The Global Decidability of $\mathsf{K} \otimes \mathsf{K}.\Diamond\blacksquare p \to \blacksquare\Diamond p$ and Related Logics

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Let L be a modal logic. We write $\Gamma \Vdash \varphi$ if φ can be derived from the tautologies of L and Γ using (MP) and (MN$_\Box$), (MN$_\blacksquare$):

$$(1) \qquad \text{(MP)} \ \frac{\chi \quad \chi \to \varphi}{\varphi} \qquad \text{(MN}_\Box) \ \frac{\chi}{\Box\chi} \qquad \text{(MN}_\blacksquare) \ \frac{\chi}{\blacksquare\chi}$$

We say that L is **globally decidable** if there is an algorithm deciding whether or not $\Gamma \vDash \varphi$ for given finite Γ and formula φ. See [1] for general reference. It is helpful to observe that $\Gamma \Vdash \Box \bigwedge \Gamma$ and $\Gamma \Vdash \blacksquare \bigwedge \Gamma$; furthermore, if $\Gamma \Vdash \varphi \to \chi$ then $\Gamma \Vdash \Diamond\varphi \to \Diamond\chi$:

$$(2) \qquad \begin{aligned} \Gamma \Vdash & \varphi \to \chi \\ \Gamma \Vdash & \neg\chi \to \neg\varphi \\ \Gamma \Vdash & \Box(\neg\chi \to \chi\varphi) \\ \Gamma \Vdash & \Box\neg\chi \to \Box\neg\varphi \\ \Gamma \Vdash & \neg\Box\neg\varphi \to \neg\Box\neg\chi \\ \Gamma \Vdash & \Diamond\varphi \to \Diamond\chi \end{aligned}$$

We fix $L := \mathsf{K}_2 \oplus \Diamond\blacksquare p \to \blacksquare\Diamond p$. The language we use here is based on just $\neg$, $\wedge$, $\Box$, $\blacksquare$, everything else being an abbreviation.

Terminology. If $\neg\Box\varphi; \Delta$ is a set of formulae, then $\neg\varphi; \Delta_\Box$ is called a **successor set**. Define

$$\Delta_\Box := \{\chi : \Box\chi \in \Delta\} \tag{3}$$

$$\Delta_\blacksquare := \{\chi : \blacksquare\chi \in \Delta\} \tag{4}$$

In general, $\Box(\Delta_\Box) \subseteq \Delta$ and $\blacksquare(\Delta_\blacksquare) \subseteq \Delta$. A **saturated** set is a set of formulae such that (a) if $\varphi \wedge \chi \in \Delta$ then $\chi; \varphi \in \Delta$, (b) if $\neg\neg\varphi \in \Delta$ then $\varphi \in \Delta$, (c) if $\neg(\varphi \wedge \chi) \in \Delta$ then either $\neg\varphi \in \Delta$ or $\neg\chi \in \Delta$. A saturation of a set is a minimal superset that is saturated. The following is clear: a set is contradictory in propositional calculus iff all saturations contain some formula and its negation. (The latter we call **directly contradictory**.)

Let $L := \mathsf{K}\otimes\mathsf{K}.\Diamond\blacksquare p \to \blacksquare\Diamond p$. This logic is Sahlqvist and globally complete for the following frames.

$$(\forall xyz)(x \lhd y \wedge x \blacktriangleleft z \to (\exists w)(y \blacktriangleleft w \wedge z \lhd w)) \tag{5}$$

We show that it is globally decidable.

For that, pick Γ and φ. We wish to decide whether or not $\Gamma \vDash \varphi$.

(Preparation.)

We start with w_0, the root and set

$$\sigma(w_0) := \Gamma \cup \{\neg\varphi\} \tag{6}$$

Now guess a saturation $\sigma^*(w_0)$ of $\sigma(w_0)$. We shall try to create a frame in the form of a tableau using a nondeterministic algorithm. The frame is constructed in cycles, where each cycle adds new successors for previously entered nodes. The first cycle consists of the set $C_0 := \{w_0\}$. Inductively, we shall create sets C_i which are pairwise disjoint. C_{i+1} will be created on the basis of C_i alone. The frame created up to C_i will be based on the set $\bigcup_{j<i+1} C_i$.

Every node x is assigned two sets, $\sigma(x)$ and $\sigma^*(x)$. The set $\sigma^*(x)$ is a downward saturation of $\sigma(x)$, and both are subsets of $\mathrm{Sf}(\Gamma; \neg\varphi)^\neg$.

(Tableau Construction.) Let $C_{i+1} := \varnothing$. In step i do the following for all $x \in C_i$.

(Step 1) For every formula $\neg\Box\chi \in \sigma^*(x)$ add a new node y to C_{i+1} with $x \lhd y$ and put

$$\sigma(y) := \neg\chi; \sigma^*(x)_\Box; \Gamma$$

Guess a saturation $\sigma^*(y)$ of $\sigma(y)$.

(Step 2) For every formula $\neg\blacksquare\chi \in \sigma^*(x)$ add a new node z to C_{i+1} with $x \blacktriangleleft z$ and put

$$\sigma(z) := \neg\chi; \sigma^*(x)_{\blacksquare}; \Gamma$$

Guess a saturation $\sigma^*(z)$ of $\sigma(z)$.

(Step 3) For every pair y, z such that $x \lhd y$ and $x \blacktriangleleft z$ add a new node u to C_{i+1} and let $y \blacktriangleleft u$ and $z \lhd u$. Put

$$\sigma(u) := \sigma^*(y)_{\blacksquare}; \sigma^*(z)_{\Box}; \Gamma$$

Guess a saturation $\sigma^*(u)$ of $\sigma(u)$.

In Step 3 we can be a bit more economical and do the following: if $\sigma^*(y) = \sigma^*(y')$ and $\sigma^*(z) = \sigma^*(z')$ then the successor u' added such that $y' \blacktriangleleft u'$ and $z' \lhd u'$ shall be identical to the successor u in that step. This makes sure that at most $|2^{\mathrm{Sf}(\Gamma;\neg\varphi)^2}|$ points get entered at this step.

The construction **closes** at a node x if we cannot assign a saturated closure for $\sigma(x)$ that does not both contain a formula and its negation (ie has a direct contradiction). Equivalently, it closes if the set $\sigma^*(x)$ is contradictory in PC. If it closes at x, then remove the node and backtrack on the last choice point. A construction **closes for good** at x if no matter how we saturate, the construction closes at x or at some point added later than x. The construction **closes** if it closes at the root. A tableau is **open** if no node is directly contradictory. A tableau is **complete** if no more rules apply to it.

Lemma 1 *Supppose that y has been created in Step 1 or in Step 2 and that $\Gamma \Vdash_L \neg \bigwedge \sigma(y)$. Then $\Gamma \Vdash_L \neg \bigwedge \sigma^*(x)$. Suppose that u has been created in Step 3 and that $\Gamma \Vdash_L \neg \bigwedge \sigma(u)$ no matter what saturation is chosen for y and z. Then $\Gamma \Vdash_L \neg \bigwedge \sigma^*(x)$.*

Proof. Suppose y has been added through 1. Assume that $\Gamma \Vdash_L \neg \bigwedge \sigma(y)$. Now

$$\sigma(y) = \neg\chi; \sigma^*(x)_{\Box}; \Gamma \tag{7}$$

and so we may rearrange this to

$$\Gamma \Vdash_L \bigwedge(\sigma^*(x)_{\Box}; \Gamma) \to \chi \tag{8}$$

$$\Gamma \Vdash_L \Box(\bigwedge \sigma^*(x)_\Box; \Gamma) \to \Box\chi \tag{9}$$

Distributing boxes we get

$$\Gamma \Vdash_L \bigwedge \Box\sigma^*(x)_\Box \wedge \bigwedge \Box\Gamma) \to \Box\chi \tag{10}$$

However, $\Gamma \Vdash_L \bigwedge \Box\Gamma$, and $\sigma^*(x) \supseteq \Box\sigma^*(x)_\Box$, so we get

$$\Gamma \Vdash_L \bigwedge \sigma^*(x) \to \Box\chi \tag{11}$$

Since $\neg\Box\chi \in \sigma^*(x)$ this means that

$$\Gamma \Vdash_L \bigwedge \sigma^*(x) \to \bot \tag{12}$$

as required. Notice that if y is entered by Step 1 then

$$\Gamma \Vdash \bigwedge \sigma^*(x) \to \Diamond\sigma(y) \tag{13}$$

Step 2 is similar. Now assume that u has been added in Step 3. Assume that

$$\Gamma \Vdash_L \neg \bigwedge \sigma(u) \tag{14}$$

Then

$$\Gamma \Vdash_L \bigwedge \Gamma \wedge \bigwedge \sigma^*(y)_\blacksquare \to \neg \bigwedge \sigma^*(z)_\Box \tag{15}$$

Applying $\blacksquare$ we get

$$\Gamma \Vdash_L \blacksquare \bigwedge \Gamma \wedge \blacksquare \bigwedge \sigma^*(y)_\blacksquare \to \blacksquare\neg \bigwedge \sigma^*(z)_\Box \tag{16}$$

$\Gamma \Vdash_L \blacksquare \bigwedge \Gamma$, so we can drop that conjunct. Now apply $\Diamond$:

$$\Gamma \Vdash_L \Diamond\blacksquare \bigwedge \sigma^*(y)_\blacksquare \to \Diamond\blacksquare\neg \bigwedge \sigma^*(z)_\Box \tag{17}$$

Using the axiom we get

$$\Gamma \Vdash_L \Diamond\blacksquare \bigwedge \sigma^*(y)_\blacksquare \to \blacksquare\Diamond\neg \bigwedge \sigma^*(z)_\Box \tag{18}$$

From this we deduce

$$\Gamma \Vdash_L \Diamond \bigwedge \sigma^*(y) \to \blacksquare\neg\Box \bigwedge \sigma^*(z)_\Box \tag{19}$$

A fortiori

$$\Gamma \Vdash_L \Diamond \bigwedge \sigma^*(y) \to \neg\blacklozenge \bigwedge \sigma^*(z) \tag{20}$$

This we rewrite as

$$\Gamma \Vdash_L \neg\Diamond \bigwedge \sigma^*(y) \vee \neg\blacklozenge \bigwedge \sigma^*(z) \tag{21}$$

Now we enter the assumption that this is independent of the actual saturation chosen. Let $S(y)$ be the set of all downward saturations of $\sigma(y)$. Then

$$\Vdash \sigma(y) \leftrightarrow \bigvee_{O \in S(y)} \bigwedge O \tag{22}$$

Similarly for $\sigma(z)$. Fix a saturation O' of $\sigma(z)$. Now by independence from the saturation chosen for $\sigma(y)$ we get

$$\Gamma \Vdash_L \{\neg\Diamond \bigwedge O \vee \neg\blacklozenge \bigwedge O' : O \in S(y)\} \tag{23}$$

From this we deduce

$$\Gamma \Vdash_L (\bigwedge_{O \in S(y)} \neg\Diamond \bigwedge O) \vee \neg\blacklozenge \bigwedge O' \tag{24}$$

Alternatively,

$$\Gamma \Vdash_L \neg \bigvee_{O \in S(y)} (\Diamond \bigwedge O) \vee \neg\blacklozenge \bigwedge O' \tag{25}$$

or even

$$\Gamma \Vdash_L \neg\Diamond(\bigvee_{O \in S(y)} \bigwedge O) \vee \neg\blacklozenge \bigwedge O' \tag{26}$$

By (22), we deduce

$$\Gamma \Vdash_L \neg\Diamond \bigwedge \sigma(y) \vee \neg\blacklozenge \bigwedge O' \tag{27}$$

We repeat the same argument with z and get, finally,

$$\Gamma \Vdash_L \neg\Diamond \bigwedge \sigma(y) \vee \neg\blacklozenge \bigwedge \sigma(z) \tag{28}$$

Thus

$$\Gamma \Vdash_L \Diamond \bigwedge \sigma(y) \to \neg\blacklozenge \bigwedge \sigma(z) \tag{29}$$

Recall however that $x \blacktriangleleft z$ and $x \lhd y$ and that $\sigma^*(y)$ is a saturated closure of

$$\Gamma; \sigma^*(x)_\Box; \neg\chi \tag{30}$$

for some χ; and similarly for z. This means that

$$\Gamma \Vdash_L \bigwedge \sigma^*(x) \to \Diamond \bigwedge \sigma(y) \tag{31}$$

and, similarly, for z we get

$$\Gamma \Vdash_L \bigwedge \sigma^*(x) \to \blacklozenge \bigwedge \sigma(z) \tag{32}$$

Hence we have

$$\Gamma \Vdash_L \bigwedge \sigma^*(x) \to \bot \tag{33}$$

contradicting our assumption. □

This construction is potentially infinite. If so, it defines an infinite open and complete tableau. Assume that a complete open tableau has been constructed according to the above. Let $F := \bigcup_{i\in\omega} C_i$ be the set of its nodes, $\lhd$ and $\blacktriangleleft$ as in the tableau. Finally, put $\beta(p) := \{x : p \in \sigma^*(x)\}$. Then by induction it is shown that for all $x \in F$ and $\chi \in \mathrm{Sf}(\varphi)$:

$$\langle F, \lhd, \blacktriangleleft, \beta, x\rangle \vDash \chi \quad \Leftrightarrow \quad \chi \in \sigma^*(x) \tag{34}$$

Moreover, the frame is an L-frame. To see this, take three nodes x, y and z such that $x \lhd y$ and $x \blacktriangleleft z$. There is a unique i such that $x \in C_i$. Thus $y, z \in C_i \cup C_{i+1}$. y and z have been added either in C_i (through Step 3) or in C_{i+1} (through Steps **??** and 2). By construction, in Step 3, the nodes y and z are present in C_{i+1}, and a node u is added such that $y \blacktriangleleft u$ and $z \lhd u$. Moreover, no new successors of x will be created at later stages.

Lemma 2 *Suppose that there is an open complete tableau for $\Gamma \vDash \varphi$. Then there is an L–model $\mathfrak{M} = \langle F, \lhd, \blacktriangleleft, \beta\rangle$ and a $x \in F$ such that $\mathfrak{M} \vDash \Gamma$ and $\langle \mathfrak{M}, x\rangle \vDash \neg\varphi$.*

On the other hand, suppose we cannot produce a complete open tableau (finite or infinite). We show that $\Gamma \vDash \varphi$. For then there is a number ν such that all constructions below ν end in a direct contradiction. One can show by induction on the height of a node that the tableau closes for good at that node, so it closes for good at the root. This means that $\Gamma \vDash \neg \bigwedge \sigma(x)$ for every node, and so in particular $\Gamma \vDash \neg \bigwedge \neg\sigma(w)$, where w is the root. Since the only choice is in the saturation, let us note the following.

Lemma 3 *Suppose for a set Δ that for every downward saturation Δ^*: $\Gamma \Vdash \neg \bigwedge \Delta^*$. Then $\Gamma \Vdash \neg \bigwedge \Delta$.*

Thus, $\Gamma \vDash \neg\neg\varphi$, as promised.

This construction is nondeterministic because we choose saturated closures and it is potentially infinite. Thus, we need to know when we can stop the construction. This is where we need to look at the cycles. The first cycle is the tableaus consisting of one node and $\sigma(x) = \{\varphi\}$. We say that $y \in C_j$ is **covered** by x if

❶ $x \in C_i$ for $i < j$,

❷ $\sigma^*(x) = \sigma^*(y)$

❸ for every z such that $y \lhd z$ there is a u such that $x \lhd u$ and $\sigma^*(z) = \sigma^*(u)$.

❹ for every z such that $y \blacktriangleleft z$ there is a u such that $x \blacktriangleleft u$ and $\sigma^*(z) = \sigma^*(u)$.

Suppose we have constructed C_0, up to C_i. The termination condition is now this: if in C_i all nodes are covered (by some node in $\bigcup_{j<i} C_j$), the tableau is declared open and the construction terminates. It is clear that given this termination condition it takes at most $2^{3\,\mathrm{Sf}(\Gamma;\varphi)}$ many cycles until the termination condition is satisfied. Moreover, for each node, Steps 1 and 2 add at most $\mathrm{Sf}(\Gamma;\varphi)$ many new points, while Step 3 adds at most $2^{\mathrm{Sf}(\Gamma;\varphi)^2}$ points. If $n > 1$ then $2n + 2^{n^2} < 2^{n^2+1}$, so each step multiplies the number of nodes by $\kappa := 2^{\mathrm{Sf}(\Gamma;\varphi)^2+1}$ at most. So in total we have at most $\kappa^{2^{3\,\mathrm{Sf}(\Gamma;\varphi)}}$ steps.

Lemma 4 *Let*

$$n := 2^{3\,\mathrm{Sf}(\Gamma;\varphi)(\mathrm{Sf}(\Gamma;\varphi)^2+1)} \tag{35}$$

If there is an open tableau of size $\geq n$ then there is an infinite open tableau.

Proof. We have established that if there is a tableau of this size, it has a cycle that adds no uncovered points. Cut back to the stage where this cycle has been added and continue as follows. Pick for every node in C_i x a covering node $x^\star$. For every formula $\neg\Box\chi \in \sigma^*(x^\star)$ there is a unique node $x^\star(\neg\Box\chi)$ added as a successor of $x^\star$. Step 1 creates a node $x(\neg\Box\chi)$ as a successor of x. It is clear that $\sigma(x(\neg\Box\chi)) = \sigma(x^\star(\neg\Box\chi))$. So we may in fact put

$$\sigma^*(x(\neg\Box\chi)) := \sigma^*(x^\star(\neg\Box\chi)) \tag{36}$$

Similarly for $\neg\blacksquare\chi' \in \sigma^*(x^\star)$. And finally, suppose there is a pair of successors y and z such that $x \lhd y$ and $x \blacktriangleleft z$. Then by ❸ and ❹ we find $y^\star$ and $z^\star$ such that $x^\star \lhd y^\star$ and $x^\star \blacktriangleleft z^\star$, and $\sigma^*(y^\star = \sigma^*(y)$, $\sigma^*(z^\star) = \sigma^*(z)$. By construction, a (unique) node $x^\star(y^\star, z^\star)$ exists such that $y^\star \blacktriangleleft x^\star(y^\star, z^\star)$ and $z^\star \blacktriangleleft x^\star(y^\star, z^\star)$. Again, we get $\sigma(x(y, z)) = \sigma(x^\star(y^\star, z^\star))$ so we may put

$$\sigma^*(x(y, z)) = \sigma^*(x^\star(y^\star, z^\star)) \tag{37}$$

We shall show that every node of C_{i+1} is again covered. Case 1. w has been added in Step 3. Then it has no successors, so only ❶ and ❷ are revelant. By construction, $w = x(y, z)$ for some y and z; then w is covered by $x^\star(y^\star, z^\star)$. Case 2. w has been added in Step 1. Then $w = x(\neg\Box\chi)$ for some $\neg\chi$. We show that w is covered by $x^\star(\neg\Box\chi)$. ❶ and ❷ are clear. ❸ is clear since there is no node u such that $w \lhd u$. (Otherwise u has been entered in Step 3 for x and so $x \blacktriangleleft w$, which is false.) ❹: suppose that $w \blacktriangleleft u$. So $u = x(w, z)$ for some z such that $x \blacktriangleleft z$. By construction, we have $\sigma^*(x^\star(w^\star, z^\star)) = \sigma^*(x(w, z))$, and $w^\star \blacktriangleleft x^\star(w^\star, z^\star))$. So this node satisfies the requirements. So, again every node is covered, and we can continue the construction in the same way ad infinitum. □

It is clear that if we follow this strategy then for all nodes to come there will always be a role model. And they will guide the construction at their turn.

Theorem 5 *L is globally decidable.*

Proof. Let Γ and φ be given. Construct a tableau with Γ global and $\neg\varphi$ local. Case 1. After some number ν of steps (which can be calculated a priori by Lemma 4), the termination condition is satisfied. Then there is a construction of a (possibly infinite) model such that Γ holds globally, and

φ is false at the root. Case 2. The construction terminates in at most ν steps and does not close. Then the tableau gives a model such that Γ holds globally and φ is false locally. Case 3. The termination condition is never reached. This means that no open tableau of depth ν exists. This means that the construction closes. This means that $\Gamma \vDash \neg\varphi$. $\Box$

Theorem 6 *The logics $L \oplus p \to \Diamond p$, $L \oplus p \to \blacklozenge p$, $L \oplus \Diamond\Diamond p \to \Diamond p$, $L \oplus \blacklozenge\blacklozenge p \to \blacklozenge p$ are all decidable.*

Proof. We use the reduction technique. Consider the functions

$$X_T^{\Box}(\Delta) := \{\Box\chi \to \chi : \Box\chi \in \mathrm{Sf}(\Delta)\} \tag{38}$$

$$X_T^{\blacksquare}(\Delta) := \{\blacksquare\chi \to \chi : \blacksquare\chi \in \mathrm{Sf}(\Delta)\} \tag{39}$$

$$X_4^{\Box}(\Delta) := \{\Box\chi \to \Box\Box\chi : \Box\chi \in \mathrm{Sf}(\Delta)\} \tag{40}$$

$$X_4^{\blacksquare}(\Delta) := \{\blacksquare\chi \to \blacksquare\blacksquare\chi : \blacksquare\chi \in \mathrm{Sf}(\Delta)\} \tag{41}$$

We show that for $M := L \oplus p \to \Diamond p$

$$\Delta \Vdash \varphi \quad \Leftrightarrow \quad \Delta; X_T^{\blacksquare}(\Delta; \varphi) \Vdash \varphi \tag{42}$$

For a proof, assume that the right hand side fails. Then there is a frame $\langle F, \lhd, \blacktriangleleft\rangle$ (not necessarily finite) a valuation β and a node x such that $\langle F, \lhd, \blacktriangleleft, \beta\rangle \vDash \Delta; X_T^{\blacksquare}(\Delta; \varphi)$ but $\langle F, \lhd, \blacktriangleleft, \beta, x\rangle \vDash \neg\varphi$. Now put $\lhd' := \lhd \cup \{\langle x, x\rangle : x \in F\}$. Then $\langle F, \lhd', \blacktriangleleft\rangle$ also is an L–frame. It also is an M–frame. By induction on the complexity of χ it is shown that for all y reachable from x:

$$\langle F, \lhd', \blacktriangleleft, \beta, x\rangle \vDash \chi \quad \Leftrightarrow \quad \langle F, \lhd, \blacktriangleleft, \beta, x\rangle \vDash \chi \tag{43}$$

(See [1] for the argument.) Similarly for $X_T^{\Box}$. For $L \oplus \blacklozenge\blacklozenge p \to \blacklozenge p$ notice that if $\langle F, \lhd, \blacktriangleleft\rangle$ is an L–frame, so is $\langle F, \lhd^+, \blacktriangleleft\rangle$. For suppose that $x \lhd^+ y$ and $x \blacktriangleleft z$. Then $x \lhd x' \lhd x'' \lhd \cdots \lhd y$. Then, since the original is an L–frame, there exists a z' such that $z \lhd z'$ and $x' \blacktriangleleft z'$. Again, there exists a z'' such that $z' \lhd z''$ and $x'' \lhd z''$. And so on. So we get an u such that $x \lhd^+ u$ and $z \blacktriangleleft u$, as required. And dually for the transitive closure of $\blacktriangleleft$. $\Box$

References

[1] Marcus Kracht. *Tools and Techniques in Modal Logic*. Number 142 in Studies in Logic. Elsevier, Amsterdam, 1999.