Highway to the Danger Zone

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Introduction

Modal logic is part of second-order logic. It can be identified with fragments of monadic second-order logic using special axioms or formulas. The use of modal logic rather than classical predicate logic (1^{st} or higher order) has to be defended. Apart from the motivations of modal logics and their suitedness to intuitive thinking – something which puts modal logic indeed on a par with predicate logic – the main motive for applying modal logics is actually the belief of almost guaranteed decidability. But general results on decidability have failed to appear except for the field of extensions of **K4**. For logics containing **K4** Kit Fine and M. Zakharyaschev have broken ground towards general decidability results via establishing fmp. For applications, usually more than one operator is needed, so their results are of limited value there. Results for polymodal logics have hitherto not been obtained. This is on the one hand connected with the fact that even for modal logics with one operator we lack sufficiently general results (with the exception of **K4**) but is also connected to the fact that several operators offer easier alleys to incompleteness and undecidability. [13; 14] has identified the reason for this twofold connection. He first finds undecidable polymodal logics by using known undecidable structures and then simulating them in the language of mono-modal logic. But his examples are quite complex and so one might still hope that some simple and yet useful class of polymodal logics is decidable. The author admits to have had such hopes with regards to the family of logics determined by universal sentences.

We will see that such hopes are unjustified. Polymodal logics are not just alleys, they are highways into undecidability. There are plenty of quite simple undecidable logics and all candidates of a natural, non-trivial class of decidable logics have been successively destroyed by negative examples.

This paper is organized as follows. In the first section we discuss Sahlqvist's Theorem and propose a special quantifier complexity hierarchy for these logics which we call the *Sahlqvist Hierarchy*. Then follows a section in which we introduce a number of undecidable Sahlqvist logics of low complexity. In the third section we discuss the simulation of polymodal logics in monomodal logics and apply this technique to the previously established logics. It follows that there are monomodal undecidable Sahlqvist logics of complexity 2, corresponding roughly to first-order formulae of type $\forall\exists$. In the last section we discuss several ramifications of this theme.

This work continues the work of Thomason and uses techniques of [5] as well as [7] on pushing up properties. It also draws on an array of results the author has obtained together with Carsten Grefe, which are reported in [4]. Only after compsing this essay I have learnt that V. Shehtman has already used Thue-processes to obtain undecidability results; his results, however, have been published only in Russian. I wish to thank Herr Grefe as well as Frank Wolter, Rajeev Goré and Valentin Shehtman for many discussions.

1 The Sahlqvist Hierarchy

Recall from [9] the following theorem.

Theorem 1 (Sahlqvist) *Let T be a modal formula which is equivalent to a conjunction of formulae of the form $\Box^m(T_1 \to T_2)$ where $m \in \omega$, T_2 is positive and T_1 is obtained from propositional variables and constants in such a way that no positive occurrence of a variable is in a subformula of the form $U_1 \vee U_2$ or $\Diamond U_1$ which is itself in the scope of some $\Box$. Then T is effectively equivalent to a first-order formula and $\mathbf{K}(T)$ is d-persistent.*

Throughout this essay all logics considered will be Sahlqvist logics. These logics have the advantage to be complete with respect to Kripke semantics. The conditions the axioms impose on the Kripke frames are elementary. In [6], the elementary conditions corresponding to Sahlqvist formulae have been characterized. It has recently become popular to look at algorithms computing first-order equivalents. It should be emphasized that contrary to what is stated in the literature the method has always been constructive. The word *effective* expresses this. Any doubts about the effectiveness of this translation should have been removed by the algorithm in [5], reproduced in [6], in which pairs $\langle \alpha(\overline{w}), \overline{P} \rangle$ are considered where $\alpha(\overline{w})$ is a first-order formula with n free variables and $\overline{P}$ an n-sequence of modal formulae. Relative to a class $\mathfrak{X}$ this pair denotes the fact that $\alpha(\overline{w})$ is equivalent to the second-order formula $(\exists \overline{p}) \bigwedge_{i \leq n} P_i(w_i)$. For different classes different rule calculi are developed which derive pairs which are valid in $\mathfrak{X}$ under this interpretation. The calculus for the union of the class of Kripke-frames with the class of descriptive frames allows to prove Sahlqvist's Theorem and also derive in tandem the corresponding first-order properties. (For details consult [6].) Sahlqvist's Theorem can be generalized straightforwardly to logics with several operators, a fact which is perhaps not so well-known. We will state the theorem in it's general form below.

For ease of understanding the workings of the Sahlqvist theorem we begin with the classification of the corresponding first-order conditions. Given that we are in the language of m-modal logics we have m modal operators $\Box_i$ ($i \leq m$) and their duals $\Diamond_i$ and on the Kripke frames m different binary accessibility relations $\lhd_i$. The standard first-order language for talking about Kripke frames uses these symbols, the usual connectives and first-order quantifiers over worlds plus equality. Rather than using the standard quantifiers we use their restricted counterparts. They are defined as follows

$$(\forall y \rhd_i x)\phi := (\forall y)(x \lhd_i y. \to .\phi) \qquad (\exists y \rhd_i x)\phi := (\exists y)(x \lhd_i y \wedge \phi)$$

From atomic formulae $x = y$, $x \lhd_i y$ we can never actually produce variable free formulae with these quantifiers. There will necessarily be at least one free variable. We are interested in formulae where there is exactly one, which we treat as implicitly universally quantified (by an unrestricted quantifier, of course). The first-order language with restricted quantifiers will here be referred to as $\mathcal{R}$. We assume also that $\mathcal{R}$ has the symbols t and f, standing for the *true* and the *false* proposition.

Particular formulas in $\mathcal{R}$ are the constant formulae. A formula is called *constant* if it is composed from the constant atomic formulae t and f. In contrast to standard predicate logic constant formulae are non-trivial. The following are constant formulae which are non-reducible.

$$(\exists y \rhd_1 x)\,\mathsf{t}, \qquad (\forall y \rhd_1 x)(\exists z \rhd_2 y)\,\mathsf{t}$$

Constant formulae correspond to constant propositions in modal logic, in our examples $\Diamond_1\top$ and $\Box_1\Diamond_2\top$. In $\mathcal{R}$ there exists no equivalent of prenex normal forms. In an $\mathcal{R}$-formula ϕ a variable is *inherently universal* if it is bound by a universal quantifier which itself is not in the scope of an existential quantifier. A formula is *positive* if it is composed from atomic formulae and constant formulae using only $\wedge$, $\vee$ and the quantifiers. Notice that the occurring constant subformulae need not be positive. The following is proved in [6].

Theorem 2 *An $\mathcal{R}$-formula is Sahlqvist iff it is equivalent to an $\mathcal{R}$-formula which is positive and in which every non-constant atomic subformula contains at least one inherently universal variable.*

We introduce a notion of a *Sahlqvist Hierarchy* that is supposed to classify the complexity of a logic in terms of the quantifier alternations that occur. Sq_0 is reserved for the constant formulae. In a formula of type Sq_n, $n > 0$, at most $n-1$ alternations of quantifier type $(\forall, \exists)$ occur, with quantifiers in constant subformulae being ignored. This deviates from the standard definition in the fact that we ignore the complexity of constant subformulae. Of course, one can introduce a standard quantifier complexity measure. However, we will propose the following notation. The symbol $Sq_n(+k)$ means that the formula is Sq_{k+n} if all quantifier alternations are counted, but merely Sq_n if the quantifiers of constant subformulae are ignored. For example

$$(\forall y \rhd_1 x)[(\exists z \rhd_2 y)\, \mathrm{t} \vee y \lhd_1 x]$$

is $Sq_1(+1)$. For the rest of this paper, we will not use this finer distinction except for formulas of level $Sq_n(+0)$ which we also call *strictly* Sq_n.

The following is a different formulation of Sahlqvist's Theorem for languages with several operators. Call a formula **strongly positive** if is composed from constant formulae and variables with only $\wedge$ and *constant restricted boxes* $\Box_i(\mathsf{C} \rightarrow$, $i \leq m$. (Technically, this allows a more general notion of strong positivity which conforms with the intended property that they commute with intersections in the valuations (see [11]). While variables may be connected with each other only by use of conjunction, they may be weakened by constant formulae before a box is prefixed. So, $\Box_1(\Diamond_2\top. \vee .p \wedge \Box_2 q)$ is strongly positive in this new definition.)

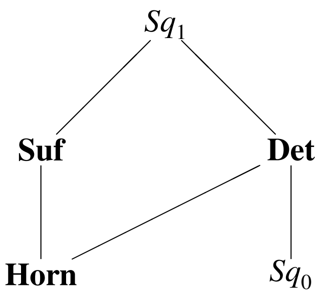
Theorem 3 *An $\mathcal{R}$-formula which is Sahlqvist corresponds to modal axioms of the form $A \rightarrow B$ where A is composed from constant formulae and strongly positive formulae using $\wedge, \vee, \Diamond_j$ ($j \leq m$), while B is composed from constant formulae and variables using $\wedge, \vee, \Diamond_j, \Box_j$ ($j \leq m$).*

It is possible to read off the position of a modal Sahlqvist formula in the hierarchy. Roughly, a $\Diamond_i$ works as a restricted existential if in the consequent, and as a restricted universal if in the antecedent. Likewise, a $\Box_i$ in the consequent is a restricted universal, but an existential in the antecedent. But these are only rules of thumb; some modal operators will not show up as quantifiers. The following is an example.

$\mathbf{Alt_1}$	$\Diamond p \wedge \Diamond q. \rightarrow .\Diamond(p \wedge q)$	$(\forall w \rhd v)(\forall x \rhd v)(x = w)$

The operator in the consequent does not introduce an existential quantifier.

All logics considered in the sequel will be of complexity Sq_2, most will be of complexity Sq_1, so universal first-order modulo ignoring the constant subformulae. Within Sq_1 we can discern some narrower classes which are of interest in their own right. First, the *subframe logics* of [2] and [15]. Elementary subframe logics are strictly Sq_1; this means that they are Sq_1 even with constant subformulae being counted. Another class are the *deterministic logics*; they are expressible in $\mathcal{R}$ without the help of disjunction – except for the constant subformulae, which may contain disjunctions. The intersection of the two classes, the deterministic subframe logics, are exactly those logics which can be axiomatized by Horn-formulae in the standard first-order logic. In $\mathcal{R}$ they are of the form $\mathfrak{A}.\phi$, where $\mathfrak{A}$ is a prefix of universal quantifiers and ϕ a single, positive atomic formula – that is, a formula of th ekind $x = y$ or $x \lhd_j y$, $j \leq m$. We call the class of logics axiomatizable by Horn-formulae **Horn-logics**. We have the following inclusion diagram.



Denote by $\mathfrak{Sq}_n$ the logics axiomatizable by a set of Sahlqvist axioms of level Sq_n and by $\mathfrak{Sq}_n^f$ the finitely axiomatizable logics in $\mathfrak{Sq}_n$.

Theorem 4 *The logics $\mathfrak{Sq}_n$ form a sublattice of the lattice $\mathcal{N}_m$ of m-modal logics, closed in $\mathfrak{Sq}_n$ under finite intersections and infinite joins. The logics $\mathfrak{Sq}_n^f$ form a subsemilattice.*

Proof. To keep notation simple we show this for the monomodal case. Closure under infinite joins is trivial. The only thing than remains to be proved is the closure under meet. To this end take $\forall x.\alpha(x), \forall x.\beta(x) \in Sq_n$. We want to show that $\forall x.\alpha(x) \vee \forall x.\beta(x) \in Sq_n$. Define formulae γ_k, $k \in \omega$ by

$$\gamma_k = (\forall x)([\bigwedge_{j \leq k}(\forall y \rhd^j x)\alpha(y)] \vee [\bigwedge_{j \leq k}(\forall y \rhd^j x)\beta(y)])$$

where $x \lhd^j y$ iff there exists a path of length j from x to y. Observe now that

$$\forall x.\alpha(x) \vee \forall x.\beta(x) \equiv \bigwedge_k \gamma_k$$

This equivalence is correct.For if a frame satisfies $(\forall x)\alpha(x) \vee (\forall x)\beta(x)$ then it also satisfies one of the two, say $(\forall x)\alpha(x)$. Thus it satisfies $(\forall x)(\forall y \rhd^j x)\alpha(y)$ for all j and consequently γ_k for all k. For the converse assume $\mathfrak{F}$ satisfies all γ_k. Take a world $w \in \mathfrak{F}$. Then for an infinite number of $k \in \omega$ we have either

$$\mathfrak{F} \models \bigwedge_{j \leq k} (\forall y \rhd^j x)\alpha(y)[w]$$

or

$$\mathfrak{F} \models \bigwedge_{j \leq k} (\forall y \rhd^j x)\beta(y)[w]$$

Let the first be the case. Denote by $\mathfrak{G}$ be the subframe generated by w in $\mathfrak{F}$. Then $\mathfrak{G} \models (\forall x)\alpha(x)$. Consequently,

$$\mathfrak{G} \models (\forall x)\alpha(x) \vee (\forall x)\beta(x)$$

This holds for all generated subframes, and so it holds for $\mathfrak{F}$ as well, since $\alpha(x), \beta(x)$ are restricted. Secondly, $\gamma_k \in Sq_n$. Two cases need to be distinguished. First case is $n = 0$. Then γ_k is constant and so in Sq_0 as well. Second case $n > 0$. Then since the formula begins with a universal quantifier and $(\forall y \rhd^j x)$ is a chain of universal quantifiers, the complexity does not rise. $\dashv$

2 Some Simple Undecidable Logics

In the domain of polymodal logics there are already some known undecidable logics. One example is the logic with three operators each of which satisfies **S5** and which commute pairwise. This logic arises from modelling 3-variable fragments of predicate logic with modal operators. The axioms are Sq_2. Here we use an extremely simple tool, that of a *Thue-process*. The algebraic equivalent of this process is known as the *word problem* in semigroups. Consider a finite presentation of a semigroup via a set $\mathfrak{G} = \{g_1, \dots, g_m\}$ of generators and a set $\mathfrak{T} = \{t_i \approx u_i | i \in n\}$ of relations among these generators. The generators plus equations form a finite presentation of the semigroup $\mathfrak{F}_{SG}(\mathfrak{G})/\mathfrak{T}$. We write $\mathfrak{T} \vdash_{bsg} r \approx s$ to denote the fact that $r \approx s$ is derivable in $\mathfrak{T}$ in the Birkhoff-calculus for semigroups. Recall that Birkhoff's calculus $\vdash_b$ for equational logic consists of the axiom (ref) $\vdash_b r \approx r$, the rules (sym) $r \approx s \vdash_b s \approx r$, (trs) $r \approx s, s \approx t \vdash_b r \approx t$, the substitution rule (sub) $s(x) \approx t(x) \vdash_b s(u) \approx t(u)$ and the rule of replacement (rep) $u \approx v \vdash_b t(u) \approx t(v)$. The calculus $\vdash_{bsg}$ for semigroups has the additional axiom (ass) $\vdash_b x(yz) \approx (xy)z$. Equivalently, $\mathfrak{T} \vdash_{bsg} r \approx s$ denotes the fact that for the canonical homomorphism $h : \mathfrak{F}_{SG}(\mathfrak{G}) \to \mathfrak{F}_{SG}(\mathfrak{G})/\mathfrak{T}$ we have $h(r) = h(s)$. The following are folklore results.

- There are $\mathfrak{T}$ over two symbols such that '$\mathfrak{T} \vdash_{bsg} r \approx s$' is undecidable.
- It is undecidable whether or not '$\mathfrak{T} \vdash_{bsg} r \approx s$' is decidable for given $\mathfrak{T}$ over two symbols.

Based on the free semigroup we can form the *canonical Thue-frame*. Its underlying set of worlds is exactly the elements of $\mathfrak{F}_{SG}(\mathfrak{G})/\mathfrak{T}$ – which can also be seen as equivalence classes of terms under $\approx$ – and the relations are $t \lhd_i u$ iff $u \approx t \cdot g_i$. This construction is actually quite known in automatic theorem proving for modal logic. We abbreviate this frame by $F(\mathfrak{T})$. We can determine the logic of this frame. To do this let us introduce some notation. A term t in the language of semigroups based on $g_1, \ldots, g_m$ can be seen as a complex modal operator based on m simple modalities, $\Diamond_1, \ldots, \Diamond_m$. Define

$$g_i^\Diamond P = \Diamond_i P, \qquad (t \cdot g_i)^\Diamond P = t^\Diamond \Diamond_i P$$

For the empty word ϵ we agree to let $\epsilon^\Diamond P = P$. Analogously, $t^\Box$ is defined. We have $t^\Box P. \leftrightarrow .\neg t^\Diamond \neg P$. Now the postulates for the logic simulating a Thue-process will be the following.

$\mathbf{Alt}_1$	$\Diamond_i p \wedge \Diamond_i q. \to .\Diamond_i(p \wedge q)$	$(\forall y \rhd_i x)(\forall z \rhd_i x)(z = y)$
$\mathbf{D}$	$\Diamond_i \top$	$(\exists y \rhd x)t$
$\mathbf{r} \approx \mathbf{s}$	$r^\Diamond p. \leftrightarrow .s^\Diamond p$	$x \cdot r = x \cdot s$

The axioms $\mathbf{Alt}_1$ and $\mathbf{D}$ ensure that each point has one and only one successor for each of the modalities $\Diamond_i$. Hence each frame for such a logic can be seen as a semigroup by taking the worlds as elements and define $x \cdot g_i$ to be the unique g_i successor of x. By induction on the term r the composition $x \cdot r$ is defined. The axiom $\mathbf{r} \approx \mathbf{s}$ says that under this identification we can reach a point via r iff we can reach it via s. This axiom is Sq_1. (But in presence of $\mathbf{Alt_1.D}$ it is effectively a subframe axiom as we will see below.)

Theorem 5 *The logic of $F(\mathfrak{T})$ is exactly*

$$\Lambda_{\mathfrak{T}} = \bigotimes_{i \le m} \mathbf{Alt_1.D}(\{r^\Diamond p. \leftrightarrow .s^\Diamond p | r \approx s \in \mathfrak{T}\})$$

Recall here from [8] the notation $\Lambda \otimes \Theta$ and $\bigotimes_{i \leq m} \Lambda_i$. The first denotes a bimodal logic with first operator satisfying Λ and the second Θ; and the second an n-modal logic where the i^{th} factor satisfies Λ_i. The proof of Theorem 5 is not hard and will be omitted.

Let us pause for a moment to reflect on the relation between Kripke frames and semigroups. If we have an m-modal logic Λ in which every operator satisfies $\mathbf{Alt_1.D}$ then any (one-generated) Λ-frame for Λ can be viewed as an m-generated semigroup. Moreover, by a slight generalization of a result by [1] we know that Λ must be complete. Furthermore, by a result of F. Wolter reported in [5] it is natural (= r-persistent) and Δ-elementary. So it is complete with respect to an elementary class of models based on m-generated semigroups. The logics $\Lambda_{\mathfrak{T}}$ are complete with respect to models based on $\mathfrak{F}_{SG}(\mathfrak{T})$. We will now show that in fact any extension of $\bigotimes_{i \leq m} \mathbf{Alt_1.D}$ can be characterized by a set of positive, universal postulates on the semigroups (or, equivalently, the Kripke frames). The logics $\Lambda_{\mathfrak{T}}$ are in addition characterized by disjunction-free conditions, which makes them slightly more special. To see this notice that the postulates $\mathbf{Alt_1}$ and $\mathbf{D}$ together are equivalent to the axiom $\Box p \leftrightarrow \Diamond p$. Moreover, $\mathbf{Alt_1}$ allows to deduce the following equivalences (and follows from them).

$$\Box(p \vee q). \leftrightarrow .\Box p \vee \Box q \qquad \Diamond(p \wedge q). \leftrightarrow \Diamond p \wedge \Diamond q$$

This allows for rather special normal forms. Given a formula P, we write it with $\Box_i$, $\Diamond_i$, $\wedge$, $\vee$ and $\neg$ in such a way that $\neg$ occurs only before variables. This is always possible. Now, $\wedge$ can be moved out of the scope of any $\Box_i$, $\Diamond_i$ and $\vee$, so that P can be written as a conjunction of conjunction-free formulae. Let Q be such a formula. Now $\vee$ can be moved out of the scope of any $\Diamond_i$ and $\Box_i$, and occurrences of $\Box_i$ can be changed into $\Diamond_i$. Thus Q is reduced to the following form

$$\bigvee_i r_i^{\Diamond} p_i \vee \bigvee_j s_j^{\Diamond} \neg q_j$$

Observe now that $s_j^{\Diamond} \neg q_j. \leftrightarrow .s_j^{\Box} \neg q_j. \leftrightarrow .\neg s_j^{\Diamond} q_j$ so that we can rewrite this further into

$$\bigwedge_j s_j^{\Diamond} q_j. \rightarrow . \bigvee_i r_i^{\Diamond} p_i$$

With $r_i^{\Diamond} p_i. \leftrightarrow .r_i^{\Box} p_i$, finally, we get this form

$$\bigwedge_j s_j^{\Diamond} q_j. \rightarrow . \bigvee_i r_i^{\Box} p_i$$

This formula is equivalent to a Sq_1 condition on frames. Consider, namely, what happens if there are i and j such that $p_i = q_j$. In that case we have a conjunct $s_j^{\diamond} q_j$ in the antecedent and a disjunct $r_i^{\Box} q_j$ in the consequent. Since we can let q_j be true at a single world, the antecedent then sets a s_j-path to this world and the consequent says that all r_i-paths (i. e. the one and only r_i-path) must end there. Thus, the two paths must be identical. Hence the formula above expresses the following universal condition, phrased here in terms of semigroups:

$$\bigvee \langle r_i \approx s_j | p_i = q_j \rangle$$

Any set of axioms expresses a set of such conditions above $\bigotimes_i \mathbf{Alt_1.D}$. In particular, Thue-processes express conditions of this type which are disjunction free.

The effect of this reduction is as follows. The problem '$\Lambda_{\mathfrak{T}} \vdash P$' can be reduced by the above manipulations to the problem

$$\Lambda_{\mathfrak{T}} \vdash \bigwedge_j s_j^{\diamond} q_j . \to . \bigvee_i r_i^{\Box} p_i$$

By completeness of both logics with respect to semigroup models, this can be checked via semigroups. Finally, the whole problem is reduced to the question

$$\mathfrak{F}_{SG}(\mathfrak{T}) \models \bigvee \langle r_i \approx s_j | p_i = q_j \rangle$$

In turn, since the statement is variable-free, this holds if any one of

$$\mathfrak{F}_{SG}(\mathfrak{T}) \models r_i = s_j$$

holds for suitable i, j. The latter is equivalent to

$$\mathfrak{T} \vdash_{bsg} r_i \approx s_j$$

where $\vdash_{bsg}$ is the Birkhoff-calculus for semigroups. This is exactly the decidability problem for the Thue-process.

Theorem 6 $\Lambda_{\mathfrak{T}}$ *is decidable iff* $\mathfrak{T}$ *is decidable.* $\dashv$

Immediately this offers the following results.

- There are finitely axiomatizable bimodal logics of complexity Sq_1 which are undecidable.

- It is not decidable whether finitely axiomatizable polymodal logics of complexity at least Sq_1 are decidable.

There are, however, many more consequences concerning logics simulating Thue-processes. Notice first of all that $\Lambda_{\mathfrak{T}}$ only marginally exceeds the class of Sq_1-logics. We can namely provide a different axiomatization as follows. Instead of $\mathbf{r} \approx \mathbf{s}$, an axiom which guarantees the existence and identity of r successors with s successors, we are going to add the axiom

$$\mathbf{r}^{\uparrow} \approx \mathbf{s}^{\uparrow} \qquad r^{\diamond} p \wedge s^{\diamond} q. \rightarrow .r^{\square} q \wedge s^{\square} p$$

which only asserts that the r successors and the s successors if they exist must be identical. In presence of the **D** axiom for both relations, this axiom is equivalent in strength to $\mathbf{r} \approx \mathbf{s}$. The logic $\Lambda_{\mathfrak{T}}^{\circ}$ is defined to be $\Lambda_{\mathfrak{T}}$ *without* the **D** axioms. This logic is strictly Sq_1 and thus a subframe logic. Any subframe logic extending $\bigotimes_{i \leq m} \mathbf{Alt_1}$ is decidable (see [15] for a proof). Thus $\Lambda_{\mathfrak{T}}^{\circ}$ is decidable.

Call a logic **globally decidable** if the problem '$\square^{\omega}\phi \vdash_{\Lambda} \psi$' is decidable. Here $\square^{\omega}\phi = \{\square^k \phi | k \in \omega\}$ and $\square^0 \phi = \phi$ and $\square^{k+1}\phi = \bigwedge_i \square_i \square^k \phi$, with $\square_i$ representing the individual boxes of the logic. (In [3] this property is called *sequential decidability* but we prefer to avoid the use of the word *sequential*.) Notice now that we have the following reduction.

Lemma 7 *Let Λ be a polymodal logic and $\Lambda(\chi)$ a strengthening of this logic by an Sq_0-axiom. Then*

$$\Psi \vdash_{\Lambda(\chi)} \phi \quad \Leftrightarrow \quad \Psi; \square^{\omega}(\bigwedge_{i \leq m} \chi) \vdash_{\Lambda} \phi$$

Proof. A pushing-up argument. Suppose the right hand side is false. Then there exists a polyframe f and a model

$$\langle f, \beta, s \rangle \models \Psi; \square^{\omega}(\bigwedge_{i \leq m} \chi); \neg\phi$$

We can assume f to be generated by s. Then it is clear that f is a frame for χ accepting Ψ but rejecting ϕ. Thus the left hand side is false. Assume now that the left hand side is false. We then have a model $\langle f, \beta, s \rangle \models \Psi; \neg\phi$. Since f is a frame for χ, we have $\langle f, \beta, s \rangle \models \square^{\omega}(\bigwedge_{i \leq m} \chi)$. Thus the right hand side is false. ⊣

Theorem 8 *Let Λ be a polymodal logic and $\chi \in Sq_0$. If Λ is globally decidable, $\Lambda(\chi)$ is globally decidable and a fortiori decidable.*

Proof. According to the previous lemma we have the following reduction

$$\Box^{\omega}\psi \vdash_{\Lambda(\chi)} \phi \quad \Leftrightarrow \quad \Box^{\omega}\psi; \Box^{\omega}(\bigwedge_{i \leq m} \chi) \vdash_{\Lambda} \phi$$

Thus, if Λ is globally decidable, so is $\Lambda(\chi)$. ⊣

Corollary 9 *Let Λ be a polymodal logic. If Λ is globally decidable, Λ.**D** is globally decidable and a fortiori decidable, where Λ.**D** denotes the extension of Λ by $\Diamond_j \top$ for $j \leq m$.* ⊣

Let us put this together. The logic $\Lambda_{\mathfrak{T}}$ arises from $\Lambda^{\circ}_{\mathfrak{T}}$ by adding all **D**-axioms. $\Lambda^{\circ}_{\mathfrak{T}}$ is decidable, $\Lambda_{\mathfrak{T}}$ is undecidable. Hence $\Lambda^{\circ}_{\mathfrak{T}}$ must be globally undecidable.

Corollary 10 *There exist bimodal subframe logics which are decidable but globally undecidable.* ⊣

In [12] a monomodal subframe logic with similar properties is contructed. Her example not only improves ours by having only one operator; the axioms are actually extremely simple, namely $\mathbf{Alt}_2$ plus

$$\bigwedge_{i \leq 4} \Diamond\Diamond q_1. \rightarrow . \bigvee_{i<j} \Diamond\Diamond(p_i \wedge p_j)$$

These postulates say that a point can have at most two immediate successors ($\mathbf{Alt_2}$), and at most three 2-step successors. These conditions allow to code so-called *recurrent tiling problems*, which present an alternative way of proving global undecidability. The only disadvantage of this logic for the present purposes is that it is not in **Det**, the class of disjunction free logics.

The previous theorem can be strengthened even further. Suppose Λ is an m-modal logic. Then define the $m+1$-modal logic $\Lambda^{\boxplus}$ by

$$\Lambda^{\boxplus} = \Lambda \otimes \mathbf{K4}(\{\boxplus p \rightarrow \Box_i p | i \leq m\})$$

Lemma 11 *Let Λ as well as $\Lambda^{\boxplus}$ be complete. If $\Lambda^{\boxplus}$ is decidable then Λ is globally decidable.*

Proof. We show that

$$\Box^{\omega}\psi \vdash_{\Lambda} \phi \quad \text{iff} \quad \psi; \boxplus\psi \vdash_{\Lambda^{\boxplus}} \phi$$

From there it indeed follows that Λ is globally decidable if $\Lambda^{\boxplus}$ is decidable and so the theorem is proved. Now, if the left hand side is false, there is a model $\langle f, \beta, s\rangle$ such that $\langle f, \beta, s\rangle \models \Box^k\psi$ for all k but $\langle f, \beta, s\rangle \nvDash \phi$. Now let $f^{\boxplus}$ arise from f by adding a new relation which interprets $\boxplus$ and is the transitive closure of the union of the $\lhd_i$. Without doubt is $f^{\boxplus}$ a $\Lambda^{\boxplus}$-frame, and $\langle f^{\boxplus}, \beta, s\rangle \models \psi; \boxplus\psi$ but $\langle f^{\boxplus}, \beta, s\rangle \nvDash \phi$. Now assume conversely that $\langle g, \beta, s\rangle \models \psi; \boxplus\psi; \neg\phi$. Then for a $\Lambda^{\boxplus}$-frame g we have $\langle g, \beta, s\rangle \models \Box^k\psi$ for all k. If $g_{\boxplus}$ results from g by removing the relation corresponding to $\boxplus$ we certainly have $\langle g, \beta, s\rangle \models \Box^k\psi$ without $\langle g_{\boxplus}, \beta, s\rangle \models \phi$; and $g_{\boxplus}$ is a Λ-frame, as required. ⊣

Theorem 12 *There are 3-modal Horn-logics which are undecidable.*

Proof. Take $\Lambda^{\circ\boxplus}_{\mathfrak{T}}$ for an undecidable Thue-process $\mathfrak{T}$. This is a Horn-logic. Then $\Lambda^{\circ}_{\mathfrak{T}}$ is globally undecidable and so by the previous lemma $\Lambda^{\circ\boxplus}_{\mathfrak{T}}$ must be undecidable. The postulates are readily checked to be Horn-definable. ⊣

These logics are characterized by universal, deterministic (i. e. $\vee$-free) and positive $\mathcal{R}$-formulae. Now it is not decidable whether $\mathfrak{T} \dashv\vdash \mathfrak{U}$ for two Thue-processes $\mathfrak{T}, \mathfrak{U}$ simply because otherwise we would be able to decide '$\mathfrak{T} \vdash r \approx s$'. Thus it is not decidable whether $\Lambda_{\mathfrak{T}} = \Lambda_{\mathfrak{U}}$, and, similarly, it is undecidable whether $\Lambda^{\circ\boxplus}_{\mathfrak{T}} = \Lambda^{\circ\boxplus}_{\mathfrak{U}}$. This has the following consequence.

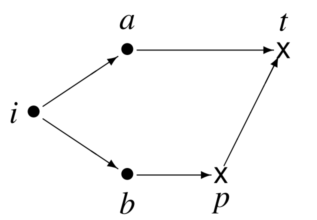
Theorem 13 *It is undecidable for two Horn-theories T, U based on at least three relations whether $T \dashv\vdash U$.* ⊣

For elementary logics this shows quite simply that it is undecidable whether two universal theories with relational symbols have the same models.

3 Simulating Polymodal Logics

The results obtained so far have established results for polymodal logics with two or three operators at least. Generally, it would be preferrable if we could also prove some (un-)decidability results for mono-modal logics. The way to obtain (mainly negative) results is by simulating frames with several modalities by frames using a single modality. This technique was established by [13]. Our simulations are different but similar in spirit.

Suppose now that we have a frame $\langle f, \lhd, \blacktriangleleft \rangle$ for the bimodal language with operators $\Box$, $\blacksquare$. We then construct a monomodal frame $\langle f^{sim}, \leqslant \rangle$ for the monomodal language based on the operator $\boxminus$ and it's dual $\diamondminus$. As in Thomason's original example, the original set of points must be blown up and a single point must be replaced by several copies. What is important in the construction is that the copies of the point must be distinguishable from each other by certain constant formulae. If that is so, the simulation is rather straightforward. Thomason's construction is more economical than ours; he needs two copies per point, the number of points in the simulating frame is exactly $2 \times \sharp f + 1$, whereas in our construction it is $5 \times \sharp f$, but it is easy to see that it can be reduced to $3 \times \sharp f + 2$. However, our construction has the advantage of being symmetrical in the operators and so we only need to consider one case out of two in each proof. Moreover, if we ignore the complexity of constant formulae – which we have chosen to do – then no difference in expenses will arise. Each point of the frame will be replaced by the following frame $\nabla = \langle \nabla, <^{\nabla} \rangle$, where $\nabla = \{a, b, i, p, t\}$. (We use a x to denote an irreflexive point and a $\bullet$ to denote a reflexive point.)



Now let us be given a (generalized) bimodal frame $\mathfrak{F} = \langle f, \lhd, \blacktriangleleft, \mathbb{F} \rangle$. Then we let $f^{sim} = f \times \nabla$. We write x^{α} rather than $\langle x, \alpha \rangle$. Also, for subsets $A \subseteq \nabla$ and $Y \subseteq F$ we let $Y^A = Y \times A$. With this convention, $f^{sim} = f^{\nabla}$. On f^{sim} we define a relation $\leqslant$ as the union of three relations, $<^{\nabla}$, $<^{\circ}$ and $<^{\bullet}$. The first derives from the blowing up of points by the frame ∇, the second codes $\lhd$ and the third one codes $\blacktriangleleft$.

$$\begin{array}{lcl} \leqslant & = & <^\nabla \cup <^\circ \cup <^\bullet \\ <^\nabla & = & \{\langle x^\alpha, x^\beta\rangle | \alpha <^\nabla \beta\} \\ <^\circ & = & \{\langle x^a, y^i\rangle | x \lhd y\} \\ <^\bullet & = & \{\langle x^b, y^i\rangle | x \blacktriangleleft y\} \end{array}$$

Thirdly, we let $\mathbb{F}^{sim}$ be all unions of sets of the form $Y^\alpha, \alpha \in \nabla$.

Proposition 14 $\mathfrak{F}^{sim} = \langle f^{sim}, \leqslant, \mathbb{F}^{sim}\rangle$ *is a (generalized) monomodal frame.*

Proof. A set of $\mathbb{F}^{sim}$ can be written as a union

$$A^a \cup B^b \cup I^i \cup P^p \cup T^t$$

with sets $A, B, I, P, T \subseteq f$. Closure under union and complement is straightforward. For closure under $\Diamond$ observe

$$\begin{array}{lcl} \Diamond A^a & = & A^a \cup A^i \\ \Diamond B^b & = & B^b \cup B^i \\ \Diamond I^i & = & I^i \cup (\Diamond I)^a \cup (\blacklozenge I)^b \\ \Diamond P^p & = & P^b \\ \Diamond T^t & = & T^p \cup T^a \end{array}$$

Thus $\mathbb{F}^{sim}$ is closed under $\Diamond$. ⊣

Alternatively, $\mathbb{F}^{sim}$ can be defined to be the smallest set containing all X^∇ for $X \subseteq f$. Namely, observe that we can write any set $Y \in \mathbb{F}^{sim}$ as the union

$$Y = (Y \cap f^a) \cup (Y \cap f^b) \cup (Y \cap f^i) \cup (Y \cap f^p) \cup (Y \cap f^t)$$

The sets $f^\alpha, \alpha \in \nabla$, are definable in any of the so constructed frames by a formula without variables, hence they are always internal. Namely, consider the following formulae.

$$\begin{array}{lcl} \mathsf{T} & = & \boxminus\bot \\ \mathsf{P} & = & \boxminus^2\bot \wedge \neg \boxminus \bot \\ \mathsf{A} & = & \Diamond \boxminus \bot \wedge \neg \boxminus^2 \bot \\ \mathsf{B} & = & \Diamond(\boxminus^2\bot \wedge \neg \boxminus \bot) \\ \mathsf{I} & = & \neg \boxminus \bot \wedge \neg \Diamond \boxminus \bot \wedge \neg \Diamond(\boxminus^2\bot \wedge \neg \boxminus \bot) \end{array}$$

Notice that $\mathsf{B} = \Diamond\mathsf{P}$, $\mathsf{I} = \neg\mathsf{T} \wedge \neg\Diamond\mathsf{T} \wedge \neg\Diamond\mathsf{P}$. In order not to complicate the notation we do not distinguish between a formula and the set it represents in a model.

Lemma 15 *The formulas* $\mathsf{A}, \mathsf{B}, \mathsf{I}, \mathsf{P}, \mathsf{T}$ *define the sets* f^a, f^b, f^i, f^p, f^t*, respectively.*

Proof. We begin with T. $x^\alpha \in \boxminus\bot$ implies $\alpha = t$ because otherwise $x^\alpha <^\nabla x^\beta$ and thus $x^\alpha \leqslant x^\beta$ for some $\beta \in \nabla$. On the other hand, x^t sees no points via $<^\circ$ and $<^\bullet$. This shows the correctness of T. Now for P. Clearly, if $x^\alpha \in \boxminus^2\bot$, then by similar arguments $\alpha = p, t$. Conversely, if $\alpha = p, t$ then $x^\alpha \in \boxminus^2\bot$. Now $\mathsf{P} = \boxminus^2\bot \wedge \neg\mathsf{T}$ and from the previous considerations on T it follows that P defines f^p. Now for A. Let $x^\alpha \in \diamondminus \boxminus \bot$. Then $x^\alpha \leqslant y^t$ for some $y \in f$, by the correctness of T. By definition of $\leqslant$ this can only hold if $x^\alpha <^\nabla y^t$ and so $\alpha = a$ or $\alpha = p$. Since $\mathsf{A} = \diamondminus \boxminus \bot \wedge \neg\mathsf{P}$ this proves the correctness of A. Now $\mathsf{B} = \diamondminus\mathsf{P}$. Let $x^\alpha \in \diamondminus\mathsf{P}$. Then, by the correctness of P, $x^\alpha \leqslant y^p$ for some $y \in f$. By definition of $\leqslant$, $x^\alpha <^\nabla y^p$ and so $\alpha = b$. Finally, $x^\alpha \in \mathsf{I}$ iff $x^\alpha \notin \mathsf{T}$, $x^\alpha \notin \mathsf{B}$ and $x^\alpha \notin \diamondminus\mathsf{T}$. The first and the second are equivalent to $\alpha \neq t, b$. Since $\diamondminus\mathsf{T}$ defines $f^a \cup f^p$, the third condition is equivalent to $\alpha \neq a, p$. Hence $\mathsf{I} = f^i$. ⊣

Now that we can simulate frames and – in effect – also polymodal algebras, we translate polymodal formulas into mono-modal formulas.

$$\begin{array}{lcl}
p^{sim} & = & p \\
(P \wedge Q)^{sim} & = & P^{sim} \wedge Q^{sim} \\
(P \vee Q)^{sim} & = & P^{sim} \vee Q^{sim} \\
(\neg P)^{sim} & = & \neg P^{sim} \\
(\Diamond P)^{sim} & = & \diamondminus(\mathsf{A} \wedge \diamondminus(\mathsf{I} \wedge P^{sim})) \\
(\blacklozenge P)^{sim} & = & \diamondminus(\mathsf{B} \wedge \diamondminus(\mathsf{I} \wedge P^{sim})) \\
(\Box P)^{sim} & = & \boxminus(\mathsf{A} \to \boxminus(\mathsf{I} \to P^{sim})) \\
(\blacksquare P)^{sim} & = & \boxminus(\mathsf{B} \to \boxminus(\mathsf{I} \to P^{sim}))
\end{array}$$

Lemma 16 (Simulation) *For all biframes* $\mathfrak{F}$ *and all bimodal formulas* P

$$\mathfrak{F} \models P \quad \Leftrightarrow \quad \mathfrak{F}^{sim} \models I \to P^{sim}$$

Proof. Simple induction on P. ⊣

We are now introducing a map $(-)^s : \mathcal{E}(\mathbf{K}{\otimes}\mathbf{K}) \to \mathcal{E}(\mathbf{K})$ defined as $\mathbf{K}{\otimes}\mathbf{K}(X)^s = \mathbf{K}(X^s)$, where $X^s = \{\mathsf{I} \to P^{sim} | P \in X\}$. As this stands, the definition of $(-)^s$ depends on a concrete axiomatization for the bimodal logic. We will however show that the choice of axioms is immaterial. Also, we define an *unsimulation* of a mono-modal logic to be $\Lambda^u = \{P | \mathsf{I} \to P^{sim} \in \Lambda\}$. While we cannot simply take the simulation of a polymodal logic to be the set of simulations of its theorems (this set happens not to be closed under the rules), the unsimulation indeed yields a logic, no matter what Λ is.

Theorem 17 *Let Λ be a monomodal logic. Then Λ^u is a bimodal logic.*

Proof. It has to be shown that Λ^u is closed under substitution, modus ponens and the two rules of necessitation. **Substitution.** Let $P \in \Lambda^u$, that is, $\mathsf{I} \to P^{sim} \in \Lambda$. Take now a substitution σ and define $\sigma^{sim}(p) = \sigma(p)^{sim}$. Then $(\mathsf{I} \to P^{sim})^{\sigma^{sim}} = \mathsf{I} \to (P^{\sigma})^{sim}$ as can be verified by induction. Since Λ is closed under substitution, $(\mathsf{I} \to P^{\sigma})^{sim} \in \Lambda$ and so $P \in \Lambda^u$. **Modus Ponens.** Let $P, P \to Q \in \Lambda^u$, that is, $\mathsf{I} \to P^{sim}, \mathsf{I} \to (P \to Q)^{sim} \in \Lambda$. Then, as $(P \to Q)^{sim} = P^{sim} \to Q^{sim}$ we also have $\mathsf{I} \to .P^{sim} \to Q^{sim} \in \Lambda$ and thus $\mathsf{I} \to Q^{sim} \in \Lambda$, by which $Q \in \Lambda^u$. **Necessitation.** Assume $P \in \Lambda^u$. Then $\mathsf{I} \to P^{sim} \in \Lambda$. Hence $\boxminus(\mathsf{I} \to P^{sim}) \in \Lambda$, and then also $\mathsf{A} \to \boxminus(\mathsf{I} \to P^{sim}) \in \Lambda$, and, finally, $(\Box P)^{sim} \in \Lambda$, by which $\mathsf{I} \to (\Box P)^{sim} \in \Lambda$, and so $\Box P \in \Lambda^u$. Analogously for $\blacksquare$. ⊣

This theorem tells us that we can simulate proofs of the bimodal calculus in the monomodal calculus and thus that the the map $(-)^s$ is independent from the axiomatization of the logic. For if $\mathbf{K}(X) = \mathbf{K}(Y)$ for different sets X, Y, then a proof of $Q \in \mathbf{K}(X)$ for $Q \in Y$ can be simulated so that $Q^s \in \mathbf{K}(X^s)$ and likewise a proof of $Q \in \mathbf{K}(Y)$ for $Q \in X$ can be simulated. It follows that $\mathbf{K}(X^s) = \mathbf{K}(Y^s)$. Now that this is established we come to another important property of the simulation, namely its faithfulness. Take two different bimodal logics Λ, Θ. Their classes of general biframes must be different, and so are then the classes of the simulating frames. The simulating frames may not be all frames for these logics, but they discriminate them nevertheless.

The next question to be addressed is the fate of Sahlqvist formulae under simulation. The following is easily checked.

Theorem 18 *If P is a Sahlqvist formula then $P^s = \mathsf{I} \to P^{sim}$ is equivalent to a Sahlqvist formula.*

Proof. Let $P = A \to B$ be Sahlqvist. $P^s = \mathsf{I} \to (A \to B)^{sim}$ is equivalent to $\mathsf{I} \wedge A^{sim}. \to .B^{sim}$. Clearly, if C is constant, so is C^{sim}. Now if S is strongly positive, so is S^{sim}. Namely, the translation of box is a constant restricted box.

$$\begin{aligned}(\Box P)^{sim} &\equiv \boxminus(\mathsf{A} \to \boxminus(\mathsf{I} \to P^{sim}))\\ (\blacksquare P)^{sim} &\equiv \boxminus(\mathsf{B} \to \boxminus(\mathsf{I} \to P^{sim}))\end{aligned}$$

Thus since A, B as well as I are constant, S^{sim} is composed from constant formulae and variables with $\wedge$ and constant restricted $\Box$. Since A is composed from strongly positive and constant formulae with the help of $\wedge$, $\vee$ and $\Diamond$ then so is A^{sim}. Finally, B^{sim} is positive, thus proving the theorem. $\dashv$

This being the case we can actually derive the elementary condition associated with simulation of a special Sahlqvist formula. To this end define the following constant $\mathcal{R}$-formulae.

$$\begin{array}{lcl}
\mathsf{T}(x) & = & (\forall y \geqslant x)\,\mathsf{f} \\
\mathsf{P}(x) & = & (\forall y \geqslant x)(\forall z \geqslant y)\,\mathsf{f} \wedge (\exists y \geqslant x)\,\mathsf{t} \\
\mathsf{A}(x) & = & (\exists y \geqslant x)(\forall z \geqslant y)\,\mathsf{f} \wedge (\exists y \geqslant x)(\exists z \geqslant y)\,\mathsf{t} \\
\mathsf{B}(x) & = & (\exists y \geqslant x)[(\forall z \geqslant y)(\forall u \geqslant z)\,\mathsf{f} \wedge (\exists z \geqslant y)\,\mathsf{t}] \\
\mathsf{I}(x) & = & (\exists y \geqslant x)\,\mathsf{t} \wedge (\forall y \geqslant x)(\exists z \geqslant y)\,\mathsf{t} \wedge \\
 & & (\forall y \geqslant x)[(\forall z \geqslant y)(\forall u \geqslant z)\,\mathsf{f} \wedge (\exists z \geqslant y)\,\mathsf{t}]
\end{array}$$

Now define the simulation of a first-order property inductively as follows.

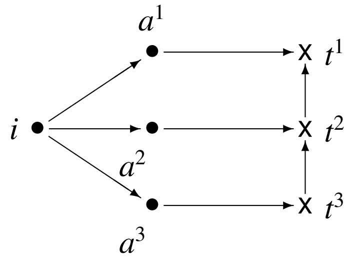
$$\begin{array}{lcl}
(x = y)^{sim} & = & x = y \\
(\phi \wedge \psi)^{sim} & = & \phi^{sim} \wedge \psi^{sim} \\
(\phi \vee \psi)^{sim} & = & \phi^{sim} \vee \psi^{sim} \\
(\neg\phi)^{sim} & = & \neg\phi^{sim} \\
((\exists y \vartriangleright x)\phi)^{sim} & = & (\exists v \geqslant x)(\mathsf{A}(v) \wedge (\exists y \geqslant v)(\mathsf{I}(y) \wedge \phi^{sim})) \\
((\exists y \blacktriangleright x)\phi)^{sim} & = & (\exists v \geqslant x)(\mathsf{B}(v) \wedge (\exists y \geqslant v)(\mathsf{I}(y) \wedge \phi^{sim})) \\
((\forall y \vartriangleright x)\phi)^{sim} & = & (\forall v \geqslant x)(\mathsf{A}(v) \to (\forall y \geqslant v)(\mathsf{I}(y) \to \phi^{sim})) \\
((\forall y \blacktriangleright x)\phi)^{sim} & = & (\forall v \geqslant x)(\mathsf{B}(v) \to (\forall y \geqslant v)(\mathsf{I}(y) \to \phi^{sim}))
\end{array}$$

Notice that we excluded a clause simulating $x \vartriangleleft y$, $x \blacktriangleleft y$. The reason is to show a slight twist in this simulation. Namely, $x \leqslant y$ is equivalent to $(\exists w \geqslant x)(w = y)$. This existential quantifier does not show up in the Sahlqvist Hierarchy because $x \vartriangleleft y$ as well as $x \blacktriangleleft y$ are atomic formulae. But the simulation of these formulae do introduce an existential quantifier. So, even when the original formula was Sq_1, the simulated formula might turn out to be Sq_2. This is the case exactly if the original formula uses atomic formulae of the type $x \vartriangleleft y$, $x \blacktriangleleft y$. Our undecidable logics are of this kind. We summarize this in the following statement.

Theorem 19 *If P is in Sq_n and n is even, then $\mathsf{I} \to P^{sim}$ is in Sq_n as well. If, however, n is odd, then $\mathsf{I} \to P^{sim}$ is in Sq_{n+1}.* $\dashv$

The process of simulation can be iterated to simulate any number of relations. The level in the Sahlqvist Hierarchy does not rise more than one, however. If n

is even, the one step simulation remains at that level; if n is odd, the level goes one up and stays there. This iterated simulation is nevertheless from an intuitive point of view quite complex. However, by redefining ∇ it is possible to achieve a simultaneous simulation of all operators. For example, with three operators, we take the following ∇.



With ∇ defined as above for an arbitrary number m of points $a^1, \dots, a^m$ and $t^1, \dots, t^m$ we can define the sets T^j and A^j as follows.

$$\begin{array}{lcl} \mathsf{T}^j & = & \boxminus^j \bot \wedge \neg \boxminus^{j-1} \bot \\ \mathsf{A}^j & = & \diamondminus \mathsf{T}^j \wedge \diamondminus^{m} \top \\ \mathsf{I} & = & \bigwedge_j \neg \diamondminus \mathsf{T}^j \end{array}$$

As before, the simulation will turn a Sq_n logic into Sq_n if n is even and into Sq_{n+1} else.

Theorem 20 *There exist finitely axiomatizable monomodal Sahlqvist logics of complexity Sq_2 which are undecidable.*

Proof. Start with the fact that for two 3-modal subframe logics $\mathbf{K}_3(X)$, $\mathbf{K}_3(Y)$ (i. e. logics with X, Y complexity $Sq_1(+0)$) it is undecidable whether $\mathbf{K}_3(X) = \mathbf{K}_3(Y)$. Then simulate X and Y. Since simulation is injective, it is undecidable whether $\mathbf{K}(X^s) = \mathbf{K}(Y^s)$. X^s, Y^s are of complexity Sq_2. $\dashv$

Consider what happens if we simulate logics extending $\mathbf{Alt_1} \otimes \mathbf{Alt_1}$. Then the simulations of the frames are frames for $\mathbf{Alt_3}$. Define a simulation of $\Lambda \supseteq \mathbf{Alt_1} \otimes \mathbf{Alt_1}$ by putting $\Lambda^\delta = \Lambda^{sim}(\mathbf{Alt_3})$. Then $\Lambda = \Theta$ iff $\Lambda^\delta = \Theta^\delta$. Hence we obtain the following result.

Theorem 21 *There are finitely axiomatizable extensions of* **Alt₃** *which are undecidable.* ⊣

4 More undecidability

These results can be sharpened in many ways to obtain counterexamples to more specific conjectures. For example. Call a formula $A \to B$ **non-descending** if the modal degree of A is at least that of B. If A and B contain no $\Box_i$ it might be reasonable to believe that a calculus based on non-descending formulae of this type is actually decidable because by reasoning forward from a formula Q we cannot increase the nestings of $\Diamond_i$ and not decrease the nestings of $\Box_i$, so that when we proved Q inconsistent by deriving a contradiction $P, \neg P$ it seems prima facie plausible that we can give good a priori estimates for P (and the other intermediate formulae). But such reasoning is unjustified. The logics $\Lambda^{\circ\boxplus}_{\mathfrak{T}}$ are axiomatized by non-descending formulae free of any $\Box_i$. One might consider whether a requirement that in a formula $A \to B$ A must have strictly greater modal depth than B would ensure decidability. Again, the answer is negative. The reasoning is rather interesting. It is based on the following theorem.

Theorem 22 *Suppose that Λ is a globally decidable m-modal logic. Then for any ϕ and any $k \in \omega$ the extension $\Lambda(\Box^k \bot \to \phi)$ is globally decidable as well.*

Let us see first its consequences. Suppose that for global decidability of a logic it is sufficient to require (among other) that for any axiom $A \to B$ the modal depth of A exceeds that of B (or exceeds $f(B)$ for some function f). Take any logic Λ axiomatized by formulae $A \to B$ where the other postulates are met but not the requirements on depth. Observe then that

$$A \to B. \leftrightarrow .(A \wedge \Box^k \bot \to B) \wedge (A \wedge \Diamond^k \top \to B)$$

so that if $A \wedge \Diamond^k \top \to B$ meets the other criteria, it also meets the depth requirements if k is large. In this way we split all axioms of Λ and add only one half and have global decidability. By the above theorem, however, adding the other halves will not destroy global decidability. So we can push decidability up. However, in this case this amounts to pushing undecidability down, destroying any of the decidability criteria based on complexity conditions at once.

Now we prove the theorem. The proof is based on the observation that for any $n \in \omega$ there is a finite set $S(n)$ of substitutions such that for any finite set of generators $\{a_1, \dots, a_n\}$ for the set algebra $\mathcal{F}$ of a refined frame $\mathfrak{F} = \langle f, \mathbb{F} \rangle$ for the valuation $\beta : p_i \mapsto a_i$ and any point $x \in f$

$$(\dagger) \qquad \langle \mathfrak{F}, \beta, x \rangle \models \{\Box^k \bot \to \phi^\sigma | \sigma \in S(n)\} \quad \Leftrightarrow \quad \mathfrak{F} \models \Box^k \bot \to \phi$$

For then it holds that for all χ, ψ based on the sentence letters $p_1, \dots, p_n$

$$\Box^\omega \chi \vdash_{\Lambda(\Box^k \bot \to \phi)} \psi \quad \Leftrightarrow \quad \Box^\omega \chi; \Box^\omega (\bigwedge_{\sigma \in S(n)} \Box^k \bot \to \phi^\sigma) \vdash_\Lambda \psi$$

For a proof just check all models on one-generated refined frames $\mathfrak{F}$ where the underlying set algebra is generated by the values of $\beta(p_1), \dots, \beta(p_n)$. By a theorem of [10], it is enough to show the theorem in the class of refined frames.

Now on to the proof of ($\dagger$). From right to left holds for any set $S(n)$. So the really interesting part is from left to right. We begin by constructing the $S(n)$. Consider the subframe $\mathfrak{C}$ based on the set C of all points x such that $\langle \mathfrak{F}, x \rangle \models \Box^k \bot$. By induction on k it can be shown that C is finite, bounded in size by a function depending only on n (and k). $\mathfrak{C}$ is a generated subframe hence refined. $\mathfrak{C}$ is therefore a full frame since it is finite. Consider now the valuation β on $\mathfrak{C}$. It is possible to show that any set $T \subseteq C$ can be presented as the extension of $\tau_T(a_1, \dots, a_n)$ under β for a suitable τ_T which is of modal degree $\leq 2k$ (see [7]). Collect in $S(n)$ all substitutions $\sigma : p_i \mapsto \tau_i(p_1, \dots, p_n)$ for formulas of depth $\leq 2k$. $S(n)$ is finite. We show ($\Rightarrow$) of ($\dagger$) with these sets. To that end, assume that $\mathfrak{F} \nvDash \Box^k \bot \to \phi$. Then there exists γ and x such that

$$\langle \mathfrak{F}, \gamma, x \rangle \models \Box^k \bot \wedge \neg \phi$$

Then $x \in C$ and so we have by the fact that $\mathfrak{C}$ is a generated subframe

$$\langle \mathfrak{C}, \gamma, x \rangle \models \neg \phi$$

Put $\sigma : p_i \mapsto \tau_{\gamma(p_i)}$. Then $\gamma(\phi) = \beta(\phi^\sigma)$ whence

$$\langle \mathfrak{C}, \beta, x \rangle \models \neg \phi^\sigma$$

And so

$$\langle \mathfrak{F}, \beta, x \rangle \models \Box^k \bot \wedge \neg \phi^\sigma$$

This demonstrates ($\dagger$) and so the theorem is proved. $\dashv$

There are refinements which are still unsolved. Call a formula a **path containment** formula if it is of the form $r^\diamond p \to s^\diamond p$. This states that the set of points reachable by r-paths is included in the set of points reachable by s-paths. The logics

$$\Gamma_{\mathfrak{T}} = \bigotimes_{i \leq m} \mathbf{K}.\mathbf{D}(\{r^\diamond p \leftrightarrow s^\diamond p | r \approx s \in \mathfrak{T}\})$$

are of this form. (Notice that $\mathbf{D} \equiv \top \to \Diamond\top$ is a path containment formula.)

Question. Do we have $\Gamma_{\mathfrak{T}} = \Gamma_{\mathfrak{U}}$ iff $\Lambda_{\mathfrak{T}} = \Lambda_{\mathfrak{U}}$?

If the answer is positive, the undecidability results hold as well for path-containment logics.

Likewise, call a formula a **path equation** if it is of the form $r^\diamond p \to s^\square p$. This axiom states that the set of points reachable by r-paths is identical to the set of points reachable by s-paths. We believe that

$$\bigotimes_{i \leq m} \mathbf{K}(\{r^\diamond p \to s^\square p | r \approx s \in \mathfrak{T}\})$$

for undecidable $\mathfrak{T}$ cannot be globally decidable.

Question. Are all polymodal logics axiomatized by path equations (globally) decidable?

The results established so far let us deduce other theorems of independent interest. They concern questions of pushing up decidability in the spirit of [7]. For monomodal logics this has proved to be a rather powerful method. For polymodal logics it would be most welcome to have analogous theorems, so that one can prove fmp or decidability for a polymodal logic by starting with the independent fusion of it's monomodal fragments and then adding one by one the postulates which mix the operators. For independent fusions these problems are largely solved in [8]. So what about pushing up properties for some non-trivial polymodal axioms? Secondly, adding master- or universal modalities is an important tool in applications for modal logics. The question which properties are preserved under the process of adding such a modality is quite an important one. We have already used a master modality to lower the bound for decidability. By that we have shown that they have a destructive force concerning decidability. These results are indepdendent of the special choice of the *master*. We could have taken a universal modality instead or add a postulate that the master is the (reflexive) transitive closure of the other relations etc.

- It is undecidable for bimodal Sq_1 logics whether adding **D** for one operator preserves decidability.

- There are subframe logics with three operators which are decidable while the addition of a postulate of the form $\Box p \to \blacksquare p$ destroys decidability.

- There are bimodal subframe logics which are decidable, while their extension by a universal modality is not.

The last statement improves on [16] who showed that fmp can be lost under addition of the universal modality. The logic of [12] which we discussed above is another example of a logic for which decidability is lost when the universal modality is added. The proofs are easy. Start with $\Lambda^{\circ}_{\mathfrak{T}}$ for an undecidable $\mathfrak{T}$. This is a subframe logic and decidable. Add one by one the postulates **D** for the individual operators. If both are added, decidability is lost. So at one point, adding **D** means losing decidability. If it is at the first step, we could strengthen the theorem to read *subframe logic* rather than Sq_0. For the second theorem use a similar argument. Here, however, the added axiom is a subframe axiom, so the property of being subframe logic is retained. For the last assertion notice that the decidability of a logic extended by a universal modality is equivalent to the global decidability of the original logic. (See [3].)

Notice that we have not established that global decidability for Thue-logics $\Lambda_{\mathfrak{T}}$ is decidable. This could be answered via the following

Question. Is it decidable for subframe logics whether adding the universal modality preserves decidability?

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