Relational semantics for the Lambek-Grishin calculus

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Abstract

We study ternary relational semantics for symmetric versions of the Lambek calculus with interaction principles due to Grishin (1983). We obtain completeness on the basis of a Henkin-style weak filter construction.[1]

1 Background, motivation

The categorial calculi proposed by Lambek and their current typelogical extensions respect an "intuitionistic" restriction: in a Gentzen presentation, Lambek sequents are of the form $\Gamma \Rightarrow B$, where B is a single formula, and Γ is a tree structure with formulas $A_1, \ldots, A_n$ at the yield. Depending on the particular calculus one works with, the antecedent structure can degenerate into a list or a multiset of formulas. The intuitionistic restriction is a serious expressive limitation when it comes to using the Lambek framework in the analysis of natural language syntax and semantics. Core phenomena such as displacement or scope construal are beyond the reach of the basic Lambek calculus; to deal with such phenomena, various extensions have been proposed based on structural rules, which can be introduced implicitly or explicitly, and with global or modally-controlled application regimes. The price one pays for such extensions is high: whereas the basic Lambek calculus has a polynomial recognition problem [3], already the simplest extension with an associative regime for $\otimes$ is known to be NP complete as shown in [8].

In a remarkable paper written in 1983, V.N. Grishin [4] has proposed a different strategy for generalizing the Lambek calculi. The starting point for Grishin's approach is a symmetric extension of the Lambek calculus: in addition to the familiar operators $\otimes, \backslash, /$ (product, left and right division), one also considers a dual family $\oplus, \oslash, \obslash$: coproduct, right and left difference.[2] The resulting vocabulary is given in (1).

$$
\begin{array}{llll}
A, B & ::= & p \mid & \text{atoms: } s \text{ sentence, } np \text{ noun phrases, } \ldots \\
 & & A \otimes B \mid B\backslash A \mid A/B \mid & \text{product, left vs right division} \\
 & & A \oplus B \mid A \oslash B \mid B \obslash A & \text{coproduct, right vs left difference}
\end{array}
\tag{1}
$$

Algebraically, the Lambek operators form a residuated triple; likewise, the $\oplus$ family forms a dual residuated triple. The *minimal* symmetric categorial grammar, which we will refer to as $\mathbf{LG}_\emptyset$,

[1]We thank Anna Chernilovskaya and the anonymous MOL'07 referees for helpful comments on an earlier version of this paper.

[2]A little pronunciation dictionary: read $B\backslash A$ as 'B under A', A/B as 'A over B', $B \obslash A$ as 'B from A' and $A \oslash B$ as 'A less B'. We follow [6] in using the notation $\oplus$ for the coproduct, which is a multiplicative operation.

consists of just the preorder axioms of (2), i.e. reflexivity and transitivity of the derivability relation, together with the (dual) residuation principles given in (3). This minimal system preserves the polynomiality of the asymmetric **NL** as shown in [2].

$$\text{(refl)} \quad A \vdash A; \qquad \text{from } A \vdash B \text{ and } B \vdash C \text{ infer } A \vdash C \quad \text{(trans)} \tag{2}$$

$$\begin{array}{ll} \text{(rp)} & A \vdash C/B \quad \text{iff} \quad A \otimes B \vdash C \quad \text{iff} \quad B \vdash A\backslash C \\ \text{(drp)} & B \oslash C \vdash A \quad \text{iff} \quad C \vdash B \oplus A \quad \text{iff} \quad C \oslash A \vdash B \end{array} \tag{3}$$

The minimal symmetric system $\mathbf{LG}_\emptyset$ doesn't have the required expressivity to address the linguistic problems mentioned in the introduction. For every theorem of the (non-associative) Lambek calculus, $\mathbf{LG}_\emptyset$ also has its image under arrow reversal. Interaction between the $\otimes$ and the $\oplus$ family, however, is limited to glueing together theorems of the two families with the transitivity rule.

What makes Grishin's work attractive from the perspective of categorial grammar, is the systematic theory he presents for extending $\mathbf{LG}_\emptyset$ with extra axioms. In section 2.7 of his paper, Grishin presents sixteen options for extending $\mathbf{LG}_\emptyset$. Eight of these represent the familiar associativity and/or commutativity postulates for $\otimes$ and symmetrically $\oplus$. Since these choices destroy sensitivity for word order and/or constituent structure, we will ignore them. The remaining eight options are principles of *interaction* relating connectives from the $\otimes$ and the $\oplus$ family. They naturally cluster in two groups of four, which we will refer to as $\mathcal{G}^\uparrow$ and $\mathcal{G}^\downarrow$.

Consider first the group $\mathcal{G}^\uparrow$ (the Class IV postulates, in Grishin's own terminology) which consists of the principles in (4). $G1$ and $G3$ have been called mixed associativity principles, $G2$ and $G4$ mixed commutativity principles. We think the use of the concepts "associativity" and "commutativity" is misleading here: as we will see below, the $\otimes$ and $\oplus$ families have individual interpreting relations of fusion and fission respectively. We prefer to refer to $G1$–$G4$ as (weak) distributivity principles.

$$\begin{array}{llll} (G1) & (A \oslash B) \otimes C \vdash A \oslash (B \otimes C) & C \otimes (B \oslash A) \vdash (C \otimes B) \oslash A & (G3) \\ (G2) & C \otimes (A \oslash B) \vdash A \oslash (C \otimes B) & (B \oslash A) \otimes C \vdash (B \otimes C) \oslash A & (G4) \end{array} \tag{4}$$

Intuitively, the interacion principles in (4) deal with the situation where a difference operation ($\oslash$ or $\oslash$) is *trapped* in a $\otimes$ context where they are inaccessible for logical manipulation. Consider first $G1$ and $G2$. On the lefthand side of the turnstile, a formula $A \oslash B$ occurs as the first or second coordinate of a product. The postulates invert the dominance relation between $\otimes$ and $\oslash$, raising the subformula A to a position where it can be shifted to the righthand side by means of the dual residuation principles of (3). $G3$ and $G4$ are the images of $G1$ and $G2$ under left-right symmetry.

Interaction principles *dual* to those in (4) are given in (5): they deal with the situation where a left or right implication is trapped within a $\oplus$ context, this time raising the A subformula to the position where it can be shifted to the lefthand side by means of the residuation principles of (3). We leave it to the reader to check that the forms Gn' in (5) are indeed derivable from the respective Gn in (4) and (2)-(3).

$$\begin{array}{llll} (G1') & (C \oplus B)/A \vdash C \oplus (B/A) & A\backslash(B \oplus C) \vdash (A\backslash B) \oplus C & (G3') \\ (G2') & (B \oplus C)/A \vdash (B/A) \oplus C & A\backslash(C \oplus B) \vdash C \oplus (A\backslash B) & (G4') \end{array} \tag{5}$$

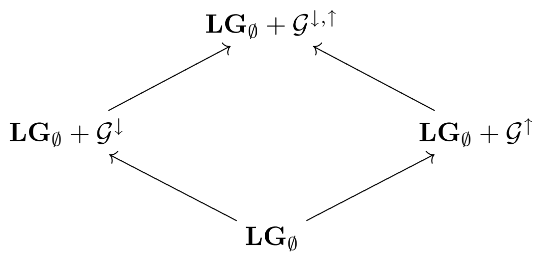


Figure 1: The Lambek-Grishin landscape

Consider next the group $\mathcal{G}^{\downarrow}$ (the Class I postulates in Grishin's classification), consisting of the interaction principles in (6) — the *converses* of the principles in group $\mathcal{G}^{\uparrow}$. $\mathcal{G}^{\uparrow}$ and $\mathcal{G}^{\downarrow}$ represent two independent options to extend $\mathbf{LG}_\emptyset$ with interaction principles. The general picture that emerges then is the landscape of Figure 1 where the minimal symmetric Lambek calculus is extended either with $G1$–$G4$ or with their converses, or with the combination of the two.

$$\begin{array}{llll} (G1^{-1}) & (A \obslash B) \otimes C \dashv A \obslash (B \otimes C) & C \otimes (B \oslash A) \dashv (C \otimes B) \oslash A & (G3^{-1}) \\ (G2^{-1}) & C \otimes (A \obslash B) \dashv A \obslash (C \otimes B) & (B \oslash A) \otimes C \dashv (B \otimes C) \oslash A & (G4^{-1}) \end{array} \quad (6)$$

Linguistic applications The choice we are making for the extension of $\mathbf{LG}_\emptyset$ is motivated by the linguistic application: we consider the full set of *structure-preserving* interaction principles, while rejecting same-sort associativity and/or commutativity options. Grishin's own paper opts for an associative regime, with both the same-sort associativity for $\otimes$ and $\oplus$, and the mixed associativity of the $\mathcal{G}^{\downarrow}$ group, i.e. $G1^{-1}$ and $G3^{-1}$ of (6). In the first thorough exposition of Grishin's work before it was translated in English, Lambek [6] also adopts the associative regime, but explores the mixed associativities of both the $\mathcal{G}^{\downarrow}$ and the $\mathcal{G}^{\uparrow}$ groups.

We give two simple illustrations of the potential of the Lambek-Grishin systems in Figure 1 to address the problems with asymmetric Lambek calculi mentioned in the introduction. The first is an example of non-local scope construal; the second is a case of non-peripheral extraction. In both cases, we start from a lexical type assignment from which the usual Lambek type is derivable, cf. the assignments in (7). What this means is that whatever could be done with the Lambek types can still be done; but thanks to the Grishin interaction principles, we will be able to do more.

$$\begin{array}{lrcl} \text{someone} & (s \oslash s) \obslash np & \vdash & s/(np\backslash s) \\ \text{which} & (n\backslash n)/((s \oslash s) \oplus (s/np)) & \vdash & (n\backslash n)/(s/np) \end{array} \quad (7)$$

In (8) one finds one of the two derivations for a sentence of the type 'Alice thinks someone left'. Whereas the Lambek assignment $s/(np\backslash s)$ is restricted to local construal in the embedded clause, the assignment $(s \oslash s) \obslash np$ also allows construal at the main clause level. It is this non-local construal that is represented by (8). By means of the interaction principle $G2$, the $(s \oslash s)$ subformula raises to the top level leaving behind a np resource *in situ*; $(s \oslash s)$ then shifts to the succedent by means of the dual residuation principle, and establishes scope via the dual application law. Notice that with only the mixed associative interactions, the $\obslash$ connective would be trapped in the $\otimes$ context, and the derivation would fail. The reader is invited to consult [1] for a detailed analysis of scope construal along these lines. Semantically, the analysis of [1] is based on a Curry-Howard interpretation of Lambek-Grishin derivations in the continuation-passing style; this interpretation associates (8) with the reading $(\exists\ \lambda x.((\mathsf{thinks}\ (\mathsf{left}\ x))\ \mathsf{a}))$ as required.

$$\dfrac{\dfrac{\dfrac{np \otimes (((np\backslash s)/s) \otimes (np \otimes (np\backslash s))) \vdash s \quad s \vdash (s \oslash s) \oplus s}{np \otimes (((np\backslash s)/s) \otimes (np \otimes (np\backslash s))) \vdash (s \oslash s) \oplus s}\ \text{trans}}{(s \oslash s) \obslash (np \otimes (((np\backslash s)/s) \otimes (np \otimes (np\backslash s)))) \vdash s}\ \text{drp}}{np \otimes (((np\backslash s)/s) \otimes (\underbrace{((s \oslash s) \obslash np)}_{\textsf{someone}} \otimes (np\backslash s))) \vdash s}\ G2 \tag{8}$$

The second example is for displacement as in '(movie which) John saw on TV'. In the derivation (9) we make use of the combined $\mathcal{G}^{\downarrow,\uparrow}$ principles, i.e. the principles (4) and their converses (6). We abbriviate $(np\backslash s)/np$ as tv (transitive verb) and $(np\backslash s)\backslash(np\backslash s)$ as adv (adverb). The (s/np) subformula is added to the antecedent via the dual residuation principle, and *lowered* to the target tv via applications of (Gn^{-1}). The tv context is then shifted to the succedent by means of the (dual) residuation principles, and the relative clause body with its np hypothesis in place is reconfigured by means of (Gn) and residuation shifting.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{np \otimes ((tv \otimes np) \otimes adv) \vdash s \quad s \vdash (s \oslash s) \oplus s}{np \otimes ((tv \otimes np) \otimes adv) \vdash (s \oslash s) \oplus s}\ \text{trans}}{tv \vdash ((np\backslash(s \oslash s))/adv) \oplus (s/np)}\ Gn, \text{rp}}{np \otimes ((tv \oslash (s/np)) \otimes adv) \vdash s \oslash s}\ \text{rp,drp}}{(np \otimes (tv \otimes adv)) \oslash (s/np) \vdash s \oslash s}\ Gn^{-1}}{np \otimes (tv \otimes adv) \vdash (s \oslash s) \oplus (s/np)}\ \text{drp} \tag{9}$$

An attractive property of the Lambek-Grishin systems in Figure 1 is that the expressivity resides entirely in the interaction principles: the composition operation $\otimes$ in itself (and the dual $\oplus$) allows no structural rules at all, which means that the resulting notion of wellformedness is fully sensitive to linear order and constituent structure of the grammatical material. It is shown in [7] that the relation of type similarity of $\mathbf{LG}_\emptyset + \mathcal{G}^\uparrow$ is as strong as similarity in (associative, commutative) **LP**: $A \sim B$ iff the images of A and B in a free Abelian group interpretation are equal. In **LP** one obtains this notion of $\sim$ by sacrificing order and constituent sensitivity; in the Lambek-Grishin setting, the same notion of similarity is obtained in a structure-preserving way.

2 Relational semantics

Let us turn now to the frame semantics for **LG**. In (10) and (11) we compare the truth conditions for the fusion and fission operations. From the modal logic perspective, the binary type-forming operation $\otimes$ is interpreted as an existential modality with ternary accessibility relation $R_\otimes$. The residual slashes are the corresponding universal modalities for the rotations of $R_\otimes$. For fission $\oplus$ and its residuals, the dual situation obtains: $\oplus$ here is the universal modality interpreted w.r.t. an accessibility relation $R_\oplus$; the coimplications are the existential modalities for the rotations of

$R_\oplus$. Notice that, in the minimal symmetric logic $\mathbf{LG}_\emptyset$, $R_\oplus$ and $R_\otimes$ are *distinct* accessibility relations. Frame constraints corresponding to the Grishin interaction postulates of the group $\mathcal{G}^\uparrow$ or $\mathcal{G}^\downarrow$ will determine how their interpretation is related.

$$\begin{array}{l} x \Vdash A \otimes B \text{ iff } \exists yz.R_\otimes xyz \text{ and } y \Vdash A \text{ and } z \Vdash B \\ y \Vdash C/B \text{ iff } \forall xz.(R_\otimes xyz \text{ and } z \Vdash B) \text{ implies } x \Vdash C \\ z \Vdash A\backslash C \text{ iff } \forall xy.(R_\otimes xyz \text{ and } y \Vdash A) \text{ implies } x \Vdash C \end{array} \tag{10}$$

$$\begin{array}{l} x \Vdash A \oplus B \text{ iff } \forall yz.R_\oplus xyz \text{ implies } (y \Vdash A \text{ or } z \Vdash B) \\ y \Vdash C \oslash B \text{ iff } \exists xz.R_\oplus xyz \text{ and } z \nVdash B \text{ and } x \Vdash C \\ z \Vdash A \obslash C \text{ iff } \exists xy.R_\oplus xyz \text{ and } y \nVdash A \text{ and } x \Vdash C \end{array} \tag{11}$$

Henkin construction To establish completeness, we use a Henkin construction. In the Henkin setting, "worlds" are (weak) *filters*: sets of formulas closed under $\vdash$. Let $\mathcal{F}$ be the formula language of (1). Let $\mathcal{F}_\vdash = \{X \in \mathcal{P}(\mathcal{F}) \mid (\forall A \in X)(\forall B \in \mathcal{F})\ A \vdash B \text{ implies } B \in X\}$. The set of filters $\mathcal{F}_\vdash$ is closed under the operations $(\cdot \mathbin{\widehat{\otimes}} \cdot)$, $(\cdot \mathbin{\widehat{\obslash}} \cdot)$ defined in (12) below. It is easy to show that $X \mathbin{\widehat{\otimes}} Y$ and $X \mathbin{\widehat{\obslash}} Y$ are indeed members of $\mathcal{F}_\vdash$.

$$\begin{array}{rcll} X \mathbin{\widehat{\otimes}} Y & = & \{C \mid \exists A, B\ (A \in X \text{ and } B \in Y \text{ and } A \otimes B \vdash C)\} & \\ X \mathbin{\widehat{\obslash}} Y & = & \{B \mid \exists A, C\ (A \notin X \text{ and } C \in Y \text{ and } A \obslash C \vdash B\}, & \text{alternatively} \\ X \mathbin{\widehat{\obslash}} Y & = & \{B \mid \exists A, C\ (A \notin X \text{ and } C \in Y \text{ and } C \vdash A \oplus B\} & \end{array} \tag{12}$$

To lift the type-forming operations to the corresponding operations in $\mathcal{F}_\vdash$, let $\lfloor A \rfloor$ be the principal filter generated by A, i.e. $\lfloor A \rfloor = \{B \mid A \vdash B\}$ and $\lceil A \rceil$ its principal ideal, i.e. $\lceil A \rceil = \{B \mid B \vdash A\}$. Writing $X^\sim$ for the complement of X, we have

$$(\dagger) \quad \lfloor A \otimes B \rfloor = \lfloor A \rfloor \mathbin{\widehat{\otimes}} \lfloor B \rfloor \qquad (\ddagger) \quad \lfloor A \obslash C \rfloor = \lceil A \rceil^\sim \mathbin{\widehat{\obslash}} \lfloor C \rfloor$$

PROOF $(\dagger)(\subseteq)$ Suppose $C \in \lfloor A \otimes B \rfloor$, i.e. $A \otimes B \vdash C$. With $A' := A$ and $B' := B$ we claim $\exists A', B'$ such that $A \vdash A'$, $B \vdash B'$ and $A' \otimes B' \vdash C$, which by (Def $\widehat{\otimes}$) means that $C \in \lfloor A \rfloor \mathbin{\widehat{\otimes}} \lfloor B \rfloor$ as desired. For the $(\supseteq)$ direction, we will prove the following lemma:

Lemma 1. $A \otimes B \in X$ implies $\lfloor A \rfloor \mathbin{\widehat{\otimes}} \lfloor B \rfloor \subseteq X$.

Since $A \otimes B \in \lfloor A \otimes B \rfloor$ by definition, we then have $\lfloor A \rfloor \mathbin{\widehat{\otimes}} \lfloor B \rfloor \subseteq \lfloor A \otimes B \rfloor$.

PROOF OF LEMMA 1. Suppose $C \in \lfloor A \rfloor \mathbin{\widehat{\otimes}} \lfloor B \rfloor$, i.e. $\exists A', B'$ such that $A' \in \lfloor A \rfloor$ i.e. $A \vdash A'$, $B' \in \lfloor B \rfloor$ i.e. $B \vdash B'$, and $A' \otimes B' \vdash C$. By Monotonicity, $A \otimes B \vdash A' \otimes B'$. By Transitivity, $A \otimes B \vdash C$. Together with $A \otimes B \in X$ this implies $C \in X$ as desired.

The $(\ddagger)$ case is entirely similar. $(\ddagger)(\subseteq)$ Suppose $B \in \lfloor A \obslash C \rfloor$, i.e. $A \obslash C \vdash B$. With $A' := A$ and $C' := C$ we claim $\exists A', C'$ such that $A' \vdash A$, $C \vdash C'$ and $A' \obslash C' \vdash B$, which by (Def $\widehat{\obslash}$) means that $B \in \lceil A \rceil^\sim \mathbin{\widehat{\obslash}} \lfloor C \rfloor$ as desired. For the $(\supseteq)$ direction, we show that the folloing holds:

Lemma 2. $A \obslash C \in X$ implies $\lceil A \rceil^\sim \mathbin{\widehat{\obslash}} \lfloor C \rfloor \subseteq X$.

Since $A \obslash C \in \lfloor A \obslash C \rfloor$ by definition, we then have $\lceil A \rceil^\sim \mathbin{\widehat{\obslash}} \lfloor C \rfloor \subseteq \lfloor A \obslash C \rfloor$.

PROOF OF LEMMA 2. Suppose $B \in \lceil A \rceil^\sim \mathbin{\widehat{\obslash}} \lfloor C \rfloor$, i.e. $\exists A', C'$ such that $A' \notin \lceil A \rceil^\sim$ i.e. $A' \vdash A$, $C' \in \lfloor C \rfloor$ i.e. $C \vdash C'$, and $A' \obslash C' \vdash B$. By Monotonicity, $A \obslash C \vdash A' \obslash C'$. By Transitivity, $A \obslash C \vdash B$. Together with $A \obslash C \in X$ this implies $B \in X$ as desired.

Canonical model Consider $\mathcal{M}^c = \langle W^c, R^c_\otimes, R^c_\oplus, V^c \rangle$ with

$W^c = \mathcal{F}_\vdash$

$R^c_\otimes XYZ$ iff $Y \mathbin{\widehat{\otimes}} Z \subseteq X$

$R^c_\oplus XYZ$ iff $Y \mathbin{\widehat{\obslash}} X \subseteq Z$

$V^c(p) = \{X \in W^c \mid p \in X\}$

Truth lemma We want to show for any formula $A \in \mathcal{F}$ and filter $X \in \mathcal{F}_\vdash$ that $X \Vdash A$ iff $A \in X$. The proof is by induction on the complexity of A. The base case is handled by V^c. Let us look first at the connectives $\oplus, \obslash, \oslash$.

Coproduct $X \Vdash A \oplus B$ iff $A \oplus B \in X$

($\Rightarrow$) Suppose $X \Vdash A \oplus B$. We have to show that $A \oplus B \in X$. By (Def $\oplus$) we have that $\forall Y, Z$ ($Y \mathbin{\widehat{\obslash}} X \subseteq Z$ and $Y \nVdash A$) implies $Z \Vdash B$. Setting $Y := \lceil A \rceil^\sim$ (therefore, $A \notin Y$ and, by IH for Y, $Y \nVdash A$) and $Z := Y \mathbin{\widehat{\obslash}} X$, the antecedent holds, implying $Z \Vdash B$. By IH and the choice of Z we then have $B \in Z$ and $B \in \lceil A \rceil^\sim \mathbin{\widehat{\obslash}} X$. By (Def $\widehat{\obslash}$) $B \in \lceil A \rceil^\sim \mathbin{\widehat{\obslash}} X$ means $\exists A_1, A_2$ such that $A_1 \notin \lceil A \rceil^\sim$, $A_2 \in X$ and $A_2 \vdash A_1 \oplus B$. $A_1 \notin \lceil A \rceil^\sim$ means $A_1 \vdash A$, hence from $A_2 \vdash A_1 \oplus B$ we get $A_2 \vdash A \oplus B$ by Transitivity. Since X is a filter, from $A_2 \in X$ and $A_2 \vdash A \oplus B$ we obtain $A \oplus B \in X$ as desired.

($\Leftarrow$) Suppose $A \oplus B \in X$. We have to show that $X \Vdash A \oplus B$, i.e. $\forall Y, Z$ ($R^c_\oplus XYZ$ and $Y \nVdash A$) implies $z \Vdash B$. Assume $R^c_\oplus XYZ$ and $Y \nVdash A$. We have to show $Z \Vdash B$. Using IH and the facts we already have ($R^c_\oplus XYZ$ and $A \notin Y$ and $A \oplus B \in X$) we conclude that $A \obslash (A \oplus B) \in Z$. But $A \obslash (A \oplus B) \vdash B$, so $B \in Z$ and by IH $Z \Vdash B$. This is what was needed to show.

Left difference $X \Vdash A \obslash B$ iff $A \obslash B \in X$

($\Rightarrow$) Suppose $X \Vdash A \obslash B$. We have to show that $A \obslash B \in X$. $X \Vdash A \obslash B$ means $\exists Y, Z$ such that $R^c_\oplus ZYX$, i.e. $Y \mathbin{\widehat{\obslash}} Z \subseteq X$, and $Y \nVdash A$ and $Z \Vdash B$. By IH $A \notin Y$ and $B \in Z$. Since also $A \obslash B \vdash A \obslash B$, from (Def $\widehat{\obslash}$) we conclude that $A \obslash B \in Y \mathbin{\widehat{\obslash}} Z$ and therefore $A \obslash B \in X$ as desired.

($\Leftarrow$) Suppose $A \obslash B \in X$. We have to show that $X \Vdash A \obslash B$. It was shown in Lemma 2 that $A \obslash B \in X$ implies $\lceil A \rceil^\sim \mathbin{\widehat{\obslash}} \lfloor B \rfloor \subseteq X$, which means we have $R^c_\oplus \lfloor B \rfloor \lceil A \rceil^\sim X$. Since $A \notin \lceil A \rceil^\sim$ and $B \in \lfloor B \rfloor$, by IH we claim $\exists Y, Z$ such that $R^c_\oplus ZYX$ and $Y \nVdash A$ and $Z \Vdash B$, which means $X \Vdash A \obslash B$ as desired.

Right difference $X \Vdash B \oslash A$ iff $B \oslash A \in X$

($\Rightarrow$) Suppose $X \Vdash B \oslash A$, i.e. $\exists Y, Z$ such that $X \mathbin{\widehat{\obslash}} Z \subseteq Y$ (Def $R^c_\oplus$) and $Y \nVdash A$ and $Z \Vdash B$, i.e. by IH $B \in Z$. To show that $B \oslash A \in X$, we reason by contradiction and assume $B \oslash A \notin X$. From this assumption and $B \in Z$ we have $(B \oslash A) \obslash B \in X \mathbin{\widehat{\obslash}} Z$ by (Def $\widehat{\obslash}$). Since $(B \oslash A) \obslash B \vdash A$, $A \in X \mathbin{\widehat{\obslash}} Z$, so also $A \in Y$. Contradiction with $Y \nVdash A$, hence the assumption $B \oslash A \notin X$ doesn't hold, as required.

($\Leftarrow$) Suppose $B \oslash A \in X$. To show that $X \Vdash B \oslash A$ we proceed by contraposition and assume $X \nVdash B \oslash A$, i.e. $\forall Y, Z$ ($R^c_\oplus ZXY$ and $Y \nVdash A$) implies $Z \nVdash B$, alternatively ($X \mathbin{\widehat{\obslash}} Z \subseteq Y$ and $Z \Vdash$

B) implies $Y \Vdash A$. Setting $Y := X \mathbin{\widehat{\oslash}} Z$ and $Z := \lfloor B \rfloor$, the antecedent holds, hence $X \mathbin{\widehat{\oslash}} Z \Vdash A$ and by IH $A \in X \mathbin{\widehat{\oslash}} Z$. By (Def $\widehat{\oslash}$) this means $\exists A_1, A_2$ such that $A_1 \notin X$, $A_2 \in \lfloor B \rfloor$ and $A_2 \vdash A_1 \oplus A$. ¿From $A_2 \in \lfloor B \rfloor$ we have $B \vdash A_2$, so by Transitivity, $B \vdash> A_1 \oplus A$, and by Dual residuation, $B \oslash A \vdash A_1$. Since $A_1 \notin X$, $B \oslash A \notin X$, contradicting our original assumption.

For the $\otimes, /, \backslash$ connectives, we refer to [5] (Theorem 3.3.2, p 75), repeated here for convenience.

Product $X \Vdash A \otimes B$ iff $A \otimes B \in X$

($\Rightarrow$) Suppose $X \Vdash A \otimes B$, i.e. $\exists Y, Z$ such that $Y \mathbin{\widehat{\otimes}} Z \subseteq X$, $Y \Vdash A$ and $Z \Vdash B$. By IH, $A \in Y$ and $B \in Z$. Since $A \otimes B \vdash A \otimes B$, by (Def $\widehat{\otimes}$) we have $A \otimes B \in X$ as desired.

($\Leftarrow$) Suppose $A \otimes B \in X$. In Lemma 1 we have shown that this implies $\lfloor A \rfloor \mathbin{\widehat{\otimes}} \lfloor B \rfloor \subseteq X$, i.e. $R^c_\otimes X \lfloor A \rfloor \lfloor B \rfloor$ by (Def $R^c_\otimes$). Since $A \in \lfloor A \rfloor$, $B \in \lfloor B \rfloor$, by IH we have $\lfloor A \rfloor \Vdash A$, $\lfloor B \rfloor \Vdash B$. By the truth condition for $\otimes$ this means $X \Vdash A \otimes B$ as desired.

Right division We do $X \Vdash A \backslash B$ iff $A \backslash B \in X$. The / case is symmetric.

($\Rightarrow$) Suppose $X \Vdash A \backslash B$, i.e. $\forall Y, Z$ if $R^c_\otimes ZYX$ and $Y \Vdash A$ then $Z \Vdash B$. Putting $Y := \lfloor A \rfloor$ and $Z := \lfloor A \rfloor \mathbin{\widehat{\otimes}} X$, since $\lfloor A \rfloor \mathbin{\widehat{\otimes}} X \subseteq \lfloor A \rfloor \mathbin{\widehat{\otimes}} X$ we have $R^c_\otimes ZYX$ by (Def $R^c_\otimes$), and since $A \in \lfloor A \rfloor$ also $\lfloor A \rfloor \Vdash A$ by IH, hence $\lfloor A \rfloor \mathbin{\widehat{\otimes}} X \Vdash B$, and by IH $B \in \lfloor A \rfloor \mathbin{\widehat{\otimes}} X$. By (Def $\widehat{\otimes}$) this means $\exists C, D$ such that $C \in \lfloor A \rfloor$ i.e. $A \vdash C$, $D \in X$ and $C \otimes D \vdash B$. By Transitivity, $A \otimes D \vdash B$ and by Residuation, $D \vdash A \backslash B$. Hence $A \backslash B \in X$ as desired.

($\Leftarrow$) Suppose $A \backslash B \in X$. We have to show that $X \Vdash A \backslash B$, i.e. $\forall Y, Z$ if $R^c_\otimes ZYX$ and $Y \Vdash A$ then $Z \Vdash B$. Suppose the antecedent holds, which means $Y \mathbin{\widehat{\otimes}} X \subseteq Z$ by (Def $R^c_\otimes$) and $A \in Y$ by IH. Together with $A \backslash B \in X$ we have $A \otimes (A \backslash B) \in Z$ by (Def $\widehat{\otimes}$). Since $A \otimes (A \backslash B) \vdash B$, also $B \in Z$. By IH $Z \Vdash B$ which means the consequent of the truth condition for $\backslash$ holds, hence $X \Vdash A \backslash B$ as desired.

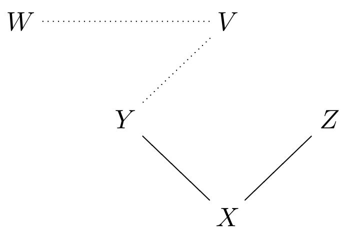
This establishes the Truth Lemma, from which completeness immediately follows.

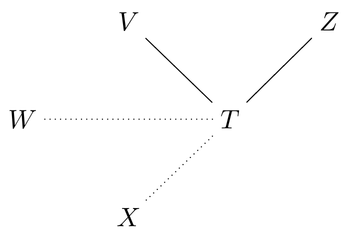
Theorem Completeness of $\mathbf{LG}_\emptyset$. If $\models A \vdash B$, then $A \vdash B$ is provable in $\mathbf{LG}_\emptyset$.

PROOF Suppose $A \vdash B$ is not provable. Then, by the Truth Lemma, $\mathcal{M}^c, A \nVdash B$. Since $\mathcal{M}^c, A \Vdash A$, we have $\mathcal{M}^c \not\models A \vdash B$, and hence $\not\models A \vdash B$.

Completeness of extensions with $\mathcal{G}^\uparrow$ and/or $\mathcal{G}^\downarrow$ In the minimal symmetric system, the $R_\otimes$ and $R_\oplus$ accessibility relations are distinct. For the extensions with Grishin interaction principles, we have frame constraints relating the interpretation of $R_\otimes$ and $R_\oplus$. Consider first the group $\mathcal{G}^\uparrow$. We take $(G1)$ as a representative: $(A \oslash B) \otimes C \vdash A \oslash (B \otimes C)$. The other axioms in the group are dealt with analogously. For $(G1)$ we have the constraint in (13) (where $R^{(-2)}xyz = Rzyx$).

$$\forall xyzwv \;\; (R_\otimes xyz \;\wedge\; R^{(-2)}_\oplus ywv) \Rightarrow \exists t \; (R^{(-2)}_\oplus xwt \;\wedge\; R_\otimes tvz) \tag{13}$$

(†) 

(‡) 

In (†) we depict $X \Vdash (A \obslash B) \otimes C$, with $W \nVdash A$, $V \Vdash B$ and $Z \Vdash C$; in (‡) $X \Vdash A \obslash (B \otimes C)$. Dotted lines represent $R^c_\oplus$, solid lines $R^c_\otimes$.

We have to show that in the Henkin model $\forall X, Y, Z, V, W$ construed as in (†), there is a fresh internal T connecting the root X to the leaves W, V, Z as in (‡). The solution $T := V \mathbin{\widehat{\otimes}} Z$ gives us $R^c_\otimes TVZ$ since $V \mathbin{\widehat{\otimes}} Z \subseteq V \mathbin{\widehat{\otimes}} Z$. To also show $R^c_\oplus TWX$, i.e. $W \mathbin{\widehat{\obslash}} T \subseteq X$, suppose $A' \in W \mathbin{\widehat{\obslash}} T$. We need to show that $A' \in X$. By (Def $\widehat{\obslash}$) $A' \in W \mathbin{\widehat{\obslash}} T$ means $\exists A_1, A_2$ such that $A_1 \notin W$, $A_2 \in T$ and $A_1 \obslash A_2 \vdash A'$. Since $T := V \mathbin{\widehat{\otimes}} Z$, $A_2 \in T$ means $\exists B_1, B_2$ such that $B_1 \in V$, $B_2 \in Z$ and $B_1 \otimes B_2 \vdash A_2$. Taking the configuration (†) together with $A_1 \notin W$ and $B_1 \in V$, we conclude $Y \Vdash A_1 \obslash B_1$ which in (†) together with $B_2 \in Z$ implies that $X \Vdash (A_1 \obslash B_1) \otimes B_2$. By the Truth Lemma, this means that $(A_1 \obslash B_1) \otimes B_2 \in X$ and since X is a filter and $(Gl1)$ an axiom, $A_1 \obslash (B_1 \otimes B_2) \in X$. But since $B_1 \otimes B_2 \vdash A_2$ we conclude that $A_1 \obslash A_2 \in X$. Together with $A_1 \obslash A_2 \vdash A'$, since X is a filter, we obtain $A' \in X$ as desired.

Consider next the group of interaction principles $\mathcal{G}^\downarrow$, the converses of $\mathcal{G}^\uparrow$. As a representative, we take $(Gl1)^{-1}$: $(A \obslash B) \otimes C \dashv A \obslash (B \otimes C)$.

This time, we have to show that in the Henkin model $\forall X, T, Z, V, W$ construed as in (‡), there is a fresh internal Y connecting the root X to the leaves W, V, Z as in (†). Let $Y := W \mathbin{\widehat{\obslash}} V$. Since $W \mathbin{\widehat{\obslash}} V \subseteq W \mathbin{\widehat{\obslash}} V$, $R^c_\oplus VWY$ holds. To show that also $R^c_\otimes XYZ$, i.e. $Y \mathbin{\widehat{\otimes}} Z \subseteq X$, suppose $A' \in Y \mathbin{\widehat{\otimes}} Z$, and let us show that $A' \in X$. By (Def $\widehat{\otimes}$), $A' \in Y \mathbin{\widehat{\otimes}} Z$ means $\exists A_2 B_1$ such that $A_2 \in Y$, $B_1 \in Z$ and $A_2 \otimes B_1 \vdash A'$. Since we had $Y := W \mathbin{\widehat{\obslash}} V$, $A_2 \in Y$ by (Def $\widehat{\obslash}$) means $\exists A_3 C_1$ such that $A_3 \notin W$, $C_1 \in V$ and $A_3 \obslash C_1 \vdash A_2$. Given that $C_1 \in V$ and $B_1 \in Z$, in the configuration (‡) we have $T \Vdash C_1 \otimes B_1$, and since $A_3 \notin W$, $X \Vdash A_3 \obslash (C_1 \otimes B_1)$. By the Truth Lemma this means that $A_3 \obslash (C_1 \otimes B_1) \in X$, and also $(A_3 \obslash C_1) \otimes B_1 \in X$, since X is a filter and we have $(Gl1)^{-1}$. Since $A_3 \obslash C_1 \vdash A_2$, we can conclude $A_2 \otimes B_1 \in X$, and since $A_2 \otimes B_1 \vdash A'$, also $A' \in X$ as desired.

3 Concluding remarks

We have established completeness for the minimal symmetric Lambek calculus $\mathbf{LG}_\emptyset$ and for its extension with interaction principles. The construction is neutral with respect to the choice between $\mathcal{G}^\uparrow$ and $\mathcal{G}^\downarrow$: it accommodates $G1$–$G4$ and the converses $G1^{-1}$–$G4^{-1}$ in an entirely similar way. In further research, we would like to consider more concrete models with a bias towards either $G1$–$G4$ or the converse principles, and to relate these models to the distinction between 'overt' and 'covert' forms of displacement, as illustrated in the examples of scope construal (8) and extraction (9).

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