Modal Horn Logics Have Interpolation

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Abstract

We shall show that the polymodal Sahlqvist logics corresponding to a set of Horn formulae have interpolation.

1 Introduction

This paper has been sparked off by a recent result by Rajeev Goré and Stéphane Demri that all grammar logics have interpolation. A **grammar logic** is a modal logic axiomatized by axioms of the form $\Box^{\sigma} p \to \Box_i p$, where σ is a string of modal operators. In this paper I show that this holds of a wider class of logics, those which are determined by a set of Sahlqvist formulae that are also Horn formulae.

2 Definitions

Notation and terminology is as in [1]. Our language is that of polymodal logic with any set B of basic modalities. We write $x \stackrel{i}{\Diamond\!\!\to} y$ in place of $x \; R(\Box_i) \; y$, and extend this to compound modalities. Compound modalities are here simply sequences $\sigma = i_0; i_1; \cdots; i_{p-1}$ where $i_j \in B$ for all $i < p$. Thus, $x \; R(\Box^{\sigma} \; y$ is synonymous $x \stackrel{\sigma}{\Diamond\!\!\to} y$. Sahlqvist logics determine first-order conditions on frames. We shall be interested in a very specific class. Recall that first-order Sahlqvist formulae have the following form. They are formed from atomic or constant formulae using $\wedge$, $\vee$ and restricted existential and universal quantifiers such that at least one variable in every matrix condition is quantified by a universal quantifier that is not in the scope of an existential. In this paper we are interested in logics whose class of frames is characterised by sets of formulae that are both Sahlqvist and Horn formulae. It is not hard to see the following. We first do a first-order characterisation and then proceed to a modal version. Call a formula a **tree formula** if it is formed from $\top$ using $\wedge$ and restricted existentials. Model theoretically such a formula χ is satisfiable in a frame iff $T(\chi)$ is embeddable where T is defined from the formula by induction as follows. (a) $T(\chi)$ is a one-point tree (the relation is empty), (b) $T(\Diamond\chi)$ is obtained by adding a node at the bottom of the tree to be the new root, (c) $T(\chi \wedge \chi')$ is obtained by fusing the two roots of $T(\chi)$ and $T(\chi')$.

Lemma 1 *A Sahlqvist formula is a Horn formula iff it is formed from any number of tree formulae and a single matrix $x\ R(\Box_i)\ y$ using $\wedge$ and restricted universal quantifiers.*

For a proof notice the following. A Horn formula has the form

$$(1) \qquad \xi = (\forall \vec{x})(\bigwedge_{i<n} \alpha_i \rightarrow \beta)$$

where α_i, $i < n$, and β are atomic. In order to convert this Sahlqvist form, the α_i must be restrictors of the variables, since they are negative. Hence they have the form $w\ R(\Box_i)\ w'$, where w' does not occur to the right of $R(\Box_j)$ in any of the α_i (but it may occur to the left). Hence, say that w' **immediately depends on** w if there is an i such that $\alpha_i = w\ R(\Box_j)\ w'$ for some w, w' and $j \in B$; let **dependency** be the transitive closure of immediate dependency. Then dependency is a tree order with root x_0. Now, let $\beta = x\ R(\Box_i)\ y$. Let $\vec{x} = x_0 \cdots x_{p-1}$. Let X be the set of variables on which x depends, Y the set of variables on which y depends, $D := X \cap Y$. Let the variables be numbered such that $X \cup Y = x_0 \cdots x_{q-1}$. Then

$$(2) \qquad \xi \equiv (\forall x_0 \cdots x_{q-1})((\exists x_q \cdots x_{p-1} \bigwedge_{i<n} \alpha_i) \rightarrow x\ R(\Box_i)\ y)$$

Now pick $i < n$. $\alpha = w\ R(\Box_j)\ w'$. Two choices arise: **Case 1.** $w' \in X \cup Y$. Then also $w \in X \cup Y$. In this case, α_i can be moved outside the scope of the existentials. Moreover, w' is bound by a universal quantifier, and α_i is its restrictor. **Case 2.** $w' \notin X \cup Y$. Then w' is bound by an existential, and α_i is its restrictor. This shows the claim.

We call a Sahlqvist Horn formula **pure** if it does not use any tree formula; equivalently, $X \cup Y = x_0 \cdots x_{p-1}$.

Definition 2 *Call L a **Horn logic** if it is Sahlqvist and determines a condition on its frames that is characterised by a set of Horn sentences.*

This class includes the following axioms: reflexivity, transitivity, symmetry, tense axioms, and grammar logics. All these formulae are also pure. The theorem below therefore generalises a theorem given in [1].

Using compound modalities, pure Sahlqvist Horn formulae can be defined as follows. They have the form

(3) $$(\forall xyzw)((x \overset{\rho}{\diamond\!\!\to} y \wedge y \overset{\sigma}{\diamond\!\!\to} z \wedge y \overset{\tau}{\diamond\!\!\to} w) \to w \overset{i}{\diamond\!\!\to} z)$$

with ρ, σ and τ are sequences. The corresponding Sahlqvist formula is as follows.

(4) $$\Box^{\rho}(\Diamond^{\sigma} p \to \Box^{\tau} \Diamond_i p)$$

We use an alternate of this formula, obtained by replacing p by $\neg$ and eliminating negations:

(5) $$\Box^{\rho}(\Diamond^{\tau} \Box_i p \to \Box^{\sigma} p)$$

We shall also call such modal formulae Horn formulae. The general case of a Sahlqvist Horn formula is as follows. Say that a constant formula χ is a tree formula if it is constant and uses only $\wedge$ and $\Diamond_i$, $i \in B$. Now, in addition to ordinary boxes allow also the use of $\Box^{\chi}_j$, which are defined as follows: $\Box^{\chi}_j \varphi := \Box_j(\chi \to \varphi)$ and call these **tree restricted basic modalities**. Extend the notion of compound modality to include tree restricted basic modalities. Then (4) and (5) can be used verbatim.

3 The Main Theorem

Theorem 3 *Modal Sahlqvist Horn logics have interpolation.*

We use the method of constructive reduction. Let L be Sahlqvist Horn. Recall that there is a function $X_L(\Delta)$ such that

(6) $$\Delta; X_L(\Delta; \varphi) \vdash_{\mathsf{K}} \varphi \qquad \Leftrightarrow \qquad \Delta \vdash_L \varphi$$

and for all Δ: $\mathrm{var}(X_L(\Delta)) \subseteq \mathrm{var}(\Delta)$. Also, the sets can be chosen finite if Δ, but we do not assume that $X_L(\Delta)$ is finite. Moreover, if X_L has the property

(7) $\quad X_L(\varphi; \delta) = X_L(\varphi) \cup X_L(\delta)$

then L has interpolation. We shall show that a function X_L satisfying (7) can be found. The proof is simple. Suppose $\varphi \vdash_L \psi$. Then $\varphi; X_L(\varphi; \psi) \vdash_L \psi$, and so $\varphi; X_L(\varphi); X_L(\psi) \vdash_{\mathsf{K}} \psi$. Now, either $X_L(\varphi)$ and $X_L(\psi)$ are already finite or else we may at this point choose finite subsets of them and continue with those in place of the original sets. From this we get $\varphi; X_L(\varphi) \vdash_{\mathsf{K}} \bigwedge X_L(\psi) \to \psi$. Now, $\mathrm{var}(\varphi; X_L(\varphi)) = \mathrm{var}(\varphi)$ and $\mathrm{var}(\bigwedge X_L(\psi) \to \psi)$. K has interpolation, and so there exists a χ such that $\mathrm{var}(\chi) \subseteq \mathrm{var}(\varphi) \cap \mathrm{var}(\psi)$ and $\varphi; X_L(\varphi) \vdash_{\mathsf{K}} \chi$ and $\chi \vdash_{\mathsf{K}} \bigwedge X_L(\psi) \to \psi$. From this we get $\varphi; X_L(\varphi) \vdash_L \chi$ and $\chi \vdash_L \bigwedge X_L(\psi) \to \psi$. Since $X_L(\Delta)$ is a set of theorems of L, $\varphi \vdash_L \chi$ as well as $\chi \vdash_L \psi$.

Let Δ be given. We know that Δ has a model based on a frame which is a tree of depth bounded by the modal depth of Δ. Let this tree be $\mathfrak{T} = \langle T, R \rangle$ with root w_0.

Lemma 4 *Let L be a Sahlqvist Horn logic. For every Kripke–frame* $\langle W, R \rangle$ *there is a least* $R^\heartsuit \supseteq R$ *such that* $\langle W, R^\heartsuit \rangle$ *is an L–frame.*

Proof. Clearly, the elementary axioms do not ask for the addition of new points; rather, they ask for the introduction of new relations between points. It is easy to define a function on the set of all functions $B \to W^2$ such that if R does not yet satisfy the axioms, then $f(R)$ consists in the addition of a single transition that is forced by the axioms. Then $R^\heartsuit$ is defined either as the limit of $f^n(R)$, or as the intersection of all f–closed sets. $\Box$

So, the process of adding relations is iterative. Notice that the Horn formulae say something like this: there is a tree T such that if this tree is embeddable into the frame then one additional relation obtains. Suppose that we start with a tree T_0. Then the first set of consequences are derived from embedding T into T_0 and then adding the new relations. This gives T_1, which may embed T in more ways, yielding additional patterns. This can be flattened into a single operation by closing the Horn formulae under derivability. This will mean that there is no easy way to choose a finite reduction set, but this is actually of no relevance for interpolation.

At this point it is perhaps instructive to see why pure Sahlqvist Horn formulae are not suitable for the proof (and why we have to move to the more complex Sahlqvist Horn formulae). Let $\delta_1 = (\forall xyz)((x \overset{a;b}{\Diamond\!\!\to} y \wedge x \overset{b}{\Diamond\!\!\to} z) \to (y \overset{c}{\Diamond\!\!\to} z))$,

$\delta_2 = (\forall xy)(x \overset{c;c}{\diamond\!\!\to} y \to x \overset{c}{\diamond\!\!\to} y)$ and $\delta_3 = (\forall xy)(x \overset{c}{\diamond\!\!\to} y \to y \overset{c}{\diamond\!\!\to} x)$. Let the following tree be given: $w_0 \overset{a}{\diamond\!\!\to} w_1 \overset{b}{\diamond\!\!\to} w_2$, $w_0 \overset{b}{\diamond\!\!\to} w_3$. Then, using δ_1 we get $w_0 \overset{c}{\diamond\!\!\to} w_3$, using δ_3 we get $w_3 \overset{c}{\diamond\!\!\to} w_0$, and finally through δ_2 we get $w_2 \overset{c}{\diamond\!\!\to} w_2$. Now, if the node w_2 had not been there, we would not have to add the transition $w_2 \overset{c}{\diamond\!\!\to} w_2$. Whether or not $w_2 \overset{c}{\diamond\!\!\to} w_2$ depends not only on the path between them but also on the existence of some additional path, here $w_0 \overset{b}{\diamond\!\!\to} w_3$.

Now we enter the definition of X_L. First, let H be the set of all Sahlqvist Horn formulae valid in L. We assume $\chi \in H$ to be of the form (5). Now set

(8) $$X_L(\Delta) := \{\Box^\rho(\Diamond^\tau\Box_i\delta \to \Box^\sigma\delta) : \Box_i\delta \in \mathrm{Sf}(\Delta), \Box^\rho(\Diamond^\tau\Box_i p \to \Box^\sigma p) \in H\}$$

This set has the property (7). (For $\Box_i\delta \in \mathrm{Sf}(\varphi;\delta)$ iff $\Box_i\delta \in \mathrm{Sf}(\varphi)$ or $\Box_i\delta \in \mathrm{Sf}(\delta)$.) We have to show that it also satisfies (6). The direction from left to right is clear. Therefore, assume

(9) $$\Delta; X_L(\Delta;\varphi) \nvDash_{\mathsf{K}} \varphi$$

Then there is a frame $\langle W, R\rangle$ such that

(10) $$\langle W, R, w_0, \beta\rangle \vDash \Delta; X_L(\Delta;\varphi); \neg\varphi$$

Let $R^\heartsuit$ be defined as follows. $R^\heartsuit$ is the set of all transitions $w \overset{i}{\diamond\!\!\to} z$ such that the following is satisfied:

(11) $$w_0 \overset{\rho}{\diamond\!\!\to} y; y \overset{\sigma}{\diamond\!\!\to} z; y \overset{\tau}{\diamond\!\!\to} w$$

Clearly, these transitions must be added in order to satisfy the first–order condition. Also, it is enough to add those, since these are all the transitions that can be deduced to exist. We show by induction on $\chi \in \mathrm{Sf}(\Delta;\varphi)$ that

(12) $$\langle W, R, \beta, z\rangle \vDash \chi \quad \Leftrightarrow \quad \langle W, R^\heartsuit, \beta, z\rangle \vDash \chi$$

There is only one problematic case, namely $\chi = \Box_i\delta$. ($\Leftarrow$) is clear. ($\Rightarrow$). So assume that we have

(13) $$\langle W, R, \beta, z\rangle \vDash \Box_i\delta$$

Pick w such that $z\ R^\heartsuit(\Box_i)\ w$. Two cases arise: $z\ R(\Box_i)\ w$ and so already $\langle W, R, \beta, w\rangle$. The second case is when the transition $z \overset{i}{\diamond\!\!\to} w$ has been added. By assumption this is the case because $\langle W, R\rangle$ satisfies

(14) $$w_0 \overset{\rho}{\diamond\!\!\to} y; y \overset{\sigma}{\diamond\!\!\to} z; y \overset{\tau}{\diamond\!\!\to} w$$

where ρ, σ and τ are sequences of tree restricted basic modalities. This means that

(15) $\langle W, R, \beta, w_0 \rangle \vDash \Box^{\rho}(\Diamond^{\tau}\Box_i\delta \to \Box^{\sigma}\delta)$

From this we get

(16) $\langle W, R, \beta, y \rangle \vDash \Diamond^{\tau}\Box_i\delta \to \Box^{\sigma}\delta$

By assumption, since $y \overset{\tau}{\Diamond\!\!\to} z$ and $\langle W, R, \beta, z \rangle \vDash \Box_i\delta$ we now have

(17) $\langle W, R, \beta, y \rangle \vDash \Box^{\sigma}\delta$

Now, since $y \overset{\sigma}{\Diamond\!\!\to} w$, we must have

(18) $\langle W, R, \beta, w \rangle \vDash \delta$

as required. It follows that for all w such that $z\ R^{\heartsuit}(\Box_i)\ w$ we have $\langle W, R, \beta, w \rangle \vDash \delta$. By induction hypothesis $\langle W, R^{\heartsuit}, \beta, w \rangle \vDash \delta$, for all w such that $x\ R(\Box_i)\ w$, from which $\langle W, R^{\heartsuit}, \beta, z \rangle \vDash \Box_i\delta$. This concludes the proof. $\Box$

References

[1] Marcus Kracht, *Tools and Techniques in Modal Logic*, Studies in Logic, no. 142, Elsevier, Amsterdam, 1999.