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ATOMIC INCOMPLETENESS OR HOW TO KILL ONE BIRD WITH TWO STONES

Abstract

We construct a variety of tense algebras that is not generated by its atomic members. Then we lift this result to the case of modal algebras.

1. Introduction

A standard counterexample in modal logic looks somewhat like this: first conctruct a fancy Kripke frame, then take (some variation on) the algebra of finite and cofinite sets on this frame. It works. Algebraically, one outstanding feature of the algebra produced that way is that it is atomic. This may suggest the following, rather informally stated, conjecture: *For any property P that does not hold of all modal logics, P can be falsified on an atomic modal algebra.* Faithful to the spirit of the great diagonalisation tradition we will exhibit a property P such that (1) if it does not hold of all modal logics then, by its very nature, it cannot be falsified on an atomic modal algebra, and (2) it indeed does not hold of all modal logics.

Let Λ be a modal logic, ϕ a formula. Further, let $\mathcal{V}(\Lambda)$ stand for the variety of modal algebras corresponding to Λ. Consider the following property:

(P) if $\phi \notin \Lambda$, then there is an atomic algebra $\mathbf{A}$ in $\mathcal{V}(\Lambda)$, with $\mathbf{A} \not\models \phi$.

Clearly, if there is a modal logic Λ that does not posses (P), then an atomic modal algebra is either not in $\mathcal{V}(\Lambda)$, or else it satisfies ϕ, hence (1) above indeed holds for our (P). All that remains is to show that (P) is not true of all modal logics Λ. Notice that (P) expressed algebraically reads: $\mathcal{V}(\Lambda)$ is generated by its atomic members. Thus, we need to show that there is a variety of modal algebras not generated by its atomic members. This we will do in two stages. First, we will explicitly construct a variety of tense algebras not generated by its atomic members. Then we employ Kracht and Wolter's simulation technique to produce a variety of modal algebras with the same property.

Since we want to move as freely as possible back and forth between algebra and logic, we will not distinguish between formulas of logics and terms of algebras. This is harmless, since formulas can be thought of as the elements of the absolutely free algebra of the appropriate type, which in turn are terms of the appropriate algebraic language. To do justice to both sides, we will always call them terms, but, on the other hand, the notation we employ will follow the symbolism from modal logic (boxes, diamonds, $\top$ and $\bot$). We assume a degree of familiarity with modal logic and some knowledge of universal algebra: we refer the reader to [3] for the former and to [1] for the latter.

2. First stone

Let $\mathcal{V}$ be a variety of boolean algebras with operators. Suppose that in the language of $\mathcal{V}$, we have a 0-ary term b (a definable constant) such that $\mathcal{V} \not\models b = \bot$. We will call *good* all these algebras in $\mathcal{V}$ which have $b > \bot$. For each algebra $\mathbf{A}$ from $\mathcal{V}$, and each unary term t, consider the property:

(A1) $\mathbf{A} \models x \neq \bot \;\&\; x \leq b \Rightarrow \bot < tx < x.$

Fact 0. If $\mathbf{A} \in \mathcal{V}$ is good and satisfies (A1), then $\mathbf{A}$ contains no atoms below b, hence is not atomic.

Proof. Recall that $\mathbf{A}$ is atomic iff for each element $a > \bot$, there is an atom below a. As $\mathbf{A}$ is good, we can take an $a \in A$ with $\bot < a \leq b$. By (A1) then, $0 < ta < a$, and thus a is not an atom.

We proceed to formulate an analogue of (A1) for varieties. To be of any use, such an analogue should require that the term t be uniformly chosen. Thus, let $\mathcal{V}$ be a variety of BAOs and t a unary term in the language of $\mathcal{V}$, such that:

(C1) all good subdirectly irreducible members of $\mathcal{V}$ have property (A1) with respect to t.

Fact 1. If $\mathcal{V}$ is as above and $\mathbf{A} \in \mathcal{V}$ is good, then $\mathbf{A}$ contains no atoms below b, hence is not atomic.

Proof. Take any $a \in A$ with $\perp < a \leq b$. Take a subdirect representation $\prod_{i\in I} \mathbf{B}_i$ of $\mathbf{A}$. Let $\langle a_i : i \in I\rangle$ be the vector of projections of a. Clearly, $a_i \leq b_i$ at every coordinate $i \in I$, and for at least one $j \in I$ we have $a_j > \perp$. Hence, $b_j \geq a_j > \perp$, and thus $\mathbf{B}_j$ is good. Condition (A1) then applies to $\mathbf{B}_j$, yielding $\perp < ta_j < a_j$. This is all we need to conclude that $\perp < ta < a$ in $\prod_{i\in I} \mathbf{B}_i$, and hence in $\mathbf{A}$. Therefore, a cannot be an atom.

Let $\mathcal{V}$ be generated by a single good algebra $\mathbf{A}$. Consider yet another condition:

(C2) all good subdirectly irreducible members of $\mathcal{V}$ belong to $SP_U(\mathbf{A})$.

Fact 2. If $\mathbf{A}$ satisfies (A1) and $\mathcal{V}$ satisfies (C2), then $\mathcal{V}$ satisfies (C1). Moreover, $\mathcal{V}$ is not generated by its atomic members.

Proof. By (C2) all good subdirectly irreducible algebras from $\mathcal{V}$ are in $SP_U(\mathbf{A})$. Since (A1) is universal it carries over to ultraproducts and subalgebras. This proves the first part. For the second, it is enough to observe that since $\mathcal{V} \not\models b = \perp$, there must be a good algebra $\mathbf{B}$ among the generators of $\mathcal{V}$. By Fact 1, such a $\mathbf{B}$ cannot be atomic.

We now proceed to construct a suitable variety $\mathcal{V}$ of tense algebras. Our $\mathcal{V}$ will be generated by the algebra that is best described in terms of the frame defined below.

For each $j \in \omega$ and $i < 2^j$ define $U_{i,j}$ to be $\{k \in \mathbf{Z} : k \equiv i \pmod{2^j}\}$, with $\mathbf{Z}$ being the usual set of integers. Put $\mathsf{U} := \{\infty\} \cup \{\overline{\mathsf{U}}_{\mathsf{i,j}} : \mathsf{i} < 2^{\mathsf{j}}, \mathsf{j} \in \omega\}$, where ∞ and $\overline{U}_{i,j}$ (for each $U_{i,j}$) are distinct objects belonging neither to $\mathbf{Z}$, nor to $\wp(\mathbf{Z})$. The set of worlds of our frame will be $W := \mathsf{U} \cup \mathbf{Z}$.

The accessibility relation $\lhd$ is defined as follows:

- $k \lhd \overline{U}_{i,j}$ iff $k \in U_{i,j}$ and $i \leq (2^j - 1)/2$,
- $\overline{U}_{i,j} \lhd k$ iff $k \in U_{i,j}$ and $i > (2^j - 1)/2$,
- $\overline{U}_{i,j} \lhd \infty$, for all $U_{i,j}$,
- $\overline{U}_{i,j} \lhd \overline{U}_{k,\ell}$ iff $U_{i,j} \subset U_{k,\ell}$.

It will be convenient to view U as the disjoint union of two sets: A (*accesssible* from $\mathbf{Z}$) and B (*barred* from $\mathbf{Z}$). Formally, $\mathsf{A} := \{\mathsf{u} \in \mathsf{U} : \mathsf{for\ some\ z} \in \mathbf{Z}, \mathsf{z} \lhd \mathsf{u}\}$, and $\mathsf{B} := \mathsf{U} - \mathsf{A}$; in particular, $\infty \in \mathsf{B}$. Put $\{ := \langle \mathcal{W}, \lhd, \rhd, \mathcal{I} \rangle$, where $\rhd := \lhd^{-1}$. Then define the family I of internal sets to be the smallest (i.e., zero-generated) tense algebra on W.

FACT 3. A subset S of W is internal iff $S = A \cup B \cup C$, where A and B are finite or cofinite subsets of respectively A and B, and C is a finite union of some $U_{i,j}$.

PROOF. We will present only a sketch of s proof and ask the reader to supply the missing details. First, observe that both $\mathbf{Z}$ and U are definable by constant terms, hence internal. In fact, $\Upsilon := \overset{\rightarrow}{\Box}\bot \vee \overset{\rightarrow}{\Diamond}\overset{\rightarrow}{\Box}\bot$ defines U, and $\zeta := \neg\Upsilon$ defines $\mathbf{Z}$. Next, each $U_{i,j} \subseteq \mathbf{Z}$ is also defined by a constant term. Namely, let $\upsilon_{i,j}$ be $\zeta \wedge \overset{\rightarrow}{\Diamond}^{j+2}\top$, if $i \leq (2^j - 1)/2$, and $\zeta \wedge \overset{\leftarrow}{\Diamond}\overset{\rightarrow}{\Diamond}^{j+1}\top$, if $i > (2^j - 1)/2$. Then $\upsilon_{i,j}$ defines $U_{i,j}$. Moreover, each $\{\overline{U}_{i,j}\} \subset \mathsf{U}$ is defined by $\overset{\rightarrow}{\Diamond}\upsilon_{i,j}$, if $i \leq (2^j - 1)/2$, or $\overset{\leftarrow}{\Diamond}\upsilon_{i,j}$, if $i > (2^j - 1)/2$. Finally, A is defined by $\Upsilon \wedge \overset{\leftarrow}{\Diamond}\zeta$ and B by $\Upsilon \wedge \neg\overset{\leftarrow}{\Diamond}\zeta$. This suffices to generate all the sets we claimed were internal by means of finite unions and complements.

FACT 4. The algebra $\mathbf{I}$ is simple and $\mathcal{V}(\mathbf{I})$ is a discriminator variety.

PROOF. Let $\overset{\leftrightarrow}{\Diamond}$ be a unary term defined by $\overset{\leftrightarrow}{\Diamond}x := \overset{\rightarrow}{\Diamond}x \vee x \vee \overset{\leftarrow}{\Diamond}x$. Write $\overset{\leftrightarrow}{\Diamond}^n x$ for $\overbrace{\overset{\leftrightarrow}{\Diamond}\ldots\overset{\leftrightarrow}{\Diamond}}^{n \text{ times}}x$. We will show that $\mathbf{I} \models a \neq \bot \Rightarrow \overset{\leftrightarrow}{\Diamond}^2 a = \top$. This suffices to define a ternary discriminator term d on $\mathbf{I}$ (see [1] for the definition, [6] for a survey of properties), putting: $d(x, y, z) := (\neg\overset{\leftrightarrow}{\Diamond}^2\neg(x \leftrightarrow y) \wedge z) \vee (\overset{\leftrightarrow}{\Diamond}^2\neg(x \leftrightarrow y) \wedge x)$. This in turn, by the properties of the discriminator, forces both the claims above.

It remains to show that $\overset{\leftrightarrow}{\Diamond}$ indeed has the relevant property on $\mathbf{I}$. To do this, we only need to prove that for every $w, v \in W$, the world w is reachable from v in at most two steps along the path composed of $\lhd$, $\rhd$, and identity relation. To start with, notice that $\overline{U}_{i,j} \lhd \infty \rhd \overline{U}_{k,\ell}$, which establishes what we need for $w, v \in \mathsf{U}$. Then, for $w, v \in \mathbf{Z}$, we have $w \lhd \overline{U}_{0,0} \rhd v$; further, $w \lhd \overline{U}_{0,0} \lhd \infty$. Finally, for $w \in \mathbf{Z}$ and $v \in \mathsf{U} - \{\infty, \overline{\mathsf{U}}_{0,0}\}$, we get $w \lhd \overline{U}_{0,0} \rhd v$.

FACT 5. The algebra $\mathbf{I}$ satisfies (A1) and the variety $\mathcal{V}(\mathbf{I})$ satisfies (C2).

PROOF. Put $\tau := \zeta \wedge \overset{\rightarrow}{\Diamond}(\overset{\rightarrow}{\Diamond}\overset{\leftarrow}{\Diamond}x \wedge \Upsilon)$. For the first part, we will show that $\mathbf{I}$ satisfies (A1) with $b = \zeta$ and $t = \tau$. Let u be a nonzero element below

ζ. Then, u is a finite union of of sets of the form $U_{i,j}$. Since diamonds and meets distribute over finite joins, it suffices to prove the claim with $u = U_{i,j}$, for some i, j. First, it is straighforward to verify that $\diamondminus U_{i,j}$ is precisely the set $S = \{\overline{U}_{k,\ell} \in \mathsf{A} : \mathsf{U}_{\mathsf{k},\ell} \subseteq \mathsf{U}_{\mathsf{i},\mathsf{j}}\}$. Note that $\overline{U}_{i,j}$ belongs to S iff $\overline{U}_{i,j} \in \mathsf{A}$ iff $i \leq (2^j - 1)/2$; a little calculation shows that otherwise $\overline{U}_{i,j+1}$ belongs to S. Then, we get that:

$$\diamondminus\diamondminus U_{i,j} \wedge \Upsilon = \diamondminus S \wedge \Upsilon = \begin{cases} \{\overline{U}_{k,\ell} \in \mathsf{A} : \mathsf{U}_{\mathsf{k},\ell} \subset \mathsf{U}_{\mathsf{i},\mathsf{j}}\}, & \text{if } \overline{U}_{i,j} \in \mathsf{A}, \\ \{\overline{U}_{k,\ell} \in \mathsf{A} : \mathsf{U}_{\mathsf{k},\ell} \subset \mathsf{U}_{\mathsf{i},\mathsf{j+1}}\}, & \text{if } \overline{U}_{i,j} \in \mathsf{B}. \end{cases}$$

Further, to compute $\zeta \wedge \diamondminus(\diamondminus\diamondminus U_{i,j} \wedge \Upsilon) = \zeta \wedge \diamondminus(\diamondminus S \wedge \Upsilon)$, let R stand for $\diamondminus S \wedge \Upsilon$. Then, we have: $\zeta \wedge \diamondminus R = \{z \in \mathbf{Z} : z \triangleleft R\} = \{z \in \mathbf{Z} :$ for some $U_{k,\ell} \subset U_{i,m},\, z \triangleleft \overline{U}_{k,\ell}\}$, with $m := j$, if $\overline{U}_{i,j} \in \mathsf{A}$, and $m := j + 1$, if $\overline{U}_{i,j} \in \mathsf{B}$. This finally yields:

$$\zeta \wedge \diamondminus R = \begin{cases} U_{i,j+1}, & \text{if } \overline{U}_{i,j} \in \mathsf{A}, \\ U_{i,j+2}, & \text{if } \overline{U}_{i,j} \in \mathsf{B}. \end{cases}$$

In both cases this is clearly a nonempty proper subset of $U_{i,j}$, which proves our claim.

For the second part, by properties of discriminator varieties, all subdirectly irreducible algebras in $\mathcal{V}(\mathbf{I})$ are simple, and they are precisely all the members of $SP_U(\mathbf{I})$. From this, (C2) is immediate.

THEOREM 1. *The variety $\mathcal{V}(\mathbf{I})$ is a minimal variety of tense algebras. It is not generated by its atomic members, in fact, it has none.*

PROOF. The first statement follows from the fact that $\mathbf{I}$ is simple and zero-generated, by means of an easy argument, which we leave out (it is spelled out e.g., in [2] Fact 3.1).

That $\mathcal{V}(\mathbf{I})$ is not generated by its atomic members follows directly from Facts 5 and 2.

For the remaining part, note first that, by properties of discriminator varieties, $\mathcal{V}(\mathbf{I}) = P_S SP_U(\mathbf{I})$. Now, "good" means "$b > \bot$", which obviously carries over to direct products, ultraproducts, and subalgebras. Thus, all nontrivial algebras in $\mathcal{V}$ are good, and the claim follows by Fact 1.

3. Second stone

In this section we will make use of *simulations* of bimodal logics by monomodal logics, defined in [5] and refined and improved upon in [4]. In full generality, simulation theory turns out to be quite a powerful tool for transfering both positive and negative results back and forth between polymodal and monomodal logics (and not only that). However, as we will need only a fraction of the theory, we will limit ourselves to bare essentials, and in a somewhat counterintuitive fashion will cast them in an algebraic form. This has the disadvantage of hiding all the intuitions behind an algebraic veil, but enables us to move directly to the point and saves space.

Let $\mathbf{A} = \langle A; \wedge, \neg, \Box_1, \Box_2, \bot, \top\rangle$ be a bimodal algebra. Define $\mathbf{A}^S$ to be $\langle A^S; \wedge, \neg, \Box, \bot, \top\rangle$, with the universe $A^S = A \times A \times \{\bot, \top\}$ and the boolean operations defined coordinatewise. Then set:

$$\Box\langle a, b, c\rangle = \begin{cases} \langle \Box_1 a \wedge b, a \wedge \Box_2 b, \top\rangle, & \text{if } c = \top, \\ \langle \bot, a \wedge \Box_2 b, \top\rangle, & \text{if } c = \bot. \end{cases}$$

So defined $\mathbf{A}^S$ is a monomodal algebra that simulates $\mathbf{A}$ in a sense that will soon become clearer. For now, notice only that the first coordinate "reproduces" the behaviour of $\Box_1$, if two others are set to $\top$, and the second does the same to $\Box_2$. Then, define the following three terms:

$$\begin{array}{rcl} \omega & := & \Box\bot \\ \alpha & := & \Diamond\Box\bot \\ \beta & := & \neg\Diamond\Box\bot \wedge \neg\Box\bot. \end{array}$$

Computed in A^S these yield respectively the values: $\langle \bot, \bot, \top\rangle$, $\langle \top, \bot, \bot\rangle$, $\langle \bot, \top, \bot\rangle$. Thus, they can be used to distinguish between the coordinates of A^S. With the help of these, we define for any term t in the bimodal language, its *simulation* t^S, by the following inductive clauses:

$$\begin{array}{rcl} x^S & := & x,\ x \text{ a variable}, \\ (\neg t)^S & := & \neg t^S, \\ (t \wedge u)^S & := & t^S \wedge u^S, \\ (\Box_1 t)^S & := & \Box(\alpha \to t^S), \\ (\Box_2 t)^S & := & \Box(\beta \to \Box(\beta \to \Box(\alpha \to t^S))). \end{array}$$

Let $t(\overline{x})$ be an n-ary term in the bimodal language, and let $\overline{a} = \langle a_0, \ldots, a_{n-1}\rangle$ be a vector from A^n. Clearly, $t^S(\overline{x})$ is then an n-ary term in the monomodal language, and $\overline{u} = \big\langle \langle a_0, \bot, \bot\rangle, \ldots, \langle a_{n-1}, \bot, \bot\rangle \big\rangle$ is a vector from $(A^S)^n$. The following is a reformulation of [3] Proposition 6.6.14.

FACT 6. In $\mathbf{A}^S$ we have: $\alpha \wedge t^S(\overline{u}) = \langle t(\overline{a}), \bot, \bot\rangle$. Moreover, $\mathbf{A} \models t = \top$ iff $\mathbf{A}^S \models \alpha \to t^S = \top$.

We call a monomodal algebra $\mathbf{B}$ a *simulation algebra* iff $\mathbf{B}$ is isomorphic to $\mathbf{A}^S$, for some bimodal $\mathbf{A}$. For a class $\mathcal{K}$ of bimodal algebras $\mathcal{K}^S$ stands for the class $\{\mathbf{A}^S : \mathbf{A} \in \mathcal{K}\}$. The next fact gathers together minimal information about simulation algebras that we will use later on. See [4], the passage from Theorem 9.5 up to Lemma 9.8, for proofs and more details.

FACT 7. Each simulation algebra $\mathbf{B}$ has the largest non-full congruence θ such that $\mathbf{B}/\theta$ is isomorphic to the two-element modal algebra with $\Box\bot = \top$. For any class $\mathcal{K}$ of bimodal algebras, we have: $SP_U(\mathcal{K}^S) = (SP_U(\mathcal{K}))^S$ and $H(\mathcal{K}^S) = (H(\mathcal{K}))^S \cup \{\mathbf{1}\}$.

Now, take the simulation algebra $\mathbf{I}^S$, and let $\mathcal{W} = \mathcal{V}(\mathbf{I}^S)$. Incidentally, it follows from Fact 7 that $\mathcal{W} = \mathcal{V}\big(\mathcal{V}(\mathbf{I})^S\big)$. Obviously, there is no hope of transfering Theorem 1 in its entirety: minimality is lost, together with discriminator and semisimplicity. Also, $\mathcal{W}$ will have atomic members. Nevertheless, the property of not being generated by atomic members, which we really are after, survives. To see that it does, we will prove that Fact 5 transfers from $\mathcal{V}(\mathbf{I})$ to $\mathcal{W}$.

FACT 8. The algebra $\mathbf{I}^S$ satisfies (A1) and the variety $\mathcal{W}$ satisfies (C2).

PROOF. For the first part, take $b = \zeta^S$ and $t = \tau^S$. Writing ζ^S in full, as $\Diamond\Box\bot \wedge \Diamond\Diamond\Box\bot \wedge \Box(\Diamond\Box\bot \to \Diamond\Diamond\Box\bot)$, we see immediately that $\zeta^S \leq \alpha$. Now by Fact 6 we have: $\mathbf{I} \models \zeta = \top$ iff $\mathbf{I}^S \models \alpha \to \zeta^S = \top$. Since $\zeta \neq \top$ in $\mathbf{I}$, we obtain $\alpha \to \zeta^S \neq \top$ in $\mathbf{I}^S$. In particular, $\zeta^S \neq \top$; hence, $\mathbf{I}^S$ is good. Then, take any $u \in A^S$, with $\bot < u \leq \zeta^S$. Since $\zeta^S \leq \alpha$, Fact 6 yields $\zeta^S = \langle \zeta, \bot, \bot\rangle$; thus $u = \langle a, \bot, \bot\rangle$, for some $a \in I$ with $\bot < a \leq \zeta$. Then, by Fact 6 again, $\tau^S(u) = \langle \tau(a), \bot, \bot\rangle$ and as $\bot < \tau(a) < a$, we obtain: $\bot < \tau^S(u) < u$. This shows that $\mathbf{I}^S$ satisfies (A1),

For the second part, take any subdirectly irreducible $\mathbf{B}$ from $\mathcal{W}$. By Jónsson Theorem, $\mathbf{B} \in HSP_U(\mathbf{I}^S)$. We will show that if $\mathbf{B} \notin SP_U(\mathbf{I}^S)$, then $\mathbf{B}$ is not good. Since, by Fact 7, all algebras in $SP_U(\mathbf{I}^S)$ are simulation algebras, we get that $\mathbf{B} = \mathbf{C}/\theta$, for some simulation algebra $\mathbf{C}$ and a nontrivial congruence $\theta \in \operatorname{Con}\mathbf{C}$. As $\mathbf{C}$ is a simulation algebra, $\mathbf{C} = \mathbf{D}^S$, for some subdirectly irreducible algebra $\mathbf{D} \in \mathcal{V}(\mathbf{I})$. By the properties of

$\mathcal{V}(\mathbf{I})$, the algebra $\mathbf{D}$ is simple. Thus, by the properties of the simulation, $\mathbf{Con}\,\mathbf{C}$ is a three element chain. If θ is full, then, obviously, $\mathbf{B}$ is not good. Suppose, θ is not full. Then θ is a maximal non-full congruence on $\mathbf{C}$. Therefore, $\Box\bot = \top$ in $\mathbf{B}$. It follows immediately that $\zeta^S = \bot$ in $\mathbf{B}$, proving that $\mathbf{B}$ is not good also in this case. This finishes the whole proof.

THEOREM 2. The variety $\mathcal{W}$ is not generated by its atomic members.

PROOF. By Fact 8 and Fact 2.

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