Efficient Allocations under Model Uncertainty in Identified Models

Trading Models

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- each model is based on certain parameter values being true along with some particular causal mechanisms being the relevant ones for the decision at hand.
- Important assumption: the parameters can be identified (ex post) by events in Ω.

The Farmer

Climate change is making it harder to be a young farmer

"We have less options to work with, so we have to get more creative."

SimonSkafar / Getty Images

grist.org: "With climate change, it's hard to put your finger on single events," says Ben Whalen, who has farmed for three years at Bumbleroot Organic Farm near Portland, Maine. "But we're accepting the reality that the weather is just going to get more extreme and unpredictable. That's the mindset that we're adopting as we start planning for the future of the farm."

The Farmer

- Consider a young farmer deciding on plans for her orchards over a 20-30 year planning horizon: e.g., what type of fruit trees to plant, what complementary investments to make.
- The decision depends on the climate forecast for the planning horizon, in particular the annual distribution of variables like rainfall, temperature, sunshine.
- Given the planning horizon, the desiderata for the investment decision is the forecast of the distribution rather than the actual realization of these variables in a particular season.
- However, due to climate change in the offing, the climate forecast, that is the forecast as to which distribution will realize, is far from confident: a set of possibilities can be identified along with a rough guess about the chance of any one of them being realized.

SCIENCE, UNCERTAINTY AND THE COVID-19 RESPONSE

Boris Johnson Coronavirus Press Conference, by Pippa Fowles / Number 10 (CC by-nc-nd 2.0)

March 16th, 2020 Ian Scoones 5 Comments

One of the abiding images of the coronavirus (COVID-19) outbreak in the UK has been the Prime Minister, Boris Johnson, looking nervous and uncomfortable, flanked by his scientific advisors at the regular press conferences. With three white men in suits in a wood-panelled room, the aim presumably was to project a sense of control and authority. The rhetoric – that the government's response was always 'led by the science' – was reinforced.

A Virus

- Think of decision making in the face of a contagion engendered by a novel virus.
- The probabilistic forecast of an epidemiological model is contingent on a host of assumptions ranging from values of parameters describing characteristics of the virus, relevant ecological factors, routes of transmission, assumptions about government policy and guidance, behavioral responses to policy and information, etc.
- Fits of the model with various historical episodes give reason to have confidence in the probabilistic forecast conditional on such parameters, public policy and mechanisms.
- rate of reproduction, mode of transmission, infectious period etc. initially unknown

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- the model is identifiable because

$$\langle W \rangle_t = \int_0^t \sigma_s^2 ds \qquad P^\sigma - a.s.$$

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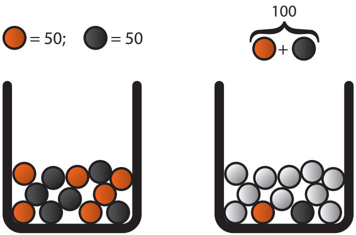
- $(\Omega, \mathcal{F})$ measurable space, states of the world
- $\mathcal{P}$ set of probability measures on $(\Omega, \mathcal{F})$, models
- $\mathcal{P}$ is identifiable, i.e. there exists a measurable mapping $k : \Omega \to \mathcal{P}$ with

$$k = P \qquad P - a.s.$$

  for all $P \in \mathcal{P}$

Examples

Ellsberg's Thought Experiment 1



Ellsberg Urn

- An urn contains 100 blue and red balls in unknown proportions
- composition of the urn is verifiable ex post
- $\omega = (c(olor), n(umberofredballs))$
- P_n: the urn contains n red balls
- $k(\omega) = P_n$

Examples

I.I.D. Experiments

- Sequence of independent and identical experiments with outcome (X_n)
- $E^{P_m} X_n = m$, mean m unknown
- Let

$$\tilde{m} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i .$$

Then $k = P_{\tilde{m}}$ identifies the unknown law

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The Smooth Model

How shall an agent evaluate uncertain consumption plans under uncertainty?

- Subjective Expected Utility: choose a belief $Q \in \mathcal{P}$ and take

$$U(X) = \mathbb{E}^Q u(X)$$

  for some Bernoulli utility function u that captures risk aversion

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- for ambiguity aversion $-\frac{\phi''(x)}{\phi'(x)} \to \infty$, we get the maxmin model
- Denti,Pomatto show that in identifiable models, the preference parameters can be uniquely identified from observed choices Cerreia, Maccheroni, Marinacci, Montrucchio, JET, 2013 establish the link to robust statistics

The Smooth Model

- Alternative representation:

$$U(X) = \int_{\mathcal{P}} v\left(c^P(X)\right)\mu(dP)$$

for $v = \phi \circ u$ and

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- second-order expected utility

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- we write $X_i = \left(X_i^P\right)$

Definitions

Definition

We say that $(X_i^P)_P$ is model-independent if we have

$$X_i^P(\omega) = X_i^Q(\omega)$$

for all states ω and all models $P, Q \in \mathcal{P}$.
We say that $(X_i^P)_P$ is ambiguity-free if $\left(X_i^P\right)_P$ is model-independent and we have

$$P[X_i^P \in \cdot] = Q[X_i^Q \in \cdot]$$

for all models $P, Q \in \mathcal{P}$ and $z \in \mathbb{R}$.

Aggregate Endowment and Allocations

Let $\bar{X}(\omega)$ be the aggregate endowment in state ω . We assume $\bar{X}$ is model-independent and drop the model index.
An allocation $\left(X_i^P\right)_i$ is feasible if $\sum_i X_i^P = \bar{X}$ for all P $P - a.s.$

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- P-conditionally efficient if for $P \in \mathcal{P}$, the allocation $\left(X_i^P\right)_i$ is Pareto efficient under model P, that is, there is no feasible allocation $\left(Y_i^P\right)_i$ such that $E^P\left(u_i\left(X_i^P\right)\right) \leq E^P\left(u_i\left(Y_i^P\right)\right)$ for every i, with at least one strict inequality.
  $\left(X_i^P\right)_{P,i}$ is conditionally efficient if it is P-conditionally efficient for all $P \in \mathcal{P}$.

The Optimization Problem

The following utilitarian welfare maximization problem characterizes efficient allocations for suitable individual weights $\lambda_i \geq 0$.

$$\begin{aligned} V\left(\bar{X}\right) = \max_{\left(X_i^P\right)_{P,i}} \quad & \sum_i \lambda_i U_i\left(\left(X_i^P\right)_P\right) \\ \text{subject to} \quad & \sum_i X_i^P \leq \bar{X} \quad \text{for all } P \in \mathcal{P} \end{aligned} \tag{1}$$

We call V the utility of the representative agent.

Conditionally Efficient Allocations

Recall the following results for expected utility economies

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- the allocation is comonotone
- if aggregate endowment is constant, efficient allocations are constant (full insurance)

Efficient Allocations

- As we allow model-contingent consumption, the problem separates across P

$$\max_{\left(X_i^P\right)_{P,i}} \sum_i \lambda_i U_i((X_i^P)_P)) = \int_{\mathcal{P}} \max_{\left(X_i^P\right)_{P,i}} \sum_i \lambda_i \phi_i \left(E^P u_i \left(X_i^P \right) \right) \mu(dP)$$

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- efficient allocations are conditionally efficient allocations!

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- *if* $(X_i^P)_i$ *is efficient under model* P*, then* $(X_i^P)_i$ *is efficient under model* Q *as well*
- *if the aggregate endowment* $\bar{X}$ *is unambiguous, then efficient allocations are also unambiguous.*

First-Order Conditions

- $$\psi(P,\omega) = \lambda_i \phi_i' \left(E^P u_i \left(X_i^P \right) \right) u_i' \left(X_i^P(\omega) \right) \tag{2}$$

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- The first-order necessary and sufficient condition for a feasible allocation $\left(X_i^P\right)_{P,i}$ to be conditionally efficient

$$\psi^P(\omega) = \eta_i^P u_i' \left(X_i^P(\omega) \right) \tag{3}$$

Representative Agent

Theorem

Define the utility possibility set

$$\mathcal{U}(P,\bar{X}) := \{v \in \mathbb{R}^I : \text{there exists a feasible allocation } (X_i) \text{ such that } v_i \leq E^P(u_i(X_i))\}. \quad (4)$$

For weights $\lambda_i > 0$*, define the function*

$$\Phi(P,\bar{X}) := \max_{(v_i)\in\mathcal{U}(P,\bar{X})} \sum_i \lambda_i \phi_i(v_i). \quad (5)$$

The representative agent's utility function (1) has the form

$$V(\bar{X}) = \int_{\mathcal{P}} \Phi(P,\bar{X})\,\mu(dP)\,.$$

Outline

Linear Risk Tolerance Economies

- For expected utility, Wilson, 1968 characterizes the class of utility functions that lead to linear risk sharing

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$$-\frac{u_i''(\xi)}{u_i'(\xi)} = \frac{1}{a_i + b\xi}, i = 1, ..., I$$

- risk tolerance, the inverse of risk aversion, is linear and the parameter b is common

$$u_i(\xi) = \begin{cases} \frac{(a_i+b\xi)^{1-1/b}}{1/b(1-1/b)} & \text{if} \quad b \neq 0, \quad b \neq 1 \\ -a_i e^{-\xi/a_i} & \text{if} \quad b = 0 \\ \log(a_i + \xi) & \text{if} \quad b = 1 \end{cases} \tag{6}$$

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- i.e. Let $\phi_i = v_i \circ u_i^{-1}$, that is, $\phi_i(t) \propto -(-t^{\gamma_i/\alpha_i})$. Ambiguity aversion is $A_i = \frac{\gamma}{\alpha_i}$.

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Let $(X_i^P)_P$ be an efficient allocation, then

1. *For each P, there is a $(\tau_i^P)_i$ such that $\sum_i \tau_i^P = 0$ and*

$$X_i^P = (\alpha/\alpha_i)\bar{X} + \tau_i^P$$

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2. *For every i there is a $(\kappa_i)_i \in \mathbb{R}^I$ such that $\sum_i \kappa_i = 0$ and for all P*

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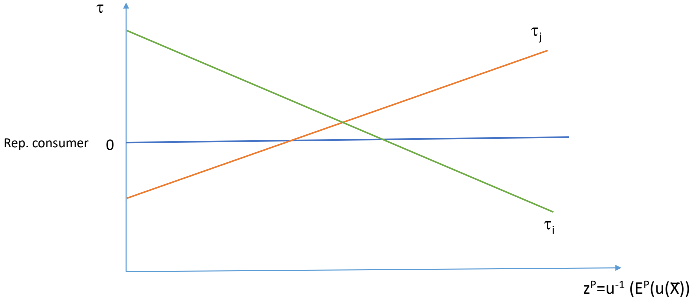
3. *The representative consumer's utility belongs to the smooth model class*

$$V(\bar{X}) = \int_{\mathcal{P}} \phi(E^P u(\bar{X}))\, \mu(dP)$$

where $\phi = v \circ u^{-1}$, v is CARA with parameter γ and $\phi(t) \propto -(-t^{\gamma/\alpha})$.

Model Insurance Payments in the CARA Case

Less ambiguity-averse consumers should be protected from the model uncertainty (the variability of the certainty equivalents of the aggregate consumption) by making their model-contingent constant term τ_i^P move in opposite directions to the certainty equivalents



i has a larger coefficient of amb. aversion than the rep. consumer. Receives a higher transfer in less optimistic models
j has a smaller coefficient of amb. aversion than the rep. consumer. Receives a higher transfer in more optimistic models

General CRRA-like Case

Theorem

Let $((X_i^P)_P)_i$ be an interim efficient allocation. Let $\zeta = \sum_i \zeta$. Then, there is a linear uncertainty sharing rule of the form

$$X_i^P = \theta_i^P(\bar{X} - \zeta) + \zeta_i.$$

A Nested Negishi–Approach For LRT Economies

- Recall that

$$U_i(X_i) = \int_{\mathcal{P}} v_i\left(u_i^{-1}(E^P u_i(X_i^P))\right)\mu(dP)$$

A Nested Negishi–Approach For LRT Economies

- Recall that

$$U_i(X_i) = \int_{\mathcal{P}} v_i \left(u_i^{-1}(E^P u_i(X_i^P)) \right) \mu(dP)$$

- Lemma: For linear risk tolerance, at the second-order level, one has to solve model by model

$$\Phi(P, \bar{X}) := \max_{(v_i):\sum c^i = c} \sum_i \lambda_i v_i(c_i) \tag{8}$$

where c is the certainty equivalent of aggregate endowment under model P

Shares θ_i^P in the Heterogeneous CRRA-Case

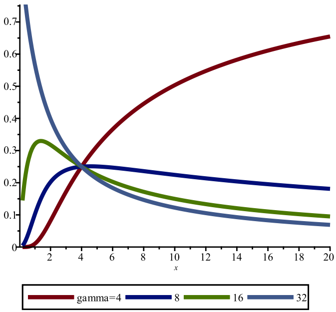


Figure: On the x-axis: welfare of nation, i.e.certainty equivalent of representative consumer for aggregate endowment. On the y-axis: share of surplus, i.e. excess endowment over subsistence levels

Shares θ_i^P in the Heterogeneous LRT case

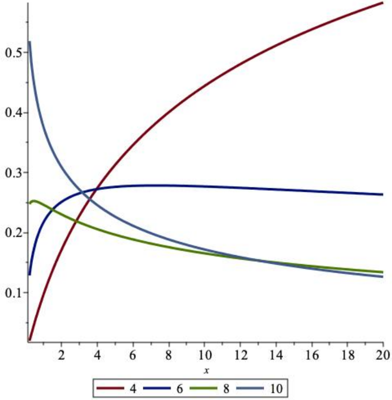


Figure: Four consumer economy with heterogeneous ambiguity aversion and common relative risk aversion 2

Outline

Pricing Kernel Puzzle

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- in Samuelson model, $\psi_t = \exp\left(-\theta W_t - \frac{\theta^2}{2} t\right)$, decreasing function of W_t (and of asset price S_t)
- empirical studies (Jackwerth (2000), Ait-Sahalia and Lo (2000)) suggest that this monotone relation does not hold true

Pricing Kernel Puzzle

- representative agent with smooth utility
- state price

$$\int_{\mathcal{P}} \phi'\left(E^P u\left(\bar{X}(s)\right)\right) u'(\bar{X}(s))P(s)\mu(dP)$$

Monotone Ordered Priors $\mathcal{P}$

- Assume that $\bar{X}(s) = s$ for every s. Write x for s.
- Assume that $\mathcal{P}$ is completely ranked according to the monotone likelihood ratio property.
- The conditional probability over models, one for each x, has MLRP:
  As x increases, the conditional probabilities are shifted from more pessimistic models to less pessimistic models, i.e., from model with smaller $E^P u\left(\bar{X}\right)$'s to those with larger $E^P u\left(\bar{X}\right)$'s.

Graph of the pricing kernel

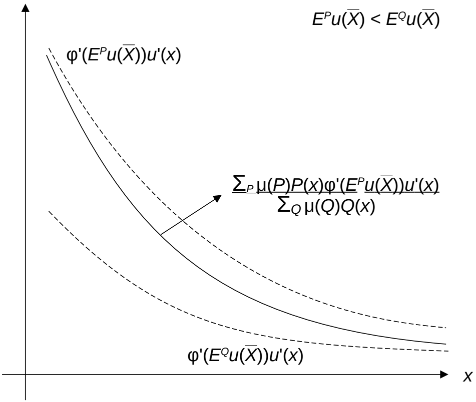

Figure: The pricing kernel is steeper than that derived solely from u'. The market price of risk, or the Hansen-Jagannathan bound, is higher.

A Regime-Switching Model

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- Aggregate endowment is lognormal. We consider a two person economy in which one agent is ambiguity neutral and the other one is very ambiguity averse.

Graph of the pricing kernel, two regimes

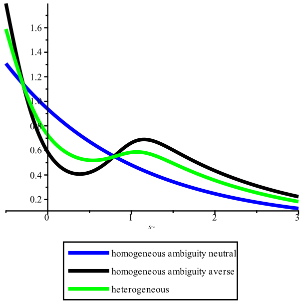

Figure: Pricing kernel in three economies: ambiguity-neutral, single agent ambiguity-averse, and mixed. Regime 1: mean 15 %, vola 10 %, Regime 2: mean -0.15 %, vola 40 %.

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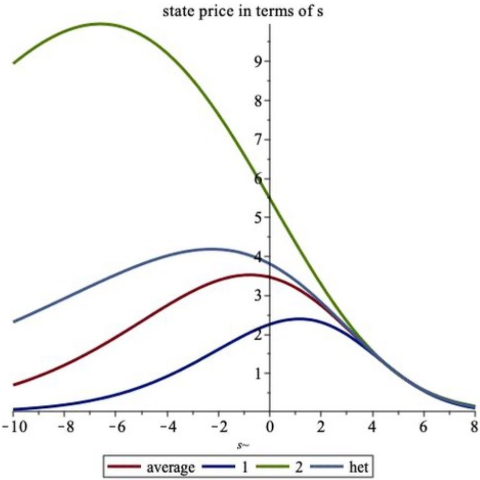
- and the variance parameter is uncertain
- In Bayesian Statistics, it is common to work with the precision, the inverse of the variance. For the precision, one commonly assumes a Gamma-distribution because the normal and the Gamma distributions form "conjugate priors"; the posterior of the precision is then also Gamma-distributed.

Graph of the pricing kernel, uncertain variance

Figure: The pricing kernel is non–monotone.



Conclusion

- We discuss efficient risk and uncertainty sharing under identifiable Knightian Uncertainty
- model-contingent trade is allowed
- efficient allocations are conditionally efficient, thus comonotone
- discussion of sharing rules under linear risk and ambiguity tolerance
- asset pricing implications