The Texas Shoot Out under Knightian Uncertainty

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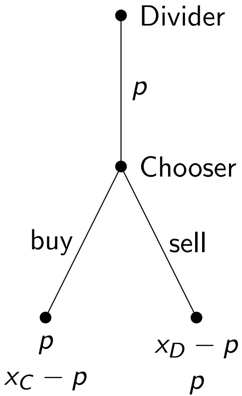
- Companies thus write an exit mechanism into their buy-sell agreement.
  $\rightsquigarrow$ Its mere existence can prevent a premature dissolving of a partnership.
  $\Rightarrow$ Can be explained by Knightian Uncertainty.

Setting

- Allocation of sole ownership of an indivisible object.
- Divider, chooser.
- Private valuations $x_D, x_C \in [0, 1]$.

The Texas Shoot Out



Chooser's response

Chooser sells the item if and only if

$$\begin{aligned} x_C - p &\leq p \\ \iff \qquad x_C &\leq 2p. \end{aligned}$$

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- independent of x_D or its distribution

Divider's decision problem (DP)

Consider a divider who anticipates chooser's reply.
If he believes chooser's valuation to be drawn from a cdf F, he tries to maximize

$$\pi_F(p \mid x_D) := (x_D - p) \cdot F(2p) + p \cdot (1 - F(2p)).$$

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- benchmark model: McAfee [McA92]
- security stragegy / maxmin: [VEW20]
  Announcing $p = \frac{x_D}{2}$ guarantees a payoff of $\frac{x_D}{2}$.

Divider's DP under Knightian Uncertainty

Let $\mathcal{G}$ be a set of cdfs divider deems possible.

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Her interim worst case expected utility from announcing p is thus

$$\pi(p \mid x_D) := \min_{G \in \mathcal{G}} \pi_G(p \mid x_D).$$

We assume divider to maximize this expression [GS89].

Lévy-Prohorov Metric

Fix a reference cdf F with pdf $f > 0$. Let $\varepsilon > 0$.

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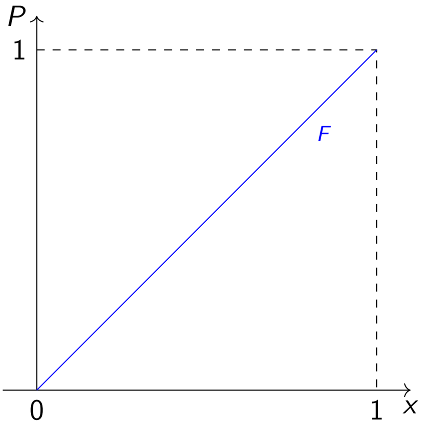
We define $\mathcal{G}$ to be the set of all cdfs within a closed ε-environment of F in the *Lévy-Prohorov metric*, i.e.

$$G \in \mathcal{G} \text{ if and only if}$$

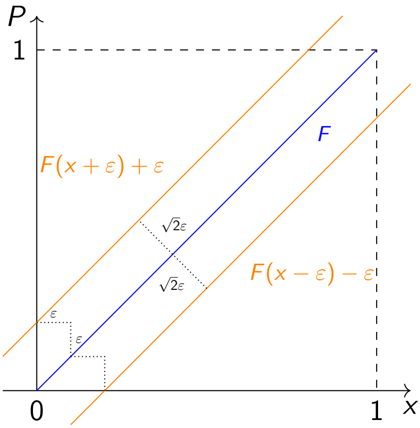
$$\inf \{\eta > 0 \,|\, F(x-\eta) - \eta \leq G(x) \leq F(x+\eta) + \eta \,\forall x\} \leq \varepsilon.$$

Lévy-Prohorov ε-environment



general

Lévy-Prohorov ε-environment



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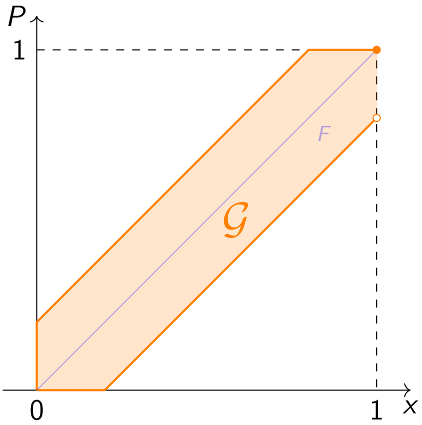

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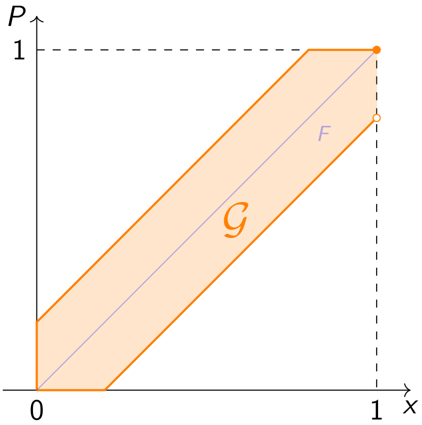

$\varepsilon = 0$

$\rightsquigarrow$ *stochastic* [McA92]

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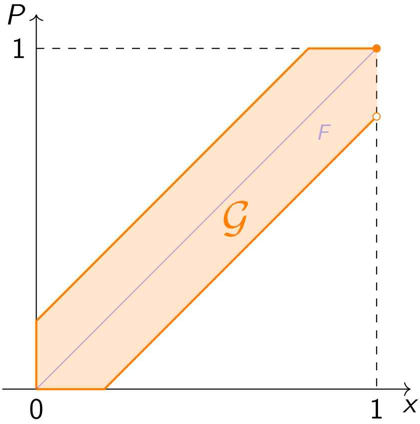
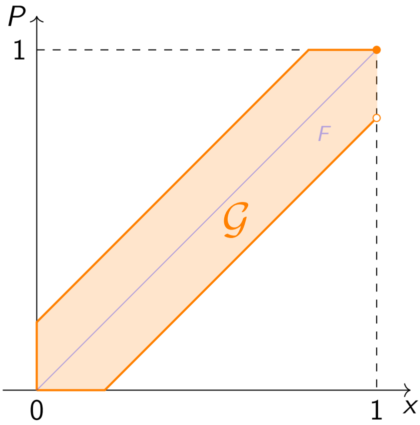


$\varepsilon = 0$
$\rightsquigarrow$ *stochastic* [McA92]

$\varepsilon \gg 0$
$\rightsquigarrow$ *full uncertainty* [VEW20]

general

Lévy-Prohorov ε-environment



$$\min_{G \in \mathcal{G}} \begin{array}{l} (x_D - p) \cdot G(2p) \\ +p \cdot (1 - G(2p)) \end{array}$$

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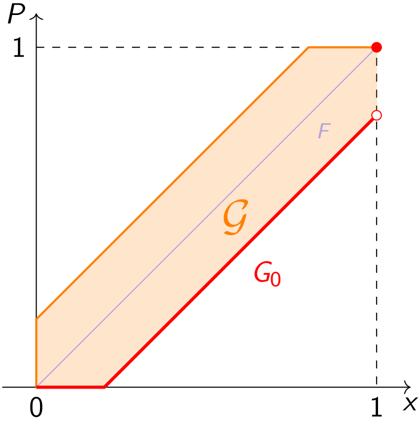

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1. $x_D > 2p$
$\rightsquigarrow G = G_0$

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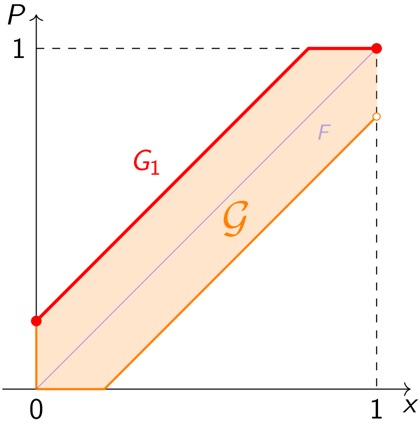

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1. $x_D > 2p$
   $\rightsquigarrow G = G_0$
2. $x_D < 2p$
   $\rightsquigarrow G = G_1$

general

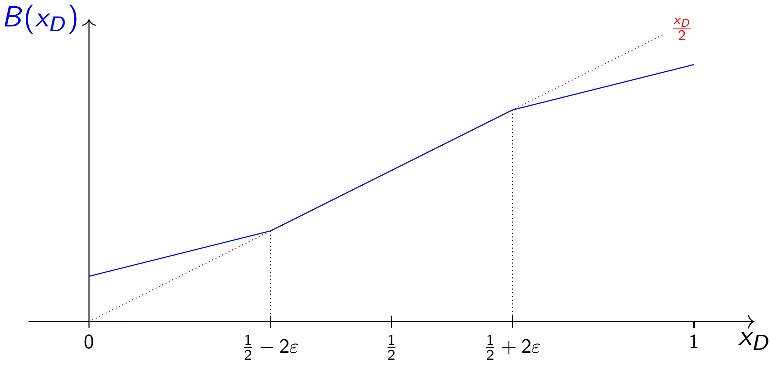
Simplified divider's DP

Having valuation x_D, divider seeks to find the p maximizing

$$\begin{aligned} &\pi(p \mid x_D) \\ =&\min_{G\in\mathcal{G}} \pi_G(p \mid x_D) \\ =&\min_{G\in\{G_0,G_1\}} \pi_G(p \mid x_D) \\ =&\begin{cases} \pi_{G_0}(p \mid x_D) & , 2p < x_D, \\ \pi_{G_1}(p \mid x_D) & , x_D \leq 2p. \end{cases} \end{aligned}$$

Optimal price announcement example

For $F \sim \mathcal{U}([0,1]), \varepsilon = 0.1$:



calc

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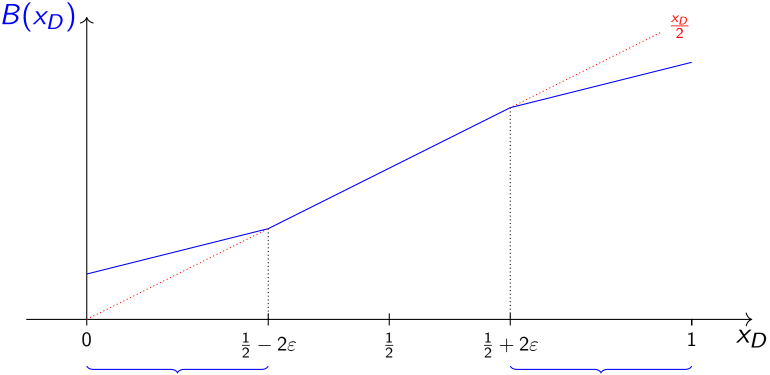

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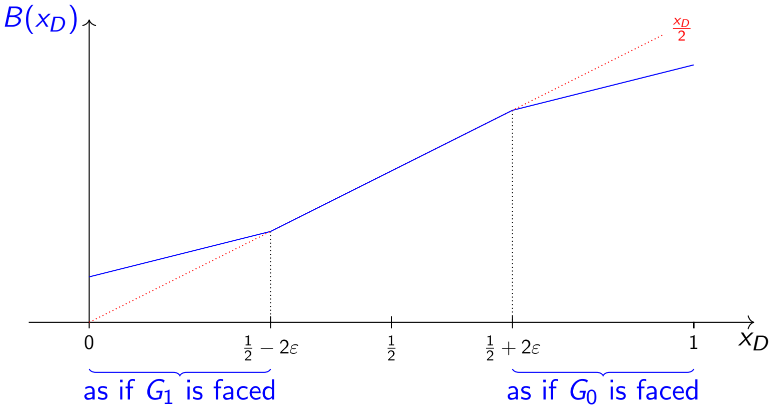

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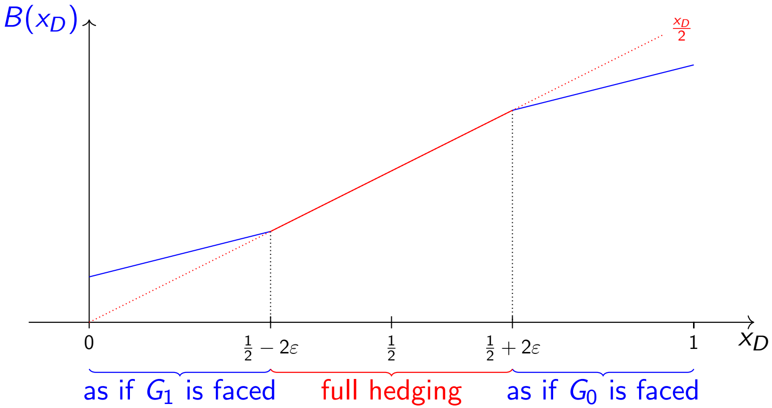

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Findings

(i) $B(x_D)$ is strictly increasing.

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(ii) Average types ($x_D \in [\underline{x}, \overline{x}]$) will fully hedge themselves.

(iii) Extreme types ($x_D \notin [\underline{x}, \overline{x}]$) try to extract additional payoff.

(a) $B(x_D) > \frac{x_D}{2}$ for $x_D < \underline{x}$,

(b) $B(x_D) < \frac{x_D}{2}$ for $\overline{x} < x_D$.

Strict quasi-concavity

Assumption

$\pi_{G_0}(. \mid x_D)$ and $\pi_{G_1}(. \mid x_D)$ are strictly quasi-concave for all x_D.

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Lemma (very rough, details)

The assumption is satisfied if

$$\frac{\partial}{\partial x}\left(x + \frac{F(x)}{f(x)}\right) - \varepsilon \cdot \left|\frac{\partial}{\partial x}\frac{1}{f(x)}\right| \geq 0,$$

$$\frac{\partial}{\partial x}\left(x - \frac{1 - F(x)}{f(x)}\right) - \varepsilon \cdot \left|\frac{\partial}{\partial x}\frac{1}{f(x)}\right| \geq 0.$$

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Example

Piecewise linear, truncated normal, triangle, Beta distributions.

pics

Optimal price announcement

Theorem

The optimal price announcement of divider is given by

$$B(x_D) = \begin{cases} p_1(x_D) & , \quad x_D < \underline{x}, \\ \frac{x_D}{2} & , \underline{x} \leq x_D \leq \overline{x} \\ p_0(x_D) & , \overline{x} < x_D, \end{cases}$$

where $p_1(x_D), p_0(x_D)$ *denote the unique and interior maxima of* $\pi_{G_1}(. \mid x_D)$ *resp.* $\pi_{G_0}(. \mid x_D)$
and $\underline{x} := F^{-1}(\frac{1}{2} - \varepsilon) - \varepsilon$, $\overline{x} := F^{-1}(\frac{1}{2} + \varepsilon) + \varepsilon$.

Allocation efficiency

The outcome of an allocation mechanism is efficient, if gives the object to the agent with the highest valuation.

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The allocation of the Texas Shoot Out under Knightian Uncertainty is efficient if the valuation of divider satisfies

$$\underline{x} \leq x_D \leq \overline{x}.$$

Interim worst case EU - Divider

A worst case expected utility maximizing divider expects to get

$$\Phi_D(x) := \pi(B(x) \mid x),$$

if he has valuation x.

Interim worst case EU - Chooser

If a worst case EU maximizing chooser faces the same ambiguity about the divider's valuation, she expects to get

$$\Phi_C(x) := \min_{G \in \mathcal{G}} \mathbb{E}_G \left[\max \{ x - B(z), B(z) \} \right],$$

if she has valuation x.

details

Chooser is better off than divider

Theorem

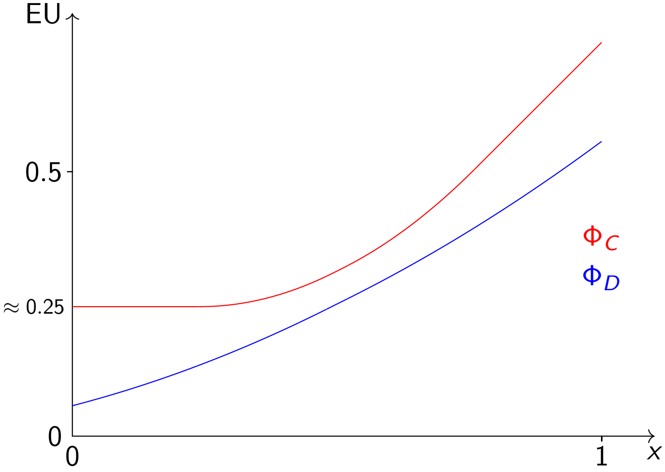
For all x we have

$$\Phi_D(x) \leq \Phi_C(x).$$

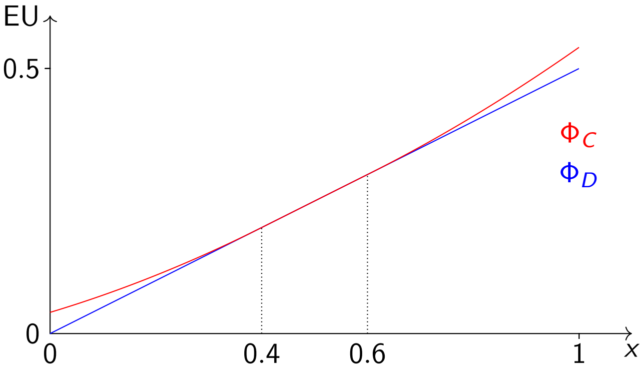
The inequality is an equality if and only if

$$x \in [F^{-1}(1-\varepsilon) - \varepsilon, F^{-1}(\varepsilon) + \varepsilon].$$

Comparison - low $\varepsilon = 0.01$



Comparison - high $\varepsilon = 0.3$



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1. The more uncertainty about the co-owners valuation is faced, the lower is the range of own valuations for a profitable exit, as well as its expected revenue. (Theorem 1)

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1. The more uncertainty about the co-owners valuation is faced, the lower is the range of own valuations for a profitable exit, as well as its expected revenue. (Theorem 1)

2. For extreme valuations you would rather be the chooser than the divider. (Theorem 2)

Thanks for your attention!

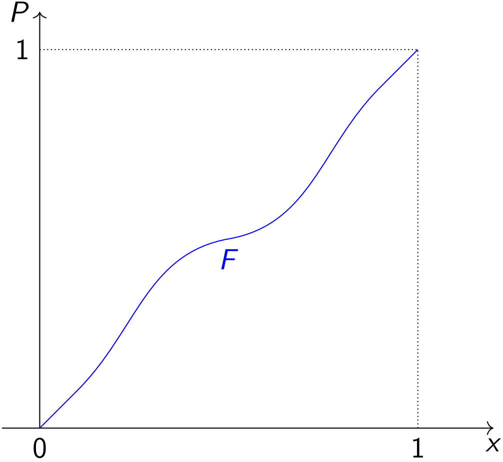
Literature

Itzhak Gilboa and David Schmeidler.
Maxmin expected utility with non-unique prior.
Journal of mathematical economics, 18(2):141–153, 1989.

R Preston McAfee.
Amicable divorce: Dissolving a partnership with simple mechanisms.
Journal of Economic Theory, 56(2):266–293, 1992.

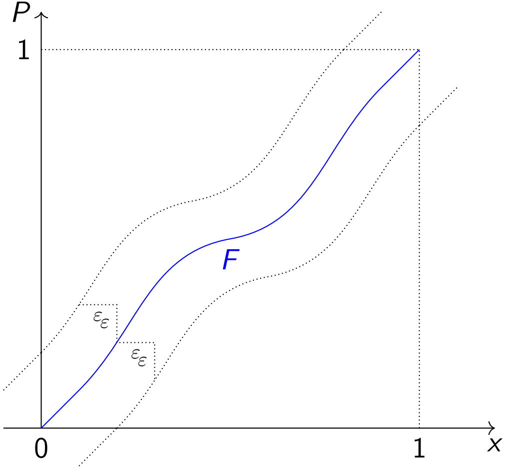
Matt Van Essen and John Wooders.
Dissolving a partnership securely.
Economic Theory, 69(2):415–434, 2020.

Lévy-Prohorov metric - general



▸ uniform

Lévy-Prohorov metric - general



▸ uniform

$B(x_D)$ explicit

Optimal price announcement for $F \sim \mathcal{U}([0,1])$ and $\varepsilon = 0.1$.

$$B(x_1) = \begin{cases} \frac{x_1}{4} - \frac{\varepsilon}{2} + \frac{1}{8} & , 0 \leq x_1 < \frac{1}{2} - 2\varepsilon, \\ \frac{x_1}{2} & , \frac{1}{2} - 2\varepsilon \leq x_1 \leq \frac{1}{2} + 2\varepsilon, , \\ \frac{x_1}{4} + \frac{\varepsilon}{2} + \frac{1}{8} & , \frac{1}{2} + 2\varepsilon < x_1 \leq 1. \end{cases}$$

back

Sufficient conditions for quasiconcavity

Lemma (7 [McA92])

If F fulfills the standard hazard rate conditions

$$\frac{\partial}{\partial x}\left(x+\frac{F(x)}{f(x)}\right)\geq 0 \quad \text{and} \quad \frac{\partial}{\partial x}\left(x-\frac{1-F(x)}{f(x)}\right)\geq 0,$$

then $u(x-p)\cdot F(2p)+u(p)\cdot(1-F(2p))$ is strictly quasiconcave in p.

Sufficient conditions for G_0, G_1

Ignoring horizontal segments of π_{G_0}, π_{G_1}, the following conditions imply strict quasiconcavity.

$$\frac{\partial}{\partial x}\left(x + \frac{F(x)}{f(x)} - \frac{\varepsilon}{f(x)}\right) \geq 0 \text{ and } \frac{\partial}{\partial x}\left(x - \frac{1-F(x)}{f(x)} - \frac{\varepsilon}{f(x)}\right) \geq 0$$

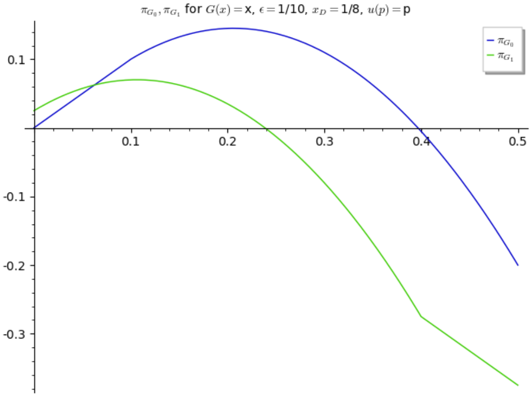
for π_{G_0} and

$$\frac{\partial}{\partial x}\left(x + \frac{F(x)}{f(x)} + \frac{\varepsilon}{f(x)}\right) \geq 0 \text{ and } \frac{\partial}{\partial x}\left(x - \frac{1-F(x)}{f(x)} + \frac{\varepsilon}{f(x)}\right) \geq 0$$

for π_{G_1}.

back

Examples of π_{G_0}, π_{G_1} for different distributions



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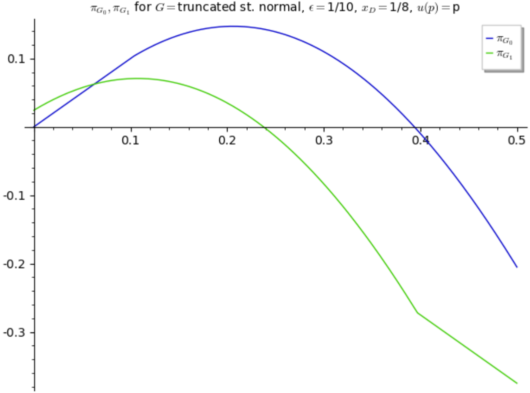

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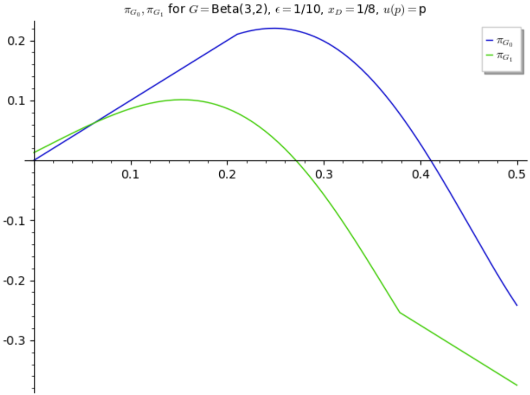

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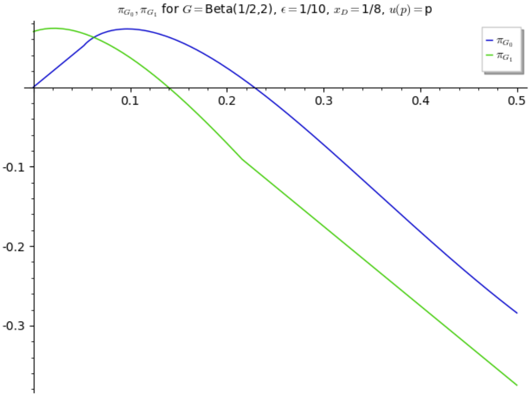

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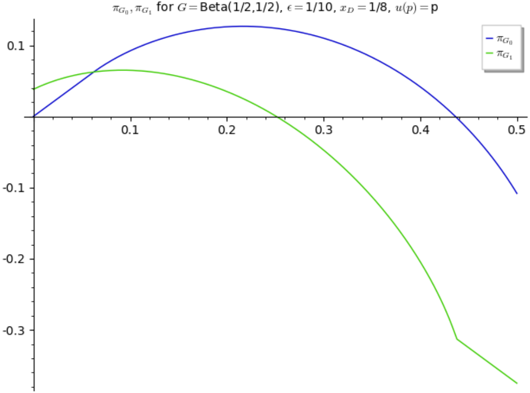

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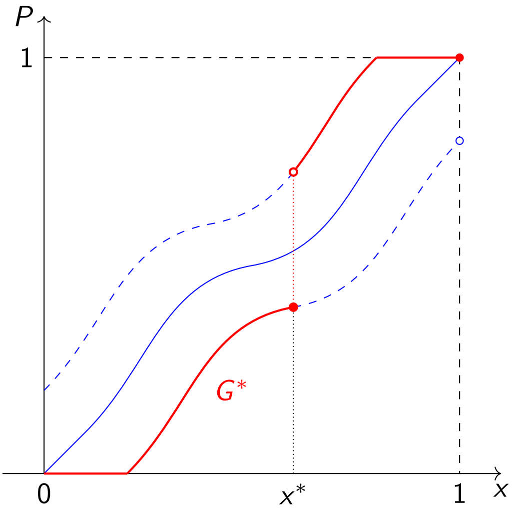
Worst cases for chooser

A chooser with valuation x_C has worst case utility

$$\Phi_C(x_C) = \begin{cases} \mathbb{E}_{G_1}[B(z)] & , x_C < \min 2B(z), \\ \mathbb{E}_{G^*(x_C)}[\max\{x_C - B(z), B(z)\}] & , x_C \in \text{range}(2B), \\ \mathbb{E}_{G_0}[x_C - B(z)] & , \max 2B(z) < x_C, \end{cases}$$

where $G^*(x_C)$ is the distribution function that switches from G_0 to G_1 at $x^* = B^{-1}(\frac{x_C}{2})$.

G^* illustration



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Ingredients

- $x_D, x_c \in X = [a, b]$ set of possible valuations
- F reference distribution function on X with positive, continuous density function f
- $u \in \mathcal{C}^2$ strictly increasing and concave utility function
- Texas Shoot Out as explained.