,i)\rangle \vDash \neg\varphi & :\Leftrightarrow \langle \mathcal{D}, (w,i)\rangle \nvDash \varphi \\
\langle \mathcal{D}, (w,i)\rangle \vDash \varphi \wedge \chi & :\Leftrightarrow \langle \mathcal{D}, (w,i)\rangle \vDash \varphi; \chi \\
\langle \mathcal{D}, (w,i)\rangle \vDash \Diamond\varphi & :\Leftrightarrow \text{for some } w' \text{ such that } w\,R\,w' : \langle \mathcal{D}, (w',i)\rangle \vDash \varphi \\
\langle \mathcal{D}, (w,i)\rangle \vDash \langle\!\langle j \rangle\!\rangle\varphi & :\Leftrightarrow \langle \mathcal{D}, (w,j)\rangle \vDash \varphi
\end{array} \tag{6}$$

Formulas are evaluated at pairs consisting of a world and an index.

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of worlds as coordinates and that the content attributed to a speaker is not obtained by diagonalisation (obviously, if the coordinate sets are different, this cannot be done). Moreover, the quotation operators used here (called deflectors) are absent in Stalnaker (1978). I thank Daniel Milne for urging me to consider that issue.

The effect of $\langle\!\langle i \rangle\!\rangle$ is to change the index of interpretation. This calls for a polymodal reformulation. The preframe $\mathcal{D}$ will be changed to a *frame*

$$\mathcal{D}^\sharp := \langle W \times I, R^\sharp, \{f_i : i \in I\}, J\rangle, \tag{7}$$

where

1. $R^\sharp := \{((w,i),(w',i)) : w\,R\,w', i \in I\}$.
2. $f_j((w,i)) := (w,j)$.
3. $J(c) := \{(w,i) : w \in J_i(c)\}$.

(The f_j are functions. However, note that as functions they are also the relations $\{((w,i),(w,j)) : (w,i) \in W \times I\}$. So we have a standard polymodal Kripke-frame, which also interprets the propositional constants.) The clauses of (6) will be transferred in the natural way. For example, here are the first and the fourth clause:

$$\begin{array}{ll} \langle \mathcal{D}^\sharp, (w,i)\rangle \vDash c & :\Leftrightarrow (w,i) \in J(c) \\ \langle \mathcal{D}^\sharp, (w,i)\rangle \vDash \Diamond\varphi & :\Leftrightarrow \text{for some } (w',i') \text{ such that} \\ & \quad (w,i)\,R^\sharp\,(w',i') : \langle \mathcal{D}^\sharp, (w',i')\rangle \vDash \varphi \end{array} \tag{8}$$

But note that by definition of $R^\sharp$, $i = i'$, so the clause for $\Diamond$ in (8) is equivalent to the one in (6). A *deflector frame* is a frame of the form $\mathcal{D}^\sharp$.

3. An example

Consider the example of the introduction. We have two constants c_9 and c_8, which are the sentences /`There are nine planets`/ and /`There are eight planets`/, respectively. There are two indices, a and p. The index a corresponds to time points to interpret language before 2006, p to time points from 2006 on. And there are three worlds. In w_0 we have the usual nine planets. In w_1, of the nine Pluto is missing; in w_2, Venus is missing instead of Pluto. (Hence, while w_0 has nine planets in the standard sense, w_1 and w_2 each have eight.) We have $W := \{w_0, w_1, w_2\}$, $R = W^2$ and $J_a(c_9) = \{w_0\}$, $J_a(c_8) = \{w_1, w_2\}$; $J_p(c_9) = \varnothing$, $J_p(c_8) = \{w_0, w_1\}$. This defines the preframe $\mathcal{P}$. We turn this into the frame

$$\mathcal{P}^\sharp = \langle W \times \{a,p\}, R^\sharp, \{f_a, f_b\}, J\rangle, \tag{9}$$

with $f_a((w,i)) := (w,a)$, $f_b((w,i)) := (w,p)$. Consider an astronomer in 1980. If he utters (1), his world is w_0 and his index is a. We find that (1) is true since $J_a(c_9)$ contains w_0. If he utters (1) in 2014, it is false since $J_p(c_9)$ does not contain w_0. However, $\langle\!\langle a \rangle\!\rangle c_9$ is true at (w_0, p), as is easily checked. It is the formal rendering of

`In the terminology of 1980, there are nine planets.` (10)

The deflector $\langle\!\langle a \rangle\!\rangle$ allows to decontextualise the interpretation.

The deflector frames dissociate the worlds of the preframes into several counterparts. Thus, while the worlds of the preframe $\mathcal{D}$ constitute different states-of-affairs, this is no longer true of the worlds of the deflector frame $\mathcal{D}^\sharp$. How is this to be explained? The idea I am pushing here is that there are two ways of looking at the notion of "state-of-affairs". In the customary meaning it denotes a way of being of the world without taking into account the language; in the other, new meaning, the language itself also is a way of being of the world. The fact that Pluto is called a planet in a but not in p is in the second sense a way of being of the worlds a and p. Thus, the worlds comprise in addition to the languageless facts also facts of the language.

This can only work nontrivially if there is a separation between a core that remains constant, and a remainder whose interpretation is free. Here the core consists in the boolean connectives, the deflectors and the modal operator $\Diamond$ whose interpretation remains fixed, while the interpretation of the constants is freely assignable. [2]

To encompass this difference we need a new terminology. We will say that the pair (w_0, p) is the same *world* as (w_0, a), the only difference being how we encode the state of affairs. This calls for a definition. A *polyphonic proposition* in a deflector frame $\mathcal{D}^\sharp$ is an arbitrary subset of $W \times I$. The proposition is *monophonic* if it is of the form $V \times I$ for some $V \subseteq W$. Standardly, a proposition is considered to be monophonic. For we wish to say that whatever a proposition is, as a set of worlds it should not depend on the code that we are using. However, this would mean that the constants "c_9" and "c_8" do not express propositions since their values are not of the required form. Indeed, in the present circumstances we would be inclined to say exactly that. Consequently, we distinguish two types of formulae: those that express a (monophonic) proposition and those that do not. An *utterance* is meant to express a proposition. Hence, when φ is uttered at index i, the proposition that is being expressed is not φ (because that may not denote a proposition) but rather $\langle\!\langle i \rangle\!\rangle \varphi$. In this way the utterance of the same expression can denote a different proposition in *the same world*. This is a general fact worthy of note.

Proposition 1. *A polyphonic proposition c is monophonic if and only if $c = \langle\!\langle i \rangle\!\rangle c$ for some (and hence for all) i.*

Now, when we use variables, it seems that we are forced to say that they denote monophonic propositions rather than polyphonic ones. This will then introduce a dichotomy between constants and propositions, since the latter are constrained in the way the former are not.

However, given the observation above, we can always explicitly reduce a proposition to become monophonic by prefixing it with a deflector.

4. The Logic of Deflector Frames

We will now proceed to an axiomatization of deflector frames. The resulting logic will not axiomatise all and only the deflector frames. Hence, we shall generalize

[2] As natural language allows to phrase deflectors using plain words, such as "in the parlance of the prime minister", the translation of deflectors of the language into those of the logical language is of course nontrivial. I shall ignore this problem here.

the terminology to include those frames that satisfy the logic of deflector frames. *Quasi-deflector frames* are frames of the form $\mathcal{F} = \langle W, R, f \rangle$, where W is a set of worlds, $R \subseteq W^2$ and f a function assigning to each $i \in I$ a function f_i such that the following holds.

① For all $w \in W$, $i, j \in I$: $f_j(f_i(w)) = f_j(w)$.
② For all $w \in W$ there is a $i \in I$ such that $w = f_i(w)$.
③ For all $w, w' \in W$, if $w \, R \, w'$ and $i \in I$, then $f_i(w) \, R \, f_i(w')$.
④ For all $w, w' \in W$ and $i \in I$, if $f_i(w) \, R \, w'$ then there is a $w'' \in W$ such that $w \, R \, w''$ and $f_i(w'') = w'$.

The first conditions says that the quoted meanings are absolute. The second says that every meaning has some index. The last two conditions state that the maps $w \mapsto f_i(w)$ are p-morphisms. To see why ① is the case, let us take a preframe $\mathcal{D}$. Then in $\mathcal{D}^\sharp$, $f_j(f_i((w,k))) = f_j((w,i)) = (w,j) = f_j((w,k))$. Using (6), this translates into $\langle\!\langle i \rangle\!\rangle \langle\!\langle j \rangle\!\rangle p \leftrightarrow \langle\!\langle j \rangle\!\rangle p$. To see this, observe that

$$\begin{array}{rl} & \langle \mathcal{D}, (w,k) \rangle \vDash \langle\!\langle i \rangle\!\rangle \langle\!\langle j \rangle\!\rangle p \\ \Leftrightarrow & \langle \mathcal{D}, (w,i) \rangle \vDash \langle\!\langle j \rangle\!\rangle p \\ \Leftrightarrow & \langle \mathcal{D}, (w,j) \rangle \vDash p \\ \Leftrightarrow & \langle \mathcal{D}, (w,k) \rangle \vDash \langle\!\langle j \rangle\!\rangle p \end{array} \tag{11}$$

The other conditions are also easily verified. The satisfaction clauses for deflector frames are standard. Observe that since the f_i are functions we have

$$\langle \mathcal{F}, \beta, w \rangle \vDash \langle\!\langle i \rangle\!\rangle \varphi \Leftrightarrow \langle \mathcal{F}, \beta, f_i(w) \rangle \vDash \varphi \tag{12}$$

A *general quasi-deflector frame* is a structure $\langle W, R, f, U \rangle$, where $\langle W, R, f \rangle$ is a quasi-deflector frame, and $U \subseteq \wp(W)$ is a collection of sets closed under complement, intersection; which for every i is closed under the operator

$$\langle\!\langle i \rangle\!\rangle a := \{ w : f_i(w) \in a \} \tag{13}$$

and which is closed under the operator

$$\Diamond a := \{ w : \exists w' : w \, R \, w' \in a \} \tag{14}$$

From this we can extract a logic of quasi-deflector frames. It is characterized by the following axioms. We use $[\![i]\!] \varphi$ to abbreviate $\neg \langle\!\langle i \rangle\!\rangle \neg \varphi$.

$$\begin{array}{ll} \text{(Ax 1)} & \langle\!\langle i \rangle\!\rangle p \leftrightarrow [\![i]\!] p \\ \text{(Ax 2)} & \langle\!\langle i \rangle\!\rangle \langle\!\langle j \rangle\!\rangle p \leftrightarrow \langle\!\langle j \rangle\!\rangle p \\ \text{(Ax 3)} & \Diamond \langle\!\langle i \rangle\!\rangle p \leftrightarrow \langle\!\langle i \rangle\!\rangle \Diamond p \\ \text{(Ax 4)} & \langle\!\langle i \rangle\!\rangle \Box p \to \Box \langle\!\langle i \rangle\!\rangle p \\ \text{(Ax 5)} & \Diamond [\![i]\!] p \to [\![i]\!] \Diamond p \\ \text{(Ax 6)} & \varphi \to \bigvee_{i \in I} \langle\!\langle i \rangle\!\rangle \varphi \end{array} \tag{15}$$

(The last axiom requires that I is finite.) (Ax 4) and (Ax 5) are derivable. (Ax 4) is obviously equivalent to $\Diamond [\![i]\!] p \to [\![i]\!] \Diamond p$ (replace p by $\neg p$, and then do contraposition). Using the fact that $[\![i]\!] p \leftrightarrow \langle\!\langle i \rangle\!\rangle p$ is derivable, we get an equivalence with $\Diamond \langle\!\langle i \rangle\!\rangle p \to \langle\!\langle i \rangle\!\rangle \Diamond p$, which is one half of (Ax 3). Similarly (Ax 5) follows from (Ax 3) by replacing $\langle\!\langle i \rangle\!\rangle$ by $[\![i]\!]$.

Notice that deflector frames are products of frames, one component dealing with the deflectors $\langle\!\langle i \rangle\!\rangle$ and the other with the modality $\Box$. The first logic is called Def_I the second L. We do allow to add any set of postulates for the modality $\Box$. If L is the logic of $\Box$, we denote by $\mathsf{Def}_I(L)$ the logic of adding the postulates of L to Def_I. As we shall show below, this logic is identical to the product of the logics Def_I and L (see (Kurucz, 2007) for an overview of products). This means that the logic of some class of quasi-deflector frames is the logic of some class of deflector frames, no matter what L is. This is a rather strong result, akin to a result by Gabbay and Shehtman (1998) on products of logics with functional operators. For definition, if L is a modal logic based on the operators taken from O and L' is a modal logic based on the modal operators taken from O' where O' is disjoint from O, write $[L, L']$ for the logic axiomatized by

(a) L and L';
(b) $\langle m\rangle\langle m'\rangle p \leftrightarrow \langle m'\rangle\langle m\rangle p$, where $m \in O$ and $m' \in O'$;
(c) $\langle m\rangle[m']p \rightarrow [m']\langle m\rangle p$, $m \in O$ and $m' \in O'$;
(d) $\langle m'\rangle[m]p \rightarrow [m]\langle m'\rangle p$, $m \in O$ and $m' \in O'$.

This logic is called the *commutator* (Kurucz (2007)). The postulates under (b) encode the commutation, the postulates under (c) and (d) encode the Church-Rosser property. (They are dual to each other, so either of (c) and (d) is sufficient.) They state that for each pair of modalities $m \in O$ and $m' \in O'$, if $w\,R(m)\,w'$ and $w\,R(m')\,w''$ there exists w''' such that $w'\,R(m')\,w'''$ and $w''\,R(m)\,w'''$. Call a frame $\langle W, R, D\rangle$, with $R : O \cup O' \rightarrow W \times W$, $D \subseteq \wp(W)$ a *product frame* if (i) $W = W_0 \times W_1$, and $(u_0, u_1)\,R(m)\,(v_0, v_1)$ iff either (iia) $m \in O$, $u_0\,R(m)\,v_0$ and $u_1 = v_1$, or (iib) $m \in O'$, $u_0 = v_0$ and $u_1\,R(m)\,v_1$. Finally, D must be a field of sets closed under the modal operators. A complete logic over $O \cup O'$ is the *product logic* $L \times L'$ if it is the logic of products of frames for L and frames for L'. The question is whether the above axioms suffice to axiomatize $L \times L'$.

Let us now return to the logic of deflector frames. Notice first that the postulates for $\mathsf{Def}_I(L)$ include the postulates for the commutator $[\mathsf{Def}_I, L]$. The remainder of this paper is devoted to showing that this axiomatises the product, whence that it *is* the logic of deflector frames. The following is easily proved.

Proposition 2. *The deflectors $\langle\!\langle i \rangle\!\rangle$ commute with $\neg$, $\wedge$, $\Diamond$, $\Box$. Moreover, every formula is equivalent in Def_I to a formula built from atomic formulae or formulae of the form $\langle\!\langle i \rangle\!\rangle p$, $\langle\!\langle i \rangle\!\rangle c$ $(i \in I)$ using only $\neg$, $\wedge$, and $\Diamond$.*

Observe namely that $\langle\!\langle i \rangle\!\rangle p \equiv [\![i]\!] p$, so that $\neg\langle\!\langle i \rangle\!\rangle p \equiv [\![i]\!]\neg p \equiv \langle\!\langle i \rangle\!\rangle \neg p$. Commutativity over $\wedge$ is clear. Moreover, we have commutativity with $\Diamond$ as an axiom. Hence, we can always push the deflectors inside. Now observe that any sequence of deflectors can be reduced to the innermost deflector.

The axioms (Ax1-6) above correspond to first-order conditions on the frames saying that for each i, the relation associated with $\langle\!\langle i \rangle\!\rangle$ is a function (Ax 1) such that $f_j(f_i(w)) = f_j(w)$ (Ax 2). (Ax 3) corresponds to the properties (16a) and

(16b), respectively.

$$(\forall w w' w'')(w\, R\, w' \wedge w'' = f_i(w') \to (\exists w''')(w''' = f_i(w) \wedge w'''\, R\, w'')) \tag{16a}$$

$$(\forall w w' w'')(w' = f_i(w) \wedge w'\, R\, w'' \to (\exists w''')(w\, R\, w''' \wedge w'' = f_i(w'''))) \tag{16b}$$

(16a) and (16b) can be simplified to (17a) and (17b).

$$(\forall w)(\forall w')(w\, R\, w' \to f_i(w)\, R\, f_i(w')) \tag{17a}$$

$$(\forall w)(\forall w')(f_i(w)\, R\, w' \to (\exists w'')(w\, R\, w'' \wedge f_i(w'') = w')) \tag{17b}$$

(Ax 4) and (Ax 5) are derivable. (It can also be checked that the Church-Rosser property must hold. Suppose namely that $w\, R\, w'$ and $w'' = f_i(w)$ then with $w''' := f_i(w')$ we have $w''\, R\, w'''$.) Thus we have managed to reproduce conditions ① – ④. Finally, (Ax 6) corresponds to the first-order condition

$$(\forall w)(\bigvee_{i \in I} f_i(w) = w) \tag{18}$$

This finishes the axiomatisation. Now we shall proceed to show that quasi-deflector frames for a logic $\mathsf{Def}_I(L)$ can be replaced by deflector frames for that same logic.

Let us be given a quasi-deflector frame $\mathcal{F}$ and a world $w_0 \in W$. Define $C_{\mathcal{F}}(w_0) := \{f_i(w_0) : i \in I\}$ and call this the *cycle* of w_0. Likewise, put $S_{\mathcal{F}}(w_0) := \{w' : w_0\, R(\Box)^*\, w'\}$, and call it the *sheaf* of w_0. (As is customary, $R(\Box)^*$ denotes the reflexive transitive closure of $R(\Box)$.) It becomes a frame $\mathcal{S}_{\mathcal{F}}(w_0)$ with the relation R^S, where $w\, R^S\, w'$ iff $w\, R\, w'$. Furthermore, let $\mathcal{C}_I$ be the following frame: $\langle I, \{g_i : i \in I\}, \wp(I)\rangle$, $g_i(i') := i$ for all $i' \in I$. We shall show that given a model for φ based on $\mathcal{F}$ at w_0 we can define a product frame over $S_{\mathcal{F}}(w_0) \times \mathcal{C}_I$ and a model for φ on this product frame. Notice that the construction even works for general frames. Thus we shall lift the restriction implicit in the definition of products that the component logics be complete.

We start with a model $\langle \mathcal{F}, \beta, w_0\rangle \vDash \varphi$. The frame $\mathcal{F}^\circ$ is defined to be the product of the frames $S_{\mathcal{F}}(w_0)$ and $\mathcal{C}_I$, with the following internal sets. For an internal set a of $\mathcal{F}$, put $a^\circ := \{(w, i) : w \in S_{\mathcal{F}}(w_0), f_i(w) \in a\}$. By definition, $a^\circ = \{(w, i) : w \in S_{\mathcal{F}}(w_0), w \in \langle\!\langle i\rangle\!\rangle a\}$.

$$\begin{aligned} (-a)^\circ &= \{(w,i) : w \in S_{\mathcal{F}}(w_0), f_i(w) \in -a\} \\ &= \{(w,i) : w \in S_{\mathcal{F}}(w_0), f_i(w) \notin a\} \\ &= -\{(w,i) : w \in S_{\mathcal{F}}(w_0), f_i(w) \in a\} \\ &= -a^\circ \end{aligned} \tag{19}$$

$$\begin{aligned} (a \cap b)^\circ &= \{(w,i) : w \in S_{\mathcal{F}}(w_0), f_i(w) \in a \cap b\} \\ &= \{(w,i) : w \in S_{\mathcal{F}}(w_0), f_i(w) \in a\} \\ &\qquad \cap \{(w,i) : w \in S_{\mathcal{F}}(w_0), f_i(w) \in b\} \\ &= a^\circ \cap b^\circ \end{aligned} \tag{20}$$

$$\begin{aligned} (\langle\!\langle j\rangle\!\rangle a)^\circ &= \{(w,i) : w \in S_{\mathcal{F}}(w_0), f_i(w) \in \langle\!\langle j\rangle\!\rangle a\} \\ &= \{(w,i) : w \in S_{\mathcal{F}}(w_0), w \in \langle\!\langle i\rangle\!\rangle\langle\!\langle j\rangle\!\rangle a\} \\ &= \{(w,i) : w \in S_{\mathcal{F}}(w_0), w \in \langle\!\langle j\rangle\!\rangle a\} \\ &= \{(w,i) : w \in S_{\mathcal{F}}(w_0), (w,j) \in a^\circ\} \\ &= \langle\!\langle j\rangle\!\rangle a^\circ \end{aligned} \tag{21}$$

For note that by (13), $\langle\!\langle j\rangle\!\rangle a^\circ = \{(w,i) : w \in S_{\mathcal{F}}(w_0), f_j((w,i)) \in a^\circ\} = \{(w,i) : w \in S_{\mathcal{F}}(w_0), (w,j) \in a^\circ\}$. And, finally,

$$\begin{aligned}(\Diamond a)^\circ &= \{(w,i) : w \in S_{\mathcal{F}}(w_0), f_i(w) \in \Diamond a\}\\ &= \{(w,i) : w \in S_{\mathcal{F}}(w_0), \exists w' : f_i(w)\, R\, w' \in a\}\\ &= \{(w,i) : w \in S_{\mathcal{F}}(w_0), \exists w' : w\, R\, w' \wedge f_i(w') \in a\}\\ &= \Diamond\{(w',i) : w' \in S_{\mathcal{F}}(w_0), f_i(w') \in a\}\\ &= \Diamond a^\circ\end{aligned} \tag{22}$$

The sets of the form a° are therefore closed under complement, intersection and the modal operators. Hence we have a general frame. By construction, it is a deflector frame. Now put $\beta^\circ(p) := \beta(p)^\circ$. From the previous considerations we get that $\beta^\circ(\varphi) = (\beta(\varphi))^\circ$. It follows that

$$\langle \mathcal{F}^\circ, \beta^\circ, (w,j)\rangle \vDash \varphi \tag{23}$$

Let us now turn to the general logic of deflector frames. Let L be the logic of $\Diamond$. Assume as above that there are no further axioms concerning deflectors. Thus, additional axioms over Def_I only contain $\Diamond$ as a modal. Denote the resulting logic by $\mathsf{Def}_I(L)$. Denote the logic of pure deflectors by Def_I. Then we have

Theorem 3. *If I is finite, $\mathsf{Def}_I(L) = \mathsf{Def}_I \times L = [\mathsf{Def}_I, L]$. If L is complete, so is $\mathsf{Def}_I(L)$.*

Proof. We have just shown that $\mathcal{F} \nvDash \varphi$ implies that $\mathcal{F}^\circ \nvDash \varphi$. If $\mathcal{F} \vDash \mathsf{Def}_I(L)$, we also have $\mathcal{F}^\circ \vDash \mathsf{Def}_I(L)$. This is seen as follows. By construction, $\mathcal{F}^\circ$ is a deflector frame, so it satisfies the postulates of Def_I. Moreover, it satisfies the postulates of L since every sheaf is isomorphic to $\mathcal{S}_{\mathcal{F}}(w_0)$, which by assumption on $\mathcal{F}$ is a frame for L. □

5. Conclusion

People do not talk alike, the meaning of words or constructions change from people to people and over time. This does not mean however that no analytic tools can be used. In this essay I have shown how we can borrow each other's language. The logic of deflectors is rather well behaved. Quoting your language is easy. The hard part, of course, is knowing what we are getting into when we do that. I have used a rather well documented case of language reform to demonstrate how we can handle different ways of talking in a fully rational way. It requires knowing when the meaning changed and how. Informal language, however, never is like that. We are often not even aware how subtle the differences are. Yet, as (Putnam, 1988) has reminded us, we often *do* borrow meanings from each other when, for example, we tacitly rely on expert opinion. There is then a place for a thorough investigation of the logic and pragmatics of deflection.

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