Modal Consequence Relations

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1 Introduction

Logic is generally defined as the science of reasoning. Mathematical logic is mainly concerned with forms of reasoning that lead from true premises to true conclusions. Thus we say that the argument from $\sigma_0; \sigma_1; \cdots; \sigma_{n-1}$ to δ is *logically correct* if whenever σ_i is true for all $i < n$, then so is δ. In place of 'argument' one also speaks of a 'rule' or an 'inference' and says that the rule is *valid*. This approach culminated in the notion of a *consequence relation*, which is a relation between sets of formulae and a single formula. A consequence relation $\vdash$ specifies which arguments are valid; the argument from a set Σ to a formula δ is valid in $\vdash$ iff $\langle \Sigma, \delta \rangle \in \vdash$, for which we write $\Sigma \vdash \delta$. δ is a *tautology* of $\vdash$ if $\varnothing \vdash \delta$, for which we also write $\vdash \delta$.

In the early years, research into modal logic was concerned with the question of finding the correct inference rules. This research line is still there but has been marginalized by the research into modal *logics*, where a logic is just a set of formulae; this set is the set of tautologies of a certain consequence relation, but many consequence relations share the same tautologies. The shift of focus in the research has to do in part with the precedent set by predicate logic: predicate logic is standardly axiomatized in a Hilbert-style fashion, which fixes the inference rules and leaves only the axioms as a parameter. Another source may have been the fact that there is a biunique correspondence between varieties of modal algebras and axiomatic extensions of K, which allowed for rather deep investigations into the space of logics, using the machinery of equational theories. This research lead to deep results on the structure of the lattice of modal logics and benefits also the research into consequence relations. Recently, however, algebraic logic has provided more and more tools that allow to extend the algebraic method to the study of consequence relations in general (see for example [59] and [14]). In particular the investigations into the Leibniz operator initiated by Blok and Pigozzi in [5] have brought new life into the discussion and allow to see a much broader picture than before.

Now, even if one is comfortable with classical logic, it is not immediately clear what the correct inferences are in modal logic. The first problem is that it is not generally agreed what the meaning of the modal operator(s) is or should be. In fact, rather than a drawback, the availability of very many different interpretations is the strength of modal logic; it gives flexibility, however at the price that there is not one modal logic, there are uncountably many. For example, $\Box$ as metaphysical necessity satisfies **S5**, $\Box$ as provability in **PA** satisfies **G**, $\Box$ as future necessity (arguably) satisfies $\mathsf{S4.3}$, and so on. This is in part because the interpretation decides which algebras are suitable (intended) and which ones are not. However, there is another parameter of variation, and this is the notion of truth itself. In the most popular interpretation, truth is truth at a world; but we could also understand it

[1] My thanks go in particular to Ian Hodkinson for his generous and detailed comments, which led to simplifications in the presentation; and to Yde Venema for many useful hints. All errors and omissions are my own responsibility.

as truth in every world of the structure. The two give rise to two distinct consequence relations, the *local* and the *global*, which very often do not coincide even though they always have the same set of tautologies.

2 Basic Theory of Modal Consequence Relations

This chapter makes heavy use of notions from universal algebra. The reader is referred to the Chapter **??** of this handbook for background information concerning universal algebra and in particular the theory of BAOs and how they relate to (general) frames. We shall quickly review some terminology. A **signature** is a pair $\langle F, \nu \rangle$, where F is a set of so–called **function symbols** or **connectives** and $\nu : F \to \omega$ a function assigning to each symbol an arity. **Terms** are expressions of this language based on variables. We shall also refer to ν alone as a signature. We shall assume that the reader is acquainted with basic notions of universal algebra, such as a ν**-algebra**. Given a map $v : X \to A$ from a set X of variables into the underlying set of A, there is at most one homomorphic extension $\overline{v} : \mathfrak{Tm}_\nu(X) \to \mathfrak{A}$, where $\mathfrak{Tm}_\nu(X)$ denotes the algebra of terms in the signature ν over the set X (whose underlying set is $\mathrm{Tm}_\nu(X)$). On a ν-algebra $\mathfrak{A}$, terms induce **term functions** in the obvious way. If we allow to expand the signature by a constant $\underline{a}$ for every $a \in A$, the term functions induced by this enriched language on $\mathfrak{A}$ are called **polynomials**. In what is to follow, F will always contain $\top$, $\wedge$ and $\neg$, and $\nu(\top) = 0$, $\nu(\neg) = 1$ and $\nu(\wedge) = 2$. Moreover, F will additionally contain connectives $\Box_i$, $i < \kappa$, called **modal operators**, which are unary unless otherwise stated. κ need not be finite. The relation corresponding to $\Box_i$ will be denoted by $\lhd_i$, unless stated otherwise. The set of variables is $V := \{p_i : i \in \omega\}$. Sets of formulae are denoted in the usual way using the colon notation: $\Delta; \chi$ abbreviates $\Delta \cup \{\chi\}$. We write $\mathrm{var}(\varphi)$ for the set of variables occurring in φ, and $\mathrm{sf}(\varphi)$ for the set of subformulae of φ. Similarly, $\mathrm{var}(\Delta)$ and $\mathrm{sf}(\Delta)$ are used for sets of formulae. A **substitution** is defined by a map $s : V \to \mathrm{Tm}_\nu(V)$. $s(\varphi)$ or φ^s denotes the effect on φ of performing the substitution s.

2.1 Consequence Relations

Definition 2.1 Let $\mathrm{Tm}_\nu(V)$ be a propositional language. A **consequence relation** over $\mathrm{Tm}_\nu(V)$ is a relation $\vdash \subseteq \wp(\mathrm{Tm}_\nu(V)) \times \mathrm{Tm}_\nu(V)$ between sets of formulae and a single formula such that

(i) $\varphi \vdash \varphi$

(ii) $\Delta \vdash \varphi$ and $\Delta \subseteq \Delta'$ implies $\Delta' \vdash \varphi$.

(iii) $\Delta \vdash \chi$ and $\Sigma; \chi \vdash \varphi$ implies $\Delta; \Sigma \vdash \varphi$.

$\vdash$ is **structural** if from $\Delta \vdash \varphi$ follows $\Delta^s \vdash \varphi^s$, where s is a substitution. $\vdash$ is **finitary** (or **compact**) if from $\Delta \vdash \varphi$ follows that there is a finite $\Delta' \subseteq \Delta$ such that $\Delta' \vdash \varphi$. A **tautology** of $\vdash$ is a formula φ such that $\vdash \varphi$. $\mathrm{Taut}(\vdash)$ is the set of tautologies of $\vdash$.

There is an alternative approach via deductively closed sets and via closure operators (see Surma [54] for a discussion of alternatives to consequence relations). Given $\vdash$, let $\Sigma^\vdash := \{\varphi : \Sigma \vdash \varphi\}$. The sets of the form Σ are called **theories** of $\vdash$. Then the following holds.

(i) $\Sigma \subseteq \Sigma^\vdash$.

(ii) $\Sigma^{\vdash\vdash} \subseteq \Sigma^\vdash$.

$\vdash$ is structural iff for all substitutions and all Σ

(1) $$\Sigma \subseteq s^{-1}((\Sigma^s)^\vdash)$$

$\vdash$ is finitary iff for all Σ

(2) $$\Sigma^\vdash = \bigcup\{\Sigma_0^\vdash : \Sigma_0 \subseteq \Sigma, \Sigma_0 \text{ finite}\}$$

A characterization of a finitary structural consequence relation in terms of its theories is as follows.

(i) The language is a $\vdash$-theory.

(ii) Every intersection of $\vdash$-theories is a $\vdash$-theory.

(iii) If T is a $\vdash$-theory, so is $s^{-1}(T)$.

(iv) If T_i, $i \in \omega$, is an ascending chain of $\vdash$-theories, $\bigcup T_i$ is a $\vdash$-theory.

For the general theory of consequence relation see [59]. For consequence relations and modal logic see [48]. In the sequel, unless otherwise stated, consequence relations are assumed to be finitary and structural. The signatures are signatures extending classical propositional logic by some (typically unary) modal operators.

One can think of a finitary consequence relation as a first order theory of formulae in the following way. A statement of the form $\Delta \vdash \varphi$ is rendered

$$(\forall \boldsymbol{x})(\bigwedge\langle T(\delta) : \delta \in \Delta\rangle) \to T(\varphi) \tag{3}$$

where T is a newly introduced predicate; the universal quantifier binds off the free variables occurring in all the formulae. Given this interpretation, the appropriate structures to interpret consequence relation in are matrices in the sense of the following definition.

Definition 2.2 A **ν-matrix** for a signature ν is a pair $\mathfrak{M} = \langle \mathfrak{A}, D\rangle$ where $\mathfrak{A}$ is a ν-algebra and $D \subseteq A$ a subset. A is called the set of **truth values** and D the set of **designated truth values**. An **assignment** or a **valuation** into $\mathfrak{M}$ is a map v from the set of variables into A. v makes φ **true** in $\mathfrak{M}$ if $\overline{v}(\varphi) \in D$; otherwise it makes φ **false**.

Given a matrix $\mathfrak{M}$ we can define a relation $\vdash_{\mathfrak{M}}$ by

$$\Delta \vdash_{\mathfrak{M}} \varphi \quad \Leftrightarrow \quad \text{for all assignments } v: \text{ If } \overline{v}[\Delta] \subseteq D \text{ then } \overline{v}(\varphi) \in D \tag{4}$$

If $\vdash \subseteq \vdash_{\mathfrak{M}}$ then we also say that $\mathfrak{M}$ is a **matrix for** $\vdash$. Given $\mathfrak{A}$, we say that D is a **filter** for $\vdash$ if D is closed under the rules; equivalently D is a filter, if $\vdash_{\langle \mathfrak{A}, D\rangle} \supseteq \vdash$. Given a class $\mathcal{S}$ of matrices (for the same signature) we define

$$\vdash_{\mathcal{S}} := \bigcap\langle \vdash_{\mathfrak{M}} : \mathfrak{M} \in \mathcal{S}\rangle \tag{5}$$

Theorem 2.3 *Let ν be a signature. For each class $\mathcal{S}$ of ν-matrices, $\vdash_{\mathcal{S}}$ is a (possibly nonfinitary) consequence relation.*

Theorem 2.4 (Wójcicki) *For every structural consequence relation $\vdash$ there exists a class $\mathcal{S}$ of matrices such that $\vdash = \vdash_{\mathcal{S}}$.*

Proof. Given the language, let S consist of all $\langle \mathfrak{Tm}_\nu(V), T\rangle$ where T is a theory of $\vdash$. First we show that for each such matrix $\mathfrak{M}$, $\vdash \subseteq \vdash_{\mathfrak{M}}$. To that end, assume $\Sigma \vdash \varphi$ and that $\overline{v}[\Sigma] \subseteq T$. Now $\overline{v}$ is in fact a substitution, and T is deductively closed, and so $\overline{v}(\varphi) \in T$ as well, as required. Now assume $\Sigma \nvdash \varphi$. We have to find a single matrix $\mathfrak{M}$ of this form such that $\Sigma \nvdash_{\mathfrak{M}} \varphi$. For example, $\mathfrak{M} := \langle \mathfrak{Tm}_\nu(V), \Sigma^\vdash\rangle$. Then with $\overline{v}$ the identity map, $\overline{v}[\Sigma] = \Sigma \subseteq \Sigma^\vdash$. However, $\overline{v}(\varphi) = \varphi \notin \Sigma^\vdash$ by definition of $\Sigma^\vdash$ and the fact that $\Sigma \nvdash \varphi$. □

If $\mathfrak{M}$ is a matrix for $\vdash$, then the set of truth values must be closed under the rules. The previous theorem can be refined somewhat. Let $\mathfrak{M} = \langle \mathfrak{A}, D\rangle$ be a logical matrix, and Θ a congruence on $\mathfrak{A}$. We write $[x]\Theta := \{y : x \;\Theta\; y\}$. The sets $[x]\Theta$ are called **blocks** of the congruence. Θ is called a **matrix congruence** if D is a union of Θ-blocks, that is, if $x \in D$ then $[x]\Theta \subseteq D$. In that case we can reduce the whole matrix by Θ and define $\mathfrak{M}/\Theta := \langle \mathfrak{A}/\Theta, D/\Theta\rangle$. The following is easy to show.

Lemma 2.5 *Let $\mathfrak{M}$ be a matrix and Θ a matrix congruence of $\mathfrak{M}$. Then $\vdash_{\mathfrak{M}} = \vdash_{\mathfrak{M}/\Theta}$.*

Call a matrix **reduced** if the diagonal, that is the relation $\Delta = \{\langle x, x\rangle : x \in A\}$, is the only matrix congruence. We can sharpen Theorem 2.4 to the following

Theorem 2.6 *For each logic $\langle \mathcal{L}, \vdash\rangle$ there exists a class S of reduced matrices such that $\vdash = \vdash_S$.*

Let $\mathcal{S}$ be a class of ν-matrices. $\mathcal{S}$ is called a **unital semantics** for $\vdash$ if $\vdash = \vdash_{\mathcal{S}}$ and for all $\langle \mathfrak{A}, D\rangle \in \mathcal{S}$ we have $|D| \leq 1$. (See Janusz Czelakowski [12,13]. A unital semantics is often called **algebraic**. This, however, is different from the notion of 'algebraic' discussed by Wim Blok and Don Pigozzi in [5].) The following is a useful fact, which is not hard to verify.

Proposition 2.7 *Let $\vdash$ have a unital semantics. Then in $\vdash$ the rules $p; q; \varphi(p) \vdash \varphi(q)$ are valid for all formulae φ.*

Notice that when a logic over a language $\mathcal{L}$ is given and an algebra $\mathfrak{A}$ with appropriate signature, the set of designated truth values must always be a deductively closed set, otherwise the resulting matrix is not a matrix for the logic. A theory is **consistent** if it is not the entire language, and **maximal consistent** if it is maximal in the set of consistent theories. Every theory is contained in a maximally consistent theory. For classical logics the construction in the proof of Theorem 2.4 can be strengthened by taking as matrices in S those containing only maximally consistent theories. For if $\Sigma \nvdash \varphi$ then $\Sigma; \neg\varphi$ is consistent and so for some maximal consistent Δ containing Σ we have $\neg\varphi \in \Delta$. Taking v to be the identity, $\overline{v}[\Sigma] = \Sigma \subseteq \Delta$, but $\overline{v}(\varphi) \notin \Delta$, otherwise Δ is not consistent.

2.2 Rules

A **rule** is a pair $\rho = \langle \Delta, \varphi\rangle$, where Δ is a set of formulae, and δ a single formula. We also write Δ/φ. If Δ is finite, we call ρ **finitary**; and if Δ is empty, we call ρ an **axiom**. ρ is a **derived** rule of $\vdash$ if $\rho \in \vdash$. ρ is **admissible** if for every substitution s: if $\Delta^s \subseteq \mathrm{Taut}(\vdash)$ then $\varphi^s \in \mathrm{Taut}(\vdash)$.

If R is a set of finitary rules, $\vdash^R$ denotes the smallest finitary, structural consequence relation that contains R. Given a consequence relation $\vdash$ and a rule ρ, $\vdash^{+\rho}$ is the least consequence relation containing $\vdash$ and ρ. $\vdash$ is called **consistent** if it is not the maximal relation. $\vdash$ is consistent iff p is not a tautology. For a consistent $\vdash$ put

$$E(\vdash) := \{n : \text{there is an } n\text{-ary rule } \rho \notin \vdash \text{ such that } \nvdash^{+\rho} p\} \tag{6}$$

$\vdash$ is called **Post-complete** if $0 \notin E(\vdash)$. It is **structurally complete** if every admissible rule is derivable.

Proposition 2.8 (Tokarz) *(1) $\vdash$ is structurally complete iff $E(\vdash) \subseteq \{0\}$. (2) $\vdash$ is maximal consistent iff it is both structurally complete and Post-complete.*

There is a special matrix, $\mathfrak{Taut} = \langle \mathfrak{Tm}_\nu(V), \varnothing^\vdash\rangle$. Recall that $\varnothing^\vdash$ are simply the tautologies of a logic.

Theorem 2.9 (Wójcicki) *$\vdash$ is structurally complete iff $\vdash = \vdash_{\mathfrak{Taut}}$.*

$\vdash^R$ can be described as follows. If s is a substitution, say that $\langle \Delta^s, \varphi^s\rangle$ is an **instance** of $\langle \Delta, \varphi\rangle$. An R-**proof** of φ from Σ is a sequence $\langle \delta_i : i < n+1\rangle$ such that $\delta_n = \varphi$, and for every $i < n+1$: either $\delta_n \in \Sigma$ or there are $j_k < i$, $k < p$, such that $\langle \{\delta_{j_k} : k < p\}, \delta_i\rangle$ is an instance of a rule from R.

Proposition 2.10 *$\Sigma \vdash^R \varphi$ iff there exists an R-proof of φ from Σ.*

We remark here that $\vdash$ is finitary iff there is a set R of finitary rules such that $\vdash = \vdash^R$. Of course, R may be infinite. $\vdash$ is **decidable** if for all finite Σ and all φ we can decide whether or not $\Sigma \vdash \varphi$. The following is from [31].

Theorem 2.11 (Harrop) *Suppose that $\mathfrak{M} = \langle \mathfrak{A}, D\rangle$ is a finite logical matrix. Then $\vdash_{\mathfrak{M}}$ is decidable.*

For example, one can use truth-tables. This procedure is generally slower than tableaux-methods, but only mildly so (see [15]).

2.3 The Deduction Theorem

The rule of **modus ponens** ($\mathrm{MP}_{\twoheadrightarrow}$) for a connective $\twoheadrightarrow$ is the rule $\langle \{p, p \twoheadrightarrow q\}, q\rangle$. ($\mathrm{MP}_{\rightarrow}$) in classical logic. There are many more connectives $\circ$ for which ($\mathrm{MP}_{\circ}$) is a derived rule, for example $\wedge$. $\twoheadrightarrow$ is said to satisfy **deduction theorem** with respect to $\vdash$ if for all Σ, φ, ψ

(7) $\qquad \Sigma; \varphi \vdash \psi \quad \Leftrightarrow \quad \Sigma \vdash \varphi \twoheadrightarrow \psi$

A consequence relation $\vdash$ is said to **satisfies the deduction theorem** (DT) for $\twoheadrightarrow$ if $\twoheadrightarrow$ satisfies ($\mathrm{MP}_{\twoheadrightarrow}$) and 7 holds. (See [14] for a survey of deduction theorems.) Given (DT) it is possible to transform any rule different from ($\mathrm{MP}_{\twoheadrightarrow}$) into an axiom preserving the consequence relation. Hence it is possible to replace the original rule calculus by a Hilbert-style calculus, where ($\mathrm{MP}_{\twoheadrightarrow}$) is the only rule which is not an axiom. Given a set of rules R, we say it has a **deduction theorem** for $\twoheadrightarrow$ if $\vdash^R$ does.

Theorem 2.12 *A Hilbert-style calculus for $\twoheadrightarrow$ has a deduction theorem for $\twoheadrightarrow$ iff $\twoheadrightarrow$ satisfies ($MP_{\twoheadrightarrow}$) and the following are axioms of $\vdash$:*

(8) $\qquad p \twoheadrightarrow (q \twoheadrightarrow p)$

(9) $\qquad (p \twoheadrightarrow (q \twoheadrightarrow r)) \twoheadrightarrow ((p \twoheadrightarrow q) \twoheadrightarrow (p \twoheadrightarrow r))$

Proof. ($\Rightarrow$) Suppose both ($\mathrm{MP}_{\rightarrow}$) and (7) hold for $\twoheadrightarrow$. Then, since $\varphi \vdash \varphi$, also $\varphi; \psi \vdash \varphi$ and (by (7)) also $\varphi \vdash \psi \twoheadrightarrow \varphi$ and (again by (7)) $\vdash \varphi \twoheadrightarrow (\psi \twoheadrightarrow \varphi)$. For (9) note that the following sequence

(10) $\qquad \langle \varphi \twoheadrightarrow (\psi \twoheadrightarrow \chi), \varphi \twoheadrightarrow \psi, \varphi, \psi \twoheadrightarrow \chi, \psi, \chi\rangle$

proves $\psi \twoheadrightarrow (\psi \twoheadrightarrow \chi); \varphi \twoheadrightarrow \psi; \varphi \vdash \chi$. Apply (DT) three times and the formula proved. ($\Leftarrow$) By induction on the length of an R-proof $\boldsymbol{\alpha}$ of ψ from $\Sigma \cup \{\varphi\}$ we show that $\Sigma \vdash \varphi \twoheadrightarrow \psi$. Suppose the length of $\boldsymbol{\alpha}$ is 1. Then $\psi \in \Sigma \cup \{\varphi\}$. There are two cases: (1) $\psi \in \Sigma$. Then observe that $\langle \psi \twoheadrightarrow (\varphi \twoheadrightarrow \psi), \psi, \varphi \twoheadrightarrow \psi\rangle$ is a proof of $\varphi \twoheadrightarrow \psi$ from Σ. (2) $\psi = \varphi$. Then we have to show that $\Sigma \vdash \varphi \twoheadrightarrow \varphi$. Now observe that the following is an instance of (9):

(11) $\qquad (\varphi \twoheadrightarrow ((\psi \twoheadrightarrow \varphi) \twoheadrightarrow \varphi)) \twoheadrightarrow ((\varphi \twoheadrightarrow (\psi \twoheadrightarrow \varphi)) \twoheadrightarrow (\varphi \twoheadrightarrow \varphi))$

But $\varphi \twoheadrightarrow ((\psi \twoheadrightarrow \varphi) \twoheadrightarrow \varphi)$ and $\varphi \twoheadrightarrow (\psi \twoheadrightarrow \varphi)$ are both instances of (8) and by applying ($\mathrm{MP}_{\twoheadrightarrow}$) twice we get $\varphi \twoheadrightarrow \varphi$. Now let $\boldsymbol{\alpha}$ be of length > 1. Then we may assume that ψ is obtained by an application of ($\mathrm{MP}_{\twoheadrightarrow}$) from some formulae χ and $\chi \twoheadrightarrow \psi$. Thus the proof looks as follows:

(12) $\qquad \ldots, \chi, \ldots, \chi \twoheadrightarrow \psi, \ldots, \psi, \ldots$

Now by induction hypothesis $\Sigma \vdash \varphi \twoheadrightarrow \chi$ and $\Sigma \vdash \varphi \twoheadrightarrow (\chi \twoheadrightarrow \psi)$. Now,

(13) $\qquad (\varphi \twoheadrightarrow (\chi \twoheadrightarrow \psi)) \twoheadrightarrow ((\varphi \twoheadrightarrow \chi) \twoheadrightarrow (\varphi \twoheadrightarrow \psi))$

is a theorem and so we get that $\Sigma \vdash \varphi \twoheadrightarrow \psi$ with two applications of ($\mathrm{MP}_{\twoheadrightarrow}$). $\Box$

For any given set Σ there exists at most one (finitary and structural) consequence relation $\vdash$ with a deduction theorem for a given connective such that Σ is the set of tautologies of $\vdash$. For assume $\Delta \vdash \varphi$ for a set Δ. Then since $\vdash$ is finitary, there exists a finite set $\Delta_0 \subseteq \Delta$ such that $\Delta_0 \vdash \varphi$. Let $\Delta_0 := \{\delta_i : i < n\}$. Put

(14) $\qquad \mathrm{ded}(\Delta_0, \varphi) := \delta_0 \twoheadrightarrow (\delta_1 \twoheadrightarrow \ldots (\delta_{n-1} \twoheadrightarrow \varphi) \ldots)$

Then, by the deduction theorem for $\twoheadrightarrow$

$$(15) \qquad \Delta \vdash \varphi \qquad \Leftrightarrow \qquad \varnothing \vdash \mathrm{ded}(\Delta, \varphi)$$

Theorem 2.13 *Let $\vdash$ and $\vdash'$ be consequence relations with* $\mathrm{Taut}(\vdash) = \mathrm{Taut}(\vdash')$. *Suppose that there exists a binary term function $\twoheadrightarrow$ such that $\vdash$ and $\vdash'$ satisfy (DT) for $\twoheadrightarrow$. Then $\vdash = \vdash'$.*

2.4 Interpolation

$\vdash$ has **interpolation** if whenever $\varphi \vdash \psi$ there exists a formula χ with $\mathrm{var}(\chi) \subseteq \mathrm{var}(\varphi) \cap \mathrm{var}(\psi)$ such that both $\varphi \vdash \chi$ and $\chi \vdash \psi$. Interpolation is a rather strong property, and generally logics fail to have it. There is a rather simple theorem which allows to prove interpolation for logics based on a finite matrix. Say that $\vdash$ has a **conjunction** if there is a term $p \wedge q$ such that the following are derivable rules: $\langle\{p, q\}, p \wedge q\rangle$ and both $\langle\{p \wedge q\}, p\rangle$ and $\langle\{p \wedge q\}, q\rangle$. In addition, if $\vdash = \vdash_{\mathfrak{M}}$ for some logical matrix we say that $\vdash$ has **all constants** if for each $\mathfrak{s} \in T$ there exists a nullary term function $\underline{\mathfrak{s}}$ such that for all valuations v $\overline{v}(\underline{\mathfrak{s}}) = \mathfrak{s}$. (Note that since $\mathrm{var}(\underline{\mathfrak{s}}) = \varnothing$ the value of $\underline{\mathfrak{s}}$ does not depend at all on v.) This rather complicated definition allows that we do not need to have a constant for each truth-value; it is enough if they are definable from the others. For example in classical logic we may have only $\top = \underline{1}$ as a primitive and then $\underline{0} = \neg\top$. An algebra is **functionally complete** if every function $A^n \to A$ is a term function of $\mathfrak{A}$; $\mathfrak{A}$ is **polynomially complete** if every function $A^n \to A$ is a polynomial function. Every functionally complete algebra is polynomially complete; the converse need not hold, since polynomials may employ constants for the elements of $\mathfrak{A}$. However, if $\mathfrak{A}$ has all constants, then it is functionally complete iff it is polynomially complete.

Theorem 2.14 *Suppose that $\mathfrak{M}$ is a finite logical matrix. Suppose that $\vdash_{\mathfrak{M}}$ has a conjunction $\wedge$ and all constants; then $\vdash_{\mathfrak{M}}$ has interpolation.*

(See [37], Theorem 1.6.4, where a proof is given.) A property closely related to interpolation is *Halldén-completeness*, named after Sören Halldén, who discussed it first in [30]. (See also [53].) $\vdash$ is called **Halldén-complete** if for all φ and ψ with $\mathrm{var}(\varphi) \cap \mathrm{var}(\psi) = \varnothing$: if $\varphi \vdash \psi$ and φ is consistent then $\vdash \psi$. 2-valued logics are Halldén-complete. Namely, assume that φ is consistent. Let $v : \mathrm{var}(\psi) \to 2$ be a valuation. Since φ is consistent there exists a $u : \mathrm{var}(\varphi) \to 2$ such that $\overline{u}(\varphi) = 1$. Put $w := u \cup v$. Since u and v have disjoint domains, this is well-defined. Then $\overline{w}(\varphi) = 1$, and so $\overline{w}(\psi) = 1$. So, $\overline{v}(\psi) = 1$. This shows that $\vdash \psi$. The following generalization is now evident.

Theorem 2.15 (Łos & Suszko) *Let $\mathfrak{M}$ be a logical matrix. Then $\vdash_{\mathfrak{M}}$ is Halldén-complete.*

In classical logic, the property of Halldén-completeness can be reformulated into a somewhat more familiar form. Namely, the property says that for φ and ψ disjoint in variables, if $\varphi \vee \psi$ is a tautology then either φ or ψ is a tautology.

Finally notice

Theorem 2.16 *Suppose that $\mathfrak{M}$ is a logical matrix and $\vdash_{\mathfrak{M}}$ has all constants. Then $\vdash_{\mathfrak{M}}$ is structurally complete and Post-complete.*

2.5 Modal Logics and Modal Consequence Relations

A **modal consequence relation** is a structural consequence relation of modal formulae which contains at least the classical tautologies and in which the rule ($\mathrm{MP}_{\to}$) is derived. Unless otherwise stated, modal consequence relations are assumed to be finitary. If in addition for every basic modality $\Box$ the rules $(\mathrm{E}_{\Box}) := \langle\{p \leftrightarrow q\}, \Box p \leftrightarrow \Box q\rangle$ are admissible, $\vdash$ is called **classical**. If the rules $(\mathrm{M}_{\Box}) := \langle\{p \to q\}, \Box p \to \Box q\rangle$ are admissible for every basic modality $\Box$, $\vdash$ is called **monotone**. Finally, if all rules $(\mathrm{MN}_{\Box}) := \langle\{p_0\}, \Box p_0\rangle$ are admissible, $\vdash$ is called **normal**. For simplicity, we refer to the set of the rules (MN$_\Box$), $\Box$ a basic modality as (MN), and treat it (somewhat inappropriately) as a single rule.

Modal logic is typically the study of modal *logics* and not that of modal *consequence relations*. The relationship is one-to-many. If $\vdash$ is a (modal) consequence relation, then

$$\text{Taut}(\vdash) := \{\varphi : \varnothing \vdash \varphi\} \tag{16}$$

is a modal logic, where a modal logic is any substitution closed set of formulae which contains all classical tautologies and (MP$_\to$). There is a converse map. Given a logic L, put

$$\vdash_L := \vdash^{L;(\text{MP}_\to)} \tag{17}$$

where L is here identified with the set of rules $\langle\varnothing, \varphi\rangle$, $\varphi \in L$. Evidently, $\Delta \vdash_L \varphi$ iff $\Delta; L \vdash^{(\text{MP}_\to)} \varphi$. By Theorem 7 $\vdash_L$ has a DT for $\to$. We shall often tacitly identify L with $\vdash_L$.

Definition 2.17 L is **classical** (**monotone**, **normal**) if $\vdash_L$ is. The smallest normal logic with κ operators is denoted by $\mathbf{K}_\kappa$. L is **quasi-normal** if L contains $\mathbf{K}_\kappa$.

We also call a consequence relation **quasi-normal** if its set of tautologies is. Call a term $t(p_0)$ a **normal operator** for L if it satisfies (a) $t(\varphi \to \chi) \to (t(\varphi) \to t(\chi)) \in L$, and (b) if $t(\varphi) \in L$ then $\Box_i t(\varphi) \in L$. There is a class of formulae that generally are normal if all basic modalities are; these are the so-called compound modalities. A term $t(p_0)$ with just one variable is called a **compound modality** if it just contains the connectives $\Box_i$, $i < \kappa$ and $\wedge$ in addition to constants; and no variable except for p_0. One can assign a relation corresponding to $t(p_0)$ on a frame $\mathfrak{F} = \langle F, \langle \lhd_i : i < \kappa\rangle$ by induction on its structure as follows.

$$\begin{aligned} R(p_0) &:= \{\langle x, x\rangle : x \in F\} \\ R(\Box_i s) &:= \lhd_i \circ R(s) \\ R(s \wedge t) &:= R(s) \cup R(t) \end{aligned} \tag{18}$$

Then for all valuations β and $x \in F$:

$$\langle \mathfrak{F}, \beta, x\rangle \vDash t(\varphi) \quad \Leftrightarrow \quad \text{for all } y \text{ such that } x\ R(t)\ y\text{: } \langle \mathfrak{F}, \beta, y\rangle \vDash \varphi \tag{19}$$

Let L be a modal logic. Then define

$$\text{CRel}(L) := \{\vdash : \text{Taut}(\vdash) = L\} \tag{20}$$

Furthermore, let $\vdash^m_L$ be the modal consequence relation containing $\vdash_L$ in which every admissible rule is derived. (It can be obtained by adding to $\vdash_L$ all admissible rules.)

Proposition 2.18 *Let L be a modal logic. Then*

$$\text{CRel}(L) = \{\vdash : \vdash_L \subseteq \vdash \subseteq \vdash^m_L\} \tag{21}$$

Moreover, $\vdash_L$ is the unique member of CRel(L) *having a deduction theorem for $\to$ and $\vdash^m_L$ is the unique member which is structurally complete.*

Now, as is reported in [37], for logics contained in **G.3**, $|\,\text{CRel}(L)| = 2^{\aleph_0}$. However, for tabular logics the situation is actually different (see also Theorem 6.14 below).

Theorem 2.19 *Let L be a tabular modal logic over a finite κ. Then* CRel(L) *is countable, and every member of* CRel(L)*, indeed every extension of $\vdash_L$, is finitely axiomatizable and decidable.*

Proof. First, a tabular logic is finitely axiomatizable. This needs some sophistication. Anticipating the results below, notice first that $\mathcal{V}(L)$ is locally finite. Then, using Corollary 2.49 we establish that $\mathrm{NExt}(L)$ is continuous, by Theorem 2.47 that $\mathrm{NExt}(L)$ has a basis, and therefore by Theorem 2.48 that $\mathrm{NExt}(L)$ strong basis. It follows with Theorem 2.50 that every extension of $\mathrm{NExt}(L)$ is finitely axiomatizable. So, $\vdash_M$ is finitely axiomatizable for every $M \supseteq L$. Also, $\mathcal{V}(L)$ is locally finite. Now, every extension of $\vdash_L$ is determined by some set of matrices verifying the axioms L. This means that they satisfy the axiom that the algebra has at most $|A|$ elements. This reduces the irreducible matrices to those of the form $\langle \mathfrak{B}, D\rangle$ where $|B| \leq |A|$, of which there are only finitely many. (The exact argument is nontrivial, see also [14], Corollary 2.5.20.) Thus, $\vdash_{\mathfrak{A}}$ has finitely many extensions. It is not difficult to show that they are all compact. Being determined by a finite set of finite algebras, they are all decidable. $\Box$

To see some more examples, consider the rule $\langle \{\Box p\}, p\rangle$. It is admissible in $\mathbf{K}$. For assume that $\varphi := p^\sigma$ is not a theorem. Then there exists a model $\langle \mathfrak{F}, \beta, x\rangle \models \neg\varphi$ based on the Kripke-frame $\langle F, \lhd\rangle$. Consider the frame $\mathfrak{G}$ based on $F \cup \{z\}$, where $x \notin F$, and the relation $\blacktriangleleft := \lhd \cup \{\langle z, y\rangle : y \in F\}$. Take $\gamma(p) := \beta(p)$. Then $\langle \mathfrak{g}, \gamma, z\rangle \models \neg\Box\varphi$. The rule $\langle \{p\}, \Diamond p\rangle$ is not admissible in $\mathbf{K}$ despite the admissibility of $\langle \{\Box p\}, p\rangle$. Take $p := \top$. $\Diamond\top$ is not a theorem of $\mathbf{K}$. Similarly, the so-called MacIntosh rule $\langle \{p \to \Box p\}, \Diamond p \to p\rangle$ is not admissible for $\mathbf{K}$. Namely, put $p := \Box\bot$. $\Box\bot \to \Box\Box\bot$ is a theorem but $\Diamond\Box\bot \to \Box\bot$ is not. Notice also that if a rule ρ is admissible in a logic L we may not conclude that ρ is admissible in every extension of L. A case in point is the rule $\langle \{\Box p\}, p\rangle$, which is not admissible in $\mathbf{K} \oplus \Box\bot$.

2.6 Lattices of Modal Consequence Relations

Every finitary consequence relation has the form $\vdash^R$ for some set R of finitary rules. Then

(22) $$\vdash^R \sqcap \vdash^S := \vdash^R \cap \vdash^S$$

(23) $$\vdash^R \sqcup \vdash^S := \vdash^{R \cup S}$$

We can even define infinitary analogs of the operations:

(24) $$\bigsqcap_{i \in I} \vdash_i := \bigcap_{i \in I} \vdash_i$$

(25) $$\bigsqcup_{i \in I} \vdash^{R_i} := \vdash^{(\bigcup_{i \in I} R)}$$

It is to be noted, though, that the infinite intersection of finitary consequence relations need not be finitary again. It is also not possible to axiomatize it in terms of the rules of the rules for the $\vdash_i$. Therefore in the sequel we shall frequently deal with lattices in which only join is infinitary.

If a finitary rule is derivable in $\vdash^S$, then it is derivable already in $\vdash^{S_0}$ for some finite S_0, since $\vdash^S$ is finitary by assumption. It follows that a consequence relation is compact iff it is finitely axiomatizable. Moreover, the lattice is algebraic, since $\vdash^R = \bigsqcup_{\rho \in R} \vdash^\rho$. Finally, $\vdash'$ is quasi-normal iff $\mathrm{Taut}(\vdash')$ is quasi-normal iff $\mathrm{Taut}(\vdash')$ contains $\mathbf{K}_\kappa$.

Proposition 2.20 *The set of modal consequence relations over a given language forms an algebraic lattice. The compact elements are exactly the finitely axiomatizable consequence relations. The lattice of quasi-normal consequence relations is the sublattice of consequence relations containing* $\vdash_{\mathbf{K}_\kappa}$.

We write $\mathrm{Ext}(\vdash)$ for the set of extensions of $\vdash$. By abuse of the notation we shall also denote the lattice over this set by $\mathrm{Ext}(\vdash)$. Similarly $\mathrm{QExt}(\vdash)$ denotes the set and the lattice of quasi-normal extensions. $\mathrm{NExt}(L)$ denotes the set and the lattice of normal extensions of a modal logic L.

Proposition 2.21 *For each quasi-normal logic L and each quasi-normal consequence relation $\vdash'$,*

(26) $$\vdash_L \subseteq \vdash' \quad \Leftrightarrow \quad L \subseteq \mathrm{Taut}(\vdash')$$

Taut$(-)$ *commutes with infinite intersections,* $\vdash_L$ *with infinite intersections and infinite joins. It follows that* $\mathrm{NExt}(\mathbf{K}_\kappa)$ *is a sublattice of* $\mathrm{Ext}(\vdash_{\mathbf{K}_\kappa})$.

Taut$(-)$ does not commute with joins. For example, let $\vdash_1 := \vdash^m_{\mathbf{G.3}}$ and $\vdash_2 := \vdash_{\mathbf{K}\oplus\Box\bot}$. Then $\mathrm{Taut}(\vdash_1 \sqcup \vdash_2) = \mathbf{K} \oplus \bot$, but $\mathbf{G.3} \sqcup \mathbf{K} \oplus \Box\bot = \mathbf{K} \oplus \Box\bot$.

Proposition 2.22 *In monomodal logic,* $\vdash_L$ *is maximal iff* L *is a coatom.*

Proof. Clearly, if $\vdash_L$ is maximal in $\mathrm{Ext}(\vdash_{\mathbf{K}})$, L must be a coatom. To show the converse, we need to show that for a maximal consistent normal logic L, $\vdash_L$ is structurally complete. (It will follow that $\mathrm{CRel}(L)$ has exactly one element.) Now, L is Post-complete iff it contains either the formula $\Box\top$ or the formula $p \leftrightarrow \Box p$. Assume that $\vdash_L$ can be expanded by a rule $\rho = \langle \Delta, \varphi \rangle$. Then, by using the axioms ρ can be transformed into a rule $\rho' = \langle \Delta', \varphi' \rangle$ in which the formulae are nonmodal. (Namely, any formula in a rule may be exchanged by a deductively equivalent formula. Either $\Box\top \in L$ and any subformula $\Box\chi$ may be replaced by $\top$, or $p \leftrightarrow \Box p \in L$ and then $\Box\chi$ may be replaced by χ.) A nonmodal rule not derivable in $\vdash_L$ is also not derivable in its boolean fragment, $\vdash^0_L$. By the maximality of the latter, adding ρ' yields the inconsistent logic. $\Box$

In polymodal logics matters are a bit more complicated. There exist $2^{\aleph_0}$ logics which are coatoms in $\mathrm{NExt}(\mathbf{K}_2)$ without their consequence relation being maximal. Moreover, in monomodal logics there exist $2^{\aleph_0}$ maximal consequence relations, which are therefore not of the form $\vdash_L$ (except for the two abovementioned consequence relations). Notice that even though a consequence is maximal iff it is structurally complete and Post-complete, Post-completeness is relative to the derivable rules. Therefore, this does *not* mean that the tautologies form a maximally consistent modal logic.

There is another consequence relation frequently associated with a logic, namely

(27) $\Vdash_L := \vdash^{L;(\mathrm{MP}_\to);(\mathrm{MN})}$

This is called the **global consequence relation**. Evidently, if (MN) is admissible, the set of tautologies is a normal logic, so $\mathrm{Taut}(\Vdash_L)$ is actually the least normal logic containing L.

Proposition 2.23 *Let L be a normal logic. Then the following are equivalent.*

(i) $\vdash_L = \Vdash_L$.

(ii) $\Vdash_L$ *admits a deduction theorem for* $\to$.

(iii) $L \supseteq \mathbf{K}_\kappa \oplus \{p \to \Box_j p : j < \kappa\}$.

(iv) *L is the logic of some set of Kripke-frames containing only one world.*

Clearly, if $\vdash_L \neq \Vdash_L$ then there are several consequence relations for a given logic. We will show now that the converse almost holds.

Proposition 2.24 *Let L be a modal logic. Then the following are equivalent.*

(a) $|\mathrm{CRel}(L)| = 1$.

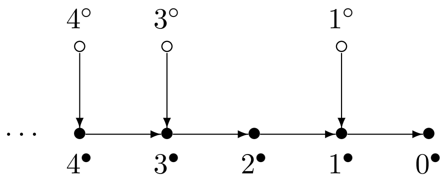
(b) $\vdash_L$ *is structurally complete.*

(c) L is the logic of a single Kripke-frame containing a single world.

(d) L is a fusion of monomodal logics of the frames $\boxed{\bullet}$ *or* $\boxed{\circ}$.

The nontrivial part is to show that (b) $\Leftrightarrow$ (c). Assume (c). Then since $\vdash_L$ is the logic of a single algebra based on two elements, and has all constants, it is structurally complete. Now let (c) fail. There are basically two cases. If L is not the logic of one-point frames, then $\vdash_L$ is anyway not structurally complete by the previous theorem. Otherwise, it is the intersection of logics determined by matrices of the form $\langle \mathfrak{A}, D \rangle$, D an open filter, $\mathfrak{A}$ the free algebra in $\aleph_0$ generators. (In fact, the freely 0-generated algebra is enough.) $\mathfrak{A}$ contains a constant c such that $0 < c < 1$. Namely, take two

Fig. 1. $\mathfrak{T}_M$, $M = \{1, 3, 4, \ldots\}$



different one point frames. Then, say, $\Box_0$ is the diagonal on one frame and empty on the other. Then $c := \blacklozenge_0 1$ is a constant of the required form. The rule $\langle\{\Diamond_0\top\}, p\rangle$ is admissible but not derivable.

The method of the last proof can be used in many different ways.

Lemma 2.25 *Let L be a logic and χ a constant formula such that neither χ not $\neg\chi$ are inconsistent. Then the rule $\rho[\chi] := \langle\{\chi\}, \bot\rangle$ is admissible for L but not derivable in $\vdash_L$.*

Since $\chi \notin L$ and $\mathrm{var}(\chi) = \varnothing$, for no substitution s, $\chi^s \in L$. Hence the rule $\rho[\chi]$ is admissible. If it is derivable in $\vdash_L$ then $\vdash_L \chi \to \bot$, by the DT. So $\neg\chi \in L$, which is not the case. So, $\rho[\chi]$ is not derivable.

Theorem 2.26 *Let L be a logic such that $\mathfrak{Fr}_L(0)$ has infinitely many elements. Then $|\,\mathrm{CRel}(L)| = 2^{\aleph_0}$.*

The idea is as follows. There is an infinite set C of constants such that $\chi \wedge \chi' \vdash_L \bot$ whenever χ, χ' are distinct members of C. The relations $\vdash^D$ for $D \subseteq C$ are all pairwise distinct.

Corollary 2.27 *Let L be a monomodal logic and $L \subseteq$ **G**.**3**. Then $|\,\mathrm{CRel}(L)| = 2^{\aleph_0}$.*

In addition, $\vdash^m_{\mathbf{G.3}}$ is maximal. This follows from the following

Theorem 2.28 *Let L be the logic of its 0-generated free algebra. Then $\vdash^m_L$ is maximal.*

Proof. Let $\Vdash \supsetneq \vdash^m_L$. Then $\mathrm{Taut}(\Vdash) \supsetneq L$. Since L is determined by its freely 0-generated algebra, there is a constant χ such that $L \subsetneq L \oplus \chi \subseteq \mathrm{Taut}(\Vdash)$. Therefore, $\chi \notin L$. (Case 1.) $\neg\chi \notin L$. Then $\rho[\chi]$ is admissible in L and so derivable in $\vdash^m_L$. Therefore $\rho[\chi] \in \Vdash$, and so $\Vdash \bot$. So, $\Vdash$ is inconsistent. (Case 2.) $\neg\chi \in L$. Then $\mathrm{Taut}(\Vdash)$ and also $\Vdash$ is inconsistent. $\Box$

We will now turn to the set of coatoms in $\mathrm{NExt}(\vdash_{\mathbf{K}})$. Let $M \subseteq \omega$. Put $T_M := \{n^\bullet : n \in \omega\} \cup \{n^\circ : n \in M\}$ and

$$(28) \qquad x \lhd y \quad \Leftrightarrow \quad \begin{cases} \quad (1.)\ x = m^\bullet, y = n^\bullet \text{ and } m > n \\ \text{or } (2.)\ x = m^\circ, y = n^\bullet \text{ and } m \geq n \\ \text{or } (3.)\ x = m^\circ, y = n^\circ \text{ and } m = n \end{cases}$$

Let $\mathbb{B}_M$ be the algebra of 0-definable sets. Put $\mathfrak{T}_M := \langle T_M, \lhd, \mathbb{B}_M\rangle$. If $M \neq N$ then $\mathrm{Th}(\mathfrak{T}_M) \neq \mathrm{Th}(\mathfrak{T}_N)$. To see this, note that every one-element set $\{n^\circ\}$ in T_M is definable by a formula $\chi(n)$ that depends only on n, not on M. First, take the formula

$$(29) \qquad \delta(n) := \Box^{n+1}\bot \wedge \neg\Box^n\bot$$

$\delta(n)$ defines the set $\{n^\bullet\}$. Now put

$$(30) \qquad \chi(n) := \Diamond\delta(n) \wedge \neg\delta(n+1) \wedge \neg\Diamond\delta(n+1)$$

It is easily checked that $\chi(n)$ defines $\{n^\circ\}$. Hence, if $n \notin M$, $\neg\chi(n) \in \mathrm{Th}\ \mathfrak{T}_M$. So, $\neg\chi(n) \in \mathrm{Th}\ \mathfrak{T}_M$ iff $n \notin M$. This establishes that if $M \neq N$, $\mathrm{Th}\ \mathfrak{T}_M \neq \mathrm{Th}\ \mathfrak{T}_N$.

Theorem 2.29 *The lattice of normal monomodal consequence relations contains $2^{\aleph_0}$ many coatoms.*

Given a normal modal logic L and a set Δ of formulae, $L \oplus \Delta$ denotes the smallest normal logic which contains L and Δ. Recall that $\mathrm{NExt}(L)$ denotes the set (and lattice) of normal logics containing L. For logics $M_i = L \oplus \Delta_i$ we have

$$\bigsqcup_{i \in I} M_i = L \oplus \bigcup_{i \in I} \Delta_i \tag{31}$$

We can also calculate the axiomatization of the intersection of two logics. Given two formulae, φ and χ, let $\varphi \dot{\vee} \chi$ denote a formula $\varphi \vee \chi^s$, where s is one-to-one and renames the variables of χ so as to makes them distinct from the variables of φ. Then

$$(L \oplus \Delta) \sqcap (L \oplus \Sigma) = L \oplus \{\boxplus\varphi \dot{\vee} \boxplus \chi : \varphi \in \Delta, \chi \in \Sigma, \boxplus \text{ a compound modality}\} \tag{32}$$

(See [48] or [37].) This can be used to show that $\mathrm{NExt}(L)$ satisfies the following infinitary distributive law

$$x \sqcap \bigsqcup_{i \in I} y_i = \bigsqcup_{i \in I} x \sqcap y_i \tag{33}$$

In particular, the usual distributivity law holds. This means that the lattice is a locale, where a **locale** is a lattice with infinitary join and finitary meet satisfying (33). Recall that the operation $\bigsqcap$ can be defined from $\bigsqcup$ as follows:

$$\bigsqcap_{i \in I} x_i := \bigsqcup \langle y : \text{for all } i \in I\text{: } y \leq x_i \rangle \tag{34}$$

A locale is **continuous** if also $L \sqcup \bigsqcap_{i \in I} M_i = \bigsqcap_{i \in I} L \sqcup M_i$. Locales $\mathrm{NExt}(L)$ are rarely continuous. An important exception is $\mathrm{NExt}(\mathbf{S4.3})$. Call an element x of a locale **meet-irreducible** (**strongly meet-irreducible**) if from $x = y \sqcap z$ follows $x = y$ or $x = z$ (if from $x = \bigsqcap_{i \in I} y_i$ follows $x = y_i$ for some $i \in I$). Call x **meet-prime** (**strongly meet-prime**) if from $x \geq y \sqcap z$ follows $x \geq y$ or $x \geq z$ (if from $x \geq \bigsqcap_{i \in I} y_i$ follows $x \geq y_i$ for some $i \in I$). Dually for **join-irreducible** and **join-prime**. If x is (strongly) meet-prime it is also (strongly) meet-irreducible. In a distributive lattice, meet-prime is equivalent to meet-irreducible, but in general a strongly meet-irreducible element need not be strongly meet-prime. However, in a locale a strongly join-irreducible element is also strongly join-prime.

Given a locale $\mathfrak{L} = \langle L, \sqcap, \bigsqcup \rangle$, let $\mathrm{Irr}(\mathfrak{L})$ be the set of strongly meet-irreducible elements of $\mathfrak{L}$. For $x \in L$ put $x^\circ := \mathrm{Irr}(\mathfrak{L}) - \uparrow x$ where

$$\uparrow x := \{y : y \geq x\} \qquad \downarrow x := \{y : y \leq x\} \tag{35}$$

It turns out that

$$(x \sqcup y)^\circ = x^\circ \cup y^\circ \tag{36}$$

$$(\bigsqcap_{i \in I} x)^\circ = \bigcap_{i \in I} x_i^\circ \tag{37}$$

Thus, $\{x^\circ : x \in L\}$ is a topology of closed sets on $\mathrm{Irr}(\mathfrak{L})$. A locale is **spatial** if it is isomorphic to the locale of open sets of a topological space.

Theorem 2.30 *The locale* $\mathrm{NExt}(L)$ *is spatial.*

To show that $\mathrm{NExt}(L)$ is spatial we need to show that the map $M \mapsto M^\circ$ is injective. For a formula $\varphi \notin M$, the set of logics not containing φ is nonempty (containing, for example, M) and has a maximal element, which we denote by $L^\star_\varphi$. (This follows from Zorn's Lemma, using the fact that

NExt(L) is algebraic. $L^\star_\varphi$ is usually not unique.) $L^\star_\varphi$ is easily seen to be strongly meet-irreducible. Now

$$M = \bigcap_{\varphi \notin M} L^\star_\varphi \tag{38}$$

The topology $\{M^\circ : M \in \mathrm{NExt}(L)\}$ satisfies the T_0-axiom: for every pair M, M' of different logics there is an open set X such that $|X \cap \{M, M'\}| = 1$. Put $M \preccurlyeq M'$ if $M^\circ \subseteq M'^\circ$. It is easy to see that $M \preccurlyeq M'$ iff $M \subseteq M'$ iff $M \leq M'$. Moreover, a closed set is lower closed, that is, if X is closed then $\downarrow X = X$. The converse need not be true. Thus, the lattice is completely reconstructible from the topology. Moreover:

Theorem 2.31 NExt(L) *is continuous iff all lower closed sets are closed.*

Indeed, if NExt(L) is continuous, then the arbitrary union of closed sets is closed. Any lower closed set is the union of sets of the form $\downarrow\{x\}$, which are all closed. More on this subject can be found in [37].

It is interesting to know which properties are at all connected with the lattice structure. Completeness, for example, is clearly closed under meet but not under join (for a counterexample see [37]). Elementarity is closed both under intersection and infinitary join. Decidability is closed under intersection, but not under join. Interpolation and Halldén-completeness show no clear connection.

2.8 Splittings

Splittings have been studied in the context of modal logics first by [4], from which most of the results below are drawn. This investigation was carried further in [47,49]. A **splitting** of a lattice $\langle L, \sqcap, \sqcup\rangle$ is a pair $\langle x, y\rangle$ such that $L = \downarrow x \cup \uparrow y$ and $\downarrow x \cap \uparrow y = \varnothing$. We say that x **splits** $\mathfrak{L}$ if there is y such that $\langle x, y\rangle$ splits $\mathfrak{L}$. We say that y is the **splitting companion** of x and write $\mathfrak{L}/x$ for y (but for logics we write L/M rather than NExt(L)/M).

Proposition 2.32 *If $\langle x, y\rangle$ is a splitting of $\mathfrak{L}$, x is strongly meet-prime and y is strongly join-prime. x splits $\mathfrak{L}$ iff it is strongly meet-prime. If $x < x'$ and $\langle x', y'\rangle$ is a splitting, then $y < y'$.*

Notice that every join-irreducible logic is join-prime. There is a useful corollary for logics. Say that M is **essentially 1-axiomatizable** over L if for every Δ: if $M = L \oplus \Delta$ then there is a $\delta \in \Delta$ such that $M = L \oplus \delta$. It is easy to see that this notion is equivalent to strong join-irreducibility. Hence we have an observation already made by [44].

Proposition 2.33 (McKenzie) *M is essentially 1-axiomatizable over L iff $M = L/N$ for some splitting logic N.*

Furthermore, this gives rise to an axiomatizability criterion. Suppose that $M = L/N$. Then $M = L \oplus \delta$ iff (a) $\delta \in M$ and (b) $\delta \notin N$. If both M and N are decidable, the problem '$M = L \oplus \delta$' is decidable. For example, $\mathbf{S5} = \mathbf{S4}/N$, where N is the logic of a four element algebra. Then clearly N is decidable; since also $\mathbf{S5}$ is decidable, the problem '$\mathbf{S5} = \mathbf{S4} \oplus \delta$' is decidable. More can be established. Also the problem '$\vdash^{+\rho}_{\mathbf{S4}} = \vdash_{\mathbf{S5}}$' is decidable. This is due to the following fact. Recall that $\vdash^m_M$ is the maximal consequence relation that has M as its set of tautologies.

Proposition 2.34 (Rautenberg) *Suppose that M induces a splitting of* NExt(L)*. Then $\vdash^m_M$ splits the lattice* Ext($\vdash_L$)*, and* Ext($\vdash_L$)$/\vdash^m_M = \vdash_{L/M}$*.*

Now, suppose a rule ρ is given. The problem whether $\rho \in \vdash^m_M$ is decidable (see Theorem 2.19). Case 1. $\rho \in \vdash^m_M$. Then $\vdash^{+\rho}_{\mathbf{S4}} \subseteq \vdash^m_M \not\supseteq \vdash_{\mathbf{S5}}$. Case 2. $\rho \notin \vdash^m_M$. Then adding $\vdash^{+\rho}_{\mathbf{S4}} \supsetneq \vdash_{\mathbf{S4}}$. Now we must check whether $\rho \in \vdash_{\mathbf{S5}}$. This is again decidable. If this holds $\vdash^{+\rho}_{\mathbf{S4}} = \vdash_{\mathbf{S5}}$. The argument generalizes to the case where M is tabular and L/M is decidable.

Lemma 2.35 *If M does not split* $\mathrm{NExt}(L)$ *there is a sequence N_i, $i \in \omega$, of logics such that $N_i \not\leq M$ but $\bigsqcap_{i\in\omega} N_i \leq M$.*

(And if M does split $\mathrm{NExt}(L)$, no such sequence can obviously exist.) If N splits $\mathrm{NExt}(L)$ it is strongly meet-prime. In particular, it is strongly meet-irreducible. It follows that $N = \mathrm{Th}\,\mathfrak{A}$, where $\mathfrak{A}$ is a subdirectly irreducible (si) algebra. However, there are examples of subdirectly irreducible algebras such that $\mathrm{Th}\,\mathfrak{A}$ is not even meet-irreducible. The algebras that induce splittings can be characterized. Call an element $x < 1$ of a si $\mathfrak{A}$ an **opremum** if for all $a < 1$ there is a compound modality $\boxplus$ such that $\boxplus a \leq x$. Intuitively, in a finite algebra an opremum is easy to find. The dual frame is generated by a single world, w, in the sense that every world is indirectly accessible from it iff the algebra is subdirectly irreducible. (This fails in the infinite, as Giovanni Sambin pointed out, see [37].) Now, the set containing everything but w, is an opremum.

Let $\Delta(\mathfrak{A})$ be the so-called **diagram** of $\mathfrak{A}$, defined by

$$\begin{aligned}(39)\qquad \Delta(\mathfrak{A}) := \; & \{p_a \vee p_b \leftrightarrow p_{a\vee b} : a, b \in A\} \\ & \cup \{p_{\neg a} \leftrightarrow \neg p_a : a \in A\} \\ & \cup \{p_{\Box_i a} \leftrightarrow \Box_i p_a : a \in A, i < \kappa\}\end{aligned}$$

Suppose that there is an algebra $\mathfrak{B}$, a valuation β, and an ultrafilter U such that $\beta(\neg p_x) \in U$ and for every compound modality $\boxplus$ and every $\delta \in \Delta(\mathfrak{A})$, $\beta(\boxplus\delta) \in U$. Then $\mathfrak{A} \in \mathsf{HS}\,\mathfrak{B}$. Moreover, (by Jónsson's Lemma), $\mathfrak{A} \in \mathsf{HSP}\,\mathfrak{B}$ iff $\mathfrak{A} \in \mathsf{HSUp}\,\mathfrak{B}$ iff every finite subset of $p_x; \bigcup\{\boxplus\Delta(\mathfrak{A}) : \boxplus \text{ a compound modality}\}$ is satisfiable. Let $\mathcal{V}(L)$ denote the variety of L-algebras. The following result appeared in its complete form in [60], generalizing theorems by [49] and [34].

Theorem 2.36 *Let $\mathfrak{A}$ be subdirectly irreducible with opremum x. The following are equivalent:*

① $\mathrm{Th}\,\mathfrak{A}$ *splits* $\mathrm{NExt}(L)$.

② *There is a finite $\Delta_0 \subseteq \Delta(\mathfrak{A})$ and a compound modality $\boxplus$ such that for every $\mathfrak{B} \in \mathcal{V}(L)$: if $\neg p_x; \boxplus\Delta_0$ is satisfiable in $\mathfrak{B}$, so is*

$$\neg p_x; \bigcup\{\boxtimes\Delta(\mathfrak{A}) : \boxtimes \text{ a compound modality}\}$$

If either obtains,

$$(40)\qquad L/\,\mathrm{Th}\,\mathfrak{A} = L \oplus \bigwedge \boxplus\Delta_0 \to p_x$$

We note that the number of variables needed to axiomatize $L/\,\mathrm{Th}\,\mathfrak{A}$ is the minimum number of variables needed to generate $\mathfrak{A}$. This can be used to show that **S4.3** cannot be axiomatized over **S4.2** (and **S4**) using less than two variables. (In tense logic, however, one variable is sufficient.)

2.9 Some Splittings

Let us first look at monomodal logics. A frame $\mathfrak{F}$ is cycle-free if there is an $n \in \omega$ such that $\mathfrak{F} \vDash \Box^n\bot$. **K** is complete with respect to all cycle-free frames. It follows from Lemma 2.35 that only logics of cycle-free frames can split $\mathrm{NExt}(\mathbf{K})$. So, let $\mathfrak{F}$ be finite, cycle free and generated by a single point. Let $\mathfrak{A}$ be the algebra of its subsets. $\mathfrak{A}$ is si. Since $\mathfrak{A} \vdash \Box^n\bot \to \Box^{n+1}\bot$, it follows that if $\Box^n\bot$ is satisfiable, then $\Box^{n+k}\bot$ is satisfiable for every $k \in \omega$. Thus, since $\Box^n\Delta(\mathfrak{A})$ is finite and implies $\Box^n\bot$, we get that $\Box^n\Delta(\mathfrak{A}) \to \Box^{n+k}\Delta(\mathfrak{A})$ for every k. So, the theory of a finite one-generated cycle free frame splits $\mathrm{NExt}(\mathbf{K})$. This argument generalizes easily for any finite number of operators.

Theorem 2.37 (Blok) *L splits* $\mathrm{NExt}(\mathbf{K})$ *iff it is the logic of a finite, one-generated cycle free frame.*

There is an easy corollary. For every splitting logic L there is a splitting logic $L' < L$. (Simply add another irreflexive point before the generator of $\mathfrak{F}$ where $L = \mathrm{Th}(\mathfrak{F})$.) Now, from Proposition 2.32 we get $\mathrm{NExt}(\mathbf{K})/L' < \mathrm{NExt}(\mathbf{K})/L$. Thus, for every strongly join-prime element there exists a strongly join-prime element strictly below it. Atoms are strongly join-irreducible, and therefore strictly join-prime, hence they are splitting companions. We have established the following result from [3].

Theorem 2.38 (Blok) $\mathrm{NExt}(\mathbf{K})$ *is atomless.*

On the other hand we have the following from [39].

Theorem 2.39 (Makinson) $\mathrm{NExt}(\mathbf{K})$ *has exactly two coatoms. Moreover, every consistent logic is below one of them.*

The coatoms are the logics of the two two-element algebras, corresponding to the one-element reflexive frame, and the one-element irreflexive frame. Take a general frame $\mathfrak{F}$. Either $\Box\bot$ is satisfiable, in which case the subframe of points satisfying $\Box\bot$ is generated, and can be contracted to the single one-generated irreflexive point, or $\Box\bot$ is not satisfiable. Then $\mathfrak{F} \vDash \Diamond\top$, so that $\mathfrak{F}$ is contractible to a one-element reflexive frame. The second fact easily follows from the following observation: if L is finitely axiomatizable, there is no infinite upgoing chain with limit L. The inconsistent logic is finitely axiomatizable, and so it is not the limit of an upgoing chain. Hence every consistent logic must be below a coatom.

Suppose now that $L = \mathbf{K}/M$ for some logic M. It so happens that $\mathrm{NExt}(L)$ may be split by N even though N does not split $\mathrm{NExt}(\mathbf{K})$. This arises exactly once: $L = \mathbf{K}/\boxed{\bullet}$, where $\boxed{\bullet}$ is the one-point irreflexive frame. Then $L = \mathbf{K.D}$, and the new splitting logic is $N = \mathrm{Th}\,\boxed{\circ}$, the logic of the one-element reflexive frame. We call L/N an **iterated splitting** of $\mathbf{K}$. L/N is actually inconsistent. However, suppose that X is a set of splitting logics of $\mathrm{NExt}(L)$. Then we may split off the logics of X in any order we like. The results is always the same. Therefore, put

(41) $$L/X := \bigsqcup \langle L/N : N \in X\rangle$$

The following theorem is much harder to establish. Let $\mathrm{Fs}(L)$ be the set of all logics that have the same Kripke-frames as L (the **Fine-spectrum** of L). Call L **intrinsically complete** if $|\,\mathrm{Fs}(L)| = 1$. The following is from [4].

Theorem 2.40 (Blok) *L is intrinsically complete iff it is inconsistent or of the form $\mathbf{K}/X$ for a set of splitting logics X. If L is not intrinsically complete, $|\,\mathrm{Fs}(L)| = 2^{\aleph_0}$.*

We say that N has a **splitting representation** over L if it has the form L/X for some set X. Although one can have $N = L/X = L/Y$ for different X and Y, there is a unique set $X^\star$ such that $N = L/X^\star$ and for every X such that $N = L/X$ we have $X \supseteq X^\star$. (The set $X^\star$ is a minimal representation of L.)

Say that a compound modality $\boxplus$ is a **master modality** for L if (a) $\boxplus p_0 \to \Box_i p_0 \in L$ for all $i < \kappa$, and (b) $\boxplus p_0 \to p_0, \boxplus p_0 \to \boxplus\boxplus p_0 \in L$. L is called **weakly transitive** if it has a master modality. Now suppose that L is weakly transitive, with master modality $\boxplus$. Then if $\mathfrak{A}$ is finite and subdirectly irreducible, it is splitting. (Actually, it is enough that $\mathfrak{A}$ is finitely presentable.) For example, the logic M of a one-generated finite frame splits $\mathrm{NExt}(\mathbf{K4})$ (and every $\mathrm{NExt}(L)$ for $L \geq \mathbf{K4}$ if only $M \geq L$). Many logics above $\mathbf{K4}$ possess a splitting representation above $\mathbf{K4}$.

We present a few applications. [18] shows that there is an infinite antichain L_i, $i \in \omega$, of logics of depth 3 in $\mathrm{NExt}(\mathbf{S4})$. Now, define the following map from subsets of ω into $\mathrm{NExt}(\mathbf{S4})$: $p : U \mapsto \mathbf{S4}/\{L_i : i \in U\}$. This map is injective. Moreover, $p(U) \leq p(V)$ iff $U \subseteq V$. So, the map is a lattice embedding. It follows not only that $\mathrm{NExt}(\mathbf{S4})$ has continuously many elements, but also that it has an infinite upgoing chain of elements.

It is known that every logic $L \supseteq \mathbf{S4.3}$ has the finite model property (see [7]). It follows that it has a representation

(42) $$L = \mathbf{S4.3}/X$$

where X is the set of logics of **S4.3**-frames which are not L-frames. Identity holds by the fact that both logics have the finite model property and the same finite models. It follows that there is a unique minimal set $X^\star$ such that $L = \mathbf{S4.3}/X^\star$. This means that there is a canonical axiomatization of every logic in terms of splitting formulae, an axiomatization base in the sense defined below.

2.10 Axiomatization Bases

The success of the canonical formulae of Michael Zakharyaschev (see [62,63]) has sparked off the question whether it is possible to find independent sets of formulae that can axiomatize any given logic above L, where L is a given modal logic (in the best case, $L = \mathbf{K}_\kappa$). The present section reviews conditions on L under which this is possible, but the outcome is, for practical purposes, rather negative: only very strong logics L have this property.

If every extension of L is of the form L/X the locale $\mathrm{NExt}(L)$ is continuous. The finitely axiomatizable logics are closed under finite union, just as the compact elements. An infinite join of finitely axiomatizable logics need not be finitely axiomatizable again. Likewise, the finite meet of finitely axiomatizable logics need not be finitely axiomatizable. However, this is the case when L is weakly transitive.

Definition 2.41 A locale is **coherent** if (i) every element is the join of compact elements and (ii) the meet of two compact elements is again compact.

Coherent locales allow a stronger representation theorem. Let $\mathfrak{L}$ be a coherent locale, $K(\mathfrak{L})$ be the set of compact elements. They form a lattice $\mathfrak{K}(\mathfrak{L}) := \langle K(\mathfrak{L}), \sqcap, \sqcup\rangle$, by definition of a coherent locale. Given $\mathfrak{K}(\mathfrak{L})$, $\mathfrak{L}$ is uniquely identified by the fact that it is the lattice of ideals of $\mathfrak{K}(\mathfrak{L})$.

Lemma 2.42 *A locale is coherent iff it is isomorphic to the locale of ideals of a distributive lattice.*

If we have a lattice homomorphism $\mathfrak{K}(\mathfrak{L}) \to \mathfrak{K}(\mathfrak{M})$ then this map can be extended uniquely to a homomorphism of locales $\mathfrak{L} \to \mathfrak{M}$. Not all locale homomorphisms arise this way, and so not all locale maps derive from lattice homomorphisms. Hence call a map $f : \mathfrak{L} \to \mathfrak{M}$ **coherent** if it maps compact element into compact elements.

Theorem 2.43 *The category* DLat *of distributive lattices and lattice homomorphisms is dual to the category* CohLoc *of coherent locales with coherent maps.*

Now if L is weakly transitive, the intersection of two finitely axiomatizable extensions is again finitely axiomatizable. Now, a logic is compact in $\mathrm{NExt}(L)$ iff it is finitely axiomatizable over L. We conclude the following theorem.

Proposition 2.44 *Let L be weakly transitive. Then* $\mathrm{NExt}(L)$ *is coherent.*

The converse need not hold. $\mathrm{NExt}(\mathbf{K.alt_1}) = \mathbf{K} \oplus \Diamond p \to \Box p$ is coherent (because every logic in this lattice is finitely axiomatizable) but the logic is not weakly transitive.

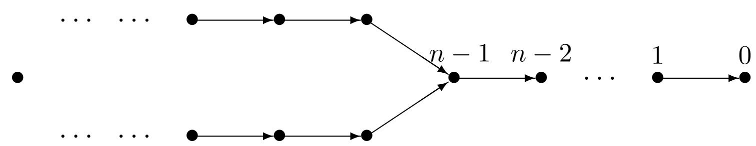
Definition 2.45 Let $\mathfrak{L}$ be a complete lattice. A set $X \subseteq L$ is a **generating set** if for every member of L is the join of a subset of X. $\mathfrak{L}$ is said to have a **basis** if there exists a least generating set. Moreover, X is a **strong basis** for $\mathfrak{L}$ if every element has a nonredundant representation, that is, for each x there exists a minimal $Y \subseteq X$ such that $x = \bigsqcup Y$.

Theorem 2.46 *Let $\mathfrak{L}$ be a locale. $\mathfrak{L}$ has a basis iff (i) $\mathfrak{L}$ is continuous and (ii) every element is the meet of $\bigsqcap$-irreducible elements. $\mathfrak{L}$ has a strong basis iff it has a basis and there exists no infinite properly ascending chain of $\bigsqcap$-prime elements.*

Let $\mathfrak{L}$ be a locale with a strong basis. Then the elements of $\mathfrak{L}$ are in one-to-one correspondence with antichains of strongly meet-prime elements (via the splitting representation, which must exist).

Theorem 2.47 *Let L be a modal logic. Then* $\mathrm{NExt}(L)$ *has a basis iff* $\mathrm{NExt}(L)$ *is continuous.*

Fig. 2. The Frame $\mathfrak{D}$



Since continuous lattices are the exception in modal logic, most extension lattices do not have a basis. We can sharpen the previous theorem somewhat to obtain stricter conditions on continuity.

Corollary 2.48 *Let L be weakly transitive and have the finite model property. Then the following are equivalent.*

(i) $\mathrm{NExt}(L)$ *has a basis.*

(ii) $\mathrm{NExt}(L)$ *has a strong basis.*

(iii) *Every extension of L has the finite model property.*

(iv) *Every extension of L is the join of co-splitting logics.*

(v) *Every join of co-splitting logics has the finite model property.*

Corollary 2.49 *Let $\mathcal{V}(L)$ be locally finite. Then* $\mathrm{NExt}(L)$ *is continuous.*

The converse does not hold. The lattice $\mathrm{NExt}(\mathbf{S4.3})$ is continuous but $\mathbf{S4.3}$ fails to be locally finite. The following once more emphasizes the importance of splittings on the structure of the lattice.

Theorem 2.50 *Let* $\mathrm{NExt}(L)$ *have a strong basis. Then the following are equivalent.*

(i) *Every extension of* $\mathrm{NExt}(L)$ *is finitely axiomatizable.*

(ii) $\mathrm{NExt}(L)$ *is finite or countably infinite.*

(iii) *There exists no infinite set of incomparable splitting logics.*

Typically the locales $\mathrm{NExt}(L)$ have no basis. We might ask, however, if for a given logic an independent axiomatization necessarily exists. This is not so. Call a set Δ of formulae **independent** if for every $\delta \in \Delta$ we have $\delta \notin \mathbf{K} \oplus (\Delta - \{\delta\})$. (For example, a basis is an independent set.) A logic L is **independently axiomatizable** if there exists an independent set Δ such that $L = \mathbf{K} \oplus \Delta$. Every finitely axiomatizable logic is independently axiomatizable. It has been shown in [8] that there exists a logic which is not independently axiomatizable. Furthermore, [35] gives an example of a logic which is not finitely axiomatizable, but all its proper extensions are. Such a logic is called **pre-finitely axiomatizable**. Here is a logic that has both properties.

Theorem 2.51 *The logic of the frame $\mathfrak{D}$ shown in Figure 2 is pre-finitely axiomatizable. It splits the lattice of extensions of $\mathbf{G}.\Omega_2$. Moreover, it is not axiomatizable by a set of independent formulae.*

Theorem 2.52 *Let $\mathfrak{A}$ be the algebra generated by the singleton sets of $\mathfrak{D}$ is not finitely presentable. Its logic splits* $\mathrm{NExt}(\mathbf{G}.\Omega_2)$.

3 The Local and the Global

3.1 Equivalential and Algebraizable Logics

In recent years, there have been a lot of results concerning the algebraizability of logics. (See [14] for a general exposition of the topics of this section.) Research has been sparked off mainly by the monograph [5]. In brief, a logic is algebraizable if the notion of truth and of consequence can be reduced faithfully to the equational calculus. Let us assume two consequence relations, $\vdash$ over language $\mathcal{L}_1$ and $\succ$ over language $\mathcal{L}_2$ are given. Let $\kappa : \mathcal{L}_1 \to \wp(\mathcal{L}_2)$ be a map from formulae in $\mathcal{L}_1$ to sets of formulae in $\mathcal{L}_2$. We write $\kappa(\Delta)$ for the union of the $\kappa(\delta)$, $\delta \in \Delta$. κ is a **transform** of $\vdash$ into $\succ$ if

(43) $\Delta \vdash \varphi \quad \Leftrightarrow \quad \text{for all } \chi \in \kappa(\varphi){:}\kappa(\Delta) \succ \chi$

If $\lambda : \mathcal{L}_2 \to \wp(\mathcal{L}_1)$ is a transform of $\succ$ into $\vdash$, κ a transform of $\vdash$ and $\succ$ we call $\langle \kappa, \lambda \rangle$ a **pair of conjugate transforms** if in addition

(44) $\varphi \vdash \chi \Leftrightarrow \lambda(\kappa(\varphi)) \vdash \chi$

(45) $\varphi \succ \chi \Leftrightarrow \kappa(\lambda(\varphi)) \succ \chi$

A consequence relation is **algebraizable** if there is a pair of conjugate transforms to a calculus of equations over the same language, and both maps commute with substitutions. Recall that there is also a first-order theory of the algebra, using the function symbols of the signature and equality ($=$). In equational logic we are mainly interested in Horn-clauses of that languages, to which we turn below.

A key element in the characterization of algebraizability is that of the Leibniz operator. A logic $\vdash$ defines the following operator $\Omega_{\mathfrak{A}}$ on an algebra $\mathfrak{A}$, called the **Leibniz operator**.

(46) $\Omega_{\mathfrak{A}}(D) := \{\langle a, b\rangle : \text{for all polynomials } p \text{ of } \mathfrak{A} : p(a) \in D \Leftrightarrow p(b) \in D\}$

Given D, $\Omega_{\mathfrak{A}}$ is the largest congruence compatible with D. $\langle \mathfrak{A}/\Omega_{\mathfrak{A}}(D), D/\Omega_{\mathfrak{A}}\rangle$ is reduced. We write Ω for the operator defined on the term algebra. As Wim Blok and Don Pigozzi have shown, many properties of the consequence relation can be defined in terms of the Leibniz-operator.

Theorem 3.1 (Blok & Pigozzi) *A consequence relation $\vdash$ is algebraizable iff*

① Ω *is monotone on the set of theories of* $\vdash$*;*

② Ω *is injective on the set of theories of* $\vdash$*; and*

③ Ω *commutes with inverse substitutions on the set of theories of* $\vdash$.

The first is to be read as follows: if T and T' are theories (deductively closed sets of formulae) and $T \subseteq T'$ then $\Omega(T) \subseteq \Omega(T')$. The latter are congruences. Similarly for the other conditions.

We shall fill the notion of algebraizability with more life. The calculus of equations can be generalized to implications. A **quasi-equation** or **quasi-identity** is an implication of the form

(47) $\sigma_0 = \tau_0 \wedge \sigma_1 = \tau_1 \wedge \ldots \wedge \sigma_{n-1} = \tau_{n-1} \to \sigma_n = \tau_n$

Alternatively, it is a Horn-clause in the first-order theory of the algebraic signature. A class of algebras is called a **quasi-variety** if it is characterized by a set of quasi-identities. The following is from [29].

Theorem 3.2 (Graetzer & Lakser) *A class of algebras is a quasi-variety iff it is closed under ultraproducts, products and subalgebras. The least quasi-variety containing a given class $\mathcal{K}$ is $\mathsf{SPP_u}(\mathcal{K})$.*

Now, in general consequence relations use the notion of truth. They are therefore said to **define truth implicitly** if for every algebra $\mathfrak{A}$ there is at most one deductively closed set D such that $\langle \mathfrak{A}, D\rangle$ is a reduced matrix for $\vdash$. An explicit definition consists in a set $\Delta(p)$ of equations such that $a \in D$ iff $\mathfrak{A} \vDash \alpha(a) = \beta(a)$ for all $\alpha(p) = \beta(p) \in \Delta(p)$. Since $\langle \mathfrak{A}, \{1\}\rangle$ is a matrix for all modal consequence relations, and reduced, a consequence relation defines truth implicitly iff (MN) is derivable.

The following definition is due to [46].

Definition 3.3 Let $\vdash$ be a consequence relation. A set of formulae $\Delta(p,q) := \{\delta_i(p,q) : i \in I\}$ is called a **set of equivalential terms for** $\vdash$ if the following holds

$$\vdash \Delta(p,p) \tag{48a}$$

$$\Delta(p,q) \vdash \Delta(q,p) \tag{48b}$$

$$\Delta(p,q); \Delta(q,r) \vdash \Delta(p,r) \tag{48c}$$

$$\bigcup_{i<\nu(f)} \Delta(p_i,q_i) \vdash \Delta(f(\boldsymbol{p}), f(\boldsymbol{q})) \tag{48d}$$

$$p; \Delta(p,q) \vdash q \tag{48e}$$

$\vdash$ is called **equivalential** if it has a set of equivalential terms, and **finitely equivalential** if it has a finite set of equivalential terms.

Theorem 3.4 *$\vdash$ is finitary and finitely equivalential iff the class of reduced matrices for $\vdash$ is a quasi-variety.*

Corollary 3.5 *Let $\vdash$ be finitary and finitely equivalential and $\mathcal{Q}$ the quasi-variety of reduced matrices for $\vdash$. Then the lattice of finitary extensions of $\vdash$ is dually isomorphic to the lattice of sub-quasi-varieties of $\mathcal{Q}$.*

A logic is algebraizable in the sense of Blok and Pigozzi if it is finitary, algebraizable and finitely equivalential. Ω is said to be **continuous** if for every upgoing chain T_i, $i \in \mu$, of theories whose limit (= union) is a theory

$$\Omega(\bigcup_{i\in\mu} T_i) = \bigcup\{\Omega\, T_i : i \in \mu\} \tag{49}$$

Continuity implies monotonicity.

Theorem 3.6 *$\vdash$ is equivalential iff Ω is monotone on the set of theories, and $s\Omega(T) \subseteq \Omega((sT)^{\vdash})$ for all substitutions s and theories T. $\vdash$ is finitely equivalential iff Ω is continuous on the set of theories of $\vdash$.*

Clearly, for any modal logic L, $\vdash_L$ is always equivalential; a set of equivalential terms is the following.

$$\Delta(p,q) := \{\boxplus(p \leftrightarrow q) : \boxplus \text{ a compound modality}\} \tag{50}$$

$\Vdash_L$ is always finitely equivalential; $p \leftrightarrow q$ is an equivalential term for $\Vdash_L$. Note that if a classical consequence relation $\vdash$ is finitely equivalential it also has an equivalential term. For if $\Delta(p,q) = \{\delta_i(p,q) : i < n\}$ is a finite set of equivalential terms for $\vdash$ then $\delta(p,q) := \bigwedge_{i<n} \delta_i(p,q)$ is an equivalential term.

For algebraizability in the Blok and Pigozzi sense we have the following.

Theorem 3.7 (Blok & Pigozzi) *Let $\vdash$ be algebraizable in the sense of Blok and Pigozzi and let $\mathcal{K}$ be the corresponding class of algebras. Then $\mathcal{K}$ is a quasi-variety and consists of all reducts of reduced matrices. Moreover, the lattice of axiomatic strengthenings is dually isomorphic to the lattice of sub-quasi-varieties of $\mathcal{K}$.*

It is to be borne in mind that there is a substantial difference between classes of matrices and classes of algebras.

3.2 Global Consequence Relations and Logics

Call a modal consequence relation **global** if the rules (MN) are derived rules. If $\vdash$ is global, then any extension contains (MN), and is also global, by structurality. Hence the lattice of global consequence relations is the lattice of extensions of $\Vdash_{\mathbf{K}}$. A modal consequence relation $\vdash$ is finitely equivalential via $p \leftrightarrow q$ iff it is global; in general, other equivalential formulae might exist, see below. A filter D for a consequence relation $\vdash$ in a modal algebra is a boolean filter. However, if $\vdash$ is global, D also satisfies $a \in D \Rightarrow \Box a \in D$ for every modality $\Box$. Such filters are called **open**. If D is open, it can be factored, and the factor algebra is unital. Hence, reduced matrices for global consequence relations have only one truth value, namely 1. It follows that truth is defined implicitly — and also explicitly via the equation $p = \top$. Thus, we can replace talk of reduced matrices with talk of algebras.

Theorem 3.8 *The lattice of global consequence relations is dually isomorphic to the lattice of quasi-varieties of modal algebras.*

Josep Font and Ramon Jansana [20] have found a way to characterize the strong consequence using the Leibniz operator. Say that a filter F on $\mathfrak{A}$ for $\vdash$ is **Leibniz** if for every $\vdash$-filter $G \subseteq F$, $\Omega_{\mathfrak{A}}(G) = \Omega_{\mathfrak{A}}(F)$. The strong consequence relation corresponding to $\vdash$ is the consequence determined by all matrices $\langle \mathfrak{A}, F\rangle$, where $\langle \mathfrak{A}, F\rangle$ is a matrix for $\vdash$ and F is a Leibniz filter. Given any filter, the largest Leibniz filter contained in F is the intersection of all filters G such that $\Omega_{\mathfrak{A}}(G) = \Omega_{\mathfrak{A}}(F)$. In the present context, this filter is the largest open filter contained in F. It consists of all elements a such that $\boxplus a \in F$ for every compound modality.

There is a difference, though, between quasi-varieties of matrices (to be considered below) and quasi-varieties of algebras. The local and global consequence relations for a logic can be characterized as follows.

Theorem 3.9 ① $\Delta \vdash_L \chi$ *iff for every generalized frame $\mathfrak{F}$ such that $\mathfrak{F} \vDash L$, every valuation β and every x: if $\langle \mathfrak{F}, \beta, x\rangle \vDash \delta$ for every $\delta \in \Delta$ then $\langle \mathfrak{F}, \beta, x\rangle \vDash \chi$.*

② $\Delta \Vdash_L \chi$ *iff for every generalized frame $\mathfrak{F}$ such that $\mathfrak{F} \vDash L$, and every valuation β: if $\langle \mathfrak{F}, \beta\rangle \vDash \delta$ for every $\delta \in \Delta$ then $\langle \mathfrak{F}, \beta\rangle \vDash \chi$*

Alternatively, $\Delta \Vdash_L \chi$ if for every algebra $\mathfrak{A} \in \mathcal{V}(L)$ and every valuation β: if $\beta(\delta) = 1$ for every $\delta \in \Delta$ then $\beta(\chi) = 1$.

$\vdash_L$ has a deduction theorem but generally, $\Vdash_L$ does not. If it does, however, the logic is weakly transitive.

Proposition 3.10 *Suppose that $\boxplus$ is a master modality for L. Then $\langle \mathfrak{F}, \beta, x\rangle \vDash \boxplus\chi$ iff χ is true in the model generated by x.*

Theorem 3.11 *$\Vdash_L$ has a deduction theorem iff L is weakly transitive.*

The notion of weak transitivity originated in the work of Wim Blok. Let $\Delta(p_0, p_1)$ be a set of terms. In weakly transitive logics, the global consequence can be reduced to the local consequence. For $\vdash_L$ is finitely equivalential if L is weakly transitive. Let $\mathrm{Cg}^{\mathfrak{A}}(a, b)$ denote the least congruence of $\mathfrak{A}$ containing the pair $\langle a, b\rangle$. Say that $\mathcal{V}$ has **elementarily definable principal congruences** if there is a first order formula $\vartheta(x, y, u, v)$ such that for all $\mathfrak{A} \in \mathcal{V}$ and $a, b, c, d \in A$, $c\ \mathrm{Cg}^{\mathfrak{A}}(a, b)\ d$ iff $\mathfrak{A} \vDash \vartheta(a, b, c, d)$. Say that $\mathcal{V}$ has **elementarily definable open filters** if there is a first order formula $\eta(x, u)$ such that for given a, c is in the open filter generated by a iff $\mathfrak{A} \vDash \eta(a, c)$. In [6] we find the following.

Theorem 3.12 *The following are equivalent.*

① *$\Vdash_L$ has a deduction theorem.*

② *L is weakly transitive.*

③ *L is finitely equivalential.*

④ *$\mathcal{V}(L)$ has elementarily definable principal congruences.*

⑤ *$\mathcal{V}(L)$ has elementarily definable open filters.*

Semisimple varieties of modal algebras are special kinds of varieties for weakly transitive logics. There is an exact characterization of semisimplicity, to be found below. Say that $\blacklozenge$ (the diamond of some compound modality $\blacksquare$) is a **dual** of $\Box$ in L if $p_0 \to \Box\blacklozenge p_0 \in L$. Frame theoretically this means that if $x\ R(\Box)\ y$ then $y\ R(\blacksquare)\ x$. If $\blacksquare$ is compound, $R(\blacksquare)$ is a finite set of finite paths in the frame. A logic is **cyclic** if every basic modality $\Box_i$ has a dual. Notice that the dual need not be basic (although a basic modality playing the role of the dual can be added conservatively). If L is cyclic, also every compound modality has a dual.

Lemma 3.13 *If L is cyclic then every finite subdirectly irreducible algebra validating L is simple.*

There are infinite algebras that are si but not finite. For example, take the set of integers and put $x \lhd y$ iff $|x-y| = 1$. Finally, let $\mathbb{O}$ be the set of finite and cofinite elements. The logic of $\mathfrak{Z} := \langle \mathbb{Z}, \lhd, \mathbb{O}\rangle$ is cyclic (with $\Diamond$ the dual of $\Box$), $\mathfrak{Z}$ is si (with opremum $\mathbb{Z} - \{0\}$), but not simple. For the set of cofinite subsets is an open filter.

Call a variety **semisimple** if every si algebra is simple. Further, say that a ternary term $t(x, y, z)$ is a **ternary discriminator** for $\mathfrak{A}$ if for all $a, b, c \in A$: $t(a, b, c) = c$ if $a = b$, and $t(a, b, c) = a$ if $a \neq b$. (See also Chapter **??** on this notion.) A variety $\mathcal{V}$ is a discriminator variety if there is a ternary term $t(x, y, z)$ which is a discriminator for all subdirectly irreducible members of $\mathcal{V}$. Notice that if t is a ternary discriminator, then $u(x) := \neg t(1, x, 0)$ has the property that $u(x) = 1$ if $x = 1$ and $u(x) = 0$ otherwise. (This is the dual notion of the one commonly used.) $u(x)$ is called a **unary discriminator**. If L is weakly transitive it has a master modality $\boxplus$. If it is also cyclic $\boxplus$ has a dual $\neg \boxtimes \neg$. We can actually assume that $\boxtimes = \boxplus$. Now look at $u(x) := \boxplus x$. By weak transitivity, the open filter generated by $a \in A$ is $\uparrow \boxplus a$. Assume that $a = \boxplus a$. Then $\boxplus \neg \boxplus a = \boxplus \neg \boxplus \neg\neg a \geq \neg a = \neg \boxplus a$, by our assumptions. So, $\uparrow \neg \boxplus a$ also is an open filter. Say that a is **open** if $a = \boxplus a$ for all basic $\boxplus$. The open elements form a boolean algebra. It follows that every si algebra is simple. It also follows that $u(x) := \boxplus x$ is a unary discriminator. The converse is much harder to establish.

Theorem 3.14 (Kracht & Kowalski) *The following are equivalent for modal logics with finitely many operators.*

(i) *$\mathcal{V}(L)$ is semisimple.*

(ii) *$\mathcal{V}(L)$ is a discriminator variety.*

(iii) *L is weakly transitive and cyclic.*

The remaining part is (i) $\Rightarrow$ (ii). Moreover, if a semisimple variety is weakly transitive, cyclicity is easy to show (because both mean that one-generated is the same as connected). So the hard part is to show that semisimple varieties are weakly transitive. We assume that the operators are $\Box_i$, $i < n$, and put

(51) $$\Box a := a \wedge \bigwedge_{i<n} \Box_i a$$

The proof is rather involved. It proceeds by first showing that all semisimple varieties of finite type of modal algebras satisfy the property (52) for $r = k$ and $l = 0$.

(52) For every $k \in \omega$ there are $r, l \in \omega$ such that $\mathcal{V} \vDash x \leq \Diamond^l \Box^k \Diamond^r x$.

Note that this is weaker than cyclicity. Now we assume that $\mathcal{V}$ satisfies (52). Define $r(i)$ to be the smallest number such that there exists an $l \in \omega$ with $\mathcal{V} \vDash \Diamond^l \Box^i \Diamond^{r(i)} x \leq x$. The function r is increasing. We define $l(i)$ to be the smallest number such that $\mathcal{V} \vDash \Diamond^{l(i)} \Box^i \Diamond^{r(i)} x \leq x$. Thus, l depends on i via $r(i)$. If $\mathcal{V}$ falsifies $\Diamond^{n+1} x = \Diamond^n x$ for each $n \in \omega$, then for each $i \in \omega$ there is a simple algebra $\mathfrak{A}_i$ in $\mathcal{V}$ and $a_i \in A_i$ such that $\Diamond^{r(i)} a_i < 1$ but $\Diamond^{r(i)+1} a_i = 1$. Now put $b_i := \neg \Diamond^{r(i)} a_i$ and fix an arbitrary $k \in \omega$. Then the following lemma holds.

Lemma 3.15 *For every $i \geq k$, we have: $\Diamond^k b_i < 1$ and $\Diamond^{l(k)+r(k)+1}\neg\Diamond^k b_i = 1$.*

Using ultraproducts one obtains an algebra $\mathfrak{B}$ and an element b such that

Lemma 3.16 *In $\mathfrak{B}$, for any $k \in \omega$ we have: $\Diamond^k b < 1$ and $\Diamond^{l(k)+r(k)+1}\neg\Diamond^k b = 1$.*

Let $\mathfrak{A} \in \mathcal{V}$ be such that there is a nonzero $a \in A$ with $\Diamond^n a < 1$ for every $n \in \omega$. For instance the free algebra $\mathfrak{Fr}_L(1)$ is such an algebra, as otherwise $\mathcal{V}$ would satisfy $\Diamond^n x = 1$ for some $n \in \omega$. Let $\alpha := \mathrm{Cg}^{\mathfrak{A}}(a, 0)$. α is neither full nor the diagonal. As α is principal, α must have a lower neighbour β in $\mathrm{Cg}(\mathfrak{A})$.

Lemma 3.17 *For every congruence β with $\beta \prec \alpha$, there is an $m \in \omega$ such that:*

(i) $\Diamond^{m+1} a \equiv_\beta \Diamond^m a$, *and*

(ii) $\neg\Diamond^m a \equiv_\beta \Diamond\neg\Diamond^m a$.

Proof. Let $\Gamma := \{\theta \in \mathrm{Cg}(\mathfrak{A}) : \theta \geq \beta, \theta \not\geq \alpha\}$. If $\Gamma = \{\beta\}$, then $\mathfrak{A}/\beta$ is si but not simple, which cannot be. Thus there is a $\theta \in \Gamma - \{\beta\}$. By congruence distributivity, $\gamma := \bigvee\Gamma \in \Gamma$. Therefore, $\mathfrak{A}/\gamma$ is subdirectly irreducible; hence simple. From this and congruence permutability it follows that $\alpha \circ \gamma = A \times A$. Thus, $(0, 1) \in \alpha \circ \gamma$, and there must be a $c \in A$ with $(0, c) \in \alpha$ and $(c, 1) \in \gamma$; hence also $(\neg c, 0) \in \gamma$. Now, $(0, c) \in \alpha$ iff for some $m \in \omega$ we have $\Diamond^m a \geq c$. Thus, $\neg\Diamond^m a \leq \neg c$ and therefore $(\neg\Diamond^m a, 0) \in \gamma$. We can then assume $c = \Diamond^m a$. By definition we have $\alpha \cap \gamma = \beta$, that is, $0/\alpha \cap 0/\gamma = 0/\beta$. Now, to prove (i), consider $\Diamond^{m+1} a \wedge \neg\Diamond^m a$. It belongs to $0/\alpha \cap 0/\gamma = 0/\beta$ and thus we obtain $\Diamond^{m+1} a \equiv_\beta \Diamond^m a$. Then, for (ii), consider $\Diamond^m a \wedge \Diamond\neg\Diamond^m a$. It too belongs to $0/\alpha \cap 0/\gamma = 0/\beta$; therefore $\neg\Diamond^m a \equiv_\beta \Diamond\neg\Diamond^m a$. □

Theorem 3.18 *If $\mathcal{V}$ satisfies (52) then $\mathcal{V}$ satisfies $\Diamond^{n+1} x = \Diamond^n x$ for some $n \in \omega$.*

Proof. Suppose $\mathcal{V}$ falsifies $\Diamond^{n+1} x = \Diamond^n x$ for all $n \in \omega$. There is then an algebra $\mathfrak{B} \in \mathcal{V}$ and an element $b \in B$ such that for all $k \in \omega$: $\Diamond^k b < 1$ and $\Diamond^{l(k)+r(k)+1}\neg\Diamond^k b = 1$. Let α be the congruence generated by $\neg b$, and take β and m as in Lemma 3.17. Then $\neg\Diamond^m b \equiv_\beta \Diamond\neg\Diamond^m b \equiv_\beta \Diamond^{l(m)+r(m)+1}\neg\Diamond^m b = 1$. Thus, $\Diamond^m b \equiv_\beta 0$ and therefore $b \equiv_\beta 0$. It follows that $\beta \geq \alpha$, contradicting the choice of β as a subcover of α. □

4 Reduction to Monomodal Logic

For each cardinality κ, there is a distinct lattice of modal consequence relations over κ operators. Surely, it would be most advantageous if one did not have to study these lattice for each individual κ. While results for the lattice $\mathrm{Ext}(\vdash_{\mathbf{K}_\kappa})$ are yet to be established, there exists fairly powerful theorems that reduce the study of $\mathrm{NExt}(\mathbf{K}_\kappa)$ for finite κ to the study of $\mathrm{NExt}(\mathbf{K}_1)$. It turns out that the locales of logics for several operators are isomorphic to certain subintervals of the locale $\mathrm{NExt}(\mathbf{K}_1)$ and that the isomorphism reflects and preserves many important properties of logic. This means that from a general perspective it is enough to obtain results for the locale of monomodal logics. The theorem that asserts this is called a *transfer theorem.* Results on monomodal logics can be extended to polymodal logics, using the transfer theorem. In practice it has turned out to be the opposite, however. Often, a counterexample to a specific conjecture can be easily constructed using several operators. Using the transfer theorem this counterexample typically yields a counterexample for monomodal logic, and so for every polymodal logic. There are certain lacunae in the theory. First, although there is a simulation of countably many operators by one (see [36]), the induced lattice map is not surjective. As for uncountably many operators, no results seem to exist. The techniques have been applied to polyadic operators and hybrid logics, and we report the results below. Again, the lattice map is not surjective, making the transfer theory less effective. Third, as we have mentioned above, the results cover logics only; no attempt has been made to reduce polymodal consequence relations to monomodal ones, though I speculate that the results will be similar.

4.1 Simulating Two Operators by One

Let $\langle F, \lhd, \blacktriangleleft, \mathbb{F}\rangle$ be a generalized bimodal frame. For a subset $B \subseteq F$ put $B_\circ := \{x_\circ : x \in B\}$ and $B_\bullet := \{x_\bullet : x \in B\}$. Here, $x_\circ$ and $x_\bullet$ are distinct copies of x, $F_\circ$ and $F_\bullet$ are disjoint and do not contain $*$.

$$
\begin{aligned}
&(53) \qquad & F^s :=& F_\circ \cup F_\bullet \cup \{*\} \\
&(54) \qquad & \leqslant := \quad & \{\langle x_\circ, y_\circ\rangle : x \lhd y\} \cup \{\langle x_\bullet, y_\bullet\rangle : x \blacktriangleleft y\} \\
&& & \cup \{\langle x_\circ, x_\bullet\rangle : x \in F\} \cup \{\langle x_\bullet, x_\circ\rangle : x \in F\} \\
&& & \cup \{\langle x_\circ, *\rangle : x \in F\} \\
&(55) \qquad & \mathbb{F}^s :=& \{B_\circ \cup C_\bullet \cup D : B, C \in \mathbb{F}, D \subseteq \{*\}\}
\end{aligned}
$$

This is a general monomodal frame. We call it the **simulating frame** of $\mathfrak{F}$. Recall that a general frame $\mathfrak{F}$ is **differentiated** if $x \neq y$ implies $x \in a$ and $y \notin a$ for some $a \in \mathfrak{F}$; that $\mathfrak{F}$ is **refined** if it differentiated and if $x \not\lhd_i y$ then there is an $a \in \mathbb{F}$ such that $x \in \Box_i a$ but $y \notin a$. Finally, $\mathfrak{F}$ is **compact** if for every filter H on $\mathbb{F}$: $\bigcap H \neq \varnothing$.

Proposition 4.1 *$\mathfrak{F}^s$ is differentiated (refined, compact) iff $\mathfrak{F}$ is.*

Proof. Notice that $F_\circ$, $F_\bullet$ and $\{*\}$ are definable by the constant formulae $\gamma_\circ := \Diamond\boxminus\bot$, $\gamma_\bullet := \neg\Diamond\boxminus\bot$, and $\gamma_* := \boxminus\bot$, respectively. (We shall also denote the sets defined by some formula by the formula itself.) Hence if $\mathfrak{F}$ is differentiated, and let $x, y \in F^s$ be different. Then if $x = *$, γ_* is the set that contains x but not y. Otherwise if $x = u_\circ$ and $y = v_\bullet$, $\gamma_\circ$ contains x but not y. Finally, if $x = u_\circ$ and $y = v_\circ$ then $u \neq v$ and there is a set O containing x but not y. Then $x \in O_\circ$, but $y \notin O_\circ$. If $\mathfrak{F}^s$ is differentiated, $\mathfrak{F}$ is differentiated. We show that if $\mathfrak{F}$ is refined, so is $\mathfrak{F}^s$. Suppose that $x \leqslant y$ does not hold. The case $x = *$ is easily dealt with. Now assume $x = u_\circ$. We deal with two representative cases. Case 1. $y = v_\circ$. Then $u \not\lhd v$. Then tightness of $\mathfrak{F}$ gives a set O such that $u \in \Box O$ but $v \notin o$. Then $x \in \boxminus(F_\bullet \cup \{*\} \cup O_\circ)$ but $y \notin O_\circ$. Case 2. $y = v_\bullet$. Then $u \neq v$. Then by differentiatedness there is a set O containing u but not v. Then $x = u_\circ \in \boxminus(F_\circ \cup \{*\} \cup O_\bullet)$ but $y \notin O_\bullet$. (Notice that for the transfer of tightness we needed differentiatedness as well.) Transfer of compactness is straightforward. $\Box$

The notion of simulation is then also defined for Kripke-frames. Denote by $\mathfrak{F}_\sharp$ the Kripke-frame underlying $\mathfrak{F}$.

Proposition 4.2 $(\mathfrak{F}^s)_\sharp = (\mathfrak{F}_\sharp)^s$.

Define

$$
(56) \qquad \boxminus_\circ \chi := \boxminus(\gamma_\circ \to \chi) \qquad \boxminus_\bullet \chi := \boxminus(\gamma_\bullet \to \chi) \qquad \boxminus_* \chi := \boxminus(\gamma_* \to \chi)
$$

Also, put

$$
(57) \qquad
\begin{aligned}
p_i^s &:= p_i \\
(\neg\varphi)^s &:= \neg(\gamma_\circ \wedge \varphi)^s \\
(\varphi \wedge \chi)^s &:= \varphi^s \wedge \chi^s \\
(\Box\varphi)^s &:= \boxminus_\circ \varphi^s \\
(\blacksquare\varphi)^s &:= \boxminus_\bullet \boxminus_\bullet \boxminus_\circ \varphi^s
\end{aligned}
$$

Finally, let β be a valuation and set $\beta^s(p) := \beta(p)_\circ$. Then the following is shown by induction.

$$
(58) \qquad \langle \mathfrak{F}, \beta, x\rangle \vDash \varphi \Leftrightarrow \langle \mathfrak{F}^s, \beta^s, x_\circ\rangle \vDash \gamma_\circ \to \varphi^s
$$

Notice that $\langle \mathfrak{F}^s, \beta^s, x_\bullet \rangle \vDash \gamma_\circ \to \varphi^s$ as well as $\langle \mathfrak{F}^s, \beta^s, * \rangle \vDash \gamma_\circ \to \varphi^s$. Not every valuation into $\mathfrak{F}^s$ is of the form β^s. However, if $\gamma(p) \cap F_\circ = \delta(p) \cap F_\circ$ then for every $x \in F^s$

$$\langle \mathfrak{F}^s, \gamma, x \rangle \vDash \gamma_\circ \to \varphi^s \Leftrightarrow \langle \mathfrak{F}^s, \delta, x \rangle \vDash \gamma_\circ \to \varphi^s \tag{59}$$

Proposition 4.3 *Let $\mathfrak{F}$ be a bimodal generalized frame. $\mathfrak{F}^s \vDash \varphi^s$ iff $\mathfrak{F} \vDash \varphi$.*

Put

$$\varphi \to \Delta := \{\varphi \to \delta : \delta \in \Delta\} \tag{60}$$

We define **Sim** to be the logic of all simulating frames.

Definition 4.4 *Let L be a bimodal logic. Then L^s is the logic of all $\mathfrak{F}^s$, where $\mathfrak{F}$ is a general frame for L.*

Theorem 4.5 *Let L be a bimodal logic. Then*

$$\Delta \vdash_L \varphi \quad \Leftrightarrow \quad \gamma_\circ \to \Delta^s \vdash_{L^s} \gamma_\circ \to \varphi^s \tag{61}$$

$$\Delta \Vdash_L \varphi \quad \Leftrightarrow \quad \gamma_\circ \to \Delta^s \Vdash_{L^s} \gamma_\circ \to \varphi^s \tag{62}$$

In particular, if $L = \mathbf{K}_2 \oplus \Delta$, $L^s = \mathbf{Sim} \oplus (\gamma_\circ \to \Delta)$.

The previous result shows that the bimodal consequence is reduced to the consequence relation based on the 'white point' of the simulating frame.

4.2 Algebraic Properties of the Simulation

Let $p : \mathfrak{F} \to \mathfrak{G}$ be a p-morphism. Define p^s by $p^s(x_\circ) := p(x)_\circ$ and $p^s(x_\bullet) := p(x)_\bullet$, and $p^s(*) := *$. It is easy to see that p^s is a p-morphism from $\mathfrak{F}^s$ to $\mathfrak{G}^s$. Conversely, let $q : \mathfrak{F}^s \to \mathfrak{G}^s$ be a p-morphism. Then $q(*) = *$, $q[F_\circ] \subseteq G_\circ$ and $q[F_\bullet] \subseteq G_\bullet$, since all sets are definable by constant formulae. Next, if $q(x_\circ) = y_\circ$ then also $q(x_\bullet) = y_\bullet$, so that q is completely defined by its action on $F_\circ$. Moreover, $q = p^s$ for some p-morphism $p : \mathfrak{F} \to \mathfrak{G}$. So, the simulation is faithful with respect to embeddings and contractions.

Notice that $(\mathfrak{F} \oplus \mathfrak{G})^s$ is not isomorphic to $\mathfrak{F}^s \oplus \mathfrak{G}^s$ (the former is connected, the latter is not). However, the two are not so different. Basically, the former has two points satisfying $\boxminus\bot$, the latter only one. Thus only the latter can be a simulation frame.

Lemma 4.6 *$(\bigoplus_{i \in I} \mathfrak{F}_i)^s$ is a contraction of $\bigoplus_{i \in I} \mathfrak{F}^s$. The contraction is the one which collapses all points satisfying $\boxminus\bot$ into one.*

The construction can be remodeled algebraically. Let $\mathfrak{A}$ be a bimodal algebra.

$$A^s := A \times A \times \{0, 1\} \tag{63}$$

$$\diamondminus \langle a, b, c \rangle := \begin{cases} \langle \Diamond a \cup b, \blacklozenge b \cup a, 0 \rangle & \text{if } c = 0 \\ \langle A, \blacklozenge b \cup a, 0 \rangle & \text{if } c = 1 \end{cases} \tag{64}$$

$\mathfrak{A}^s$ is the simulating algebra for $\mathfrak{A}$. It is easy to verify that if $\mathfrak{A}$ is the algebra of subsets of $\mathfrak{F}$, $\mathfrak{A}^s$ is the algebra of subsets of $\mathfrak{F}^s$, and conversely. If $h : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism, so is $h^s : \mathfrak{A}^s \to \mathfrak{B}^s$. Moreover, if $q : \mathfrak{A}^s \to \mathfrak{B}^s$ is a homomorphism, then $q = p^s$ for some $p : \mathfrak{A} \to \mathfrak{B}$. So, we have an isomorphism of the category of bimodal algebras and of the category of simulation algebras.

We are interested in the varieties generated by simulation algebras. $\mathcal{V}(\mathbf{Sim})$ is the variety generated by all simulation algebras. It is easier to look at the frames. Take a nonempty generated subframe $\mathfrak{G}$ of $\mathfrak{F}^s$. It is easy to see that it must be of the form $\mathfrak{H}^s$. Simply take $H := G \cap F_\circ$.

However, the empty subframe is not of that form. So, with the exception of the empty frame every subframe of $\mathfrak{F}^s$ is a simulation frame. It follows that $\mathrm{Cg}(\mathfrak{A}^s) \cong \mathrm{Cg}(\mathfrak{A}) + 1$, where the latter denotes the addition of a new top element to $\mathrm{Cg}(\mathfrak{A})$.

Proposition 4.7 *$\mathfrak{A}$ is subdirectly irreducible iff $\mathfrak{A}^s$ is.*

Next, let $p : \mathfrak{F}^s \to \mathfrak{G}$ be a contraction. It is easy to see that $p(x_\circ) = p(y_\bullet)$ cannot hold; also, $p(x_\circ) \neq p(*) \neq p(x_\bullet)$. Moreover, $p(x_\circ) \leqslant p(y_\bullet)$ iff $x = y$ iff $p(y_\bullet) \leqslant p(x_\circ)$; and if $p(x_\circ) = p(y_\circ)$ then $p(x_\bullet) = p(y_\bullet)$, and conversely. So, $\mathfrak{G} = \mathfrak{H}^s$ for some $\mathfrak{H}$. It follows that $\mathrm{Sub}(\mathfrak{A}^s) \cong \mathrm{Sub}(\mathfrak{A})$. Finally, we have noticed that $(\prod_{i \in I} \mathfrak{A}_i)^s \in \mathsf{P}(\prod_{i \in I} \mathfrak{A}_i^s)$.

Now, if $\mathcal{K}$ is a class of bimodal algebras, denote by $\mathcal{K}^\sigma := \{\mathfrak{A}^s : \mathfrak{A} \in \mathcal{K}\}$. Also, denote by $\mathcal{K}_{si}$ the class of subdirectly irreducible members of $\mathcal{K}$.

Proposition 4.8 *If $\mathcal{V}$ is a variety of bimodal algebras, $(\mathcal{V}_{si})^\sigma = (\mathcal{V}^\sigma)_{si}$.*

Now let $\mathcal{V}^s$ be the variety generated by $\mathcal{V}^\sigma$. For any variety generated by simulation algebras, the subdirectly irreducible members are simulation algebras. Hence, any subvariety of $\mathcal{V}(\mathbf{Sim})$, with the exception of the trivial variety, is of the form $\mathcal{V}^s$.

Theorem 4.9 *The map $\mathcal{V} \mapsto \mathcal{V}^s$ is an isomorphism from the lattice of varieties of bimodal algebras onto the lattice of nontrivial subvarieties of $\mathcal{V}(\mathbf{Sim})$.*

We now turn to the axiomatization of the simulations. Put

$$\begin{aligned} &*(x) := \neg(\exists y)(x \leqslant y) \\ (65) \qquad &\circ(x) := (\exists y)(x \leqslant y \wedge *(y)) \\ &\bullet(x) := \neg * (x) \wedge \neg \circ (x) \end{aligned}$$

A monomodal frame is a simulation frame iff it satisfies the following elementary formulae (here, $\exists!$ is short for: 'there exists exactly one'):

(66a) $\quad (\forall x)(\circ(x) \to (\exists! y)(x \leqslant y \wedge \bullet(y)))$

(66b) $\quad (\forall x)(\bullet(x) \to (\exists! y)(x \leqslant y \wedge \circ(y)))$

(66c) $\quad (\exists! x)(*(x))$

(66a) and (66b) are modally definable, but (66c) is not. It turns out that the class of **Sim**-frames is the class of frames satisfying (66a) and (66b). The axiomatization can be derived from the correspondence between first-order and modal formulae. Moreover, **Sim** is R-persistent.

4.3 Unsimulation

Let $\mathfrak{F}$ be a **Sim**-frame. We can identify the sets F° of points satisfying $\circ(x)$, and $F^\bullet$ of points satisfying $\bullet(x)$. If $x \in F^\circ$ let $x^\dagger$ be the unique successor in $F^\bullet$, and if $x \in F^\bullet$ then let $x^\dagger$ be the unique successor in F°. Now put $\mathfrak{F}_s := \langle F_s, \lhd, \blacktriangleleft, \mathbb{F}_s \rangle$, where

(67a) $\quad F_s := F^\circ$

(67b) $\quad \lhd := \leqslant \cap F_s^2$

(67c) $\quad \blacktriangleleft := \{\langle x^\dagger, y^\dagger \rangle : x \leqslant y, x, y, \in F_\bullet\}$

(67d) $\quad \mathbb{F}_s := \{a \in \mathbb{F} : a \subseteq F_s\}$

Hence, any logic containing **Sim** is complete with respect to simulation frames.

Proposition 4.10 *Let $\mathfrak{M}$ be a monomodal frame, $\mathfrak{B}$ a connected bimodal frame. Then $\mathfrak{M} \cong (\mathfrak{M}_s)^s$ and $\mathfrak{B} \cong (\mathfrak{B}^s)_s$.*

For each variable p we introduce three variables $p_\circ$, $p_\bullet$, p_*. We call the new set the **extended set of variables**. Think of $p_\circ$ as 'p is true at the region of $\gamma_\circ$ worlds'; $p_\bullet$ as 'is true at the region of $\gamma_\bullet$ worlds' and p_* as 'is true at the region of γ_* worlds'. For formulae χ we define the formulae $\chi_\circ$, $\chi_\bullet$ and χ_* by mutual recursion (and think of them as interpreted in the same way as the new variables).

$$
\begin{array}{llll}
 & (\neg\varphi)_\alpha & := & \neg\varphi_\alpha & \alpha \in \{\bullet, \circ *\} \\
 & (\varphi \wedge \chi)_\alpha & := & \varphi_\alpha \wedge \chi_\alpha & \alpha \in \{\bullet, \circ, *\} \\
(68) & (\Diamond\varphi)_\circ & := & \varphi_\bullet \vee \Diamond(\varphi_\circ) \vee \varphi_* & \\
 & (\Diamond\varphi)_\bullet & := & \varphi_\circ \vee \blacklozenge\varphi_\bullet & \\
 & (\Diamond\varphi)_* & := & \bot &
\end{array}
$$

Let β be a valuation on $\mathfrak{F}^s$. Define β_s on $\mathfrak{F}$ so that for all variables p (assuming $x_* = x$ and $\alpha \in \{\circ, \bullet, *\}$):

(69) $\langle \mathfrak{F}, \beta_s, x\rangle \vDash p_\alpha \quad \Leftrightarrow \quad \langle \mathfrak{F}^s, \beta, x_\alpha\rangle \vDash p$

Then it is established by induction on the formulae that

(70) $\langle \mathfrak{F}^s, \beta, x_\alpha\rangle \vDash \varphi \quad \Leftrightarrow \quad \langle \mathfrak{F}, \beta_s, x\rangle \vDash \varphi_\alpha \quad \Leftrightarrow \quad \Leftrightarrow \langle \mathfrak{F}^s, \beta, x_\alpha\rangle \vDash \varphi^s_\circ(p/p_\circ, \Diamond_\bullet p/p_\bullet, \Diamond_* p/p_*)$

Now, every valuation on $\mathfrak{F}$ of the extended set of variables is of the form β_s on $\mathfrak{F}^s$. Thus we obtain $\langle \mathfrak{F}^s, x_\alpha\rangle \vDash \varphi$ iff $\langle \mathfrak{F}, x\rangle \vDash \varphi_\alpha$, for every $\alpha \in \{\circ, \bullet, *\}$. Finally, this gives

(71) $\mathfrak{F}^s \vDash \varphi \quad \Leftrightarrow \quad \mathfrak{F} \vDash \gamma_\circ \to \varphi_\circ; \gamma_\bullet \to \varphi_\bullet; \gamma_* \to \varphi_*$

Therefore, let

(72) $\varphi_s := (\gamma_\circ \to \varphi_\circ) \wedge (\gamma_\bullet \to \varphi_\bullet) \wedge (\gamma_* \to \varphi_*)$

Theorem 4.11 *Let $L = \mathbf{Sim} \oplus \Delta$ be consistent. Put $L_s := \mathbf{K}_2 \oplus \Delta_s$. Then $(L_s)^s = L$. Additionally,*

(73) $\Delta \vdash_L \varphi \quad \Leftrightarrow \quad \Delta_s \vdash_{L_s} \varphi_s \qquad$ *and* $\qquad \Delta \Vdash_{L_s} \varphi \quad \Leftrightarrow \quad \Delta_s \Vdash_{L_s} \varphi_s$

Proposition 4.12 Th $\mathfrak{F}^s = \mathbf{Sim} \oplus (\text{Th } \mathfrak{F})^s$.

4.4 The Main Theorem

Let StSim be the category of differentiated monomodal Sim-frames, with γ_* containing a single point. The morphisms are the p-morphisms. Let Dif_2 be the category of differentiated bimodal frames with p-morphisms as maps.

Theorem 4.13 StSim *and* Dif_2 *are naturally equivalent. The map* $(-)_s$ *is a functor from* StSim *to* Dif_2*,* $(-)^s$ *a functor from* Dif_2 *to* StSim*. Moreover, there is a natural transformation from the identity on* Dif_2 *to* $((-)^s)_s$ *and a natural transformation from the identity on* StSim *to* $((-)_s)^s$*.*

From Theorem 4.5 and 4.11 follows that $L \mapsto L^s$ preserves and reflects finite and recursive axiomatizability. Moreover, decidability in a bimodal logic L can be translated using LOGSPACE into decidability in L^s; and similarly for monomodal logics. The complexity of the problems is therefore preserved and reflected.

Next, if $L = \mathrm{Th}\,\mathcal{K}$ then $L^s = \mathrm{Th}\,\mathcal{K}^s$ and conversely. It follows that completeness, finite model property, tabularity are preserved and reflected.

Second, suppose that L is Df-persistent. Let $\mathfrak{M}$ be a differentiated monomodal frame for L^s. Then $\mathfrak{M}_s$ is differentiated and $\mathfrak{M} \cong (\mathfrak{M}_s)^s$, which is therefore differentiated. It follows that $\mathfrak{M}_s \vDash L$, so that the underlying Kripke-frame $(\mathfrak{M}_s)_\sharp$ is an L^s-frame. Since $(\mathfrak{M}_s)_\sharp \cong (\mathfrak{M}_\sharp)_s$, we have $\mathfrak{M}_\sharp \vDash L^s$. Similarly for R-persistence.

Now we turn to interpolation. Using the algebraic characterization of interpolation (Theorem 5.4) the preservation and reflection of interpolation is actually straightforward to show. There are also direct ways. Suppose that the bimodal logic L has interpolation. Now let $\varphi \vdash_{L^s} \psi$. We know that for every formula ρ,

$$(74) \qquad \gamma_\circ \vdash_{L^s} \rho \leftrightarrow \sigma(\rho_s)^s$$

where $\sigma : p_\circ \mapsto p, p_\bullet \mapsto \Diamond_\bullet p, p_* \mapsto \Diamond_* p$. Then $\gamma_\circ \to \varphi \vdash_{L^s} \gamma_\circ \to \psi$, and so by Theorem 4.5, $\varphi_s \vdash_L \psi_s$. There exists a formula χ such that $\mathrm{var}(\chi) \subseteq \mathrm{var}(\varphi_s) \cap \mathrm{var}(\psi_s)$ and $\varphi_s \vdash_L \chi \vdash_L \psi_s$. Now, χ is in the variables $p_\circ, p_\bullet, p_*$ for $p \in \mathrm{var}(\varphi) \cap \mathrm{var}(\psi)$, and this applies as well to χ^s. Furthermore,

$$(75) \qquad \gamma_\circ \to (\varphi_s)^s \vdash_{L^s} \chi^s \vdash_{L^s} \gamma_\circ \to (\psi_s)^s$$

so that

$$(76) \qquad \gamma_\circ \to \varphi \equiv \gamma_\circ \to \sigma((\varphi_s)^s) \vdash_{L^s} \sigma(\chi^s) \vdash_{L^s} \gamma_\circ \to \sigma((\psi_s)^s) \equiv \gamma_\circ \to \sigma((\psi_s)^s)$$

Put $\chi^\circ := \gamma_\circ \to \sigma(\chi^s)$. Likewise formulae χ' and χ'' can be found such that

$$(77) \qquad \gamma_\bullet \to \varphi \vdash_{L^s} \gamma_\bullet \to \sigma(\chi'^s) \vdash_{L^s} \gamma_\bullet \to \psi$$

$$(78) \qquad \gamma_* \to \varphi \vdash_{L^s} \gamma_* \to \sigma(\chi''^s) \vdash_{L^s} \gamma_* \to \psi$$

Then $\chi^\bullet := \gamma_\bullet \to \Diamond_\circ \sigma(\chi'^s)$, and $\chi^* := \gamma_* \to \sigma(\chi'')$. Then $\chi^\circ \wedge \chi^\bullet \wedge \chi^*$ is the desired interpolant. The proof works analogously for global interpolation and transfer of local and global Halldén-completeness.

Now look at Sahlqvist formulae. By a theorem of [37] a modal logic is Sahlqvist iff it can be axiomatized by formulae of the form $\varphi \to \psi$, where ψ and φ is composed from compound modalities using only $\wedge$, $\vee$ and diamonds. (Compound modalities are also called **strongly positive**.) From this it follows immediately that if $\varphi \to \psi$ is Sahlqvist, so is $(\varphi \to \psi)^s = \varphi^s \to \psi^s$. (The original formulation allows a prefix of boxes but this does not define a larger class of logics.) The converse is similar. We have $(\varphi \to \psi)_s = \varphi_s \to \psi_s$, and the unsimulation translates boxes into boxes and diamonds into diamonds. Finally, note that simulation and unsimulation commute with ultraproducts.

Theorem 4.14 (Kracht & Wolter) *The map $L \mapsto L^s$ is an isomorphism from the locale of bimodal logics onto the interval* $[\mathbf{Sim}, \mathrm{Th}\, \boxed{\bullet}\,]$ *in the locale of monomodal logics. Moreover, the following properties of logics are invariant under this map:*

(i) *decidability, PSPACE-computability,*

(ii) *elementarity, Df-persistence, R-persistence, being Sahlqvist,*

(iii) *finite model property, completeness, compactness,*

(iv) *local and global interpolation.*

Halldén-completeness is actually *not* preserved under simulation. For example, the logic $\mathbf{D} \otimes \mathbf{D}$ is Halldén-complete (being the fusion of two Halldén-complete logics). However, its simulation has more than two constants, so it cannot be Halldén-complete (see below Theorem 5.10).

In general, for $\kappa \in \omega$ there is a similar isomorphism from $\mathrm{NExt}(\mathbf{K}_\kappa)$ onto an interval $[\mathbf{Sim}_\kappa, \mathrm{Th}(\mathfrak{Ch}_{\kappa-1})]$, where $\mathbf{Sim}_\kappa$ is the simulating logic for κ-modal frames, and $\mathfrak{Ch}_{\kappa-1} = \langle\{0, 1, \ldots, \kappa - 1\}, \prec\rangle$ $i \prec j$ iff

$j = i+1$ if the chain of $\kappa - 1$ many points. The underlying set is $F \times \{0, 1, \ldots, \kappa - 1\}$ plus the points of the chain, and we put $\langle x, i\rangle \leqslant \langle y, j\rangle$ iff (a) $x = y$ or (b) $i = j$ and $x \lhd_i y$. Additionally, every point $\langle x, i\rangle$ sees the point $i - 1$. Finally, $i \leqslant j$ iff $i \prec j$. All aforementioned properties are invariant under this simulation. For countable κ, [36] describes an embedding of $\mathrm{NExt}\,\mathbf{K}_\kappa$ into (not necessarily onto) an interval in $\mathrm{NExt}\,\mathbf{K}_1$.

4.5 Simulating Polyadic and Nonstandard Operators by Monadic Operators

Assume that L is a complete logic. Now add two modal operators $\boxminus$ and $\boxbar$, together with the following axioms: **G3** for $\boxbar$, **K4.3** for $\boxminus$, $p \to \boxminus\Diamond p$, $p \to \boxbar\Diamond p$ and $(\boxminus p \wedge \boxbar p \wedge p) \to \boxplus p$ for every basic modality $\boxplus$. Thus, $\boxminus$ and $\boxbar$ are tense duals, the relation for $\boxminus$ is a well-order, and if $x\ R(\boxplus)\ y$ for any compound modality $\boxplus$ then either $x = y$ or $x\ R(\boxminus)\ y$ or $y\ R(\boxminus)\ x$. Call this logic L^w. The difference operator of [16,51] and the universal modality are now definable on all connected frames by

(79) $\qquad [\neq]\chi := \boxminus\chi \wedge \boxbar\chi \qquad [u]\chi := \chi \wedge [\neq]\chi$

Also, [2] has introduced a logic using a special type of variables, called **nominals**, which must be interpreted by singleton sets. Logics that admit both standard variables and nominals are called **hybrid**, see Chapter **??** of this handbook. It turns out that with the difference modality the standard languages have the same expressive power as the hybrid ones. Consider a variable p. Put

(80) $\qquad n(p) := [u](p \leftrightarrow [\neq]\neg p)$

β satisfies $n(p)$ on a Kripke-frame iff the value of p is a singleton set. ([28] call $Op := p \wedge [\neq]p$ the 'only' operator. It says that p is true 'only here'.) In absence of the difference modality, the nominals give extra expressive power. Consider the operator $\blacksquare$ defined by the following axiom:

(81) $\qquad n(p) \to ((\blacklozenge p \to \Box\neg p) \wedge (\Diamond p \to \blacksquare\neg p))$

Then a frame satisfies this axiom iff $\blacktriangleleft$ is the complement of $\lhd$ (the inaccessibility relation of [33]). Notice that the following holds.

Theorem 4.15 (Gargov & Goranko) *A class of frames is definable using the language with nominals and the universal modality iff it is definable using the difference operator.*

Recall the characterization of the first-order properties axiomatizable by means of Sahlqvist-formulae. Using the inaccessibility relation and the universal modality we can not only express unrestricted quantification (on connected frames) but also negative formulae. This means that all first-order conditions over binary relations are now expressible (over the logic of these structures with enough modal operators) in which an atomic formula contains at least one universally quantified variable whose quantifier is not in the scope of an existential. This last restriction can be circumvented through the introduction of new modalities (to mimic the Skolem functions) on condition that the variable depends only on one other variable. All these codings proceed by adding more operators, not more points. For example, **ZFC** without foundation can be so axiomatized, see [36]. The infinity axiom can be expressed much more succinctly than in that paper. Simply require

(82) $\qquad (\exists x)(\varnothing \in x \wedge (\forall y)(y \in x \to (\exists z)(y \in z \in x)))$

The outer existential can be massaged away by introducing a constant, and the second existential can be dealt with using a Skolem function. Foundation of course is axiomatizable using the **G**-axioms for $\ni^+$. Even full class comprehension is axiomatizable.

If one is interested in simulating polyadic operators then it is not enough to just add relations (similarly if one wants to simulate predicate logic with a signature containing at least ternary relations symbols). One approach was outlined in [38]. A better one is presented in [27]. It is enough to look at the case of a single binary modal operator ∇. **Kripke-frames** are pairs $\langle F, R\rangle$ where R is a ternary relation. A **generalized frame** is a triple $\mathfrak{F} = \langle F, R, \mathbb{F}\rangle$ where $\mathbb{F} \subseteq \wp(F)$ is a field of sets closed under

$$\mu_R(A_1, A_2) := \{x \in F : (\exists v_1 \in A_1)(\exists v_2 \in A_2)R(x, v_1, v_2)\} \tag{83}$$

Satisfaction of a formula is defined as follows.

$$\langle \mathfrak{F}, \beta, x\rangle \vDash \nabla(\varphi_1, \varphi_2) \quad \Leftrightarrow \quad \text{there are } v_1, v_2 \text{ such that } R(x, v_1, v_2) \text{ and } \langle \mathfrak{F}, \beta, v_1\rangle \vDash \varphi_1; \langle \mathfrak{F}, \beta, v_2\rangle \vDash \varphi_2 \tag{84}$$

For a triple $\boldsymbol{x}$, let $\boldsymbol{x}_i$ be the ith component of $\boldsymbol{x}$. Given $\mathfrak{F}$, assume $F \cap R = \varnothing$ and put

$$\begin{aligned} F^\bullet &:= F \cup R \\ R_i &:= \{\langle \boldsymbol{x}, \boldsymbol{x}_i\rangle : \boldsymbol{x} \in R\} \\ S &:= \{\langle \boldsymbol{x}_0, \boldsymbol{x}\rangle : \boldsymbol{x} \in R\} \\ \mathbb{F}^\bullet &:= \{a \cup (R \cap \textstyle\bigcup_{i<n} a_i \times b_i \times c_i) : a, a_i, b_i, c_i \in \mathbb{F}, n < \omega\} \\ \mathfrak{F}^\bullet &:= \langle F^\bullet, S, R_0, R_1, R_2, \mathbb{F}^\bullet\rangle \end{aligned} \tag{85}$$

$\mathfrak{F}^\bullet$ is a general frame and $\mathbb{F}^\bullet$ is generated by $\mathbb{F}$. The set R is definable by $\mu := \langle R_0\rangle\top$. The set F in $F^\bullet$ is called the set of **base points**. It too is definable. The simulation is now defined as follows.

$$\begin{aligned} p^\bullet &:= p \\ (\neg\varphi)^\bullet &:= \neg\varphi^\bullet \\ (\varphi_1 \wedge \varphi_2)^\bullet &:= \varphi_1^\bullet \wedge \varphi_2^\bullet \\ (\nabla(\varphi_1, \varphi_2))^\bullet &:= \langle S\rangle(\langle R_1\rangle\varphi_1^\bullet \wedge \langle R_2\rangle\varphi_2^\bullet) \end{aligned} \tag{86}$$

The translation $(\cdot)^\bullet$ preserves truth of formulae at base points. So we get

Proposition 4.16 $\mathfrak{F} \vDash \varphi \quad \Leftrightarrow \quad \mathfrak{F}^\bullet \vDash \neg\mu \to \varphi^\bullet$.

We remark that $\mathfrak{F}^\bullet$ is differentiated (descriptive) iff $\mathfrak{F}$ is.

Unsimulation is less straightforward. Let $\langle M, S, R_0, R_1, R_2, \mathbb{M}\rangle$ be a monadic frame. Put

$$\begin{aligned} M_\bullet &:= \{x : x \vDash \neg\mu\} \\ T &:= \{\boldsymbol{x} : \exists v : v\, R_0\, \boldsymbol{x}_0, v\, R_1\, \boldsymbol{x}_1 \text{ and } v\, R_2\, \boldsymbol{x}_2\} \\ \mathbb{M}_\bullet &:= \{a \cap M_\bullet : a \in \mathbb{M}\} \end{aligned} \tag{87}$$

It is straightforward to check that for a ternary frame $\mathfrak{F} \cong (\mathfrak{F}^\bullet)_\bullet$. A useful observation is this.

Proposition 4.17 *Let L be a dyadic logic and $\mathcal{K}$ a class of* $\mathbf{Sim}^\bullet$*-frames. If $L^\bullet$ is complete with respect to $\mathcal{K}$ then L is complete with respect to $\mathcal{K}_\bullet$.*

Now let $\mathbf{Sim}^\bullet$ be the logic of general frames of the form $\mathfrak{F}^\bullet$. The simulation map sends a dyadic logic L to the logic

$$L^\bullet := \mathbf{Sim}^\bullet \oplus \{\neg\mu \to \varphi^\bullet : \varphi \in L\} \tag{88}$$

This map turns out to be a lattice homomorphism. It is injective but not surjective, unlike in the monadic case (in finite signature). Given an extension of $\mathbf{Sim}^\bullet$, let $L_\bullet$ be the logic generated by all formulae valid on all unsimulations of descriptive L-frames.

Proposition 4.18 *The following holds for a dyadic logic L and an extension M of* $\mathbf{Sim}^\bullet$.

① $M_\bullet = \{\varphi : \neg\mu \to \varphi^\bullet \in M\}$.

② $L \subseteq M_\bullet$ *iff* $L_\bullet \subseteq M$.

③ $(M_\bullet)^\bullet \subseteq M$.

④ $(L^\bullet)_\bullet = L$.

The following is shown in [27].

Theorem 4.19 (Goguadze & Piazza & Venema) *The map $L \mapsto L^\bullet$ is a lattice homomorphism into the lattice* $\mathrm{NExt}(\mathbf{Sim}^\bullet)$. *It preserves and reflects*

- *finite and recursive axiomatizability,*
- *completeness, finite model property, tabularity,*
- *canonicity,*
- *first-order definability.*

It preserves Sahlqvist axiomatizability and it reflects decidability.

5 Interpolation

This section uses some algebraic notions that are either covered at the beginning of this chapter or in Chapter **??**. Notice also the discussion on interpolation and fusion in Chapter **??**, as well as simulations and fusion discussed in the previous section.

5.1 Algebraic Characterization

Definition 5.1 *A modal logic L has* ***local interpolation*** *if $\vdash_L$ has interpolation; it has* ***global interpolation*** *if $\Vdash_L$ has interpolation.*

Since $\vdash_L$ has a deduction theorem, local interpolation can also formulated as follows: if $\varphi \to \psi \in L$ there is a χ such that $\mathrm{var}(\chi) \subseteq \mathrm{var}(\varphi) \cap \mathrm{var}(\psi)$ and $\varphi \to \chi, \chi \to \psi \in L$. This property is also known as **Craig interpolation**. Notice that interpolation is a property of the consequence relation not of the logic.

Proposition 5.2 *If L has local interpolation it also has global interpolation.*

Proof. Assume $\varphi \Vdash_L \psi$. Then for some compound modality $\boxplus$, $\boxplus\varphi \vdash_L \psi$. By assumption there is a χ in the joint variables such that $\boxplus\varphi \vdash_L \chi \vdash_L \psi$. It follows that $\varphi \Vdash_L \chi \Vdash_L \psi$. □

Definition 5.3 *A variety $\mathcal{V}$ of modal algebras has the* ***amalgamation property*** *if for every triple $\mathfrak{A}_0$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ from $\mathcal{V}$ and embeddings $\iota_1 : \mathfrak{A}_0 \rightarrowtail \mathfrak{A}_1$, $\iota_2 : \mathfrak{A}_0 \rightarrowtail \mathfrak{A}_2$ there is a $\mathfrak{B} \in \mathcal{V}$ and embeddings $\varepsilon_1 : \mathfrak{A}_1 \to \mathfrak{B}$, $\varepsilon : \mathfrak{A}_2 \to \mathfrak{B}$ such that $\varepsilon_1 \circ \iota_1 = \varepsilon_2 \circ \iota_2$. $\mathcal{V}$ has the* ***superamalgamation property*** *if in addition to the above for every $a_1 \in A_1$ and $a_2 \in A_2$: (a) if $\varepsilon_1(a_1) \leq \varepsilon_2(a_2)$ then there is a $c \in A_0$ such that $a_1 \leq \varepsilon_1(c)$ and $\varepsilon_2(c) \leq a_2$ and (b) if $\varepsilon_1(a_1) \geq \varepsilon_2(a_2)$ then there is a $c \in A_0$ such that $a_1 \geq \varepsilon_1(c)$ and $\varepsilon_2(c) \geq a_2$.*

(89)
$$\begin{tikzcd}
 & \mathfrak{A}_1 \arrow[dr, dotted, "\varepsilon_1"] & \\
\mathfrak{A}_0 \arrow[ur, "\iota_1"] \arrow[dr, "\iota_2"'] & & \mathfrak{A}_3 \\
 & \mathfrak{A}_2 \arrow[ur, dotted, "\varepsilon_2"'] &
\end{tikzcd}$$

Theorem 5.4 (Maksimova) *Let L be a modal logic.*

(i) *L has local interpolation iff $\mathcal{V}(L)$ has the superamalgamation property.*

(ii) *L has global interpolation iff $\mathcal{V}(L)$ has the amalgamation property.*

We sketch a proof of the second claim. Suppose that L has global interpolation, and let $\mathfrak{A}_0$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ plus two embeddings be given. We define $\mathfrak{F}_i := \mathfrak{Fr}_L(A_i)$ and $\mathfrak{F}_3 := \mathfrak{Fr}_L(A_1 \cup A_3)$. The embeddings form a commuting square as in (89). The identity map is a surjective homomorphism $\pi_i : \mathfrak{F}_i \twoheadrightarrow \mathfrak{A}_i$. For $i = 1, 2$ put $T_i := \{\varphi : \pi_i(\varphi) = 1\}$. Let $T := \{\chi : T_1 \cup T_2 \Vdash_L \chi\}$. Then the following holds for $\varphi \in F_1$, $\psi \in F_2$:

(90) $$T \Vdash_L \varphi \to \psi \quad \Leftrightarrow \quad (\exists \chi \in F_0)(\varphi \to \chi \in T_i \text{ and } \chi \to \psi \in T_j)$$

For if $T \Vdash_L \varphi \to \psi$ then there are finite $\Gamma_i \subseteq T_i$ and a compound modality such that $\boxplus\Gamma_1; \boxplus\Gamma_2 \vdash_L \varphi \to \psi$, giving $\Gamma_1; \varphi \Vdash_L \boxplus\Gamma_2 \to \psi$. There is an interpolant $\chi \in F_0$, from which we deduce $\varphi \to \chi \in T_1$ and $\chi \to \psi \in T_2$.

Put $\varphi \,\Theta\, \psi$ iff $T \Vdash_L \varphi \leftrightarrow \psi$. This is a congruence on $\mathfrak{F}_3$ and we put $\mathfrak{A}_3 := \mathfrak{F}_3/\Theta$. Using the above property it is shown that for $i = 1, 2$ and $\varphi \in F_i$: $\varphi \,\Theta\, \top$ implies $\varphi \in T_i$. So, the natural map $\mathfrak{F}_i \twoheadrightarrow \mathfrak{A}_3$ factors through π_1, giving a map $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{A}_3$ with the desired properties.

Conversely, assume that $\mathcal{V}(L)$ has the amalgamation property. Let $\varphi = \varphi(\boldsymbol{p}, \boldsymbol{r})$ and $\psi = \psi(\boldsymbol{r}, \boldsymbol{q})$ be given such that no global interpolant exists. We shall show that $\varphi \nVdash_L \psi$. Let $\mathfrak{F}_0 := \mathfrak{Fr}_L(\boldsymbol{r})$, $\mathfrak{F}_1 := \mathfrak{Fr}_L(\boldsymbol{p}, \boldsymbol{r})$, $\mathfrak{F}_2 := \mathfrak{Fr}_L(\boldsymbol{q}, \boldsymbol{r})$, and $\mathfrak{F}_3 := \mathfrak{Fr}_L(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r})$. Let O_1 be an open filter in $\mathfrak{F}_1$ containing φ and O_2 an open filter containing $\neg\psi$ and $O_1 \cap F_0$. Then $O_3 := O_2 \cap F_0 = O_1 \cap F_0$. Let Θ_i be the congruence associated with O_i, $\mathfrak{A}_i := \mathfrak{F}_i/\Theta_i$. Then for all $\chi, \chi' \in F_0$, $\chi \,\Theta_1\, \chi'$ iff $\chi \,\Theta_2\, \chi'$. So we have embeddings $\iota_i : \mathfrak{A}_0 \rightarrowtail \mathfrak{A}_i$. Now we get an algebra $\mathfrak{B}$ and maps $\varepsilon_i : \mathfrak{F}_i \to \mathfrak{B}$ such that $\varepsilon_1 \circ \iota_1 = \varepsilon_2 \circ \iota_2$. Define v by $v(p) := \varepsilon([p]\Theta_1)$ if $p \in \mathrm{var}(\varphi)$ and $v(p) := \varepsilon_2([p]\Theta_2)$ if $p \in \mathrm{var}(\psi)$. Since $\varphi \in O_1$, $v(\boxplus\varphi) = 1$ for all compound modalities. Since $\overline{v}(\neg\psi) \neq 1$, $\varphi \nVdash_L \psi$.

We remark here that the proof established that the category of L-algebras has pushouts for monomorphisms.

Theorem 5.5 (Maksimova) *There are exactly seven consistent logics containing* **Grz** *which have interpolation. There are at most 50 consistent logics containing* **S4** *which have global interpolation and at most 37 logics having interpolation.*

The first result is from [40], the second from [41]. We sketch a proof, restricting our attention to global interpolation. The first step is to notice that if a modal logic L has interpolation, then so does the intermediate logic determined by this class of frames (under the Gödel translation). Notice the following. For a (general) frame $\mathfrak{F}$, define the skeleton $S(\mathfrak{F})$ by reducing every cluster to size 1. If $L \supseteq$ **S4** is determined by the class $\mathcal{K}$ of general frames, then the intermediate logic associated with it is determined by the class $\{S(\mathfrak{F}) : \mathfrak{F} \in \mathcal{K}\}$. It is not hard to see that if $\mathcal{K}$ has amalgamation then so does $S(\mathcal{K})$. Therefore, the first step is to characterize the intermediate logics which have interpolation.

It is best to use the dual characterization in terms of frames: a necessary condition for a logic to have global interpolation is that if $p_1 : \mathfrak{F}_1 \to \mathfrak{F}_0$ and $p_2 : \mathfrak{F}_2 \to \mathfrak{F}_0$ are surjective p–morphisms of L–frames there is an L–frame $\mathfrak{G}$ and p–morphisms $q_1 : \mathfrak{G} \to \mathfrak{F}_1$ and $q_2 : \mathfrak{G} \to \mathfrak{F}_2$ such that $p_1 \circ q_1 = p_2 \circ q_2$. Call $\mathfrak{G}$ a **fibred product** of the $\mathfrak{F}_i$. Now suppose that the logic contains only frames of depth $\leq n$, where $n > 2$, and that it contains the chain of length n, which is the frame $\mathfrak{L}_n = \langle \{x_i : i < n\}, \lhd \rangle$ with $x_i \lhd x_j$ iff $i \leq j$. Now define two maps: $p_1(x_i) = x_i$ if $i < n - 1$ and $p_1(x_{n-1}) = x_{n-2}$; $p_2(x_i) = x_{i-1}$ if $i > 0$, and $p_2(0) = x_0$. It is easily seen that there is no fibred product $\mathfrak{G}$ of depth n; there only is one of depth $n + 1$. This observation leads to the following result.

Table 1
Intermediate Logics with Interpolation

Name	Axiomatization	Characteristic Frames
Int		all **Grz**–frames
LC	$(p \to q) \vee (q \to p)$	all linear frames
$\mathbf{BD}_2$	$p \vee (p \to (q \vee \neg q))$	all frames of depth 2
KC	$\neg p \vee \neg\neg p$	all confluent frames
$\mathbf{BD}_2.\mathbf{BW}_2$	$p \vee (p \to (q \vee \neg q)), (p \to q) \vee (q \to p) \vee (p \leftrightarrow \neg q)$	$\mathfrak{B}_2$
$\mathbf{LC}_2$	$\neg p \vee \neg\neg p, p \vee (p \to (q \vee \neg q))$	the two element chain
PC	$p \vee \neg p$	the one element frame
Inc	p	no frames

Lemma 5.6 *Let $L \supseteq$ **S4** have global interpolation. If $\mathfrak{L}_3$ is an L–frame, then every $\mathfrak{L}_n$, $n \in \omega$, is an L–frame.*

Notice that every frame of depth n can be mapped onto $\mathfrak{L}_n$, so that if L contains a frame of depth at least 3, it has frames of any given depth. This can be generalized. Let $\mathfrak{F} = \langle F, \lhd \rangle$ and $\mathfrak{G} = \langle G, \blacktriangleleft \rangle$ be frames with $F \cap G = \varnothing$; then let $\mathfrak{F} \oslash \mathfrak{G} := \langle F \cup G, \prec \rangle$, where $x \prec y$ iff (a) $x, y \in F$ and $x \lhd y$ or (b) $x \in F$, $y \in G$ or (c) $x, y \in \mathfrak{G}$ and $x \blacktriangleleft y$. Further, let $\circ$ denote the one–element reflexive frame.

Lemma 5.7 *Let $L \supseteq$ **S4** have global interpolation. If $\mathfrak{F} \oslash \circ \oslash \circ$ is an L–frame, then so is $\mathfrak{F}(\oslash\circ)^n \oslash \circ$ for every $n \in \omega$.*

Notice that for every frame, it is possible to collapse the points of depth j into a single points, if one is doing that for all $j < m$, m given. This means that the previous theorem restricts the set of logics with interpolation enormously.

Now we turn to branching. Let $\mathfrak{B}_n := \langle \{x_i : i < n+1\}, \lhd \rangle$, where $x_0 \lhd x_i$ for every $i < n+1$, and $x_i \lhd x_j$ iff $i = j$ when $i > 0$. Similarly, it is established that if L has interpolation and contains $\mathfrak{B}_3$ then it contains all $\mathfrak{B}_n$. A related result is that if L has a frame in which a node branches into 3 immediate successors, then it has unbounded branching. Finally, let $\mathfrak{K}_n := \{y_i : i < n\} \cup \{x, z\}, \lhd \rangle$, where for all $i < n$ we have $x \lhd x, z, y_i$, $y_i \lhd y_i, z$ and $z \lhd z$ and no other relations hold. First consider the p–morphisms $q_0, q_1 : \mathfrak{K}_2 \to \mathfrak{L}_3$ defined by $q_0(x) \neq q_0(y_1) \neq q_0(y_0) = q_0(z)$ and $q_1(x) \neq q_1(y_0) \neq q_1(y_1) = q_1(z)$. The fibred product is the frame $\mathfrak{K}_3$. Iterating this argument gives us that all $\mathfrak{K}_n$ must be L–frames. Next consider the p–morphisms $p_0, p_1 : \mathfrak{K}_2 \to \mathfrak{L}_2$ defined by $p_0(y_0) = p_0(y_1) = p_0(x) \neq p_0(z)$ and $p_1(x) \neq p_1(y_0) = p_1(y_1) = p_1(z)$. The fibred product is a frame of depth 3 which has the structure $\circ \oslash (\circ \oplus \circ \oplus \circ) \oslash (\circ \oplus \circ \oplus \circ) \oslash \circ$. This means that as soon as $\mathfrak{K}_2$ is an L–frame, more and more frames can be shown to be L–frames, so that we can eventually conclude that $L =$ **Int**.

These results shall suffice to motivate the result that at most 7 consistent intermediate logics have interpolation. They are listed in Table 1. Now we turn to **S4**. We repeat the strategy with the clusters. Let $\mathfrak{Cl}_n = \langle \{x_i : i < n\}, \lhd \rangle$ with $x_i \lhd x_j$ for all $i, j < n$ be the n–element cluster. It can be shown that if a logic has a frame with a final 3–element cluster, then it has final clusters of arbitrary size. Similarly for nonfinal clusters. Now let $\mathcal{K}$ be a class of **Grz**–frames that has superamalgamation. Then we can derive the following nine possibilities for classes of **S4**–frames: (a) allow the final clusters to be of size 1, 2 or limitless, (b) allow the nonfinal clusters to be of size 1, 2 or limitless. Applied to any of the 7 consistent logics we derive a maximum of 63 combinations of logics that have global interpolation. Some of these combinations are meaningless (for example, allowing the nonfinal clusters for **PC** to be proper), so that the list can be further reduced.

The notion of Halldén-completeness also splits into a local and a global version.

Definition 5.8 A modal logic L is **locally Halldén-complete** if $\vdash_L$ is Halldén-complete; L is **globally Halldén-complete** if $\Vdash_L$ is Halldén-complete.

Global Halldén-completeness is also called the **pseudo relevance property**. The following is clear: if L is locally or globally Halldén-complete, it has up to equivalence at most two constants. For let φ be constant. $\varphi \vdash_L \varphi$, which by Halldén-completeness yields that φ is either inconsistent or a tautology. L has at most two constants iff $\Diamond\top \in L$ (iff $L \supseteq \mathbf{K.D}$) or $\Box\bot \in L$ (iff L is inconsistent or the logic of the one point irreflexive frame).

Definition 5.9 A variety $\mathcal{V}$ of modal algebras has **fusion** if for every pair $\mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{V}$ there is a $\mathfrak{B} \in \mathcal{V}$ and embeddings $\varepsilon_1 : \mathfrak{A}_1 \to \mathfrak{B}$, $\varepsilon_2 : \mathfrak{A}_2 \to \mathfrak{B}$. $\mathcal{V}$ has **superfusion** if in addition to the above for every $a \in A_1 - \{0\}$ and every $b \in A_2 - \{1\}$ we have $\varepsilon_1(a) \not\leq \varepsilon_2(b)$.

It is not hard to see that $\mathcal{V}(L)$ has fusion iff it has finite coproducts.

Theorem 5.10 (Maksimova) *Let L be a modal logic.*

(i) *L is locally Halldén-complete iff $\mathcal{V}$ has superfusion and the zero-generated algebras has at most two elements.*

(ii) *L is globally Halldén-complete iff $\mathcal{V}$ has fusion and the zero-generated algebras has at most two elements.*

The proof is essentially the same as in the case of interpolation. For the algebra $\mathfrak{Fr}_L(0)$ consists of two elements. For given two algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$, if they are nontrivial there are maps $\iota_i : \mathfrak{Fr}_L(0) \rightarrowtail \mathfrak{A}_i$. Using the same proof we obtain an algebra $\mathfrak{B}$ and embeddings $\varepsilon_i : \mathfrak{A}_i \rightarrowtail \mathfrak{B}$ such that $\varepsilon_1 \circ \iota_1 = \varepsilon_2 \circ \iota_2$.

5.2 Proving Interpolation

Besides the algebraic characterization there are at least two other methods to prove interpolation. The first is based on tableau calculi and is basically due to [50]. This method can only be used if the tableau rules meet certain structural criteria. We show here only the case of **K**. Given a tableau calculus for L we do the following. Suppose that $\varphi \vdash_L \psi$. Then $\varphi; \neg\psi$ is L-inconsistent. So it has a closing tableau. We label the formulae in the tableau ^{a} if they derive from the formula φ, and ^{c} if they derive from $\neg\psi$. (If χ is a subformula of both, we create two copies of χ, namely χ^a and χ^c.) From the closing tableau we construct *two* closing tableaux, one for $\varphi^a; (\neg\chi)^c$, and one for $\chi^a; \neg\psi^c$. Moreover, χ will be based on the common variables of φ and ψ. This gives $\varphi \vdash_L \chi$ and $\chi \vdash_L \psi$. Here is the calculus.

$$
(91) \qquad
\begin{array}{llll}
(w) & \dfrac{\Delta; \Sigma}{\Delta} & (\neg E) & \dfrac{\Delta; \neg\neg\varphi}{\Delta; \varphi} \\[2ex]
(\wedge E) & \dfrac{\Delta; \varphi \wedge \psi}{\Delta; \varphi; \psi} & (\vee E) & \dfrac{\Delta; \neg(\varphi \wedge \psi)}{\Delta; \neg\varphi \mid \Delta; \neg\psi} \\[2ex]
(\Box E) & \dfrac{\Box\Delta; \neg\Box\varphi}{\Delta; \neg\varphi} & &
\end{array}
$$

A **K-tableau** is a tree $\mathcal{C}$ constructed according to these rules. $\mathcal{C}$ **closes** if all leaves are of the form $p; \neg p$, p a variable, or $\bot$. Suppose $\mathcal{C}$ closes. The construction of the interpolant is bottom up. We show: *If $\Delta^a; \Sigma^c$ has a closed tableau there is a formula χ such that $\Delta^a; (\neg\chi)^c$ and $\Sigma^c; \chi^a$ both have a closed tableau.* The proof is by induction on the length of a closing tableau for $\Delta^a; \Sigma^c$. χ will be an interpolant for the sequent $\Delta \vdash \neg\Sigma$, where $\neg\Sigma$ is read disjunctively.

There are six cases for the leaves. (1) $p^a; (\neg p)^a$, (2) $p^a; (\neg p)^c$, (3) $(\neg p)^a; p^c$, (4) $p^c; (\neg p)^c$, (5) $\bot^a$ and (6) $\bot^c$. In Case (1), choose $\chi := \bot$, and the first tableau will end in $p^a; (\neg p)^a; (\neg\bot)^c$, the second in $\bot^a$. In Case (2), choose $\chi := p$. The first tableau will consist in $p^a; (\neg p)^c$, the second in $(\neg p)^c; p^a$. The Cases (3) and (4) are dual. In Case (5), let $\chi := \bot$, in Case (6) $\chi := \neg\bot$. Now, suppose that the last step has been an application of $(\Box E)$. With labeling, the step one of the following.

$$(92) \qquad \frac{(\Box\Delta)^a; (\Box\Sigma)^c; (\neg\Box\varphi)^a}{\Delta^a; \Sigma^c; (\neg\varphi)^a} \qquad \frac{(\Box\Delta)^a; (\Box\Sigma)^c; (\neg\Box\varphi)^c}{\Delta^a; \Sigma^c; (\neg\varphi)^c}$$

We deal first with the left hand case. By inductive hypothesis there is a closing tableau for $\Delta^a; (\neg\varphi)^a; (\neg\chi)^c$ and a closing tableau for $\chi^a; \Sigma^c$. The following steps are now valid.

$$(93) \qquad \frac{(\Box\Delta)^a; (\neg\Box\varphi)^a; (\Box\neg\chi)^c}{\Delta^a; (\neg\varphi)^a; (\neg\chi)^c} \qquad \frac{(\Box\Sigma)^c; (\neg\Box\neg\chi)^a}{\Sigma^c; \chi^a}$$

The desired interpolant is $\neg\Box\neg\chi$.

Now we look at the right hand case. By inductive hypothesis there is a formula χ in the common variables and a closing tableau for $\Sigma^c; (\neg\varphi)^c; \chi^a$ and one for $\Delta^a; (\neg\chi)^c$. Now look at the following tableaux.

$$(94) \qquad \frac{(\Box\Delta)^a; (\neg\Box\chi)^c}{\Delta^a; (\neg\chi)^c} \qquad \frac{(\Box\Sigma)^c; (\Box\chi)^a; (\neg\Box\varphi)^c}{\Sigma^c; \chi^a; (\neg\varphi)^c}$$

So, $\Box\chi$ is the desired interpolant. The other induction cases are dealt with similarly.

For extensions of **K** the tableau methods have proved not so useful. The criterion of [50] is not so easy to apply. Here is another method. Call a function X from sets formulae to sets of formulae a **local reduction function** from logic L to logic M if the following holds for all Δ and φ:

(i) $X(\Delta) \subseteq L$.

(ii) If Δ is finite, so is $X(\Delta)$.

(iii) $\mathrm{var}(X(\Delta)) \subseteq \mathrm{var}(\Delta)$.

(iv) $\Delta \vdash_L \varphi$ iff $\Delta; X(\Delta; \varphi) \vdash_M \varphi$.

A **global reduction function** satisfies the same conditions, with iv replaced by

$$(95) \qquad \Delta \Vdash_L \varphi \text{ iff } \Delta; X(\Delta; \varphi) \Vdash_M \varphi$$

The following are global reduction functions to **K**. (For a correct formulation, we assume that the primitive function symbols are $\top$, $\wedge$, $\neg$, and $\Box$. All other symbols are abbreviations. $\mathrm{sf}(\Delta)$ denotes the set of subformulae of formulae from Δ.)

$$(96) \qquad X_{\mathbf{4}}(\Delta) := \{\Box\chi \to \Box\Box\chi : \Box\chi \in \mathrm{sf}(\Delta)\}$$
$$(97) \qquad X_{\mathbf{T}}(\Delta) := \{\Box\chi \to \chi : \Box\chi \in \mathrm{sf}(\Delta)\}$$
$$(98) \qquad X_{\mathbf{B}}(\Delta) := \{\neg\chi \to \Box\neg\Box\chi : \Box\chi \in \mathrm{sf}(\Delta)\}$$
$$(99) \qquad X_{\mathbf{alt_1}}(\Delta) := \{\neg\Box\chi \to \Box\neg\chi : \Box\chi \in \mathrm{sf}(\Delta)\}$$

It is easy to see that reduction functions always exist if $M \subseteq L$. For let $X(\Delta) \subseteq L$. Then from $\Delta; X(\Delta; \varphi) \vdash_M \varphi$ follows $\Delta \vdash_L \varphi$. Conversely, if $\Delta \vdash_L \varphi$, there is a finite proof of φ from Δ. It involves a set $T(\Delta; \varphi)$ of finitely many axioms of L, all of which use only variables from Δ. (To see this, take any proof of φ. If the proof contains a variable q not occurring in φ, replace it uniformly throughout the proof by $\top$. This transforms the proof into a new proof not containing q.) Let $X(\Delta)$ be the union of all these sets $T(\Delta'; \varphi)$ such that $\Delta'; \varphi = \Delta$. This is a reduction function from M to L.

Observe that if X is a local reduction function, then it is also a local reduction function. And if X is a global reduction function, there is a function p from sets of formulae to natural numbers such that $Y(\Delta) := \Box^{\leq p(\Delta)} X(\Delta)$ is a local reduction function. And if Y is a local reduction function, X is a global reduction function.

Definition 5.11 *A reduction function X* ***splits*** *if $X(\varphi \to \psi) = X(\varphi) \cup X(\psi)$.*

Theorem 5.12 (Kracht) *Suppose that there is a splitting global reduction function from L to M. If M has local (global) interpolation, then so does L. If M is locally (globally) Halldén-complete, so is L.*

Proof. Suppose that $\varphi \vdash_L \psi$. Then $\vdash_L \varphi \to \psi$ and so $\Vdash_L \varphi \to \psi$. Hence $X(\varphi \to \psi) \Vdash_M \varphi \to \psi$, and by assumption on X, $X(\varphi); X(\psi) \Vdash_M \varphi \to \psi$. There is a compound modality $\boxplus$ such that $\boxplus X(\varphi); \boxplus X(\psi) \vdash_M \varphi \to \psi$, from which $\boxplus X(\varphi); \varphi \vdash_L \bigwedge \boxplus X(\psi) \to \psi$. We have $\mathrm{var}(\boxplus X(\varphi); \varphi) = \mathrm{var}(\varphi)$ and $\mathrm{var}(\boxplus X(\psi) \to \psi) = \mathrm{var}(\psi)$. L has local interpolation, so there is a χ in the joint variables of φ and ψ such that

$$(100) \qquad \boxplus X(\varphi); \varphi \vdash_M \chi \vdash_M \bigwedge \boxplus X(\psi) \to \psi$$

From this follows $\varphi \vdash_L \chi \vdash_L \psi$. The proof of global interpolation is similar. Likewise for Halldén-completeness. □

The following general result holds (analogous theorems hold for the other functions shown in (96) – (99) with respect to transitive closure (for $X_{\mathbf{4}}$) and reflexive closure (for $X_{\mathbf{T}}$)).

Theorem 5.13 *Let L be a complete logic whose class of frames is closed under symmetric closure. Then $L.\mathbf{B}$ is complete for symmetric L-frames. If L has the finite model property, so does $L.\mathbf{B}$. If L has interpolation, so does $L.\mathbf{B}$.*

Proof. Assume that $\Delta \vdash_{L.\mathbf{B}} \varphi$. Put $n := \mathrm{dp}(\Delta; \varphi)$. We show that

$$(101) \qquad \Delta; \Box^{\leq n} X_{\mathbf{B}}(\Delta; \varphi) \vdash_L \varphi$$

(In other words, we show that $Y_{\mathbf{B}}(\Delta) := \Box^{\leq \mathrm{dp}(\Delta)} X_{\mathbf{B}}(\Delta)$ is a local reduction function from $L.\mathbf{B}$ to L.) Clearly, if (101) holds, we have $\Delta \vdash_{L.\mathbf{B}} \varphi$. Assume that it fails. Then there is an L-frame $\langle F, \lhd \rangle$ and a model

$$(102) \qquad \langle F, \lhd, \beta, x \rangle \vDash \Delta; \Box^{\leq n} X_{\mathbf{B}}(\Delta; \varphi); \neg\varphi$$

Let $\blacktriangleleft$ denote the symmetric closure of $\lhd$. Then $\langle F, \blacktriangleleft \rangle$ is an $L.\mathbf{B}$-frame, by assumption. We claim that for every $\chi \in \mathrm{sf}(\Delta; \varphi)$ and every y accessible in at most $n - \mathrm{dp}(\chi)$ steps from x using $\lhd$ (or in fact $\blacktriangleleft$):

$$(103) \qquad \langle F, \blacktriangleleft, \beta, y \rangle \vDash \chi \quad \Leftrightarrow \quad \langle F, \lhd, \beta, y \rangle \vDash \chi$$

The only critical step is $\chi = \Box\chi'$. ($\Rightarrow$) is clear. ($\Leftarrow$). Suppose that we have $\langle F, \blacktriangleleft, \beta, y \rangle \vDash \neg\Box\chi'$. Then there is a z such that $y \blacktriangleleft z$ and $\langle F, \blacktriangleleft, \beta, z \rangle \vDash \neg\chi'$. (1) $y \lhd z$. Then the induction hypothesis yields $\langle F, \lhd, \beta, z \rangle \vDash \neg\chi'$, and the claim follows. (2) $y \not\lhd z$. Then $z \lhd y$. Moreover, $\langle F, \lhd, \beta, z \rangle \vDash \neg\chi' \to \Box\neg\Box\chi'$, and so $\langle F, \lhd, \beta, y \rangle \vDash \neg\Box\chi'$.

Finite model property is easy; for interpolation just observe that the global reduction function is splitting. □

Notice that it follows that any combination of symmetry, transitivity and reflexivity is covered by these theorems. This can be generalized to polymodal logics. Finally, observe that all the reduction functions split.

Theorem 5.14 (Kracht) *Let L be a polymodal logic characterized by any combination of reflexivity, symmetry and transitivity for any of the modal operators. Then L has the finite model property and interpolation.*

This covers among other **K4** and **S4** and fusions thereof. Similarly, passing from a monomodal logic to its minimal tense extension preserves interpolation if the logic is complete. For $\mathbf{alt_1}$ one needs to assume that L is a subframe logic.

Using similar techniques, one can show the following.

Theorem 5.15 *Let $L \supseteq$ **K4** be a subframe logic with interpolation. Then L.**G** and L.**Grz** are subframe logics which have interpolation.*

Finally, one can also prove the following observation.

Proposition 5.16 (Rautenberg) *Let χ be a constant formula. Then if L has local (global) interpolation, so does $L \oplus \chi$.*

5.3 Beth Properties

As is known from predicate logic, interpolation is related to the Beth-property. However, in modal logic the relationship is somewhat more complex.

Definition 5.17 L has the **local Beth-property** if the following holds. Suppose that $\varphi(p, \boldsymbol{q})$ is a formula and

$$\varphi(p, \boldsymbol{q}); \varphi(r, \boldsymbol{q}) \vdash_L p \leftrightarrow r \tag{104}$$

Then there exists a formula $\chi(\boldsymbol{q})$ such that

$$\varphi(p, \boldsymbol{q}) \vdash_L p \leftrightarrow \chi(\boldsymbol{q}) \tag{105}$$

If (104) is satisfied, $\varphi(p, \boldsymbol{q})$ is called a **local implicit definition** of p. If $\chi(\boldsymbol{q})$ satisfies (105), it is called the **corresponding explicit definition** of p.

There is a stronger property, the **local projective Beth property**. Here it is required that if

$$\varphi(p, \boldsymbol{q}, \boldsymbol{r}_1); \varphi(p', \boldsymbol{q}, \boldsymbol{r}_2) \vdash_L p \leftrightarrow p' \tag{106}$$

there exists a formula $\chi(\boldsymbol{q})$ such that

$$\varphi(p, \boldsymbol{q}, \boldsymbol{r}_1) \vdash_L p \leftrightarrow \chi(\boldsymbol{q}) \tag{107}$$

The global notions are defined similarly. Notice that a local implicit definition is also a global implicit definition. Hence if L has the local Beth-property it also has the global Beth-property.

Theorem 5.18 (Maksimova) *A classical modal logic has local interpolation iff it has the local Beth property.*

Proposition 5.19 (Maksimova) *Let L be a classical modal logic. If L has interpolation then it has the global Beth-property.*

The logic **G.3** has the global Beth-property but fails to have global interpolation. The logic $\mathbf{S4.1.2} \cap \mathbf{S5}$ has the global projective Beth-property but fails to have interpolation ([42]).

A rather different property is shown by logics above **G**. Say that a formula $\psi(\boldsymbol{q})$ is a **fixed point** of $\varphi(p, \boldsymbol{q})$ for p in L if

$$\vdash_L \psi(\boldsymbol{q}) \leftrightarrow \varphi(\psi(\boldsymbol{q}), \boldsymbol{q}) \tag{108}$$

If a logic satisfies this it is said to have the **fixed point property**. It is clear that if $L \subseteq M$ then if $\psi(\boldsymbol{q})$ is a fixed point for $\varphi(p, \boldsymbol{q})$ for p in L it is one in M, too. The following is known as the **fixed point theorem**.

Theorem 5.20 (Sambin, de Jongh) *Suppose that $\varphi(p, \boldsymbol{q})$ is a formula in which every occurrence of p is in the scope of a box. Then $\varphi(p, \boldsymbol{q})$ has a fixed point for p in* **G**.

Proof. The conditions on $\varphi(p, \boldsymbol{q})$ imply that on any finite **G**-frame, the valuation for p is fixed by that for $\boldsymbol{q}$. So, $\varphi(p, \boldsymbol{q})$ globally implicitly defines p. **G** has local interpolation and so the global Beth property, whence $\psi(\boldsymbol{q})$ exists. ◻

It follows that all extensions of **G** have the fixed point property.

Theorem 5.21 (Maksimova) *All extensions of* **G** *have the global Beth-property.*

Call a formula φ **$\boldsymbol{q}$-boxed** if every occurrence of a variable from $\boldsymbol{q}$ is in the scope of some modal operator.

Lemma 5.22 *Let L be a logic containing* **G**. *Let q_i, $i < n$, be distinct variables and p a variable not contained in $\boldsymbol{q}$. For a set $S \subseteq n$ define χ_S by $\chi_S := \bigwedge_{i \in S} q_i \wedge \bigwedge_{i \in n-S} \neg q_i$. Suppose that $\varphi(p, \boldsymbol{q})$ is $\boldsymbol{q}$-boxed and*

$$\vdash_L \chi_S \to \varphi(p, \boldsymbol{q}). \tag{109}$$

Then already $\vdash_L \varphi(p, \boldsymbol{q})$.

Lemma 5.23 *Let $\varphi(p, \boldsymbol{q})$ be a formula. Then there exist $\boldsymbol{q}$-boxed formulae $\psi_1(p, \boldsymbol{q})$, $\psi_2(p, \boldsymbol{q})$, $\chi_1(p, \boldsymbol{q})$ and $\chi_2(p, \boldsymbol{q})$ such that*

$$\vdash_{\mathbf{G}} \varphi(p, \boldsymbol{q}) \leftrightarrow ((p \vee \psi_1(p, \boldsymbol{q})) \wedge (\neg p \vee \psi_2(p, \boldsymbol{q}))) \tag{110}$$

$$\vdash_{\mathbf{G}} \varphi(p, \boldsymbol{q}) \leftrightarrow ((p \wedge \chi_1(p, \boldsymbol{q})) \vee (\neg p \wedge \chi_2(p, \boldsymbol{q}))) \tag{111}$$

Now suppose that $\varphi(p, \boldsymbol{q})$ is a global implicit definition of p in L, and $L \supseteq \mathbf{G}$. Then $\varphi(p, \boldsymbol{q}); \varphi(r, \boldsymbol{q}) \Vdash_L p \leftrightarrow r$. Using Lemma 5.23 we get $\boldsymbol{q}$-boxed formulae $\chi_1(p, \boldsymbol{q})$ and $\chi_2(p, \boldsymbol{q})$ such that

$$\vdash_L \varphi(p, \boldsymbol{q}) \leftrightarrow ((p \wedge \chi_1(p, \boldsymbol{q})) \vee (\neg p \wedge \chi_2(p, \boldsymbol{q}))) \tag{112}$$

Write

$$\boxdot\varphi := \varphi \wedge \Box\varphi \tag{113}$$

Since we also have (by transitivity of L) that

$$\vdash_L \boxdot\varphi(p, \boldsymbol{q}) \wedge \boxdot\varphi(r, \boldsymbol{q}) \to (p \leftrightarrow r) \tag{114}$$

we now get

$$\vdash_L (\Box\varphi(p, \boldsymbol{q}) \wedge \Box\varphi(r, \boldsymbol{q}) \wedge p \wedge \chi_1(p, \boldsymbol{q}) \wedge \neg r \wedge \chi_2(r, \boldsymbol{q})) \to (p \to r) \tag{115}$$

This formula has the form $(\mu \wedge p \wedge \neg r) \to (p \to r)$, where μ is $\boldsymbol{q}$-boxed. This is equivalent to $\neg\mu \vee \neg p \vee r$, or $(p \wedge \neg r) \to \neg\mu$. By use of Lemma 5.22 we deduce that $\vdash_L \neg\mu$, that is,

$$\vdash_L \Box\varphi(p, \boldsymbol{q}) \wedge \Box\varphi(r, \boldsymbol{q}) \to (\chi_1(p, \boldsymbol{q}) \to \neg\chi_2(r, \boldsymbol{q})) \tag{116}$$

We substitute p for r and obtain

$$(117) \qquad \vdash_L \Box\varphi(p, \boldsymbol{q}) \to (\chi_1(p, \boldsymbol{q}) \to \neg\chi_2(p, \boldsymbol{q}))$$

Now from this and (112) it follows after some boolean manipulations

$$(118) \qquad \vdash_L \boxdot\varphi(p, \boldsymbol{q}) \to \boxdot(p \leftrightarrow \chi_1(p, \boldsymbol{q}))$$

By the fixed point theorem for $\mathbf{G}$ there is a $\psi(\boldsymbol{q})$ such that

$$(119) \qquad \vdash_{\mathbf{G}} \boxdot(p \leftrightarrow \chi_1(p, \boldsymbol{q})) \to (p \leftrightarrow \psi(\boldsymbol{q}))$$

So we obtain

$$(120) \qquad \vdash_L \boxdot\varphi(p, \boldsymbol{q}) \to (p \leftrightarrow \psi(\boldsymbol{q}))$$

which is nothing but

$$(121) \qquad \varphi(p, \boldsymbol{q}) \Vdash_L p \leftrightarrow \psi(\boldsymbol{q})$$

So $\psi(\boldsymbol{q})$ is an explicit definition. [1] take a slightly different approach. They show

Theorem 5.24 (Areces & Hoogland & de Jongh) *Let L be a transitive logic in which the rule*

$$(122) \qquad \boxdot p \to (\Box q \to q)/ \boxdot p \to q$$

is admissible. Then L has the local Beth property iff it satisfies the fixed point theorem.

Notice that the admissibility of (122) implies that the Löb-rule (123) is admissible.

$$(123) \qquad \Box p \to p/p$$

For if $\Box\varphi \to \varphi$ is a theorem, so is $\boxdot\top \to (\Box\varphi \to \varphi)$. By (122), $\boxdot\top \to \varphi$ is a theorem, whence $\varphi \in L$. The following is folklore.

Theorem 5.25 *A transitive logic contains* $\mathbf{G}$ *iff it satisfies the Löb rule.*

Proof. Suppose $L \supseteq \mathbf{G}$ and $\Box\varphi \to \varphi \in L$. Then $\Box(\Box\varphi \to \varphi) \in L$, from which $\Box\varphi \in L$. Now, using $(\mathrm{MP}_\to)$ on the assumption, $\varphi \in L$. Conversely, assume that the Löb rule is admissible. Put $\chi := \Box(\Box p \to p)$, $\psi := \Box p$. We need to show that $\chi \to \psi$ is a theorem of L.

$$(124) \qquad \vdash_{\mathbf{K4}} \Box(\chi \to \psi) \to (\Box\chi \to \Box\psi)$$
$$(125) \qquad \vdash_{\mathbf{K4}} \chi \to (\Box\psi \to \psi)$$
$$(126) \qquad \vdash_{\mathbf{K4}} \chi \to \Box\chi$$
$$(127) \qquad \vdash_{\mathbf{K4}} \Box(\chi \to \psi) \to (\chi \to \psi)$$

Since the Löb rule is admissible in $L \supseteq \mathbf{K4}$, $\chi \to \psi \in L$. □

On the other hand, the same method can be used to show that if $L \supseteq \mathbf{G}$ then (122) is admissible. Suppose namely that $\boxdot\varphi \to (\Box\chi \to \chi)$ is a theorem. Then so is $\Box \boxdot \varphi \to \Box(\Box\chi \to \chi)$. From this we get $\Box \boxdot \varphi \to \Box\chi$ with the $\mathbf{G}$-axiom. But $\boxdot\varphi \to \Box \boxdot \varphi \in \mathbf{G}$, and so $\boxdot\varphi \to \Box\chi$, which together with the premiss yields $\boxdot\varphi \to \chi$. Therefore, the coverage of Theorem 5.24 is not larger than that of Theorem 5.20 and 5.21.

L has **uniform interpolation** if

① given φ and variables $\boldsymbol{q}$ there exists a formula χ such that $\mathrm{var}(\chi) \subseteq \boldsymbol{q}$ and for all formulae ψ such that $\varphi \vdash_L \psi$ and $\mathrm{var}(\varphi) \cap \mathrm{var}(\psi) = \boldsymbol{q}$ we have $\varphi \vdash_L \chi \vdash_L \psi$ (**uniform preinterpolation**) and

② given ψ and variables $\boldsymbol{q}$ there exists a formula χ such that $\mathrm{var}(\chi) \subseteq \boldsymbol{q}$ and for all formulae φ such that $\varphi \vdash_L \psi$ and $\mathrm{var}(\varphi) \cap \mathrm{var}(\psi) = \boldsymbol{q}$ we have $\varphi \vdash_L \chi \vdash_L \psi$ (**uniform postinterpolation**).

By classical logic, L has uniform preinterpolation iff it has uniform postinterpolation. Notice that uniform interpolation of L can be used to define second order quantification inside the modal language. Let L^q be the extension of L by propositional quantifiers. Now, $(\forall p)\varphi \vdash_L \varphi$ is always valid. Moreover, if $\mathrm{var}(\psi) = \mathrm{var}(\varphi) - \{p\}$ and $\psi \vdash_L \varphi$ then also $\psi \vdash_{L^q} (\forall p)\varphi$. So, $(\forall p)\varphi$ is up to equivalence *the* uniform preinterpolant. If L has uniform interpolation, there is a preinterpolant χ in the variables $\mathrm{var}(\varphi) - \{p\}$. Hence, $(\forall p)\varphi$ is equivalent to χ, and L^q reduces to L in expressivity. This idea has been one of the reasons to study uniform interpolation (see [45]). The logics **K**, **Grz** and **G** have uniform interpolation, **S4** fails to have uniform interpolation (see [56] and [26]). Furthermore, the following is known about fusions, see [61].

Theorem 5.26 (Wolter) *If L and L' have uniform interpolation, so does $L \otimes L'$.*

Notice that if $\varphi_1 \vdash_L \psi$ and $\varphi_2 \vdash_L \psi$, and if χ_i are interpolants for φ_i and ψ, then $\chi_1 \vee \chi_2$ is an interpolant for both:

$$\varphi_1 \vdash_L \chi_1 \vdash_L \chi_1 \vee \chi_2 \vdash_L \psi \tag{128}$$

So if a logic has interpolation and there are up to equivalence only finitely many formulae in n variables then L has uniform interpolation as well ([61]).

Theorem 5.27 (Wolter) *Let L have interpolation. If $\mathcal{V}(L)$ is locally finite then L also has uniform interpolation.*

We shall sketch a proof that **K** has uniform interpolation. For example, we show that it has uniform preinterpolation. The proof uses tableau calculi again. By induction on the length of Σ we prove the following: *Let $\boldsymbol{q}$ be a set of variables. There is a χ in the variables $\boldsymbol{q}$ such that for any Δ such that* $\mathrm{var}(\Delta) \cap \mathrm{var}(\Sigma) = \boldsymbol{q}$*, given a closing tableau for $\Delta^a; \Sigma^c$ both $\Delta^a; (\neg\chi)^c$ and $\chi^a; \Sigma^c$ have a closing tableau.* In other words, the interpolant is determined by the ^{c}-set alone (in addition to the set of shared variables). The proof of this fact is actually not hard. We look again at the proof sketched above. Suppose that the tableau closes. It closes in six possible situations. (1) $p^a; (\neg p)^a$, (2) $p^a; (\neg p)^c$, (3) $(\neg p)^a; p^c$, (4) $p^c; (\neg p)^c$, (5) $\bot^a$ and (6) $\bot^c$. In Case (5) $\chi := \bot$ and in Case (6) $\chi := \neg\bot$ satisfy the requirements. Consider the other cases. (A) $p \in \boldsymbol{q}$. Then only (2) and (3) can arise. The interpolant is completely determined by knowing Σ. (B) $p \notin \boldsymbol{q}$. Then (1) or (4) arise. Again, the interpolant is determined solely by knowing Σ.

We consider briefly the other cases. If (w) has applied, the interpolant χ for the lower sequent is an interpolant for the upper sequent. Clearly, it only depends on the upper ^{c}-set if that was true for the lower ^{c}-set. The same happens with $(\neg E)$ and $(\wedge E)$. Next we look at $(\vee E)$.

$$\frac{\Delta^a; (\neg(\varphi \wedge \psi))^a; \Sigma^c}{\Delta^a; (\neg\varphi)^a; \Sigma^c \;\big|\; \Delta^a; (\neg\psi)^a; \Sigma^c} \tag{129}$$

By inductive hypothesis there is an interpolant χ depending only on Σ^c, not on the ^{a}-set. Therefore we can use the same formula as follows:

$$(130)\qquad \begin{array}{c|c} \multicolumn{2}{c}{\Delta^a; (\neg(\varphi \wedge \psi))^a; \chi^c} \\ \hline \Delta^a; (\neg\varphi)^a; \chi^c & \Delta^a; (\neg\psi)^a; \chi^c \end{array}$$

This tableau closes. Now suppose that the rule application is

$$(131)\qquad \begin{array}{c|c} \multicolumn{2}{c}{\Delta^a; (\neg(\varphi \wedge \psi))^c; \Sigma^c} \\ \hline \Delta^a; (\neg\varphi)^c; \Sigma^c & \Delta^a; (\neg\psi)^c; \Sigma^c \end{array}$$

By inductive hypothesis there are interpolants χ_1 and χ_2 independent of Δ for the left and right hand side. Now we get

$$(132)\qquad \begin{array}{c|c} \multicolumn{2}{c}{\Delta^a; (\neg(\chi_1 \wedge \chi_2))^c} \\ \hline \Delta^a; (\neg\chi_1)^c & \Delta^a; (\neg\chi_2)^c \end{array} \qquad \begin{array}{c|c} \multicolumn{2}{c}{(\chi_1 \wedge \chi_2)^a; (\neg(\varphi \wedge \psi))^c; \Sigma^c} \\ \hline (\chi_1 \wedge \chi_2)^a; (\neg\varphi)^c; \Sigma^c & (\chi_1 \wedge \chi_2)^a; (\neg\psi)^c; \Sigma^c \\ \hline \chi_1^a; \chi_2^a; (\neg\varphi)^c; \Sigma^c & \chi_1^a; \chi_2^a; (\neg\psi)^c; \Sigma^c \\ \hline \chi_1^a; (\neg\varphi)^c; \Sigma^c & \chi_2^a; (\neg\psi)^c; \Sigma^c \end{array}$$

Both tableaux close by assumption. The desired interpolant is $\chi_1 \wedge \chi_2$. So far the interpolant did not depend on what Δ is.

The rule $(\Box E)$ is the last and most complex to consider. Here we face two options: either it was applied to an ^{a}-formula, and then $\neg\Box\neg\chi$ is the new interpolant, or it was applied to a ^{c}-formula, and then the interpolant is $\Box\chi$. Case (1). There is no Δ such that $(\Box E)$ can be applied to an ^{a}-formula. Then the preinterpolant is $\neg\Box\neg\chi$. Case (2). There is no Δ such that $(\Box E)$ can be applied to a ^{c}-formula. Then $\Box\chi$ is the interpolant. Case (3). There is Δ_1 such that $(\Box E)$ can be applied to an ^{a}-formula, and Δ_2 such that $(\Box E)$ can be applied to a ^{c}-formula. Then $\neg\Box\neg\chi \vee \Box\chi$ is the desired interpolant. For by assumption, $(\neg\Box\neg\chi)^a; \Sigma^c$ and $(\Box\chi)^a; \Sigma^c$ both close, and so does therefore $(\neg\Box\neg\chi \vee \Box\chi)^a; \Sigma^c$. And given Δ, either $\Delta^a; (\neg\Box\neg\chi)^c$ closes or $\Delta^a; (\Box\chi)^c$. However, this means that $\Delta^a; (\neg(\neg\Box\neg\chi \vee \Box\chi))^c$ closes.

6 Admissible Rules

The study of admissible rules in modal logic has been the topic of the monograph by Vladimir Rybakov, [52], from which most of the results of this section are taken. Studying admissibility can be taken to mean the study of the consequence relations $\vdash_L^m$, where $\vdash_L^m$ is the largest consequence relation whose set of tautologies is L. For in this consequence relation every admissible rule is derived. Thus, we may either speak of characterizing the consequence $\vdash_L^m$ or about the admissible rules of $\vdash_L$, or, for that matter, L itself. We shall prefer the latter. Historically, the first breakthrough was the solution by Rybakov to Problem 40 of the list of 102 problems by Harvey Friedman [21]. It asked whether admissibility of a rule in **Int** is decidable, which by way of the Gödel translation can be turned into a problem of **Grz**, see Theorem 6.5. Based on this, Rybakov has extended the results to cover large classes of extensions of **K4**, giving criteria of when admissibility of rules is decidable, and when $\vdash_L^m$ is finitely axiomatizable.

6.1 General Theory

We start with some general considerations. Let $\Delta = \{\delta_i : i < m\}$. A modal algebra satisfies the rule $\langle\Delta, \varphi\rangle$ iff it satisfies the Horn-formula

$$(133)\qquad \bigwedge_{i<m} \delta_i = \top \to \varphi = \top$$

Admissibility can be characterized as follows. Let L be a logic. $\langle \Delta, \varphi \rangle$ is admissible in L iff for all m

$$(134) \qquad \mathfrak{Fr}_L(n) \vDash \bigwedge_{i<m} \delta_i = \top \to \varphi = \top$$

where $\mathfrak{Fr}_L(n)$ denotes the freely n-generated L-algebra. For notice that for every valuation h into $\mathfrak{Fr}_L(n)$ there are formulae σ_i, $i \in \omega$, such that $h(p_i) = \sigma_i$. Hence with κ the map induced by the identity and $\sigma : p_i \mapsto \sigma_i$, $h(\varphi) = \kappa(\varphi^\sigma)$. So, if $h(\psi) = 1$ in $\mathfrak{Fr}_L(n)$ there is a substitution σ for which $\kappa(\varphi^\sigma) = 1$, which means that $\varphi^\sigma \in L$. Equivalently we have

Proposition 6.1 *Let $\Delta = \{\delta_i : i < n\}$. $\langle \Delta, \varphi \rangle$ is admissible in L iff $\mathfrak{Fr}_L(\omega) \vDash \bigwedge_{i<n} \delta_i = \top \to \varphi = \top$.*

We shall restrict our attention to extensions of **K4**. The problem whether admissibility of a rule is decidable in intuitionistic logic can be turned into a question of modal logics. Let us note that each rule can be brought into the form $\langle \{\chi_1\}, \chi_2 \rangle$, also written χ_1/χ_2. Now, call a substitution s a **unifier** for χ in L if $\vdash_L s(\chi)$. Then the rule χ_1/χ_2 is admissible in L if every unifier for χ_1 in L is also a unifier for χ_2. Thus admissibility can be checked by inspecting the unifiers of a given formula. In a logic L, say that s is **general than** s', in symbols $s \leq s'$ if there is a substitution t such that $t(s(p)) \leftrightarrow s'(p) \in L$. Classical logic enjoys the property that if a formula has a unifier, it also has a unique most general unifier and it can effectively be found ([43]). Given that, admissibility can be checked in boolean logic as follows. Determine the most general unifier, say s, for χ_1. Now decide whether $s(\chi_2)$ is a theorem. This fails in intuitionistic logic for the reason that there is no single most general unifier. The strategy can however be generalized. Suppose for any given formula χ we can compute a finite set Π_χ of minimal unifiers, then we can decide admissibility if the logic is decidable. (If L is undecidable, admissibility is a fortiori undecidable.)

[24] gives a proof along these lines that admissibility is decidable in intuitionistic logic. The methods are similar for modal logic. A formula χ is called **projective** if there is a unifier s such that for all $p \in \mathrm{var}(\chi)$:

$$(135) \qquad \chi \vdash p \leftrightarrow s(p)$$

It is possible to construct such a unifier. Let S be a subset of $\mathrm{var}(\chi)$. Define θ^S_χ by

$$(136) \qquad \theta^S_\chi(p) := \begin{cases} \chi \to p & \text{if } p \in S \\ \chi \wedge p & \text{otherwise} \end{cases}$$

This substitution satisfies (135) but is not necessarily a unifier. Define an enumeration S_i, $i < k$, $k := 2^{|\,\mathrm{var}(\chi)|}$ on the subsets of $\mathrm{var}(\chi)$ so that if $S_i \subseteq S_j$, then $i \leq j$. Next put

$$(137) \qquad \theta_\chi := \theta^{Sk-1}_\chi \circ \theta^{S_{k-2}}_\chi \circ \cdots \circ \theta^{S_0}_\chi$$

Theorem 6.2 (Ghilardi) *θ_χ is a unifier for χ iff χ is projective.*

This serves as a test for projectivity. Define $c(\chi)$ to be the maximum nesting of $\to$ (alternatively, $c(\chi)$ is the $\to$-depth of χ). There are only finitely many formulae χ over a given finite set of variables such that $c(\chi) \leq n$, for any given n. (The other connectives are $\wedge$, $\vee$ and $\neg$. Obviously, this requires showing that from a given set of formulae, there is a bounded number of formulae that can be built using $\neg$, $\vee$ and $\wedge$.) Say that a set U of substitutions is **complete** for χ in L if for every unifier t for χ there is an $s \in U$ such that $s \geq t$. To check the admissibility of a rule χ/φ in L it is enough to be able to determine whether a formula has a unifier and if so to be able to construct a finite complete set for it. For then it is enough to check the complete set for φ against that for χ.

Theorem 6.3 (Ghilardi) *Every unifiable formula has a finite complete set of unifiers in* **Int***.*

This set is found as follows. Let

$$S_\chi := \{\psi : \mathrm{var}(\psi) \subseteq \mathrm{var}(\chi), \psi \text{ projective and } c(\psi) \leq c(\chi)\} \tag{138}$$

This set is shown to be finite. Then $\{\theta_\psi : \psi \in S_\chi\}$ is complete. What we really need, though, is a set of substitutions that is a basis, where S is a **basis** iff it is complete and for every $s, t \in S$, if $s \neq t$ then $t \not\leq s$. To get a basis, let Π_χ b any subset of S_χ for which (i) if $\psi_1, \psi_2 \in \Pi_\chi$ and $\psi_1 \vdash \psi_2$ then $\psi_1 = \psi_2$ and (ii) for every $\psi \in S_\chi$ there is a $\psi' \in \Pi_\chi$ such that $\psi \vdash \psi'$. Such a set obviously exists and is easy to construct on the basis of S_χ. The set $\{\theta_\psi : \psi \in \Pi_\chi\}$ is a basis for χ. Now the rule χ/χ' is admissible iff for every $\psi \in \Pi_\chi$: $\psi \vdash \chi'$.

Let us briefly mention some relations with modal logic. Consider the dual of $\mathfrak{Fr}_L(n)$, the weak canonical frame $\mathfrak{Can}_L(n)$. Let $\mathfrak{Ch}_L(n)$ be the subframe of all points of finite depth. This is also called the n-**characterizing frame**, while $\langle \mathfrak{Ch}_L(n), \kappa\rangle$, $\kappa : p_i \mapsto p_i$, is called the n-**characterizing model**.

Lemma 6.4 *Assume that $L \supseteq$* **K4** *has the finite model property. Then the rule $\delta_0, \ldots, \delta_{m-1}/\varphi$ is admissible in L iff for all n,*

$$\mathfrak{Ch}_L(n) \vDash \bigwedge_{i<m} \delta_i = \top \to \varphi = \top \tag{139}$$

Recall the Gödel-translation T from intuitionistic logic to modal logic. Let L be a superintuitionistic logic. Put $\sigma(L) := \mathbf{Grz} \oplus T(L)$.

Theorem 6.5 *The rule $\delta_0, \ldots, \delta_{n-1}/\varphi$ is admissible in L iff the translation $T(\delta_0), \ldots, T(\delta_{n-1})/T(\varphi)$ is admissible in $\sigma(L)$.*

6.2 Frame Characterization of Admissibility

A logic has **branching below** m if whenever in some frame for L there is a cluster with d immediate successor clusters, then whenever we find d incomparable clusters in $\mathfrak{Ch}_L(n)$, there is a cluster C having these clusters as its immediate successor clusters. The effective m-drop point property is still more cumbersome to define. To understand it, recall the selection procedure of Fine and Zakharyaschev (see [19] and [62]). This procedure extracts a finite model out of a given model $\mathfrak{M}$ on the basis of a set Σ of formulae closed under subformulae. Denote this frame by $X(\mathfrak{M}, \Sigma)$, and by $X_m(\mathfrak{M}, \Sigma)$ the model containing both $X(\mathfrak{M}, \Sigma)$ and the points of depth at most m. (We are assuming that the model is based on finitely many generators.) Crucially, this procedure does not preserve the truth of all formulae, since we are taking not necessarily generated subframes, but it does preserve the truth of all formulae from Σ. For cofinal subframe logics this shows that they have the finite model property. The m-**drop point property** says the following. Suppose that we have a finite n-generated L-model $\mathfrak{M}$ and that it is large. Then it contains a submodel $\mathfrak{W} \supseteq X_m(\mathfrak{M}, \Sigma)$ which is contractible onto a L-frame of no more than $g(x, y)$ elements, where g is a recursive function and $x = |\Sigma|$, and y the number of points of depth at most m in $\mathfrak{M}$.

Theorem 6.6 (Rybakov) *Suppose that L is a logic containing* **K4***. Suppose further that*

(i) *L has fmp,*

(ii) *L has branching below m for some $m \in \omega$, and*

(iii) *L has the effective m-drop point property for some $m \in \omega$.*

Let ρ be a rule with k variables. Then ρ is admissible in L iff it is valid in the algebra of all subsets of the Kripke-frame underlying the k-characterizing frame. Furthermore, suppose that there is an algorithm which decides for a finite frame whether it is an L-frame. Then there exists an algorithm deciding whether a given inference rule is admissible for L.

The proof of this theorem uses the selection procedure. It shows that if ρ is refutable in the n-characterizing model then we can construct a model whose size we can estimate a priori and in which ρ is refuted as well. This model also has the so-called **view-realizing** property. Conversely, if such a model exists, ρ is refutable in the n-characterizing model. The proof of the latter statement is the most involved, but it seems that it can be simplified using the technique of homogenization proposed in [37].

Let I_n be an axiom saying that the frames are of width at most n (that is, have no antichain of length $n+1$).

Corollary 6.7 *Admissibility of rules is decidable in the modal systems* **K4**, **S4**, **GL**, **Grz**, **S5**, *and in the logics $L \oplus I_n$, where L is any of the aforementioned logics.*

[25] apply the method of [24] and develop criteria for extensions L of **K4**. Let us be given a formula χ. There are infinitely many substitutions s such that χ^s is a theorem of L. Say that unification in L is **filtering** if for any two unifiers s_0 and s_1 for a formula χ there is a unifier t such that $t \leq s_0, s_1$. Evidently, if unification is filtering in L then a complete set is either infinite or contains just one member. (If the latter is always the case, L is called **unitary**.)

Theorem 6.8 (Ghilardi & Sacchetti) *Let $L \supseteq$ **K4**. Then unification is filtering iff $L \supseteq \mathbf{K4} \oplus \mathbf{2}^+ := \mathbf{K4} \oplus \neg \boxdot \neg \boxdot p \to \boxdot \neg \boxdot p$.*

The additional axiom is similar $\mathbf{2} = \Diamond\Box p \to \Box\Diamond p$, only that we use $\boxdot$ in place of $\Box$. So, above **S4** this axiom reduces to **2**. The condition also has algebraic analogs. Say that an algebra $\mathfrak{A}$ is **projective** in a variety $\mathcal{V}$ if there is a free algebra $\mathfrak{Fr}_{\mathcal{V}}(X)$ and maps $p : \mathfrak{Fr}_{\mathcal{V}}(X) \to \mathfrak{A}$ and $m : \mathfrak{A} \to \mathfrak{Fr}_{\mathcal{V}}(X)$ such that $p \circ m = 1_{\mathfrak{A}}$. Say that an algebra is **finitely presented** in $\mathcal{V}$ if there is a finite set X and a finite set E of equations such that $\mathfrak{A} \cong \mathfrak{Fr}_{\mathcal{V}}(X)/\Theta(E)$, where $\Theta(E)$ is the smallest congruence containing E.

Theorem 6.9 (Ghilardi & Sacchetti) *Unification in L is filtering iff finitely presented projective L–algebras are closed under binary products.*

Let $\mathfrak{F}_i = \langle F_i, \lhd \rangle$ be a family of L–frames. $(\bigoplus_{i \in I} \mathfrak{F}_i)^\circ$ and $(\bigotimes_{i \in I} \mathfrak{F}_i)^\bullet$ are defined like the disjoint union, except that a root world is added, which is irreflexive in the first, and reflexive in the second case. Finally, $\mathrm{irr}((\bigoplus_{i \in I} \mathfrak{F}_i)^\circ)$ and $\mathrm{irr}((\bigoplus_{i \in I} \mathfrak{F}_i)^\bullet)$ are obtained by identifying all final clusters (assuming that they are isomorphic). Now L has the **2–glueing property** if whenever L has a Kripke–frame containing an irreflexive point (a reflexive point) and $\mathfrak{F}_i$, $i \in I$, are L–frames whose final clusters are isomorphic, then $\mathrm{irr}((\bigoplus_{i \in I} \mathfrak{F}_i)^\circ)$ and $(\mathrm{irr}((\bigoplus_{i \in I} \mathfrak{F}_i)^\circ))$ is an L–frame.

Theorem 6.10 (Ghilardi & Sacchetti) *Unification is unitary in $L \supseteq$ **K4** if L contains $\mathbf{K4} \oplus \mathbf{2}^+$, has the finite model property and has the 2–glueing property.*

In particular, for every L satisfying these conditions the admissibility of inference rules is decidable, which implies that the logics are decidable. For clearly, if admissibility of a rule in L is decidable L must be decidable. But the converse need not hold.

Theorem 6.11 (Chagrov) *There is a logic which is decidable, but admissibility of rules is undecidable.*

6.3 Axiomatizing the Admissible rules

There also is a question whether the admissible rules can actually be axiomatized. In the present terms this means axiomatizing $\vdash^m_{\mathbf{Grz}}$. One speaks of a **basis** for the set of admissible rules. In [52], a series $\mathfrak{E}_n$, $n \in \omega$, of frames is defined.

(i) $E^1_n := \{x^0_0\}$, $x^0_0 \lhd_0 x^0_0$.

(ii) $E^2_n := E^1_1 \cup \{x^1_i : i < 2^n + 2\}$. $x^i_j \lhd_2 x^{i'}_{j'}$ iff $i' = 0$ or $i = i'$ and $j = j'$.

(iii) Let H be the set of all antichains of $E^i_n - E^{i-1}_n$. $E^{i+1}_n := E^i_n \cup \{x^i_h : h \in H\}$. $\lhd_{i+1}$ satisfies (a) $\lhd_{i+1} \restriction E^i_n = \lhd^i$, (b) $x^i_h \lhd_{i+1} x^j_k$ iff $j = i-1$ and $k \in h$ or there is a x^{i-1}_p such that $p \in h$ and $x^{i-1}_p \lhd_i x^j_k$.

(140) $$\mathfrak{E}_n := \langle \bigcup_i E^i_n, \bigcup_i \lhd_i \rangle$$

Furthermore, the following is established.

Theorem 6.12 (Rybakov) *Let $L \supseteq$ **S4** be a logic with the following properties.*

① *For all n: $\mathfrak{F}_n \vDash L$.*

② *L has the finite model property and branching below 1.*

③ *L has the effective m-drop point property.*

Then $\vdash^m_L$ cannot be axiomatized by finitely many rules.

Similar criteria are established for superintuitionistic logics and logics containing **K4**. What is important is the following consequence.

Theorem 6.13 (Rybakov) *The logics* **S4**, **S4.1** *and* **S4.2** *have no finite basis for the admissible rules.*

[52] also shows that the logics **K4**, **K4.1**, **K4.2**, and **G** have no basis for admissible rules.

6.4 Decidability of the Admissibility of a Rule

Lemma 6.4 can be strengthened. A rule is admissible in L with finite model property iff it is valid in $\mathfrak{Ch}_L(n)$, where n is the number of variables occurring in the rule. We obtain the following.

Theorem 6.14 (Rybakov) *Let $L \supseteq$ **K4** be finitely axiomatizable. Suppose that $\mathcal{V}(L)$ is locally finite. Then the admissibility of a given rule in L is decidable.*

6.5 Structural Completeness

For a class $\mathcal{K}$ of algebras, $\mathcal{K}^Q$ denotes the least quasi-variety containing $\mathcal{K}$. The following is a useful criterion.

Theorem 6.15 (Rybakov) *A modal logic $L \subseteq$ **K4** is structurally complete iff every subdirectly irreducible $\mathfrak{A} \in \mathcal{V}(L)$ is contained in $(\mathfrak{Fr}_L(\omega))^Q$. This is the case iff $\mathcal{V}(L) = (\mathfrak{Fr}_L(\omega))^Q$.*

If L is a logic that has the finite model property then the free algebra $\mathfrak{Fr}_L(\omega)$ is a subalgebra of the product of the finite subdirectly irreducible L-algebras. Under this condition, a logic L is structurally complete iff every finite subdirectly irreducible L-algebra is embeddable into the algebra $\mathfrak{Fr}_L(\omega)$ (or some $\mathfrak{Fr}_L(n)$, $n \in \omega$). Suppose $\mathfrak{A}$ is a finite, subdirectly irreducible **K4**-algebra. Then $\mathfrak{A}$ has an opremum ω. Now, for each element a of $\mathfrak{A}$ let p_a be a variable and let $r(\mathfrak{A})$ be the following rule:

(141) $$r(\mathfrak{A}) := \frac{\{p_{a*b} \leftrightarrow p_a * p_b : a, b \in A\} \cup \{p_{\circ a} \leftrightarrow \circ p_a : a \in A\} \cup \{p_1\}}{p_\omega}$$

where $*$ runs through all the basic binary connectives and $\circ$ through all the basic unary connectives. This is the **quasi-characteristic** inference rule of $\mathfrak{A}$. Now the following holds:

Theorem 6.16 (Citkin) *Let $\mathfrak{A}$ be a finite, subdirectly irreducible* **K4***-algebra. Then for any* **K4***-algebra $\mathfrak{B}$, $r(\mathfrak{A})$ is invalid in $\mathfrak{B}$ iff $\mathfrak{A}$ is isomorphically embeddable into $\mathfrak{B}$.*

This technique is reminiscent of the technique of splittings (see [10] and [11]).

It is not hard to show that no **K4**-algebra with at least two elements is embeddable into $\mathfrak{Fr}_{\mathbf{K4}}(\omega)$. Armed with this result one can show that there are infinitely many admissible rules which are independent from each other. One has to show only that there are infinitely many simple, finite **K4**-algebras. On the other hand, the set of admissible quasi-characteristic rules of **S4** and **Grz** have a finite basis. In the latter case the generalized Mints' rule alone forms a basis:

$$\frac{[(p \to q) \to (q \vee r)] \vee u}{[((p \to q) \to q) \vee ((p \to q) \to r)] \vee u} \tag{142}$$

For **S4** we need in addition to the modal translation of this rule two more, one of which is the quasi-characteristic rule of the two element cluster, which is equivalent to the rule $\Diamond p, \Diamond\neg p/q$, which we have already met above.

This can be brought to bear on extensions of **S4.3** in the following way.

Lemma 6.17 *Let L be a modal logic containing* **S4.3** *and $\mathfrak{A}$ a finite, subdirectly irreducible L-algebra. Then $\mathfrak{A} \times \mathbf{2}$ is a subalgebra of $\mathfrak{Fr}_L(\omega)$, where* **2** *is the two-element* **S4***-algebra.*

Lemma 6.18 *The rule $\Diamond p, \Diamond\neg p/q$ is valid in $\mathfrak{A}$ iff the algebra of the two element cluster is not embeddable into $\mathfrak{A}$.*

Now, any extension L of **S4.3** is finitely axiomatizable and has the finite model property, by results of [7] and [17]. L has the property of branching below 1 and the effective m-drop point property for some m. It follows that the admissibility of inference rules is decidable for L. Second, if we add the rule $\Diamond p, \Diamond\neg p/q$ then the resulting consequence relation axiomatizes the quasi-variety containing all finite L-algebras of the form $\mathfrak{A} \times \mathbf{2}$. Since L is determined by such algebras, we see that this quasi-variety contains $\mathfrak{Fr}_L(\omega)$. Moreover, since the smallest quasi-variety containing $\mathfrak{Fr}_L(\omega)$ must contain these algebras, the two are equal.

Theorem 6.19 (Rybakov) *Let $L \supseteq$* **S4.3***. Then $\vdash_L^m$ is axiomatized over $\vdash_L$ by (MN) and $\Diamond p, \Diamond\neg p/q$.*

We derive that **Grz.3** is structurally complete, since the rule $\Diamond p, \Diamond\neg p/q$ is actually derivable in $\Vdash_{\mathbf{Grz.3}}$. It follows that **LC** is also structurally complete, since $\mathbf{Grz.3} = \sigma(\mathbf{LC})$.

Call a logic L **hereditarily structurally complete** if all its extensions are structurally complete. L is **structurally precomplete** if it is not structurally complete, but all its proper extensions are.

Theorem 6.20 (Rybakov) *There are exactly 20 structurally precomplete logics containing* **K4***, and they are all tabular of the form* $\mathrm{Th}(\mathfrak{H}_i)$*, $i < 20$.*

The Kripke-frames $\mathfrak{H}_0 - \mathfrak{H}_{19}$ mentioned in the theorem are known. (They are for example of width and depth at most 3.) We derive the following.

Corollary 6.21 *$L \supseteq$* **K4** *is hereditarily structurally complete iff $L \supseteq \mathbf{K4}/\{\mathfrak{H}_i : i < 20\}$.*

From this, results on **S4** and **Int** can be immediately derived (since the frames are explicitly known). All these logics must be of width 2.

7 Further Topics

There are notions of consequence that are not included in this study that we shall mention here only briefly. First, a **multiple conclusion rule** is a pair $\langle \Delta, \Theta \rangle$ of sets of formulae. A multiple conclusion rule is **derived** in $\vdash$ if whenever Δ is made true by a substitution, that substitution makes one member of Θ true. It is **admissible** in L if for every substitution such that $\Delta^\sigma \subseteq L$ we have $\theta^\sigma \in L$ for at least one $\theta \in \Theta$. A case in point is the pair $\langle \{p \vee q\}, \{p, q\} \rangle$. This rule is admissible in intuitionistic logic but not in classical logic. Its reflex in modal logic is the rule $\langle \{\Box p \vee \Box q\}, \{\Box p, \Box q\} \rangle$. A modal logic has the **disjunction property** if this rule is admissible. Since the disjunction property does not specify which of the alternatives holds, it is not characterizable in terms of single-conclusion rules. In [32] a modal logic is said to **provide the rule of disjunction** if all of the rules $\langle\{\Box \bigvee_{i<n} p_i\}, \{\Box p_i : i < n\}\rangle$ are admissible, and it is shown that if a logic provides the rule of disjunction then the canonical frame is generated by a single point, which is the set $\{\neg\Box\varphi : \varphi \notin L\}$. More on the disjunction property can be found in [9]. The multiple conclusion rules are more general than ordinary rules, which we may call also **single conclusion rules**.

There is also a **strong rule of disjunction**:

$$\langle\{\bigvee_{i<n} \Box p_i\}, \{p_i : i < n\}\rangle \tag{143}$$

(see [57]) and the **rule of margins**

$$\langle\{p \to \Box p\}, \{p, \neg p\}\rangle \tag{144}$$

(see [58]).

Another kind of rule is presented by the **irreflexivity rule**.

$$\frac{\neg(p \to \Diamond p) \to \varphi}{\varphi} \qquad \text{provided } p \text{ does not occur in } \varphi \tag{145}$$

This rule has been proposed in [22]. It is called irreflexivity rule since in tense logic adding that rule to a logic L gives the logic of the irreflexive frames of L. In ordinary modal logic this does not go through ([55]), unless one adds infinitely many of them (see [23]). See also the chapter on hybrid logics. The difference between this rule and the standard or multiple conclusion rules is the reference to variables, which we know from predicate logic but is quite uncommon elsewhere in propositional logic. The possibility of defining negative properties of frames using rules has been explored in [55]. mm

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