Asymptotics of Highly Composite Numbers

Achim Flammenkamp
Fakultät für Mathematik, Universität Bielefeld, Germany
achim@uni-bielefeld.de

For any number $n \in \mathbb{N}$ let

$$n = \prod_{i=1}^{\infty} p_i^{e_i} = \prod_{i=1}^{m} p_i^{e_i} \quad \text{with} \quad \forall\, i \in \mathbb{N}: \; e_i \in \mathbb{N}_0, \; p_i \in \mathbb{P} \; \text{and} \; \forall\, j < i: \; p_j < p_i$$

$$\text{and} \qquad m = \max\{i \in \mathbb{N} : e_i > 0\} \;\; \text{with} \;\; 0 = \max\{\}$$

be its canonical prime factorization. Then the number of its divisors is

$$d(n) = \prod_{i=1}^{m} (1 + e_i) \quad .$$

Notation:

$$p_0 := 1 \qquad \text{and} \qquad \forall\, x \in \mathbb{R}^+ : \quad \ln x := \log_{\mathrm{e}} x \;\; , \;\; \mathrm{lb}\, x := \log_2 x$$

Definition 1
A number $n \in \mathbb{N}$ is called highly composite *iff* $\forall\, n' < n : \quad d(n') < d(n)$.

Thus the sequence of the **Highly composite numbers** is the sequence of the recordholder of the d-function. It follows immediatly for any $n \in$ HCN, the set of all Highly composite numbers, holds

$$\forall\, i, j \in \mathbb{N}: \quad i \geq j \;\Rightarrow\; e_i \leq e_j \quad .$$

S. Ramanujan ([2]) proved, that iff $n \in$ HCN then $\forall\, i \in \mathbb{N}$ holds

$$\lfloor \frac{\ln p_m}{\ln p_i} \rfloor \leq e_i \leq 2 \lfloor \frac{\ln p_{m+1}}{\ln p_i} \rfloor \quad . \tag{1}$$

With

Definition 2 $\quad \forall\, n \in \text{HCN}\; \forall\, k \in \mathbb{N}: \quad m_k := \max\{i \in \mathbb{N} : e_i \geq k\}$

these bounds (1) can simply be improved to

$$\forall\, i \in \mathbb{N}: \quad \max_{k\in\mathbb{N}}\{k\lfloor\frac{\ln p_{m_k}}{\ln p_i}\rfloor\} \;\leq\; e_i \;\leq\; \min_{k\in\mathbb{N}}\{(k+1)\lceil\frac{\ln p_{m_k+1}}{\ln p_i}\rceil - 2\}$$

These inequalities give the asymptotically correct result $e_i \sim \frac{1}{\ln 2}\frac{\ln p_m}{\ln p_i}$, which Ramanujan also proved. Preciser, $\forall\, n \in \mathrm{HCN}\; \forall\, i \in \mathbb{N}$ holds

$$\frac{\ln p_i}{\ln(1+\frac{1}{e_i+0.5})} \sim \frac{\ln p_m}{\ln 2} \quad \text{respectively} \quad e_i \sim \frac{1}{2^{\frac{\ln p_i}{\ln p_m}}-1} - \frac{1}{2} \quad .$$

Definition 3

$$\forall\, n \in \mathrm{HCN}: \quad x_{min}(n) := \max\{\frac{\ln p_i}{\ln(1+\frac{1}{e_i})} \;:\; 1 \leq i \leq m\}$$

$$\forall\, n \in \mathrm{HCN}: \quad x_{max}(n) := \min\{\frac{\ln p_i}{\ln(1+\frac{1}{e_i+1})} \;:\; 1 \leq i \leq m+1\}$$

Theorem 1
If $n \in \mathrm{HCN}$ *and* $x_{min}(n) < x_{max}(n)$ *then* $\forall\, x \in [x_{min}, x_{max}[$ *holds*

$$e_i = \left\lfloor \frac{1}{\sqrt[x]{p_i}-1} \right\rfloor \quad \forall\, i \in \mathbb{N} \tag{2}$$

Definition 4
A member of HCN *is called a* **superior highly composite number***, if* $\exists\, x \in \mathbb{R}^+$ *satisfying (2). Denote by* SHCN *the set of these numbers and define*

$$\forall\, x \in \mathbb{N}: \qquad \pi(x) := |\{p \leq x \;:\; p \in \mathbb{P}\}|$$

Theorem 2
$\forall\, x \in \mathbb{R}^+\; \exists\, n \in \mathbb{N}$ *whose exponents* e_i *are defined by condition (2) with* $n \in \mathrm{SHCN}$*. Especially for these* n *holds:* $p_m \leq 2^x < p_{m+1}$ *. Furthermore, define* $\kappa = \sum_{i=1}^{m} e_i$ *then we have*

$$\kappa = \sum_{i=1}^{\lfloor\frac{x}{\ln 2}\rfloor} \pi(p_{m_i}) = \sum_{i=1}^{\lfloor\frac{x}{\ln 2}\rfloor} \pi\big((1+\tfrac{1}{i})^x\big) \tag{3}$$

Each $\kappa \in \mathbb{N}$ is also the unique index of a $n \in \mathrm{SHCN}$ because the equation

$$\frac{x \ln p_i}{x \ln p_j} = \frac{\ln(1+\frac{1}{a})}{\ln(1+\frac{1}{b})} \quad \text{with} \quad a, b \in \mathbb{N} \tag{4}$$

has only solutions if $i = j$ (then of course with any $a = b$).

Definition 5

$$\forall\, x \in \mathbb{R}^+ : \qquad \mathrm{Li}(x) := \int_2^x \frac{dt}{\ln t}$$

$$\forall\, x \in \mathbb{C} : \quad \zeta(x) := \sum_{k=1}^{\infty} \frac{1}{k^x} \;=\; \frac{1}{2^{1-x}-1} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^x} \;=\; 2 \sin \frac{x\pi}{2}\, \frac{\Gamma(1-x)}{(2\pi)^{1-x}} \zeta(1-x)$$

Now let us remember a few known conjectures respectively theorems:

Conjecture 1

$$\pi(x) \sim \mathrm{Li}(x) + O(\sqrt{x} \ln x) \qquad \text{„}\textit{Riemann Hypothesis}\text{''}$$

$$\pi(x) \sim \mathrm{Li}(x) + O\Big(\sqrt{\frac{x \ln\ln x}{\ln x}}\Big) \qquad \text{„}\textit{Cramèr's Estimation}\text{''}$$

H. von Mangoldt [6] proved the following theorems, namely

Theorem 3

$$\sum_{\substack{p \in \mathbb{P} \\ p^k \leq x}} \ln p = x - \ln 2\pi - \ln \sqrt{1 - \frac{1}{x^2}} + \sum_{\substack{0 \leq \Re\rho \leq 1 \\ \zeta(\rho)=0}} \frac{x^\rho}{\rho}$$

and

Theorem 4

$$T \in \mathbb{R}^+ : \quad |\{t \in \mathbb{R}^+ : \; \zeta(\sigma + 2\pi i t) = 0 \;, \; 0 < \sigma < 1, \; t \leq T\}| \sim T \ln \frac{T}{\mathrm{e}} + O(\ln T)$$

whereas Riemann made his famous conjecture regarding the zeroes of the ζ-function, namely

Conjecture 2

$$\forall\, \sigma, t \in \mathbb{R} \;\; \textit{with} \;\; 0 \leq \sigma \leq 1 \;\; \textit{and} \;\; \zeta(\sigma + it) = 0 \quad \Rightarrow \quad \sigma = \frac{1}{2}$$

The above two theorems together with this conjecture imply

Conjecture 3

$$\sum_{\substack{p \in \mathbb{P} \\ p \leq x}} \ln p = \sum_{h \in \mathbb{N}} \mu(h) \sum_{\substack{p \in \mathbb{P} \\ p^k \leq x^{\frac{1}{h}}}} \ln p \;\sim\; x + O(\sqrt{x} \ln^2 x)$$

Using the Riemann Hypothesis one can show that

Conjecture 4

$$p_{i+1} - p_i \sim O(\sqrt{p_i} \ln p_i) \tag{5}$$

Since 1994 it is known [4] that $\forall \varepsilon \in \mathbb{R}^+ : \; p_{i+1} - p_i \sim O(p_i^{0.535+\varepsilon})$ *and the true order of the "maximal" gap size is conjectured to be* $\sim \ln^2 p_i$*, preciser:*

$$p_{i+1} - p_i = n \quad \Rightarrow \quad \ln p_i \sim \sqrt{n} + \ln \sqrt{n} + O(1)$$

Theorem 5

$$\forall\, p_i \in [4, 2^{60}] \; \subset \mathbb{P} : \qquad \frac{p_{i+1} - p_i}{(\ln p_i)(\ln p_{i+1})} \;<\; 1$$

Theorem 6

$$\forall\, k \in \mathbb{N} : \quad \frac{\ln p_{m_k}}{\ln p_{m_1}} \sim \mathrm{lb}(1 + \frac{1}{k}) + \frac{1}{O(\sqrt{p_{m_k}})}$$

Proof: Let $n \in$ HCN with $n = \prod_{j=1} \prod_{i=m_{j+1}+1}^{m_j} p_i^j$. Next we choose

$$n'(h, \ell) = n \frac{\prod_{i=m_1+1}^{m_1+\ell} p_i}{\prod_{i=m_k+1-h}^{m_k} p_i} \quad \text{and} \quad n^*(h, \ell) = n \frac{\prod_{i=m_k+1}^{m_k+\ell} p_i}{\prod_{i=m_1+1-h}^{m_1} p_i}$$

with $h, \ell \in \mathbb{N}$ to our disposal. Because $n \in$ HCN we have

$$d(n') > d(n) \;\Rightarrow\; n' > n \quad \text{and} \quad d(n^*) > d(n) \;\Rightarrow\; n^* > n$$

meaning

$$2^\ell k^h > (k+1)^h \;\Rightarrow\; \prod_{i=1}^{\ell} p_{m_1+i} > \prod_{i=1}^{h} p_{m_k-i+1}$$

and

$$(k+1)^\ell > k^\ell 2^h \;\Rightarrow\; \prod_{i=1}^{\ell} p_{m_k+i} > \prod_{i=1}^{h} p_{m_1-i+1}$$

resulting in

$$\ell + h \,\mathrm{lb}\, k > h \,\mathrm{lb}(k+1) \;\Rightarrow\; \sum_{i=1}^{\ell} \ln p_{m_1+i} > \sum_{i=1}^{h} \ln p_{m_k-i+1}$$

and

$$\ell \,\mathrm{lb}(k+1) > h + \ell \,\mathrm{lb}\, k \;\Rightarrow\; \sum_{i=1}^{\ell} \ln p_{m_k+i} > \sum_{i=1}^{h} \ln p_{m_1-i+1}$$

Estimating these sums by their maximal and minimal summand, we get

$$\frac{\ell}{h} > \mathrm{lb}\,\frac{k+1}{k} \;\Rightarrow\; \ell \ln p_{m_1+\ell} > h \ln p_{m_k-h+1}$$

and

$$\mathrm{lb}\,\frac{k+1}{k} > \frac{h}{\ell} \;\Rightarrow\; \ell \ln p_{m_k+\ell} > h \ln p_{m_1-h+1}$$

Transforming the right sides further, we get

$$\ell \ln p_{m_1} + \ell \ln(1 + \frac{p_{m_1+\ell} - p_{m_1}}{p_{m_1}}) > h \ln p_{m_k} + h \ln(1 - \frac{p_{m_k} - p_{m_k-h+1}}{p_{m_k}})$$

and for the second conclusion

$$\ell \ln p_{m_k} + \ell \ln(1 + \frac{p_{m_k+\ell} - p_{m_k}}{p_{m_k}}) > h \ln p_{m_1} + h \ln(1 - \frac{p_{m_1} - p_{m_1-h+1}}{p_{m_1}})$$

Using now the prime-gap conjecture (5), we get next for the first conclusion

$$\ell \ln p_{m_1} + \ell \frac{O(\sqrt{p_{m_1}} \ln p_{m_1})}{p_{m_1}} > h \ln p_{m_k} - h \frac{O(\sqrt{p_{m_k}} \ln p_{m_k})}{p_{m_k}}$$

and for the other

$$\ell \ln p_{m_k} + \ell \frac{O(\sqrt{p_{m_k}} \ln p_{m_k})}{p_{m_k}} > h \ln p_{m_1} - h \frac{O(\sqrt{p_{m_1}} \ln p_{m_1})}{p_{m_1}}$$

Simplifying both relation further, we get

$$\frac{\ell}{h} + \frac{\ell}{h} O(\frac{1}{\sqrt{p_{m_1}}}) > \frac{\ln p_{m_k}}{\ln p_{m_1}} - \frac{\ln p_{m_k}}{\ln p_{m_1}} O(\frac{1}{\sqrt{p_{m_k}}})$$

and

$$\frac{\ln p_{m_k}}{\ln p_{m_1}} + \frac{\ln p_{m_k}}{\ln p_{m_1}} O(\frac{1}{\sqrt{p_{m_k}}}) > \frac{h}{\ell} - \frac{h}{\ell} O(\frac{1}{\sqrt{p_{m_1}}})$$

Now, chose in the first case $\frac{\ell}{h} = \mathrm{lb}\frac{k+1}{k} + \varepsilon$ and in the second case $\frac{h}{\ell} = \mathrm{lb}\frac{k+1}{k} - \varepsilon$ such that both pre-conditions with $\varepsilon \in \mathbb{R}^+$ become true, we get

$$(\mathrm{lb}\frac{k+1}{k} + \varepsilon)(1 + \frac{1}{O(\sqrt{p_{m_1}})}) > \frac{\ln p_{m_k}}{\ln p_{m_1}}(1 - \frac{1}{O(\sqrt{p_{m_k}})})$$

and

$$\frac{\ln p_{m_k}}{\ln p_{m_1}}(1 + \frac{1}{O(\sqrt{p_{m_k}})}) > (\mathrm{lb}\frac{k+1}{k} - \varepsilon)(1 - \frac{1}{O(\sqrt{p_{m_1}})})$$

Let now n, n', n^* grow larger and larger, such that also $m_1 - m_2$ and $m_k - m_{k+1}$ become larger and larger, we can chose larger and larger values h and ℓ resulting in ε tends to 0. With the notice that surely $\frac{\ln p_{m_k}}{\ln p_{m_1}} \sim O(1)$, we finally arrive at

$$\mathrm{lb}\frac{k+1}{k} + \frac{1}{O(\sqrt{p_{m_1}})} > \frac{\ln p_{m_k}}{\ln p_{m_1}} + \frac{1}{O(\sqrt{p_{m_k}})} > \mathrm{lb}\frac{k+1}{k} - \frac{1}{O(\sqrt{p_{m_1}})}$$

which is equivalent to

$$\frac{\ln p_{m_k}}{\ln p_{m_1}} \sim \mathrm{lb}\frac{k+1}{k} + \frac{1}{O(\sqrt{p_{m_k}})}$$

$. \square$

Corollary 1

$$\forall\, k \in \mathbb{N}: \qquad p_{m_k} \sim \left(1 + O\left(\frac{\ln p_m}{\sqrt{p_{m_k}}}\right)\right) p_m^{\mathrm{lb}(1+\frac{1}{k})} \sim p_m^{\mathrm{lb}(1+\frac{1}{k})} + p_m^{\frac{1}{2}\mathrm{lb}(1+\frac{1}{k})}\, O(\ln p_m)$$

Theorem 7

$$n \in \mathrm{HCN}: \quad \ln n \sim p_m + p_m^{\mathrm{lb}\,\frac{3}{2}} + O(\sqrt{p_m}\ln^2 p_m)$$

Proof:

$$\ln n = \sum_{i=1}^{m} e_i \ln p_i = \sum_{i=1}^{m_1} \ln p_i + \sum_{i=1}^{m_2} \ln p_i + \sum_{i=1}^{m_3} (e_i - 2)\ln p_i$$

$$\geq \sum_{\substack{p\in\mathbb{P}\\ p\leq p_{m_1}}} \ln p + \sum_{\substack{p\in\mathbb{P}\\ p\leq p_{m_2}}} \ln p \;\sim\; p_{m_1} + O(\sqrt{p_{m_1}}\ln^2 p_{m_1}) + p_{m_2} + O(\sqrt{p_{m_2}}\ln^2 p_{m_2})$$

Using $p_m = p_{m_1}$ and $\ln p_{m_2} \sim \mathrm{lb}\frac{3}{2}\ln p_m$ we get

$$\ln n \geq p_m + p_m^{\mathrm{lb}\,\frac{3}{2}} + O(\sqrt{p_m}\ln^2 p_m)$$

Because the sequence of the $n \in$ HCN is strictly monoton increasing the subsequence of the $n' \in$ SHCN must have the same asymtotic in its order of growth as far as we doesn't get finer than the spacing of successive SHCN, which is bounded from p_m by above. Thus we have $\ln n \sim \ln n' + O(\ln p_m)$.

$$\sum_{i=1}^{m_3}(e_i - 2)\ln p_i \leq \sum_{i=1}^{m_3}\Big(\frac{1}{\sqrt[x]{p_i}-1} - 2\Big)\ln p_i = \sum_{i=1}^{m_3}\Big(\frac{1}{\mathrm{e}^{\frac{\ln p_i}{x}}-1} - 2\Big)\ln p_i \leq \sum_{i=1}^{m_3}$$

$$\left(\frac{x}{\ln p_i} - 2\right)\ln p_i \leq \sum_{i=1}^{m_3} x - 2\ln p_i = m_3 x - 2\sum_{i=1}^{m_3}\ln p_i \leq m_3\,\mathrm{lb}\,p_{m+1} - 2\sum_{\substack{p\in\mathbb{P}\\ p\leq p_{m_3}}}\ln p <$$

$$\frac{p_{m_3}}{\ln p_{m_3} - 1}\frac{\ln p_{m+1}}{\ln 2} - 2p_{m_3} + O(\sqrt{p_{m_3}}\ln^2 p_{m_3}) \sim p_{m_3} O\left(\frac{\ln p_m}{\ln 2\ln p_{m_3}} - 2\right) \sim p_{m_3}O(1)$$

Putting everthing together we finally get

$$\ln n \sim \ln n' + O(\ln p_m) \leq p_m + p_m^{\mathrm{lb}\,\frac{3}{2}} + O(\sqrt{p_m}\ln^2 p_m) + p_{m_3}O(1) + O(\ln p_m)$$

and because $p_{m_3} \sim p_m^{\mathrm{lb}\,\frac{4}{3}} < p_m^{\frac{1}{2}}$ we get the necessary upper bound. $\square$

Corollary 2

$$\forall\, n \in \text{HCN}: \quad p_m \sim \ln n - \ln^{\operatorname{lb}\frac{3}{2}} n + O(\ln^{\frac{1}{2}} n\ \ln^2 \ln n)$$

$$\Rightarrow \quad m = \pi(p_m) \sim \pi(\ln n - \ln^{\operatorname{lb}\frac{3}{2}} n) \sim \operatorname{Li}(\ln n) - \frac{\ln^{\operatorname{lb}\frac{3}{2}} n}{\ln\ln n}$$

Theorem 8

$$n \in \text{HCN}: \quad \operatorname{lb} d(n) \sim \sum_{i=1}^{k-1} \operatorname{lb}(1+\frac{1}{i})\, \pi(p_m^{\operatorname{lb} 1+\frac{1}{i}})$$

Proof: Firstly the trival direction with fixed $k \in \mathbb{N}$:

$$\operatorname{lb} d(n) = \operatorname{lb} \prod_{j=1}^{m} (1+e_j) = \sum_{i=1}^{k-1} (m_i - m_{i+1}) \operatorname{lb}(1+i) + \sum_{j=1}^{m_k} \operatorname{lb}(1+e_j)$$

$$= \sum_{i=1}^{k-1} m_i \operatorname{lb}(1+\frac{1}{i}) + \sum_{j=1}^{m_k} \operatorname{lb}\frac{1+e_j}{k} \geq \sum_{i=1}^{k-1} m_i \operatorname{lb}(1+\frac{1}{i}) = \sum_{i=1}^{k-1} \pi(p_{m_i}) \operatorname{lb}(1+\frac{1}{i})$$

To get a sufficient upper bound we estimate the neglected term with $n' \in$ SHCN using (2):

$$\sum_{j=1}^{m_k} \operatorname{lb}(1+e_j) = \frac{1}{\ln 2} \sum_{j=1}^{m_k} \ln(1 + \left\lfloor \frac{1}{\sqrt[x]{p_j} - 1} \right\rfloor) \leq \frac{1}{\ln 2} \sum_{j=1}^{m_k} \ln(1 + \frac{1}{\sqrt[x]{p_j} - 1})$$

Next we use the estimation $\ln(1+x) < \sqrt[2.6]{x} < x^{0.4}$ which holds $\forall\, x \in \mathbb{R}^+$.

$$\frac{1}{\ln 2} \sum_{j=1}^{m_k} \ln(1 + \frac{1}{\sqrt[x]{p_j} - 1}) < \frac{1}{\ln 2} \sum_{j=1}^{m_k} \Big(\frac{1}{\sqrt[x]{p_j} - 1}\Big)^{0.4} = \frac{1}{\ln 2} \sum_{j=1}^{m_k} \frac{1}{(\mathrm{e}^{\frac{\ln p_j}{x}} - 1)^{0.4}}$$

$$< \frac{1}{\ln 2} \sum_{j=1}^{m_k} \Big(\frac{x}{\ln p_j}\Big)^{0.4} < \frac{m_k}{\ln 2} \Big(\frac{x}{\ln 2}\Big)^{0.4} < \frac{m_k}{\ln 2} \Big(\frac{\operatorname{lb} p_{m+1}}{\ln 2}\Big)^{0.4} < \frac{m_k}{\ln^2 2} (\ln p_{m+1})^{0.4}$$

So, putting everything together, we get

$$\operatorname{lb} d(n') < \sum_{i=1}^{k-1} \pi(p_{m_i}) \operatorname{lb}(1+\frac{1}{i}) + \pi(p_{m_k})(\frac{\sqrt{\ln p_m}}{\ln^2 2} - \operatorname{lb} k)$$

and finally together with the lower bound and $p_{m_i} \sim p_m^{\operatorname{lb} 1+\frac{1}{i}} + O(1)\, p_m^{\frac{1}{2}\operatorname{lb} 1+\frac{1}{i}} \ln p_m$ we arrive at

$$\operatorname{lb} d(n) \sim \sum_{i=1}^{k-1} \operatorname{lb}(1+\frac{1}{i}) \pi(p_m^{\operatorname{lb} 1+\frac{1}{i}}) + O(\sqrt{\ln p_m}\, \pi(p_m^{\operatorname{lb} 1+\frac{1}{k}})) + O(\pi(p_m^{\frac{1}{2}\operatorname{lb}\frac{3}{2}} \ln p_m))$$

Because for $k > \sqrt{6}+2$ yields $\mathrm{lb}(1+\frac{1}{k}) < \frac{1}{2}\,\mathrm{lb}\frac{3}{2}$, we lastly have

$$\mathrm{lb}\, d(n) \sim \sum_{i=1}^{4} \mathrm{lb}(1+\frac{1}{i})\pi(p_m^{\mathrm{lb}\, 1+\frac{1}{i}}) + O(p_m^{\frac{1}{2}\,\mathrm{lb}\,\frac{3}{2}})$$

$\square$

Corollary 3

$$n \in \mathrm{HCN}: \quad \mathrm{lb}\, d(n) \sim \mathrm{Li}(\ln n) + \frac{1}{\mathrm{lb}\frac{3}{2}}\,\frac{\ln^{\mathrm{lb}\frac{3}{2}} n}{\ln^2 \ln n} + O(\sqrt{n}\ln n)$$

Next, two conjectures follow about the order of appearance of the first respectively last Highly composite number for fixed largest prime p_m.

Conjecture 5

$$\forall\, k \in \mathbb{N}\; \forall\, n \in \mathrm{HCN}\; \exists\, n' \in \mathrm{HCN}: \quad n' > n\;,\;\; m_k(n') = m_k(n)+1$$

$$\forall\, k \in \mathbb{N}\; \forall\, n \in \mathrm{HCN}, m_k(n){>}0\; \exists\, n' \in \mathrm{HCN}: \quad n' < n\;,\;\; m_k(n') = m_k(n)-1$$

Worded differently: if there is a HCN containing p_i^k then there will be a larger HCN with p_{i+1}^k and, supposed $i{>}0$, there is a smaller HCN containing p_{i-1}^k .

Conjecture 6
Let $n', n^* \in \mathrm{HCN}$ *and* $d(n^*) > 2d(n')$. *Then holds with* $n \in \mathrm{HCN}$

$$\forall\, n \in]n', n^*]: \; m(n) > m(n') \quad \Rightarrow \quad \forall\, n > n': \; m(n) > m(n') \quad .$$

Some remarks to the two previous conjectures: given a fixed maximal prime index m then it seems highly evident, that the smallest HCN with this m equals always another HCN with largest prime index $m-1$ multiplyed by p_m, and similarily, the largest HCN with this m is always a different HCN with the same prime index m multiplyed by a prime smaller p_m. This last observation would even imply the validity of the last conjecture for the bound $d(n^*) > 1.5\, d(n')$.

Now let us give some simple upper and lower bounds on $d(n)$ with

Theorem 9

$$\forall\, n \in \mathrm{HCN}: \quad 2^{m(n)} \le d(n) \le \mathrm{e}^{m(n)}$$

Proof: The lower bound follows trivially from $d(n) = \prod_{i=1}^{m}(e_i+1) \ge 2^m$.
$\square$

Theorem 10

$$\forall\, x \in [1, 2^{40}[\setminus \{1421, 1422\} \;\subset \mathbb{R}^+ : \quad x - 2\sqrt{x} \;<\; \sum_{\substack{p \le x \\ p \in \mathbb{P}}} \ln p \;\le\; x - \frac{1}{4}\sqrt{x}$$

$$\forall\, x \in [1, 2^{40}[\;\subset \mathbb{P} : \qquad x - \frac{28}{15}\sqrt{x} \;<\; \sum_{\substack{p \le x \\ p \in \mathbb{P}}} \ln p \;\le\; x - \frac{1}{4}\sqrt{x}$$

Corollary 4

$$\forall\, n \in \text{HCN } \textit{with } \ln n < 2^{40} :$$

$$p_m - \frac{28}{15}\sqrt{p_m} < \sum_{i=1}^{m} \ln p_i \le \sum_{i=1}^{m} e_i \ln p_i - \sum_{i=1}^{m_2} \ln p_i = \ln n - p_{m_2} + \frac{28}{15}\sqrt{p_{m_2}} \quad (6)$$

By inspecting all HCN with $\ln n < 10^4$ we conjecture that holds

Conjecture 7

$$\forall\, n \in \text{HCN } \textit{with } n > 55440 : \quad p_m < \ln n$$

$$\forall\, n \in \text{HCN } \textit{with } n > 43243200 : \quad m_1 - m_2 \ge 4$$

$$\forall\, n \in \text{HCN } \textit{with } \ln n > 85.92 : \quad m_2 > m_3$$

Some bounds on $p_{m_1}, p_{m_2}, p_{m_3}$ follow which hold for every HCN:

$$m_2 > m_3 \quad \Rightarrow \quad p_{m_2}\, p_{m_3} \le p_{m_1+1} \quad \text{and} \quad p_{m_1} \le p_{m_2+1}\, p_{m_3+1} \qquad (7)$$

$$p_{m_2}\, p_{m_2-1}\, p_{m_2-2} < p_{m_1+1}\, p_{m_1+2}$$

$$p_{m_2}\, p_{m_2-1} < p_{m_3+1}\, p_{m_3+2}\, p_{m_3+3}$$

$$p_{m_2}\, p_{m_2-1}\, p_{m_2-2}\, p_{m_2-3}\, p_{m_2-4} < p_{m_1+1}\, p_{m_1+2}\, p_{m_1+3}$$

The bound (7) is particular useful because of its sharpeness. I implemented an algorithm to generate all Highly composite numbers up to a given upper limit. Mainly, the algorithm checks the exponent range given by (1) where p_m is bound from above by (6) and insure the validity of (7). To avoid the usage of very time-expensive arbitrary precision integer calculations, all the HCN were handled with their logarithm value. The useful representation

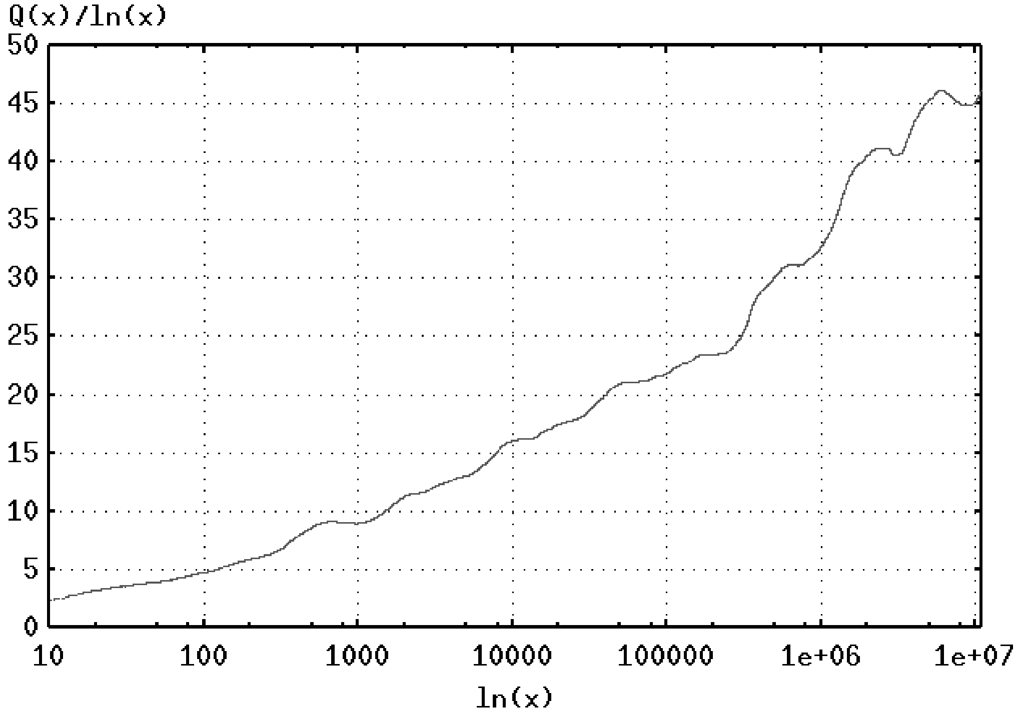
$$\operatorname{lb} n = \sum_{i=1}^{m} e_i \operatorname{lb} p_i = \sum_{k=1}^{e_1} \sum_{i=1}^{m_k} \operatorname{lb} p_i = e_1 - e_2 + \sum_{k=1}^{e_2} \sum_{i=1}^{m_k} \operatorname{lb} p_i$$

shows that we need for any HCN at most $e_2 + 1$ additions to calculate $\operatorname{lb} n$ given already the sums of the logarithm up to any prime. To ensure the necessary precision of these non-integer values, all the calculations involving the logarithms were done using interval arithmetic.

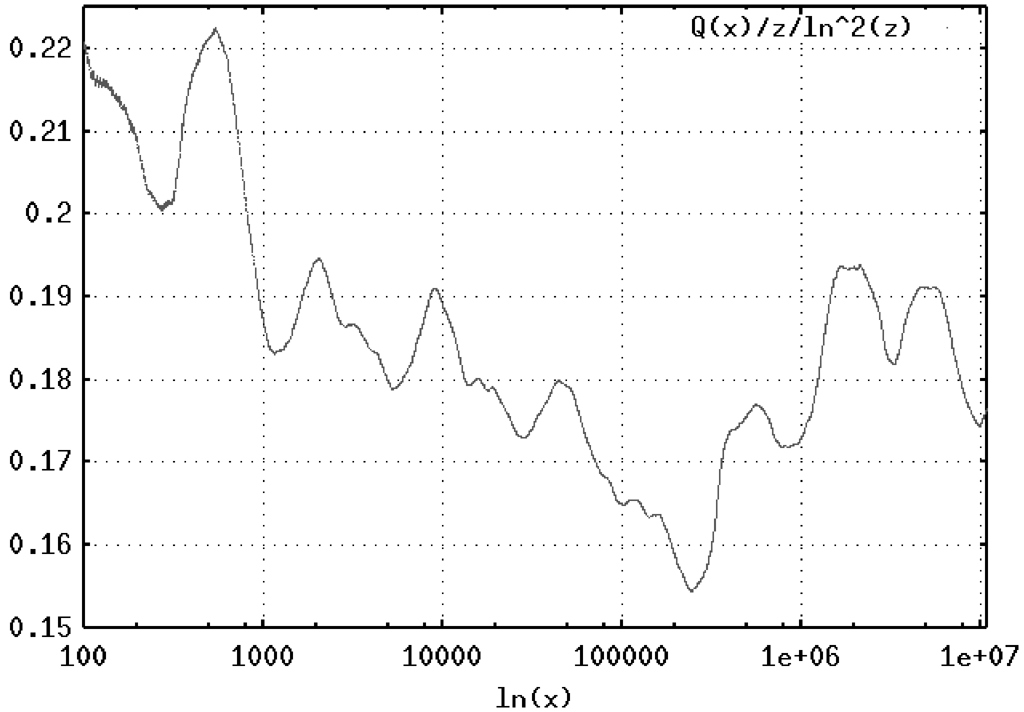
Define $Q(x) := |\{n \leq x : n \in \mathrm{HCN}\}|$. Then one knows that asymptotically holds

$$\exists\, c_1, c_2 \in \mathbb{R}^+ : \quad \ln^{1+c_1} x < Q(x) < \ln^{1+c_1+c_2} x$$

from the work of Erdös [1] and Nicholas [7].



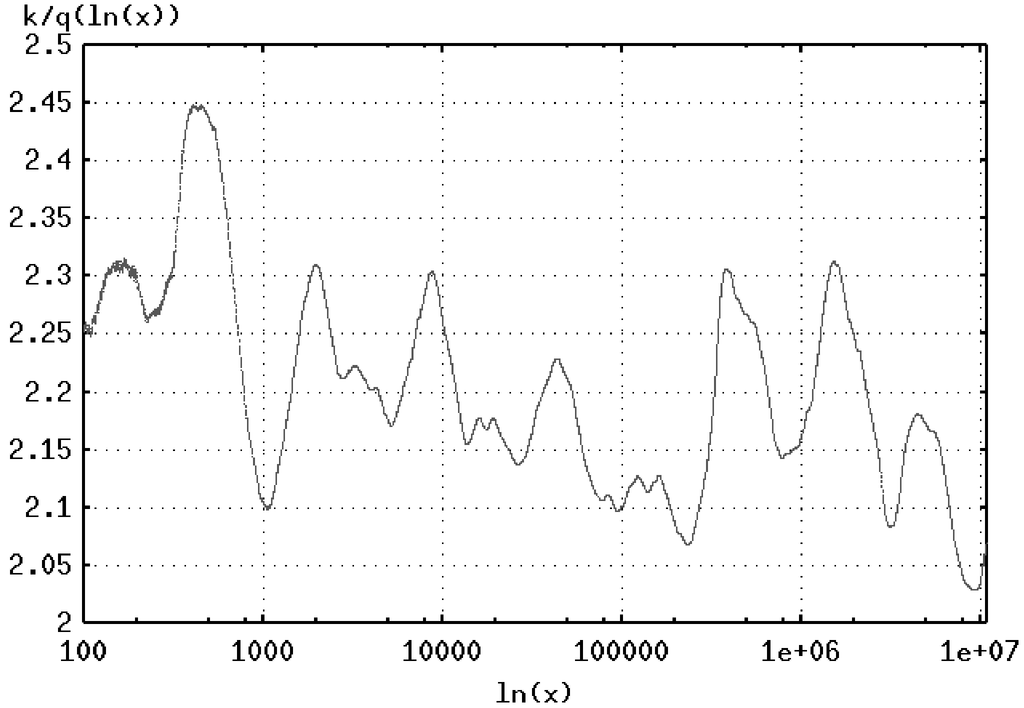
To get a feeling of the fluctuations of $Q(x)$ — which are due to the fluctuations in the sequence of prime numbers —, we make the transformation $z = \ln x$ and plot $\frac{Q(e^z)}{z \ln^2 z}$ for $z \in [100, 10^7]$:



Notice that the ordinate has a logarithmic scale! To "average" the function $Q(z)$, we define the average number of HCNs below a given e^z to be $q(z) = \frac{1}{z}\int_0^z Q(e^t)\,dt$. Because $Q(z)$ is a staircase function this may be writen as

$$q(\ln n_k) = \frac{1}{\ln n_k}\sum_{i=1}^{k} i \ln\frac{n_i}{n_{i-1}} = k - \frac{1}{\ln n_k}\sum_{i=0}^{k-1}\ln n_i \quad \text{with } n_i \in \text{HCN}$$

if we agree on $n_0 = 1$. If we expect asymptotically $Q(x) \approx f(x)\ln^c x$ then we will expect $q(\ln n_k) \approx \frac{k}{1+c}$, provided $f(x) \sim o(\ln x)$. Next follows the diagram showing $\frac{k}{q(\ln n_k)}$ versus $\ln x$:



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