Normal monomodal logics can simulate all others

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Abstract

This paper shows that non–normal modal logics can be simulated by certain polymodal normal logics and that polymodal normal logics can be simulated by monomodal (normal) logics. Many properties of logics are shown to be reflected and preserved by such simulations. As a consequence many old and new results in modal logic can be derived in a straightforward way, sheding new light on the power of normal monomodal logic.

This paper is dedicated to our teacher, Wolfgang Rautenberg

§1. Introduction.

A simulation of a logic Λ by a logic Θ is a translation of the expressions of the language for Λ into the language of Θ such that the consequence relation defined by Λ is reflected under the translation by the consequence relation of Θ. A well–known case is provided by the Gödel translation, which simulates intuitionistic logic by Grzegorczyk's logic (cf. [12] and [6]). Such simulations not only yield technical results but may also provide a deeper understanding of the simulated logic. This is certainly the case with Gödel's translation which in effect translates the intuitionistic connectives by a modal rendering of the semantic acceptance clauses. In this paper we will use the simulation technique to obtain two types of reductions. One is the reduction of normal logics with several operators to mono–modal logics. The other is a reduction of non–normal logics to normal bimodal logics.

The first results concerning simulations of modal logics were the results of [29] and [30] where simulations are used to obtain substantial negative results in modal logic. Thomason shows how to simulate polymodal logics — even second order logic — in monomodal logic. Since counterexamples can be constructed much easier using several operators, this offers a rather easy method for systematically creating counterexamples in monomodal logic. Unfortunately, simulations were shown to preserve only negative properties of logics such as incompleteness, lack of finite model property etc. There is an array of problems raised in [15] and [14] which cannot be attacked this way because they require completeness properties to be preserved as well. We show here that this is in fact the case. We will apply this to solve some open problems in modal logic. Moreover, using these techniques it is proved in [19] that there exist logics which have finite model property locally but are globally incomplete. Other problems, such as the decidability of finite model property or of decidability itself have a straightforward solution in bimodal logics (using word problems). By appealing to the simulation method, the same is proved for monomodal logic. (See also [18].) This shows quite clearly the usefulness of simulations as a tool in modal logic. Moreover, we will show that the simulation defined by Thomason gives rise to an isomorphism from the lattice of bimodal normal logics onto an interval in the lattice of monomodal normal logics.

Not much is known about non–normal modal logics. Neighbourhood–semantics, which is usually applied to investigate them, does not allow to analyse modal formulas as first order properties. General completeness results, as for Sahlqvist–logics in the case of normal polymodal logics, are not known so far. In this situation it seems reasonable to investigate non–normal modal logics by simulating them as polymodal normal ones. This paper defines simulations of this type for all monotonic modal logics as well as a large class of classical modal logics. By applying Sahlqvist's Theorem to the bimodal interpretations we get a general completeness result for monotonic modal logics. Applied to normal logics the simulation gives us new insights into the relation between Kripke–semantics and neighbourhood–semantics. Conversely, undecidability results for monotonic modal logics can be used to derive corresponding results for normal bimodal systems. To obtain the required undecidability results for monotonic modal logics equational theories of lattices are interpreted in them.

The positive results on simulations show that there is no essential difference between the classes of monomodal normal logics, monotonic logics, and polymodal logics.

This paper is structured as follows. We begin by defining the notion of a simulation of a consequence by another and prove some general results. After that we define our notions and notation from modal logic. The paper then splits into two parts. The first is dedicated to the reduction of non–normal logics to bimodal normal logics. The second treats Thomason–type reductions of polymodal logics.

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§2. Simulations.

A **propositional language** consists of a (mostly) denumerable set of propositional variables $p_1, p_2, \ldots$ and a finite set of connectives $f_1, f_2, \ldots, f_n$. For propositional languages $\mathcal{L}^1, \mathcal{L}^2$ over the same set of variables an **interpretation** of $\mathcal{L}^1$ in $\mathcal{L}^2$ is a map which assigns to the variables uniformly a formula of $\mathcal{L}^2$ and to each formula $f(P_1, \ldots, P_k)$ of $\mathcal{L}^1$ uniformly a formula of $\mathcal{L}^2$. More precisely, an interpretation $(-)^F : \mathcal{L}^1 \longrightarrow \mathcal{L}^2$ must satisfy

$$(f(P_1, \ldots, P_k))^F = (f(p_1, \ldots, p_k))^F[P_1^F/p_1, \ldots, P_k^F/p_k]$$

for all connectives f of $\mathcal{L}^1$, $P_1, \ldots, P_k \in \mathcal{L}^1$, and variables $p_1, \ldots, p_k$. And it must satisfy, for all variables p, q,

$$q^F \;=\; p^F[q/p].$$

These definitions have some noteworthy consequences. First of all, a variable p is translated into an expression p^F which contains at most the variable p, that is, $var(p^F) \subseteq \{p\}$. For if $q \neq p$ we have $p^F = q^F[p/q]$, and so we find that $p \in var(p^F)$ iff $q \in var(q^F)$. Moreover, for $r \notin \{p, q\}$ we also have $r \in var(p^F)$ iff $r \in var(q^F)$. This can only hold if $var(p^F) = \{p\}$ for all p. Likewise, for any expression P we have $var(P^F) \subseteq var(P)$.

A **consequence (relation)** over a language $\mathcal{L}$ is a relation $\vdash$ between subsets of $\mathcal{L}$ and individual formulas satisfying the following postulates.

(ax) If $P \in X$ then $X \vdash P$.
(mon) If $X \subseteq Y$ and $X \vdash P$ then $Y \vdash P$.
(trs) If $X \vdash P$ for all $P \in Y$ and $Y \vdash Q$ then $X \vdash Q$.
(str) If $X \vdash P$ and σ is a substitution then $\sigma(X) \vdash \sigma(P)$.

The consequence $\vdash$ satisfies **replacement** if

(rep) $P_1 \dashv\vdash P_2$ implies $Q[P_1/p] \dashv\vdash Q[P_2/p]$.

(Here and in what follows $P_1 \dashv\vdash P_2$ abbreviates the conjunction of $P_1 \vdash P_2$ and $P_2 \vdash P_1$.) Now consider a consequence $\vdash_1$ over $\mathcal{L}^1$, a consequence $\vdash_2$ over $\mathcal{L}^2$ and an interpretation F of $\mathcal{L}^1$ in $\mathcal{L}^2$. Then $\vdash_2$ **simulates** $\vdash_1$ with respect to F, if for all $\Gamma \subseteq \mathcal{L}_1$ and $P \in \mathcal{L}_1$,

$$\Gamma \vdash_1 P \text{ iff } \Gamma^F \vdash_2 P^F.$$

The following is a fundamental property of simulations.

Proposition 1 *Suppose that $\vdash_2$ simulates $\vdash_1$ with respect to some interpretation F. Then if $\vdash_2$ is decidable, so is $\vdash_1$. Moreover, the complexity class of the decision problem for $\vdash_1$ is at most that of $\vdash_2$.*

For a proof just observe that by definition the problem $\Gamma \vdash_1 P$ is equivalent to $\Gamma^F \vdash_2 P^F$. Since the translation is linear in the size and increases the length only by a constant factor, the complexity class of the simulated problem is (at most) that of $\vdash_2$.

A priori, any connective can be translated by an arbitary expression. However, under mild conditions we can show that the interpretation of boolean connectives must be an expression equivalent to that boolean connective. In the case of modal logics this means that under these conditions only the modal operators receive a nontrivial interpretation. Call an interpretation F **atomic** if $p^F = p$ for all propositional variables p. In this case we will often write $f_F(P_1, \dots P_k)$ instead of the somewhat longwinded

$$f(p_1, \dots, p_k)^F[P_1/p_1, \dots, P_k/p_k]$$

Proposition 2 *Suppose that $\wedge, \neg \in \mathcal{L}^i$ and that $\vdash_i$ are consequences over $\mathcal{L}^i$, for $i \in \{1, 2\}$. Assume that $\vdash_i$, $i \in \{1, 2\}$, restricted to the language with connectives $\{\wedge, \neg\}$ both coincide with classical propositional logic, and that $\vdash_2$ simulates $\vdash_1$ with respect to an atomic interpretation F. Then (i) $p \wedge q \dashv\vdash_2 p \wedge_F q$ and (ii) $\neg p \dashv\vdash_2 \neg_F p$.*

Proof. (i) We have $p \wedge q \vdash_2 \{p, q\} \vdash_2 p \wedge_F q \vdash_2 \{p, q\} \vdash_2 p \wedge q$. (ii) It is readily checked that P is $\vdash_1$–inconsistent iff P^F is $\vdash_2$–inconsistent. Hence, $\{p, \neg_F p\}$ is $\vdash_2$–inconsistent, since $\{p, \neg p\}$ is $\vdash_1$–inconsistent. Hence $\neg_F p \vdash_2 \neg p$. It remains to show that $\{\neg\neg_F p, \neg p\}$ is $\vdash_2$–inconsistent. Using $q \vdash_2 ((p \wedge q) \vee (\neg p \wedge q))^F$ and (i) we have

$$\begin{aligned} & \neg\neg_F p \wedge \neg p \\ & \vdash_2 (p \wedge_F (\neg\neg_F p \wedge \neg p)) \vee_F (\neg_F p \wedge_F (\neg\neg_F p \wedge \neg p)) \\ & \vdash_2 (p \wedge_F \neg p) \vee_F (\neg_F p \wedge_F \neg\neg_F p). \end{aligned}$$

But the last formula is $\vdash_2$–inconsistent since both $p \wedge_F \neg p$ as well as $\neg_F p \wedge_F \neg\neg_F p$ are $\vdash_2$–inconsistent. $\dashv$

§3. Basic Facts and Terminology.

§3.1. Classical and Monotonic Logics.

The language $\mathcal{L}_n$ of n–modal propositional logic has the connectives $\wedge, \neg$ and $\Box_i$, $i < n$. The symbols $\vee$, $\rightarrow$, $\top$, and $\bot$ have the usual meaning. Sometimes we use fancy symbols such as $\blacksquare$, $\boxminus$ etc. instead of indexed boxes. A **classical** (n–)modal logic is a subset of $\mathcal{L}_n$ which contains all classical tautologies and is closed under substitutions, modus ponens and $p \leftrightarrow q / \Box_i p \leftrightarrow \Box_i q$ $(i < n)$. The smallest classical n–modal logic is denoted by $\mathbf{E}_n$. The smallest classical modal logic containing a classical modal logic Λ and a set of formulas Γ is denoted by $\Lambda + \Gamma$. Classical (n–)modal logics containing

$$\Box_i(p \wedge q) \rightarrow \Box_i q, \text{ for } i < n,$$

are called **monotonic** (n–)modal logics. The smallest (n–)monotonic logic is denoted by $\mathbf{M}_n$. Monotonic (n–)modal logics containing

$$\Box_i p \wedge \Box_i q \rightarrow \Box_i(p \wedge q) \text{ and } \Box_i \top, \text{for } i < n,$$

are called **normal** (n–)modal logics. The smallest n–modal logic is denoted by $\mathbf{K}_n$. We shall often write $\mathbf{K}$ for $\mathbf{K}_1$, $\mathbf{E}$ for $\mathbf{E}_1$ and $\mathbf{M}$ for $\mathbf{M}_1$. Recall that normal modal logics are precisely those classical modal logics which contain $\Box_i(p \rightarrow q) \rightarrow (\Box_i p \rightarrow \Box_i q)$ and are closed under (mn): $p/\Box_i p$, $i < n$.

The consequence relation associated with a classical modal logic Λ is defined by

$\Gamma \vdash_\Lambda P$ iff P is derivable from $\Lambda \cup \Gamma$ by modus ponens.

Clearly $\vdash_\Lambda$ satisfies replacement and we can apply Proposition 2. A modal logic Θ simulates a modal logic Λ with respect to an interpretation F if $\vdash_\Theta$ simulates $\vdash_\Lambda$ with respect to F.

We shall introduce neighbourhood semantics for monotonic 1–modal logics (monotonic modal logics, for short). (For more detailed introductions into neighbourhood semantics consult e.g. [8] and [9].) Let g be a set and N a map with domain g which assigns to each $x \in g$ a set of subsets of g. Then $\langle g, N\rangle$ is called a **neighbourhood–frame** (N–**frame**). The set of **neighbourhoods** of $\langle g, N\rangle$ is defined by putting

$$\mathcal{C}[\langle g, N\rangle] := \{a \subseteq g : (\exists x \in g)(a = N(x))\}.$$

Now let $\mathbf{G}$ be a set of subsets of g with $\mathcal{C}[\langle g, N\rangle] \subseteq \mathbf{G}$ which is closed under intersection, complement and

$$\Box^g a := \{y \in g | (\exists b \in N(y))(b \subseteq a)\}. \tag{1}$$

Then $\mathcal{G} = \langle g, N, \mathbf{G}\rangle$ is called a **general neighbourhood–frame** (**general** N–**frame**). (Clearly, N–frames are identified with general N–frames satisfying $\mathbf{G} = 2^g$.) Valuations β in general N–frames are homomorphisms from the algebra of formulas into the boolean algebra of sets, $\mathbf{G}$, which satisfy $V(\Box\phi) = \Box^g V(\phi)$. Mostly we shall consider general N^h–frames, i.e. general N–frames satisfying

$$(\forall a, b \in \mathbf{G})(\forall y \in g)(a \in N(y) \wedge a \subseteq b \Rightarrow b \in N(y)). \tag{2}$$

Note that in N^h–frames we have the following equation, for all $a \in \mathbf{G}$.

$$\Box^g a = \{y \in g | a \in N(y)\}. \tag{3}$$

Completeness of a classical modal logic with respect to general N–frames is defined as usual and we have

Proposition 3 *Each monotonic modal logic is determind by a class of general N^h–frames. Conversely, each class of general N–frames determines a monotonic modal logic.*

As usual, a monotonic logic is called **complete** if it is determined by N–frames (or, equivalently, by N^h–frames). It is readily checked that a monotonic modal logic Λ is complete with respect to a class $\mathcal{K}$ of general N–frames iff for all finite $\Gamma \subseteq \mathcal{L}_1$ and $P \in \mathcal{L}_1$: $\Gamma \vdash_\Lambda P$ iff for all $\mathcal{G} \in \mathcal{K}$, for all valuations β and all $x \in g$: $\langle \mathcal{G}, \beta, x\rangle \models \Gamma \Rightarrow \langle \mathcal{G}, \beta, x\rangle \models P$.

§3.2. Normal Logics.

For normal modal logics the semantics reduces considerably in its complexity as we can now have relations instead of neighbourhoods. A **1–frame** is a generalized monomodal frame, a **2–frame** is a generalized bimodal frame. A similar convention is used for **3–frames**, **polyframes**. A **kripke** n–**frame** is a pair $\mathfrak{f} := \langle f, \langle \lhd_i | i < n\rangle\rangle$, where f is a set and $\lhd_i \subseteq f^2$ a binary relation over f for each $i < n$. An n–**frame** is a pair $\langle \mathfrak{f}, \mathbf{F}\rangle$ where $\mathfrak{f} = \langle f, \langle \lhd_i | i < n\rangle\rangle$ is an kripke n–frame, and $\mathbf{F}$ a system of subsets of f closed under intersection, complements and the operations

$$\Box_i a := \{t | (\forall u)(t \lhd_i u \Rightarrow u \in a)\}$$

A subset of f is called **internal** in $\mathfrak{F}$ if it is a member of $\mathbf{F}$. A valuation into $\mathfrak{F}$ is a function β assigning to each variable an internal set. In the usual way, $\langle \mathfrak{F}, \beta, x\rangle \models P$ is defined by induction over P. Furthermore, we write $\langle \mathfrak{F}, x\rangle \models P$ if for all valuations β, $\langle \mathfrak{F}, \beta, x\rangle \models P$, and $\mathfrak{F} \models P$ if $\langle \mathfrak{F}, x\rangle \models P$ for all $x \in f$. Given a point $x \in f$ we call the **transit** of x in $\mathfrak{F}$ the set of points y such that there exists a chain $\langle x_i | i < p\rangle$ with $x = x_0$, $y = x_{p-1}$ and such that for all $i < p-1$ there exists a $j_i < n$ with $x_i \lhd_{j_i} x_{i+1}$. If the transit of x is the entire underlying set, f, then x is called a **root** of $\mathfrak{F}$ and $\mathfrak{F}$ is called **rooted at** x; $\mathfrak{F}$ is **rooted** if there exists a root. An n–**morphism** between n–frames $\mathfrak{F}$ and $\mathfrak{G}$ is a map $\phi : f \to g$ satisfying three conditions. (p1) If $x \lhd_i y$ for $x, y \in f$ then $\phi(x) \lhd_i \phi(y)$. (p2) If $\phi(x) \lhd_i w$ for $x \in f$, $w \in g$ there exists a $y \in f$ such that $x \lhd_i y$ and $\phi(y) = w$. (p3) For each internal set b of $\mathfrak{G}$ the preimage $\phi^{-1}[b] := \{x|\phi(x) \in b\}$ is an internal set of $\mathfrak{F}$. An n–**modal algebra** is a structure $\mathfrak{A} = \langle A, 1, -, \cap, \langle \Box_i | i < n\rangle\rangle$ for which the reduct to $\{\cap, -, 1\}$ is a boolean algebra, and such that for every $i < n$ and $a, b \in A$ we have $\Box_i 1 = 1$ and $\Box_i(a \cap b) = \Box_i a \cap \Box_i b$. Valuations are functions β from the set of variables into A. β can be naturally extended to a homomorphism from the free algebra of formulae into $\mathfrak{A}$, which we also denote by β. If a formula P receives the value 1 under β we write $\langle \mathfrak{A}, \beta\rangle \models P$; moreover, $\mathfrak{A} \models P$ if for all β, $\langle \mathfrak{A}, \beta\rangle \models P$. We put $\mathsf{Th}\,\mathfrak{A} := \{P|\mathfrak{A} \models P\}$; this is called the **theory** of $\mathfrak{A}$. For a class $\mathfrak{K}$ of algebras we put

$$\mathsf{Th}\,\mathfrak{K} := \bigcap_{\mathfrak{A}\in\mathfrak{K}} \mathsf{Th}\,\mathfrak{A}$$

This is a normal n–modal logic. Conversely, given an n–modal logic Θ, we let $\mathsf{Alg}\,\Theta$ be the class of algebras $\mathfrak{A}$ such that $\mathfrak{A} \models P$ for all $P \in \Theta$. $\mathsf{Alg}\,\Theta$ is a variety; in other words, it is closed under products, taking subalgebras, and taking homomorphic images. The map Alg defines a dual isomorphism from the lattice of n–modal logics onto the lattice of varieties of n–modal algebras; its inverse is Th.

With an n–frame $\langle f, \langle \lhd_i : i < n\rangle, \mathbf{F}\rangle$ we associate a n–algebra $\mathfrak{F}_+ = \langle \mathbf{F}, 1, -, \cap, \langle \Box_i | i < n\rangle\rangle$ and a kripke (n–)frame $\mathfrak{F}_\sharp = \langle f, \langle \lhd_i : i < n\rangle\rangle$. For a kripke frame $\mathfrak{f}$, $\mathfrak{f}^\sharp = \langle \mathfrak{f}, 2^f\rangle$ is an n–frame. n–frames of this form are called **full**. An n–algebra $\mathfrak{A}$ defines a canonical n–frame over the set $pt(\mathfrak{A})$ of ultrafilters ('points') by letting $U \lhd_i V \Leftrightarrow (\forall \Box_i a \in U) a \in V$. Furthermore, the system of sets is the system of all sets of the form $\widehat{a} = \{U \in pt(\mathfrak{A})| a \in U\}$. This frame is denoted by $\mathfrak{A}^+$. This representation of algebras by frames is known as *Stone Representation*. It is known that $(\mathfrak{f}^\sharp)_\sharp \cong \mathfrak{f}$ and $(\mathfrak{A}^+)_+ \cong \mathfrak{A}$. An n–frame is **differentiated** if whenever $x \neq y$ there exists an internal set a such that $x \in a$ but $y \notin a$. $\mathfrak{F}$ is called **tight** if whenever $x \mathbf{6}_i\ y$ there exists an internal set a such that $x \in \Box_i a$ but $y \notin a$; and $\mathfrak{F}$ is **compact** if for every $U \in pt(\mathfrak{F}_+)$ we have $\bigcap U \neq \emptyset$. An n–frame is **refined** if it is differentiated and tight, and **descriptive** if it is refined and compact. As is well known, an n–frame is descriptive iff it is isomorphic to a frame of the form $\mathfrak{A}^+$. The classes of diffentiated, tight, refined, full (=kripke), finite and descriptive n–frames are denoted by ${}^n\mathfrak{Df}$, ${}^n\mathfrak{Ti}$, ${}^n\mathfrak{R}$, ${}^n\mathfrak{Krp}$, ${}^n\mathfrak{Fin}$ and ${}^n\mathfrak{D}$. Finally, the class of all n–modal is denoted by ${}^n\mathfrak{G}$. We will drop the superscripts whenever possible.

A logic Λ is a subset of the set of formulae over a fixed set V of variables. We can always take $V_\kappa = \{p_i | i < \kappa\}$, κ some cardinal number. Defined in this way, Λ depends on κ. It should be clear, however, that we can always choose a presentation of Λ in the form $\mathbf{K} \oplus X$, for some X with $var(X) \subseteq V_{\aleph_0}$. We call Λ n–**axiomatizable** if there exists a set X such that $var(X) \subseteq V_n$ and $\Lambda = \mathbf{K} \oplus X$. Λ is **finitely** (**recursively**) **axiomatizable** if X can be chosen finite (recursive).

Definition 4 *Let $\mathfrak{X}$ be a class of n–frames and Λ an n–modal logic. Λ is called* ***compact*** *(**complete**) with respect to $\mathfrak{X}$ if for every (finite) set X and every formula P, if X $\mathbf{0}_\Lambda$ P there exists a model $\langle \mathfrak{F}, \beta, x \rangle$ such that $\mathfrak{F} \in \mathfrak{X}$, and $\mathfrak{F} \models \Lambda$ and $\langle \mathfrak{F}, \beta, x \rangle \models X; \neg P$.*

A logic is **compact** iff it is compact with respect to ${}^n\mathfrak{Krp}$ in the sense of the definition above. Let us also discuss some other specializations of this definition. Every logic is compact with respect to ${}^n\mathfrak{D}$, the class of descriptive frames, by Stone representation and the fact that for every consistent set X there exists a Λ–algebra $\mathfrak{A}$, a valuation β and a point U such that $\beta(X) \subseteq U$. Hence every logic is complete with respect to ${}^n\mathfrak{D}$.

With a logic Λ we can associate another deducibility relation that also has Λ as its set of theorems. It is denoted by $\Vdash_\Lambda$ and called the **global consequence relation**, as opposed to the **local consequence relation**, which is $\vdash_\Lambda$. We put $X \Vdash_\Lambda P$ if P can be deduced from X using the rules of modus ponens and (mn): $X \Vdash Q / X \Vdash \Box_i Q$, $i < n$. Thus, while (mn) is in general only an admissible rule of normal modal logics, it is a derived rule of $\Vdash_\Lambda$. To understand the meaning of the term *global*, let us note the following. Denote by $X^\Box$ the closure of X under (mn). It consists of all formulae of the form αP, where α is a sequence of modal operators and $P \in X$.

Proposition 5 *Let Λ be a normal modal logic. Then $X \Vdash_\Lambda P$ iff $X^\Box \vdash_\Lambda P$.*

A proof can be found in [25]. Now we say that X holds **globally** in a model $\langle \mathfrak{F}, \beta, x \rangle$ if $\langle \mathfrak{F}, \beta, x \rangle \models X^\Box$. This is the same as saying that X holds everywhere in the submodel generated by x.

Definition 6 *Let Λ be an n–modal normal logic and $\mathfrak{X}$ a class of n–frames. Λ is called* ***globally compact*** *(**complete**) with respect to $\mathfrak{X}$ if for every (finite) set X and every formula P the following holds. If X $\mathbf{1}_\Lambda$ P there exists a model $\langle \mathfrak{F}, \beta, x \rangle$ such that $\mathfrak{F} \in \mathfrak{X}$, $\mathfrak{F} \models \Lambda$ and $\langle \mathfrak{F}, \beta, x \rangle \models X^\Box; \neg P$.*

In general, if a logic is globally complete with respect to a class $\mathfrak{X}$ it is also locally complete. This is easy to show. However, the converse may fail to hold. For example, there exist logics which have the local finite model property but not the global finite model property. (See [33].) However, the following has been shown in [32].

Theorem 7 *A logic is globally compact iff it is locally compact.* $\dashv$

Further, a logic Λ is **globally decidable** if the problem '$P \Vdash_\Lambda Q$' is decidable for every pair P and Q of formulae. The notion of persistence with respect to sets of formulas, which arises from correspondence theory, also allows to distinguish global from local notions. A logic Λ is called **locally $\mathfrak{X}$–persistent** if for every $\mathfrak{F} = \langle \mathfrak{f}, \mathbf{F} \rangle$ from $\mathfrak{X}$ if $\langle \mathfrak{F}, x \rangle \models \Lambda$ then $\langle \mathfrak{f}, x \rangle \models \Lambda$. And Λ is called **globally $\mathfrak{X}$–persistent** if $\langle \mathfrak{f}, \mathbf{F} \rangle \models \Lambda$ implies $\mathfrak{f} \models \Lambda$ for all $\langle \mathfrak{f}, \mathbf{F} \rangle \in \mathfrak{X}$. Here the local property implies the global one. For if Λ is locally $\mathfrak{X}$–persistent and $\langle \mathfrak{f}, \mathbf{F} \rangle \models \Lambda$, then for every x $\langle \langle \mathfrak{f}, \mathbf{F} \rangle, x \rangle \models \Lambda$, from which $\langle \mathfrak{f}, x \rangle \models \Lambda$ for every x and so $\mathfrak{f} \models \Lambda$. A logic Λ is κ–**canonical** for κ a given cardinal number, if for every $\lambda < \kappa$ the frame underlying the Stone representation of the freely λ–generated Λ–algebra satisfies the theorems of Λ. A logic is **canonical** if it is κ–canonical for every κ. By a theorem of [27], a logic is canonical iff it is $\mathfrak{D}$–persistent.

§ 4. Simulations of Monotonic Logics.

We call a monotonic modal logic Λ N–**compact** if, for all $\Gamma \subseteq \mathcal{L}_1$ and $P \in \mathcal{L}_1$: $\Gamma \vdash_\Lambda P$ iff, for all N–frames g for Λ and all valuations β and $x \in g$, $\langle \mathcal{G}, \beta, x \rangle \models \Gamma$ implies $\langle \mathcal{G}, \beta, x \rangle \models P$.

Proposition 8 *Suppose that a monotonic modal logic Θ simulates a monotonic modal logic Λ with respect to an atomic interpretation F and let $\boldsymbol{P}$ be one of the following properties:*

- *decidability,*
- *completeness with respect to N–frames,*
- *finite model property,*
- *N–compactness.*

Then Λ has $\boldsymbol{P}$ if Θ has $\boldsymbol{P}$.

Proof. Decidability is clear. Let now Θ be complete. We may assume that boolean connectives are interpreted as boolean connectives, by Proposition 2. Suppose now that $\neg P \notin \Lambda$. Then $\neg P^F \notin \Theta$ and there exists a Θ–frame $\langle g, N' \rangle$ such that $\langle \langle g, N' \rangle, \beta, x \rangle \models P^F$. We define a function N on g as follows. For $y \in g$ and $a \subseteq g$ let

$$a \in N(y) \Leftrightarrow \langle \langle g, N' \rangle, \beta', y \rangle \models \Box_F p,$$

where β' is a valuation such that $\beta'(p) := a$. It follows by induction that

$$\langle \langle g, N \rangle, \gamma, y \rangle \models Q \text{ iff } \langle \langle g, N' \rangle, \gamma, y \rangle \models Q^F,$$

for all valuations γ, all $y \in g$ and all $Q \in \mathcal{L}_1$. Hence $\langle \langle g, N \rangle, \beta, x \rangle \models P$ and $\langle g, N \rangle$ is a Λ–frame. The other statements can be proved analogously. $\dashv$

Let $D : \mathcal{L}_1 \to \mathcal{L}_2$ be the atomic interpretation defined by

$$(p \wedge q)^D = p \wedge q, \ (\neg p)^D = \neg p, \ (\Box p)^D = \Diamond_1 \Box_2 p.$$

For a monotonic modal logic Λ let $\mathcal{S}(\Lambda)$ denote the set of normal bimodal logics which simulate Λ with respect to D. The aim of the following investigation is to get some insight into the structure of the map $\Lambda \mapsto \mathcal{S}(\Lambda)$. We start with three simulations of N–frames $\langle g, N\rangle$ as bimodal Kripke–frames, which we denote by $\langle g, N\rangle^{mn_i}$, for $i = 1, 2, 3$. The idea of the construction is as follows. Put $\mathcal{C} = \mathcal{C}[\langle g, N\rangle]$. The set of points of $\langle g, N\rangle^{mn_1}$ is $g \cup \mathcal{C}$ and the relations are defined by xR_1y iff $y \in N(x)$ and yR_2x iff $x \in y$. Then each neighbourhood $C \in \mathcal{C}$ is interpreted as a new point and the term $x \in \Box a$ in $\langle g, N\rangle$ corresponds to $x \in \Diamond_1\Box_2 a$ in $\langle g, N\rangle^{mn_1}$, for $a \subseteq g$ and $x \in g$. It follows that a point in g satisfies a formula P in $\langle g, N\rangle$ iff it satisfies P^D in $\langle g, N\rangle^{mn_1}$. So everything is fine with the points in g. There remains, however, a difficulty with the new points in $\mathcal{C}$. For suppose that we want to simulate $\Theta = \mathbf{M} + \Box\top$. Then, for a Θ–frame $\langle g, N\rangle$, the frame $\langle g, N\rangle^{mn_1}$ should satisfy $\Box\top^D = \Diamond_1\Box_2\top$, which is certainly not the case with the points in $\mathcal{C}$ in $\langle g, N\rangle^{mn_1}$ since they have no R_1–successors. For this reason we have to add new points in order to obtain appropriate frames $\langle g, N\rangle^{mn_i}$, $i = 2, 3$. In order to obtain those new successors we will need the following result (see [25]).

Proposition 9 *Each consistent monotonic logic is included in the theory of one of the following N–frames:*

$$\mathcal{F}_1 = \langle\{0\}, \emptyset\rangle, \;\; \mathcal{F}_2 = \langle\{0\}, \langle 0, \{\emptyset\}\rangle\rangle, \;\; \mathcal{F}_3 = \langle\{0\}, \langle 0, \{\{0\}\}\rangle\rangle$$

We remark that two of these theories are well–known. Namely, the theory of $\mathcal{F}_2$ coincides with the theory of the irreflexive point (in relational semantics) and the theory of $\mathcal{F}_3$ coincides with the theory of the reflexive point (in relational semantics). So the only non normal logic among those theories is the theory of $\mathcal{F}_1$ which coincides with $\mathbf{M} + \Diamond p$. Now define

$$\begin{aligned}
\mathfrak{F}_1 &= \langle\{0\}, \emptyset, \emptyset\rangle \\
\mathfrak{F}_2 &= \langle\{0,1,2\}, \{\langle 0,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle\}, \emptyset\rangle \\
\mathfrak{F}_3 &= \langle\omega, \{\langle n, n+1\rangle | n \in \omega\}, \{\langle n+1, n\rangle | n \in \omega\}\rangle
\end{aligned}$$

Lemma 10 *For all $\Gamma \subseteq \mathcal{L}_1$:*
If $\mathcal{F}_1 \models \Gamma$ then $\mathfrak{F}_1 \models \Gamma^D$.
If $\mathcal{F}_2 \models \Gamma$ then $\mathfrak{F}_2 \models \Gamma^D$.
If $\mathcal{F}_3 \models \Gamma$ then $\mathfrak{F}_3 \models \Gamma^D$.

We omit the simple proof. Of course, $\mathcal{F}_3$ has a finite simulation. We have chosen this simulation because we will need the fact, that 0 has no predecessor in the first relation and no successor in the second relation in $\mathfrak{F}_i$.

We now come to the exact definition of the simulating frames $\langle g, N\rangle^{mn_i}$, $i = 1, 2, 3$. ($\langle g, N\rangle^{mn_1}$ was defined above. In order to have a homogeneous notation we shall introduce an isomorphic copy below.) Let $\langle g, N\rangle$ be an N–frame and put $\mathcal{C} = \mathcal{C}[\langle g, N\rangle]$. Take a Kripke–frame $\mathfrak{F}_i = \langle f, S_1, S_2\rangle$ from Lemma

10 and define $\langle \otimes f, \otimes S_1, \otimes S_2 \rangle$ by

$$\begin{array}{rcl} \otimes f & := & \mathcal{C} \times f. \\ \otimes S_1 & := & \{\langle \langle C, y\rangle, \langle C, z\rangle\rangle | C \in \mathcal{C}, y S_1 z\}. \\ \otimes S_2 & := & \{\langle \langle C, y\rangle, \langle C, z\rangle\rangle | C \in \mathcal{C}, y S_2 z\}. \end{array}$$

Now let $\langle g, N\rangle^{mn_i} = \langle h, R_1, R_2\rangle$, where

$$\begin{array}{rcl} h & := & g \cup \otimes f. \\ R_1 & := & \{\langle x, \langle C, 0\rangle\rangle | x \in g, C \in N(x)\} \cup \otimes S_1. \\ R_2 & := & \{\langle \langle C, 0\rangle, x\rangle | C \in \mathcal{C}, x \in C\} \cup \otimes S_2. \end{array}$$

For a frame h with valuation β and a subframe g of h we denote the valuation $\gamma(p) := \beta(p) \cap g$ by $\beta | g$ and call it the **restriction** of β to g.

Lemma 11 *Let $i \in \{1, 2, 3\}$. For all valuations β in $\langle g, N\rangle^{mn_i}$ and $Q \in \mathcal{L}_1$ the following holds:*

(1) For all $x \in g$

$\langle \langle g, N\rangle, \beta | g, x\rangle \models Q$ *iff* $\langle \langle g, N\rangle^{mn_i}, \beta, x\rangle \models Q^D$

(2) For all $x \in \otimes f$

$\langle \langle \otimes f, \otimes S_1, \otimes S_2\rangle, \beta | \otimes f, x\rangle \models Q^D$ *iff* $\langle \langle g, N\rangle^{mn_i}, \beta, x\rangle \models Q^D$

Proof. Assume that $\langle g, N\rangle^{mn_i} = \langle h, R_1, R_2\rangle$. (1) By induction on Q. The interesting step is $Q = \Box P$. Suppose that $\langle g, \beta | g, x\rangle \models \Box P$. Then there exists a $C \in N(x)$ with $\langle g, \beta | g, y\rangle \models P$ for all $y \in C$. Consider $\langle C, 0\rangle \in h$. The R_2–successors of $\langle C, 0\rangle$ are precisely the points in C. By induction hypothesis we have $\langle h, \beta, y\rangle \models P^D$ for all $y \in C$. It follows that $\langle h, \beta, \langle C, 0\rangle\rangle \models \Box_2 P^D$. Hence $\langle h, \beta, x\rangle \models \Diamond_1 \Box_2 P^D = Q^D$. Conversely, assume that $\langle h, \beta, x\rangle \models \Diamond_1 \Box_2 P^D$. Then there exists $C \in N(x)$ with $\langle h, \beta, \langle C, 0\rangle\rangle \models \Box_2 P^D$. Then $\langle h, \beta, y\rangle \models P^D$ for all $y \in C$ and, by induction hypothesis, $\langle g, \beta | g, y\rangle \models P$ for all $y \in C$. Thus $\langle g, \beta | g, x\rangle \models \Box P$. (2) is proved analogously. One only has to use the fact that $\langle C, 0\rangle$ has no S_1–predecessor. $\dashv$

Theorem 12 *For a monotonic modal logic Λ the following properties are equivalent:*

(i) Λ is complete with respect to neighbourhood–semantics.

(ii) $\mathcal{S}(\Lambda)$ contains a logic which is complete with respect to Kripke–semantics.

(iii) There is an atomic interpretation F and a logic Θ simulating Λ with respect to F which is complete with respect to Kripke–semantics.

Proof. (i) $\Rightarrow$ (ii) Choose an i such that Λ is included in the theory of $\mathcal{F}_i$. Such a frame exists, by Proposition 9. Now form the the class

$$\mathcal{K} = \{\langle g, N\rangle^{mn_i} : \langle g, N\rangle \models \Lambda\}.$$

We show that $\Theta \in \mathcal{S}(\Lambda)$, for the theory Θ of $\mathcal{K}$. Let $\langle g, N\rangle^{mn_i} = \langle g \cup \otimes f, R_1, R_2\rangle$ and suppose that $\langle \langle g, N\rangle^{mn_i}, \beta, x\rangle \not\models P^D$. If $x \in g$, then $P \notin \Lambda$, by Lemma 11

(1). If $x \in \otimes f$, then $P \notin \Lambda$, by Lemma 11 (2) and Lemma 10. Thus, $\Theta \in \mathcal{S}(\Lambda)$. Θ is complete with respect to Kripke semantics, by the definition. (ii) $\Rightarrow$ (iii) is trivial and (iii) $\Rightarrow$ (i) is Proposition 8. $\dashv$

If we apply this result to a normal monomodal logic Λ it has the consequence that Λ is complete with respect to neighbourhood–semantics iff a bimodal simulation is complete with respect to Kripke–semantics. Recall now that there are normal modal logics which are complete with respect to neighbourhood–semantics but incomplete with respect to Kripke semantics (consult [11]). Hence there are Kripke–incomplete normal logics which have a bimodal simulation which is Kripke–complete. The explanation of this phenomenon is simple: A normal modal logic is complete with respect to neighbourhood–semantics iff the corresponding variety of modal logics is generated by full modal algebras, that is algebras $\langle 2^g, \cap, -, \Box\rangle$, where $\Box g = g$ and $\Box(a \cap b) = \Box a \cap \Box b$. But a logic is complete with respect to Kripke–semantics iff the corresponding variety is generated by full algebras which satisfy the continuity axiom

$$\text{(Con)} \qquad \Box \bigcap\{a_i | i \in I\} = \bigcap\{\Box a_i | i \in I\}$$

Now there are bimodal algebras $\langle 2^g, \cap, -, \Box_1, \Box_2\rangle$ such that $\Box_1, \Box_2$ satisfy (Con) while $\Diamond_1\Box_2$ does not satisfy (Con).

Our next step is to show that $\mathcal{S}(\Lambda)$ is always non–empty. To prove this we simulate general N^h–frames. Let $\mathcal{G} = \langle g, N, \mathbf{G}\rangle$ be a general N–frame and take $\langle g, N\rangle^{mn_i} = \langle h, R_1, R_2\rangle$, for some $i \in \{1, 2, 3\}$. It remains to define the internal sets B on this frame. We shall need a set B such that

$$\{b \cap g : b \in B\} = \mathbf{G}. \tag{4}$$

We start with the definition of the restriction of B to $\otimes f$. Define $B_1 \subseteq 2^{\mathcal{C}\times\{0\}}$ as follows: $b \in B_1$ iff there exist finite subsets $I_1, \dots, I_k$ and $J_1, \dots, J_k$ of $\mathbf{G}$ with $b = b_1 \cup \ldots \cup b_k$, where

$$\begin{array}{rcl} b_j & = & \bigcap\{\{\langle C, 0\rangle | C \subseteq a, C \in \mathcal{C}\} | a \in I_j\} \\ & \cap & \bigcap\{\{\langle C, 0\rangle | C \cap a \neq \emptyset, C \in \mathcal{C}\} | a \in J_j\}. \end{array}$$

Here $\mathcal{C} = \mathcal{C}[\langle g, N\rangle]$. Now take the smallest set $B_2 \subseteq 2^{\otimes f - (\mathcal{C}\times\{0\})}$ such that

$$\otimes\mathfrak{F} := \langle \otimes f, \otimes S_1, \otimes S_2, \{b \cup c | b \in B_1, c \in B_2\}\rangle$$

is a general frame. (Clearly, $B_1 \cup B_2$ is the closure of B_1 under the operations $\Box^{\otimes f}$, intersection, and complement.) Finally define

$$B := \{a \cup b \cup c | a \in \mathbf{G}, b \in B_1, c \in B_2\}.$$

and let $\mathcal{G}^{mn_i} := \mathfrak{H} = \langle h, R_1, R_2, B\rangle$.

Lemma 13 *If $\mathcal{G}$ is a general N^h–frame then $\mathfrak{H} = \mathcal{G}^{mn_i}$ is a general frame, for all $i \in \{1, 2, 3\}$.*

Proof. Closure of B with respect to intersection and complement is obvious. Suppose that $a \cup b \cup c \in B$ with $a \in \mathbf{G}$, $b \in B_1$, $c \in B_2$. We denote $\Diamond_j^{\mathfrak{H}}$ by $\Diamond_j$, $j = 1, 2$.
(1) Closure under $\Diamond_2$. We have

$$\begin{aligned} \Diamond_2(a \cup b \cup c) &= \Diamond_2 a \cup \Diamond_2(b \cup c) \\ \Diamond_2 a &= \{\langle C, 0\rangle | C \in \mathcal{C}, C \cap a \neq \emptyset\} \in B_1 \\ \Diamond_2(b \cup c) &= \Diamond_2^{\otimes\mathfrak{F}}(b \cup c) \in \{b \cup c | b \in B_1, c \in B_2\}. \end{aligned}$$

(2) Closure under $\Diamond_1$. Clearly $\Diamond_1 a = \emptyset$ and $\Diamond_1 c = \Diamond_1^{\otimes\mathfrak{F}} c$. Let $b = b_1 \cup \ldots \cup b_k$ with

$$\begin{aligned} b_i \quad = \quad & \bigcap\{\{\langle C, 0\rangle | C \subseteq a, C \in \mathcal{C}\} | a \in I_i\} \\ \cap \quad & \bigcap\{\{\langle C, 0\rangle | C \cap a \neq \emptyset, C \in \mathcal{C}\} | a \in J_i\} \end{aligned}$$

Then

$$\begin{aligned} & x \in \Diamond_1 b_i \\ \Leftrightarrow\ & (\exists C \in N(x))(\forall a \in J_i)(C \subseteq \textstyle\bigcap I_i \text{ and } C \cap a \neq \emptyset) \\ \Leftrightarrow\ & (\forall a \in J_i)(\textstyle\bigcap I_i \in N(x) \text{ and } \textstyle\bigcap I_i \cap a \neq \emptyset). \end{aligned}$$

There are two cases.
Case 1. There is an $a \in J_i$ with $\bigcap I_i \cap a = \emptyset$. Then $\Diamond_1 b_i = \emptyset$.
Case 2. Else. Then $\Diamond_1 b_i = \{x \in g | \bigcap I_i \in N(x)\} = \Box^{\mathcal{G}} \bigcap I_i \in \mathbf{G}$. $\dashv$

Theorem 14 *Let $\Lambda = \boldsymbol{M} + \Gamma$ be a monotonic logic. Then $\Theta = \boldsymbol{K}_2 + \Gamma^D$ simulates Λ with respect to D.*

Proof. The implication $P \in \Lambda \Rightarrow P^F \in \Theta$ follows immediately from the fact that Θ is closed with respect to the rule $p \to q / \Diamond_1 \Box_2 p \to \Diamond_1 \Box_2 q$. Now suppose that $\neg P \notin \Lambda$. Then there is a general N^h–frame $\mathcal{G} = \langle g, N, \mathbf{G}\rangle$ for Λ, a valuation β and $x \in g$ with $\langle \mathcal{G}, \beta, x\rangle \models P$. Take an $i \in \{1, 2, 3\}$ such that $\mathcal{F}_i$ is a frame for Λ and consider the frame $\mathcal{G}^{mn_i} = \langle h, R_1, R_2, B\rangle$. Then, by the construction of B, β is a valuation of $\mathcal{G}^{mn_i}$. By Lemma 11, $\langle \mathcal{G}^{mn_i}, \beta, x\rangle \models P^D$. So the theorem is shown if $\mathcal{G}^{mn_i}$ is a frame for Θ. Let $Q \in \Lambda$. By Lemma 10 and Lemma 11 (2), Q^D is valid in all new points (i.e., the points which are not in g) in $\mathcal{G}^{mn_i}$. Q^D holds in all points in g since each valuation of $\mathcal{G}^{mn_i}$ can be restricted to $\mathcal{G}$, by the definition of B, and we can apply Lemma 11 (1). $\dashv$

Corollary 15 *Let $\Gamma \subseteq \mathcal{L}_1$ be a set of formulas of the form $P \to Q$, where P is of the form $\bigwedge\langle \Box p_i | i \leq n\rangle \wedge \bigwedge\langle q_i | i \leq m\rangle$ and Q is built from propositional variables by using $\wedge, \vee, \Box, \Diamond$ only. Then $\boldsymbol{M} + \Gamma$ is N–compact. Also, $\boldsymbol{M} + \Gamma$ is complete with respect to N–frames.*

Proof. It follows from the proof in [17] that Sahlqvist's Theorem [26] holds for normal polymodal logics. For $P \to Q \in \Gamma$ the formula $(P \to Q)^D$ is a polymodal Sahlqvist–formula. Hence $\mathbf{M} + \Gamma$ is N–compact, by Proposition 8. $\dashv$

As examples of complete (and even N–compact) modal logics we get the standard systems $\mathbf{M} + \Box p \to \Diamond p$, $\mathbf{M} + \Box p \to p$, $\mathbf{M} + p \to \Box\Diamond p$, $\mathbf{M} + \Box p \to \Box\Box p$.

Corollary 16 *Let $\Gamma \subseteq \mathcal{L}_1$ be a finite set of constant formulae. Then $\boldsymbol{M} + \Gamma$ has the finite model property.*

Proof. In this case Γ^D is a finite set of constant formulae. It is shown in [18] that $\mathbf{K}_2 + \Gamma^D$ has the finite model property. $\dashv$

A simple observation allows us to improve Corollary 15. Let $(-)^d : \mathcal{L}_1 \to \mathcal{L}_1$ be the atomic interpretation defined by $(\Box p)^d = \Diamond p$. Then $\mathbf{M}+\Gamma^d$ simulates $\mathbf{M}+\Gamma$, for all Γ and $\mathbf{M}+\Gamma$ is complete iff $\mathbf{M}+\Gamma^d$ is complete. Now let Γ be a set of formulas of the form $P \to Q$, where P is of the form $\bigwedge\langle \Diamond p_i | i \leq n\rangle \wedge \bigwedge\langle q_i | i \leq m\rangle$ and Q is built from propositional variables by using $\wedge, \vee, \Box, \Diamond$ only. Then $\mathbf{M}+\Gamma$ is complete with respect to N–frames, by Corollary 15.

§ 5. Simulations by Tense logics.

In general it is rather difficult to describe the set $\mathcal{S}(\Lambda)$. Here we show that an interesting part of the lattice of monotonic logics can be simulated by tense logics. (Consult e.g. [4], [13], or [34] for information on tense logics.) Recall that the smallest tense logic is the normal bimodal logic $\mathbf{K}.t := \mathbf{K}_2 + \{p \to \Box_1\Diamond_2 p,\ p \to \Box_2\Diamond_1 p\}$. Clearly all monotonic modal logics which are simulated by tense logics contain the axioms $\Box p \to p$ and $\Box p \to \Box\Box p$. Our aim is to prove that the converse holds as well. Define $\mathbf{MT4} := \mathbf{M} + \{\Box p \to p, \Box p \to \Box\Box p\}$. All extensions of $\mathbf{MT4}$ are complete with respect to general N^h–frames $\mathcal{G} = \langle g, N, \mathbf{G}\rangle$ satisfying the following conditions:

(i) For all $x \in g : C \in N(x) \Rightarrow x \in C$.

(ii) For all $x \in g : C \in N(x) \Rightarrow \{y \in g | C \in N(y)\} \in N(x)$.

Note that $\{y \in g | C \in N(y)\} = \Box^g C$, for all $C \in \mathbf{G}$, by equation (3). Put $\mathcal{C} = \mathcal{C}[\langle g, N\rangle]$ and define $\mathcal{C}^t := \{\Box^g C | C \in \mathcal{C}\}$ and let $\mathcal{G}^t := \langle g, N^t, \mathbf{G}\rangle$, where $D \in N^t(x)$ iff $D \in \mathcal{C}^t$ and $x \in D$. $\mathcal{G}^t$ has the following properties.

(s1) For all $x \in g : N^t(x) \subseteq N(x)$

(s2) For all $x \in g, C \in N(x)$ there is a $D \in N^t(x)$ with $D \subseteq C$.

(**s2**) is clear. Now (**s1**). Suppose that $D \in N^t(x)$. Then there exists a $y \in g$ with $C \in N(y)$ and $D = \Box^g C$. We have $x \in D$. Hence

$$x \in \Box^g C = \Box^g\Box^g C = \{z \in g | \Box^g C \in N(z)\}.$$

Thus $\Box^g C \in N(x)$. It follows immediately from (**s1**) and (**s2**) that the theories of $\mathcal{G}$ and $\mathcal{G}^t$ coincide. Note, however, that $\mathcal{G}^t$ is not a N^h–frame, in general. Let us call a frame of the form $\mathcal{G}^t$ a **special** general frame. Then we get

Proposition 17 *All extensions of **MT4** are complete with respect to special general N–frames. If an extension of **MT4** is complete with respect to N–frames then it is complete with respect to special N–frames.*

Theorem 18 *Suppose that Λ is a monotonic modal logic above **MT4** and that Λ is complete with respect to N–frames. Then there is a Kripke–complete logic containing $\boldsymbol{K}.t$ in $\mathcal{S}(\Lambda)$.*

Proof. Let $\Lambda \supseteq \mathbf{MT4}$ and take an $i \in \{1, 2, 3\}$ such that $\mathcal{F}_i \models \Lambda$. Clearly $i \neq 2$ since $\Box p \to p \in \Lambda$. Now put

$$\mathcal{K} = \{\langle g, N^t\rangle^{mn_i} : \langle g, N\rangle \models \Lambda, \langle g, N\rangle \text{ is a } N^h\text{–frame }\}.$$

Denote by Θ the theory of $\mathcal{K}$. We have, as in the proof of Theorem 12, that $\Theta \in \mathcal{S}(\Lambda)$. Moreover, $R_1 = R_2^{-1}$, for all $\langle g, R_1, R_2\rangle \in \mathcal{K}$ since $C \in N^t(x)$ iff $x \in C$ and $C \in \mathcal{C}^t$ and since both $\mathfrak{F}_1$ and $\mathfrak{F}_3$ are tense frames. Hence $\Theta \supseteq \mathbf{K}.t$. $\dashv$

Theorem 19 *Suppose that $\Lambda = \boldsymbol{MT4} + \Gamma$. Then $\boldsymbol{K}.t + \Gamma^D$ simulates Λ with respect to D.*

Proof. The proof is completely analogous to the proof of Theorem 14. The only difference is in the proof that $\langle g, N^t, \mathbf{G}\rangle^{mn_i}$ is a general frame whenever $\langle g, N, \mathbf{G}\rangle$ is a N^h–frame satisfying (i) and (ii). The crucial step is to show that $\Diamond_1 b_i \in B$ for

$$\begin{array}{rcl} b_i & = & \bigcap\{\{\langle C, 0\rangle | C \subseteq a, C \in \mathcal{C}^t\} | a \in I_i\} \\ & \cap & \bigcap\{\{\langle C, 0\rangle | C \cap a \neq \emptyset, C \in \mathcal{C}^t\} | a \in J_i\}. \end{array}$$

But we have $x \in \Diamond_1 b_i \Leftrightarrow \exists C \in N^t(x) : C \subseteq \bigcap I_i$ and $C \cap a \neq \emptyset$ for all $a \in J_i \Leftrightarrow \Box^g \bigcap I_i \in N(x)$ and $\Box^g \bigcap I_i \cap a \neq \emptyset$ for all $a \in J_i$.
Case 1. There is an $a \in J_i$ with $\Box^g \bigcap I_i \cap a = \emptyset$. Then $\Diamond_1 b_i = \emptyset$.
Case 2. Else. Then $\Diamond_1 b_i = \{x | \Box^g \bigcap I_i \in N(x)\} = \Box^g \Box^g \bigcap I_i \in \mathbf{G}$. $\dashv$

The frames of the form $\langle g, N^t, \mathbf{G}\rangle^{mn_1}$ are, in a certain sense, quite simple. Suppose that $\langle g, N^t, \mathbf{G}\rangle^{mn_1} = \langle g, R_1, R_2, B\rangle$. Then $R_1 = R_2^{-1}$ and y has no R_1–successor whenever xR_1y, for some x. So we have $\langle g, N^t, \mathbf{G}\rangle^{mn_1} \models \Box_1\Box_1\bot$. Define

$$\mathbf{G}.\mathbf{J}_1.t := \mathbf{K}.t + \Box_1\Box_1\bot.$$

Note that the monomodal fragments of $\mathbf{G}.\mathbf{J}_1.t$ are almost trivial: they have the finite model property and all their proper extensions are determined by finite frames (i.e. $\mathbf{G}.\mathbf{J}_1.t$ is pretabular), as is easily shown. We immediately get the following corollary, which shows the complexity of the lattice of tense logics.

Theorem 20 *Suppose that $\Lambda = \boldsymbol{MT4} + \Gamma$ is a monotonic logic with $\Lambda \subseteq \boldsymbol{M} + \Diamond p$. Then $\boldsymbol{G}.\boldsymbol{J}_1.t + \Gamma^D$ simulates Λ with respect to D.* $\dashv$

§ 6. SIMULATING EQUATIONAL THEORIES OF LATTICES.

Extensions of **MT4** are suitable as modal descriptions of closure operators. Let Cl be a closure operator on a set g. Then we have for $\Diamond := Cl$ that $a \subseteq b$ implies $\Diamond a \subseteq \Diamond b$ and $a \subseteq \Diamond a$ and $\Diamond\Diamond a = \Diamond a$. It follows that the theory of $\langle 2^g, \cap, -, \Diamond \rangle$ is a monotonic logic above **MT4**. Conversely, if the theory of a neighbourhood–frame $\langle g, N \rangle$ is above **MT4**, then $\langle g, Cl \rangle$ with $Cl(a) := \Diamond a = \{z \in g | a \cap C \neq \emptyset$ for all $C \in N(z)\}$ is a closure operator on g. We use this observation to get some undecidability results for tense logics.

Define the following translation of the language of lattices $\mathcal{L}$ into the language of modal logic:

$$\begin{array}{lll} p^{Mod} & := & \Diamond p \\ (P \wedge Q)^{Mod} & := & P^{Mod} \wedge Q^{Mod} \\ (P \vee Q)^{Mod} & := & \Diamond(P^{Mod} \vee Q^{Mod}) \end{array}$$

For a closure operator Cl on a set g let L_{Cl} denote the lattice of Cl–closed subsets of g. A simple induction proves

Lemma 21 *Let Cl be a closure operator on g and $\chi_1, \chi_2 \in \mathcal{L}$. Let $\Diamond := Cl$. Then $L_{Cl} \models \chi_1 = \chi_2$ iff $\langle 2^g, \cap, -, \Diamond \rangle \models \chi_1^{Mod} \leftrightarrow \chi_2^{Mod}$.*

Let L denote the equations in $\mathcal{L}$ defining lattices. For Φ a set of equations in $\mathcal{L}$ and $\chi_1, \chi_2 \in \mathcal{L}$ we write $L \cup \Phi \models \chi_1 = \chi_2$ iff $\chi_1 = \chi_2$ follows from $L \cup \Phi$ in equational logic. The closure of $L \cup \Phi$ under $\models$ is denoted by $L + \Phi$. In order to simulate equational theories of lattices as modal logics we need the following observation.

Proposition 22 *All varieties of lattices are generated by complete lattices.*

Proof. It is readily checked that all varieties of lattices are generated by lattices with maximal and minimal elements. Consider a lattice $\mathcal{A}$ with maximal and minimal elements. There is an embedding of $\mathcal{A}$ into the complete lattice $I(\mathcal{A})$ of ideals of $\mathcal{A}$. Hence the proposition is shown if $\mathcal{A} \models P = Q \Rightarrow I(\mathcal{A}) \models P = Q$ holds for all $P, Q \in \mathcal{L}$. Assume $\mathcal{A} \models P = Q$ but $I(\mathcal{A}) \not\models P \leq Q$. Then there exist $I_1, \ldots, I_k \in I(\mathcal{A})$ with $P[I_1, \ldots, I_k] \not\subseteq Q[I_1, \ldots I_k]$. Take an $a \in P[I_1, \ldots, I_k]$ which is not in $Q[I_1, \ldots, I_k]$. There exist $a_i \in I_i$ with $a \leq P[a_1, \ldots, a_k]$. But then $a \leq P[a_1, \ldots, a_k] = Q[a_1, \ldots, a_k] \in Q[I_1, \ldots, I_k]$ and therefore $a \in Q[I_1, \ldots, I_k]$. We have a contradiction. $\dashv$

Theorem 23 *For all equational theories of lattices $L + \Phi$ the following properties are equivalent:*

(i)	$\chi_1 = \chi_2$	$\in L + \Phi$.
(ii)	$\chi_1^{Mod} \leftrightarrow \chi_2^{Mod}$	$\in \boldsymbol{MT4} + \{P^{Mod} \leftrightarrow Q^{Mod} \mid P = Q \in \Phi\}$.
(iii)	$(\chi_1^{Mod} \leftrightarrow \chi_2^{Mod})^D$	$\in \boldsymbol{K}.t + \{(P^{Mod} \leftrightarrow Q^{Mod})^D \mid P = Q \in \Phi\}$.
(iv)	$(\chi_1^{Mod} \leftrightarrow \chi_2^{Mod})^D$	$\in \boldsymbol{G.J}_1.t + \{(P^{Mod} \leftrightarrow Q^{Mod})^D \mid P = Q \in \Phi\}$.

Proof. It is easy to see that $\mathbf{MT4}+\{P^{Mod} \leftrightarrow Q^{Mod} | P = Q \in \Phi\} \subseteq \mathbf{MT4}+\Diamond p$. Hence (ii) ,(iii) and (iv) are equivalent. The implication from (i) to (ii) is clear. Now suppose that $\chi_1 = \chi_2 \notin L + \Phi$. By Proposition 22, there exists a complete lattice $\mathcal{D}$ satisfying Φ with $\mathcal{D} \not\models \chi_1 = \chi_2$. Take a set g and a closure operator Cl on g with $\mathcal{D} \simeq L_{Cl}$. (Such a closure operator exists for complete lattices; consult [3].) Let $\Diamond = Cl$. Then $\langle 2^g, \cap, -, \Diamond\rangle \not\models \chi_1^{Mod} \leftrightarrow \chi_2^{Mod}$, by Lemma 21. Hence (ii) implies (i). $\dashv$

One can use this result in both directions. From the decidability of $\mathbf{K}.t$ follows the decidability of the equational theory of lattices. Furthermore, the finite model property of $\mathbf{K}.t$ implies that the variety of all lattices is generated by finite lattices. But, of course, these results are well–known in lattice theory. The above theorem together with the proof show however that in principle we can investigate strong tense logics in order to analyse varieties of lattices. Conversely, the theorem shows once more the complexity of tense logic. Recall that there exist $2^{\aleph_0}$ varieties of lattices. Hence there exist $2^{\aleph_0}$ normal extensions of $\mathbf{G}.\mathbf{J}_1.t$.

Corollary 24 *For all normal bimodal logics Λ with $\boldsymbol{K}_2 \subseteq \Lambda \subseteq \boldsymbol{G}.\boldsymbol{J}_1.t$ the logic $\Lambda + ((p \wedge (q \vee (p \wedge r)))^{Mod})^D \leftrightarrow (((p \wedge q) \vee (p \wedge r))^{Mod})^D$ is undecidable.*

Proof. $L + \{p \wedge (q \vee (p \wedge r)) = (p \wedge q) \vee (p \wedge r)\}$ corresponds to the variety of modular lattices. By a result of Freese [10] the equational theory of modular lattices is undecidable. Now the corollary follows from Theorem 23. $\dashv$

§ 7. Simulating classical modal logics.

We have shown that monotonic modal logics can be simulated by normal bimodal logics in a natural way. The question arises whether similar techniques can be applied to classical systems. Here we have partial results only. To define semantics for classical modal logics we just have to put

$$\Box a = \{x \in g : a \in N(x)\},$$

for any N–frame $\langle g, N\rangle$ and $a \subseteq g$. A valuation V is now a homomorphism from the algebra of formulas into the boolean algebra of subsets of g such that $V(\Box\phi) = \Box V(\phi)$. Completeness of a classical modal logics with respect to N–frames is defined as usual. $\mathbf{E}$ is the logic of all N–frames, as is well known. We define an atomic interpretation F of $\mathcal{L}_1$ into the 3–modal language $\mathcal{L}_3$ with $\Box_1, \Box_2$ and $\Box_3$.

$$\begin{array}{lcl} p_i^F & := & p_i \\ (P \wedge Q)^F & := & P^F \wedge Q^F \\ (\neg P)^F & := & \neg P^F \\ (\Box P)^F & := & \Diamond_1(\Box_2 P^F \wedge \Box_3 \neg P^F) \end{array}$$

Theorem 25 *(1) Let Λ be a classical logic contained in a monotonic modal logic. Then the following properties are equivalent.*

(i) Λ *is complete with respect to* N*–frames.*
(ii) *There is a 3–modal normal logic which simulates* Λ *with respect to* F *and which is complete with respect to Kripke–semantics.*

(2) Let $\Lambda = \boldsymbol{E} + \Gamma$ *be complete with respect to* N*–frames. Further suppose that* Λ *is a subset of a monotonic logic. Then* $\boldsymbol{K}_3 + \Gamma^F$ *simulates* Λ *with respect to* F*.*

Proof. Let $\langle g, N\rangle$ be a N–frame for Λ and $\mathcal{C} = \mathcal{C}[\langle g, N\rangle]$. Take an $i \in \{1, 2, 3\}$ such that $\mathcal{F}_i \models \Lambda$. Now construct $\mathfrak{F} = \langle g, N\rangle^{mn_i} = \langle g \cup \otimes f, R_1, R_2\rangle$. We define a relation R_3 on $g \cup \otimes f$ by putting xR_3y iff $x = \langle C, 0\rangle$ and $y \in g - C$, for a $C \in \mathcal{C}$. It follows by induction that for all $Q \in \mathcal{L}_1$ and valuations β of h:

$$\text{For all } x \in g: \ \langle\langle g, N\rangle, \beta, x\rangle \models Q \text{ iff } \langle\langle h, R_1, R_2, R_3\rangle, \beta|g, x\rangle \models Q^F.$$

Now the proof is analogous to the proof in the monotonic case and left to the reader. $\dashv$

§ 8. The Thomason–Simulation.

In a series of papers, S. K. Thomason has shown how to simulate normal polymodal logics by normal monomodal logics. We will present this construction and prove some very strong theorems about it. Take a bimodal frame $\mathfrak{F} = \langle f, \lhd, \blacktriangleleft, \mathbf{F}\rangle$. Then put $f^{sim} := f \times \{w, b\} \cup \{*\}$, where w, b, $*$ are new symbols. We write x^w instead of $\langle x, w\rangle$ and x^b instead of $\langle x, b\rangle$. We also put $f^t := \{*\}$ and use the notation x^t for $*$. This will shorten a lot of casewise distinctions. The names are mnemonic; w for *white*, b for *black* and t for *terminal.* We let α and β range over the set $\{w, b, t\}$. Also, for subsets $a \subseteq f$ we write a^w for $a \times \{w\}$ and a^b for $a \times \{b\}$. a^t will on this convention be empty if a is, and otherwise $a^t = \{*\}$. (There is a small quirk concerning the possibility that f is empty. In that case, f^w and f^b are empty as well, while $f^t = \{*\}$. This exceptional case never causes any trouble in the theorems. So it will never be excluded, although one has to take care with the notation here.) On f^{sim} we define the relation $\prec$ as the union of three relations, $\prec^\nabla$, $\prec^\circ$ and $\prec^\bullet$.

$$\begin{array}{lcl}
\prec & := & \prec^\nabla \cup \prec^\circ \cup \prec^\bullet \\
\prec^\nabla & := & \{\langle x^w, *\rangle | x \in f\} \cup \{\langle x^w, x^b\rangle | x \in f\} \cup \{\langle x^b, x^w\rangle | x \in f\} \\
\prec^\circ & := & \{\langle x^w, y^w\rangle | x \lhd y\} \\
\prec^\bullet & := & \{\langle x^b, y^b\rangle | x \blacktriangleleft y\}
\end{array}$$

$\prec^\nabla$ provides a skeleton for the monomodal frame in which $\prec^\circ$ codes $\lhd$ and $\prec^\bullet$ codes $\blacktriangleleft$. Finally, we put $\mathbf{F}^{sim}$ to be all unions of sets of the form a^w, a^b, a^t where $a \in \mathbf{F}$. (If f is empty, the set of internal sets is simply $\wp(\{*\})$. Then $\mathfrak{F}^{sim}$ is a Kripke–frame.)

Proposition 26 $\mathfrak{F}^{sim} = \langle f^{sim}, \prec, \mathbf{F}^{sim}\rangle$ *is a (generalized) monomodal frame.*

Proof. Assume $f \neq \emptyset$. A set of $\mathbf{F}^{sim}$ can be written as a union

$$a^w \cup b^b \cup c^t$$

where $a, b, c \in \mathbf{F}$. Closure under union and complement is now straightforward. For closure under $\diamondsuit$ observe that

$$\begin{array}{rcl} \diamondsuit a^w & = & a^b \cup (\Diamond a)^w \\ \diamondsuit b^b & = & b^w \cup (\blacklozenge b)^b \\ \diamondsuit c^t & = & \left\{ \begin{array}{ll} f^w & \text{if } c \neq \emptyset \\ \emptyset & \text{if } c = \emptyset \end{array} \right. \end{array}$$

Thus $\mathbf{F}^{sim}$ is closed under $\diamondsuit$. $\dashv$

We call frames of the form $\mathfrak{F}^{sim}$ simply *simulation frames*. It is worth remembering that simulation frames are always connected via $\prec$, and so they are not decomposable into a disjoint union. Moreover, any point in a simulation frame sees a dead end (i. e. a point without successors) in at most two steps. The sets f^w, f^b and f^t are definable in any of the thus constructed frames by a formula without variables, hence they are always internal. Consider, namely, the following formulae.

$$\begin{array}{rcl} \mathsf{t} & := & \boxminus\bot \\ \mathsf{w} & := & \diamondsuit\mathsf{t} \\ \mathsf{b} & := & \neg\mathsf{t} \wedge \neg\mathsf{w} \end{array}$$

Notice that $\mathsf{t} \vee \mathsf{w} \vee \mathsf{b} \in \mathbf{K}$. It is not hard to verify that $\langle \mathfrak{F}^{sim}, x\rangle \models \mathsf{t}$ iff $x \in f^t$, that $\langle \mathfrak{F}^{sim}, x\rangle \models \mathsf{w}$ iff $x \in f^w$ and $\langle \mathfrak{F}^{sim}, x\rangle \models \mathsf{b}$ iff $x \in f^b$. Note that if we have a 1–morphism $\phi : \mathfrak{F} \to \mathfrak{G}$ and a constant formula d then $\langle \mathfrak{F}, x\rangle \models \mathsf{d}$ iff $\langle \mathfrak{G}, \phi(x)\rangle \models \mathsf{d}$. It follows that if $\mathfrak{F}$ and $\mathfrak{G}$ are nonempty 2–frames and $\phi : \mathfrak{F}^{sim} \to \mathfrak{G}^{sim}$ then $\phi[f^w] \subseteq g^w$, $\phi[f^b] \subseteq g^b$ and $\phi[f^t] \subseteq g^t$. If $\phi : \mathfrak{F} \to \mathfrak{G}$ is a 2–morphism, we define its simulation, ϕ^{sim}, via

$$\phi^{sim}(x^\alpha) := \phi(x)^\alpha$$

Theorem 27 $(-)^{sim}$ *is a covariant functor from the category* ${}^2\mathfrak{Frm}$ *of 2–frames into the category* ${}^1\mathfrak{Frm}$ *of 1–frames.*

Proof. Let $\mathfrak{F}$ and $\mathfrak{G}$ be 2–frames and $\phi : \mathfrak{F} \to \mathfrak{G}$ a 2–morphism. We have to show that $\phi^{sim} : f^{sim} \to g^{sim}$ as defined above is a 1–morphism. The cases where $\mathfrak{F}$ or $\mathfrak{G}$ is empty are easily handled, so we assume now that they are nonempty. To check (p1), let $x^\alpha \prec y^\beta$ for some $x, y \in f$. Then, by a straightforward checking of cases we get $\phi(x)^\alpha \prec \phi(y)^\beta$, whence $\phi^{sim}(x^\alpha) \prec \phi^{sim}(y^\beta)$. Now for the second condition, (p2), let $\phi^{sim}(x^\alpha) \prec u^\beta$, that is, $\phi(x)^\alpha \prec u^\beta$. We have to find a y^γ such that $x^\alpha \prec y^\gamma$ and $\phi^{sim}(y^\gamma) = \phi(y)^\gamma = u^\beta$. Clearly, $\gamma = \beta$. By the construction of $\mathfrak{F}^{sim}$, $\alpha \neq t$. Suppose that $\alpha = w$. If $\beta = t$, then we can put $y^\beta := x^t$. For we have $x^w \prec x^t$ and $\phi^{sim}(x^t) = u^\beta$, as required. Assume now $\beta \neq t$. If $\beta = b$, then $u = \phi(x)$, by construction of $\mathfrak{G}^{sim}$. Let $y := x$. Since $x^w \prec y^b$, that case is finished as well. Let finally be $\beta = w$.

Then, by construction, $\phi(x) \lhd u$. Since ϕ is a 2–morphism, there exists a y such that $\phi(y) = u$ and $x \lhd y$. Then $x^w \prec y^w$, and $\phi^{sim}(y^w) = u^w$, as required. The case $\alpha = b$ is similar. Thus (p2) is proved. Now, finally we have to show (p3). To that end, let $d := a^w \cup b^b \cup c^t$ be an internal set of $\mathfrak{G}^{sim}$. Then

$$(\phi^{sim})^{-1}[d] = (\phi^{-1}[a])^w \cup (\phi^{-1}[b])^b \cup (\phi^{-1}[c])^t \quad ,$$

which is internal. For by assumption ϕ is a 2–morphism, and so $\phi^{-1}[a]$ as well as $\phi^{-1}[b]$ are internal in $\mathbf{F}$, and so $(\phi^{-1}[a])^w$ as well as $(\phi^{-1}[b])^b$ is internal in $\mathfrak{F}^{sim}$. Moreover, $(\phi^{-1}[c])^t$ is internal in $\mathfrak{F}^{sim}$ for any c. ⊣

Theorem 28 *$\mathfrak{F}^{sim}$ is refined (compact, full) iff $\mathfrak{F}$ is.*

Proof. The case where $\mathfrak{F}$ is empty is easily dealt with; thus we assume from now on that it is not empty. Suppose that $\mathfrak{F}$ is refined. We show that $\mathfrak{F}^{sim}$ is refined. **(1.)** *$\mathfrak{F}^{sim}$ is differentiated.* Let $x^\alpha, y^\beta \in f^{sim}$ and $x^\alpha \neq y^\beta$. Then either $x \neq y$ or $\alpha \neq \beta$. The case where $\alpha = t$ or $\beta = t$ is easily dealt with. Assume next $\alpha = \beta$ ($\neq t$). Then $x \neq y$ and so there exists a set $c \in f$ such that $x \in c$, $y \in -c$. Then $x^\alpha \in c^\alpha$, $y^\alpha \in (-c)^\alpha \subseteq -c^\alpha$. Thus the case $\alpha \neq \beta$ remains. Here we may assume without loss of generality $\alpha = w$. Then $x^\alpha \in f^w$ but $y \notin f^w$, and f^w is internal. This shows that $\mathfrak{F}^{sim}$ is differentiated. **(2.)** *$\mathfrak{F}^{sim}$ is refined.* Let $x^\alpha \mathbf{\Sigma}\, y^\beta$. We have to find a set a such that $x^\alpha \in \boxminus a$ but $y^\beta \in -a$. If $\alpha = t$ then $a := \emptyset$ is a good choice, and if $\beta = t$ and $\alpha = b$ then $a := f^w \cup f^b$ is a good choice. Thus we are left with $\alpha, \beta \neq t$. If $\alpha = \beta = w$ then we must have $x \mathbf{6}\, y$ and so there exists a set $c \in \mathbf{F}$ such that $x \in \Box c$ but $y \in -c$. Now put $a := c^w \cup f^b \cup f^t$. Then $y^w \in -a$ but $x^w \in \boxminus a$. If $\alpha = \beta = b$ then not $x \blacktriangleleft y$ and so there exists a c such that $y \in -c$ and $x \in \blacksquare c$. Put $a := f^w \cup c^b$. Then $x^b \in \boxminus a$ but $y^b \in -a$. The case $\alpha \neq \beta$ remains (and both are $\neq t$). Here we have $x^\alpha \mathbf{\Sigma}\, y^\beta$ exactly when $x \neq y$. Assume that $\alpha = w$ and $\beta = b$; the other case is analogous. Since $\mathfrak{F}$ is differentiated we have a set c such that $x \in c$ but $y \in -c$. Put $a := f^w \cup c^b \cup f^t$; we get $x^w \in \boxminus a$ and $y^b \notin a$, as required. Hence $\mathfrak{F}^{sim}$ is tight, and thus refined. Now assume conversely that $\mathfrak{F}^{sim}$ is refined. Then $\mathfrak{F}$ is refined as well. For if $x \neq y$, then in particular $x^w \neq y^w$ and we get by assumption a set c such that $x^w \in c$ but $y^w \notin c$. Let $c = a_1^w \cup a_2^b \cup a_3^t$ for some $a_1, a_2, a_3 \in \mathbf{F}$. It follows that $x \in a_1$ but $y \notin a_1$, and so $\mathfrak{F}$ is differentiated. Next, if $x \mathbf{6}\, y$ then $x^w \mathbf{\Sigma}\, y^w$ and we get a set c such that $x^w \in \boxminus c$ but $y^w \notin c$. Decompose c in the same way as before. Then it follows that $x \in \Box a_1$ but $y \notin a_1$. The case of $\blacktriangleleft$ is similar. Next we take compactness. Assume that $\mathfrak{F}$ is compact; let $U \subset \mathbf{F}^{sim}$ be an ultrafilter. Then three cases can arise. (1.) $f^t \in U$. In that case $U = \{a | * \in a\}$. Clearly, $\bigcap U \neq \emptyset$. (2.) $f^w \in U$. Then $f^t, f^b \notin U$, and so there exists an ultrafilter V on $\mathbf{F}$ such that $c \in U$ exactly if $c = c_1^w \cup c_2^b \cup c_3^t$ with $c_1 \in V$, $c_2, c_3 \in \mathbf{F}$. It is then clear that if $x \in \bigcap V$ we must have $x^w \in \bigcap U$. Since $\mathfrak{F}$ is compact, there exists such a x. (3.) $f^b \in U$. Reason as in the case (2.). Now assume conversely that $\mathfrak{F}^{sim}$ is compact, and take an ultrafilter $V \subset \mathbf{F}$. Then $U := \{a_1^w \cup a_2^b \cup a_3^t | a_1 \in V, a_2, a_3 \in \mathbf{F}\}$ is an ultrafilter on $\mathbf{F}^{sim}$. By assumption, there exists $u \in \bigcap U$. It is not hard to see that $u = x^w$ for some $x \in f$. Hence $x \in \bigcap V$. Thus $\mathfrak{F}$ is compact. It remains the property of *fullness*. However, this is straightforward. ⊣

Now let $\mathfrak{A}$ be a 2–algebra. Then we may define $\mathfrak{A}^{sim}$ by $((\mathfrak{A}^+)^{sim})_+$. However, we can actually spell out this construction as follows. Assume that $\mathfrak{A} = \langle A, 1, -, \cap, \Box, \blacksquare\rangle$; then we put $A^{sim} := A \times A \times 2$ and

$$\boxminus\langle a, b, c\rangle := \begin{cases} \langle b \cap \Box a, a \cap \blacksquare b, \{1\}\rangle & \text{if } c = \{1\} \\ \langle 0, a \cap \blacksquare b, \{1\}\rangle & \text{if } c = \emptyset \end{cases}$$

Then let $\mathfrak{A}^{sim} := \langle A^{sim}, 1, -, \cap, \boxminus\rangle$. That $\mathfrak{A}^{sim}$ as just defined is indeed isomorphic to $((\mathfrak{A}^+)^{sim})_+$ is a matter of direct verification. Namely, $\mathfrak{A}^+$ is based on the set $pt(\mathfrak{A})$. $(\mathfrak{A}^+)^{sim}$ contains the points U^w, U^b for $U \in pt(\mathfrak{A})$, and an additional point denoted by U^t. (If $\mathfrak{A}$ has one element only, $\mathfrak{A}^+$ is empty. That case is to be treated separately.) A general internal set is of the form $\widehat{a}^w \cup \widehat{b}^b \cup \widehat{c}^t$, where $a, b, c \in B$, by construction. Hence define $h : \langle a, b, c\rangle \mapsto \widehat{a}^w \cup \widehat{b}^b \cup \widehat{c}^t$. This map is an isomorphism between $\mathfrak{A}^{sim}$ and $((\mathfrak{A}^+)^{sim})_+$. We will use w, b and t and A^α, $\alpha \in \{w, b, t\}$, in an analogous way for algebras.

Now we will axiomatize the modal theory of the simulation frames. To do this, we introduce a shorthand notation. We write $\boxminus_w P$ for $\boxminus(\mathsf{w} \to P)$ as well as $\diamondminus_w P$ for $\diamondminus(\mathsf{w} \wedge P)$. A similar notation is used with respect to b and t. Since $\mathsf{w} \vee \mathsf{b} \vee \mathsf{t}$ is a theorem of **K** we have in **K** the following theorems

$$\boxminus P \leftrightarrow \bigwedge_{\alpha \in \{w,b,t\}} \boxminus_\alpha P \qquad \diamondminus P \leftrightarrow \bigvee_{\alpha \in \{w,b,t\}} \diamondminus_\alpha P$$

Take the following formulae:

$$\begin{array}{lll} (a) & \mathsf{w}. \to .\diamondminus_b p \leftrightarrow \boxminus_b p, & \mathsf{b}. \to .\diamondminus_w p \leftrightarrow \boxminus_w p \\ (b) & \mathsf{w} \wedge p. \to . \boxminus_b \boxminus_w p, & \mathsf{b} \wedge p. \to . \boxminus_w \boxminus_b p \\ (c) & \diamondminus_t p \to \boxminus_t p, & \mathsf{w} \wedge \diamondminus_t p. \to . \boxminus_w \diamondminus_t p \\ & \mathsf{w} \wedge \diamondminus_w \diamondminus_t p. \to .\diamondminus_t p, & \mathsf{w} \wedge \diamondminus_t p. \to . \boxminus_b \boxminus_b \boxminus_w \diamondminus_t p \end{array}$$

Denote by **Sim** the logic obtained by adding these formulae as normal axioms to **K**. If $\mathfrak{F}$ is a frame for **Sim**, we denote by f^w (f^b, f^t) the set of points satisfying w (b, t).

Proposition 29 *Let $\mathfrak{F}$ be a nonempty refined frame for **Sim**. Suppose that $\mathfrak{F}$ is rooted. Then there exists a bimodal frame $\mathfrak{G}$ such that $\mathfrak{F}$ is isomorphic to $\mathfrak{G}^{sim}$. Moreover, $\mathfrak{G}$ is unique up to isomorphism.*

Proof. Since $\mathfrak{F}$ is not empty, there exists a point $x \in f$. First we show that for every point x in f^w there exists a unique point y in f^b such that $x \prec y$, and that then also $y \prec x$. The existence of y is immediate from the first postulate of (a) (e. g. by substituting $\top$ for p). However, assume there exists a $y' \neq y$ such that $x \prec y'$ and $y' \in f^b$. Then there is a set c such that $y \in c$ but $y' \notin c$. Now put $\beta(p) := c$. Then $\langle \mathfrak{F}, \beta, x\rangle \models \mathsf{w} \wedge \diamondminus_b p$, but $\langle \mathfrak{F}, \beta, x\rangle \models \neg \boxminus_b p$. This violates the first axiom of (a). Now, given y there exists by the second axiom of (a) a z such that $y \prec z$ and $z \in f^w$. By the first axiom of (b) and the fact that the frame is refined we obtain that $z = x$. Analogously we show that for every point y from f^b there exists a unique $x \in f^w$ such that $y \prec x$, and that for this x we have $x \prec y$ as well.

Now we show that there is exactly one point in f^t. Let us first prove $f^t \neq \emptyset$. If $f^t = f$ then f^t is nonempty, since f is rooted. Now assume that there exists a $x \in f - f^t$. Then $x \in f^w \cup f^b$. If $x \in f^w$ then $x \prec y$ for a $y \in f^t$ and if $x \in f^b$ then there exists a $y \in f^w$ such that $x \prec y$. Thus f^t is nonempty.

We show that f^t contains not more than one point. To that end, let r be the root of $\mathfrak{F}$. If $r \in f^t$, we have succeeded. For then $\mathfrak{F}$ contains only one point. If $r \in f^b$ then there exists $z \in f^w$ such that $r \prec z \prec r$, and so z is a root of $\mathfrak{F}$ as well. Thus let us assume that $r \in f^w$. We will show that for all $x \in f^w$, and all z, z' from f

$$(\dagger): \text{ If } \ r \prec z \in f^t, x \prec z' \in f^t, \text{ then } z = z'.$$

From (†) we immediately obtain that f^t contains only one point. To prove (†) let $x \in f^w$. Then by the fact that there is a path from r to x (and the previous considerations concerning the constitution of $\mathfrak{F}$) there exists a sequence $\langle y_i | i \leq n \rangle$ such that $y_0 = r$, $y_n = x$ and such that all y_i are in f^w, and either $y_i \prec y_{i+1}$ (a link of type (α)) or there exists $z_i, z_{i+1} \in f^b$ such that $y_i \prec z_i \prec z_{i+1} \prec y_{i+1}$ (a link of type (β)). We prove that for all $i < n$, the points y_i and y_{i+1} have the same successors in f^t. **Case 1.** y_i and y_{i+1} form a link of type (α). Then $y_i \prec y_{i+1}$. The axiom $\mathsf{w}. \rightarrow .\Diamond_w \Diamond_t p \rightarrow \Diamond_t p$ forces that each successor of y_{i+1} in f^t is also a successor of y_i and the second (c) postulate shows that each successor of y_i in f^t is also a successor of y_{i+1}. **Case 2.** y_i and y_{i+1} form a link of type (β). Then the axiom $\mathsf{w} \wedge \Diamond_t p. \rightarrow . \boxminus_b \boxminus_b \boxminus_w \Diamond_t p$ shows that each successor of y_i in f^t is also a successor of y_{i+1}. Hence y_i and y_{i+1} have the same successors in f^t since both have precisely one successor in f^t, by the first (c) postulate. It follows by induction that x and r have the same successors in f^t. (†) follows from the fact that all point in f^w have not more than one successor in f^t.

Now put

$$\begin{array}{rcl} g & := & f^w \\ \lhd & := & \prec \cap (f^w)^2 \\ \blacktriangleleft & := & \{\langle x, y\rangle | (\exists \overline{x}, \overline{y} \in f^b)(x \prec \overline{x} \prec \overline{y} \prec y)\} \cap (f^w)^2 \\ \mathbf{G} & := & \{a \cap g | a \in \mathbf{F}\} \end{array}$$

It is easy to verify that $\mathfrak{G}^{sim}$ is isomorphic to $\mathfrak{F}$. $\dashv$

§ 9. Unsimulation.

It is clear that there exist frames for **Sim** which are not simulation frames; simply take the disjoint union of two simulation frames. However, this is in some sense the only exception.

Proposition 30 *A refined **Sim**–frame $\mathfrak{F}$ is a simulation frame iff* t *can be satisfied at exactly one point iff $\mathfrak{F}$ is not a disjoint union of two proper generated subframes. A **Sim**–algebra $\mathfrak{A}$ is a simulation algebra iff* t *is an atom iff $\mathfrak{A}$ is directly indecomposable.*

Proof. If $\mathfrak{F}$ is empty, or if the algebra is trivial, the above equivalences hold. So this case will not be considered below. The interesting part is the following. Assume that t is satisfiable at more than one point, say at x_1 and x_2, where $x_1 \neq x_2$. Since $\mathfrak{F}$ is refined, there exists a set c such that $x_1 \in c$ but $x_2 \notin c$. Put $a := \mathsf{t} \cap c$ and $b := \mathsf{t} - c$. Then $a \cap b = \emptyset$ and $a \cup b = f^t$. Now put $g_1 := a \cup \blacklozenge a \cup \blacklozenge\blacklozenge a$, $g_2 := b \cup \blacklozenge b \cup \blacklozenge\blacklozenge b$. g_1 and g_2 are internal. Both are successor closed sets. Moreover, it is straightforwardly checked that $g_1 \cup g_2 = f$ as well as $g_1 \cap g_2 = \emptyset$. (By duality, each point sees in at most two steps a point in t. Moreover, each point can see at most one such point.) Hence we have a decomposition of $\mathfrak{F}$ into a disjoint union of two generated subframes. From this the first equivalences follow. The assertion concerning the algebra $\mathfrak{A}$ are proved analogously. If t is not an atom there exists a set a such that $0 < a < \mathsf{t}$. Now put $b := \mathsf{t} \cap (-a)$. Next let $d_1 := a \cup \blacklozenge a \cup \blacklozenge\blacklozenge a$ and $d_2 := b \cup \blacklozenge b \cup \blacklozenge\blacklozenge b$. The maps $h_1 : x \mapsto x \cap d_1$ and $h_2 : x \mapsto x \cap d_2$ are boolean homomorphisms and can be shown to be also homomorphisms of the modal algebras. Finally, $x = y$ iff $h_1(x) = h_1(y)$ as well as $h_2(x) = h_2(y)$, as can be verified. Hence, $\mathfrak{A}$ is directly decomposable. ⊣

(A note on the proof. That $\mathfrak{A}$ is directly decomposable is seen in some more detail as follows. We have two homomorphisms, h_1 and h_2, with corresponding congruences Θ_1 and Θ_2. We have $\Theta_1 \cap \Theta_2 = \{\langle x, x\rangle : x \in A\}$, the least congruence, and $\Theta_1 \circ \Theta_2 = A \times A$, the largest congruence. The congruences permute, since the variety of modal algebras has permuting congruences. Thus $\mathfrak{A}$ is a direct product of $\mathfrak{A}/\Theta_1$ and $\mathfrak{A}/\Theta_2$.) The next proposition shows that the defect of a **Sim**–frame is rather easily removed.

Proposition 31 *Let $\mathfrak{F}$ be a refined nonempty frame for **Sim**. The mapping collapsing the set* t *into a single point is a p–morphism of $\mathfrak{F}$ onto a simulation frame.*

Proof. t is a generated subset. Hence the mapping is a p–morphism on the underlying kripke–frame. Moreover, since t is internal, the mapping is actually a p–morphism of the frame. The image of the map is a simulation frame by Proposition 30. ⊣

We now define the *unsimulation* of a refined **Sim**–frame $\mathfrak{F}$ as follows. We put

$$\begin{array}{lcl} f_{sim} & := & f^w \\ \lhd & := & \prec \cap (f_{sim})^2 \\ \blacktriangleleft & := & \{\langle x, y\rangle | (\exists \overline{x}, \overline{y} \in f^b)(x \prec \overline{x} \prec \overline{y} \prec y)\} \cap (f_{sim})^2 \\ & & \\ \mathbf{F}_{sim} & := & \{a \cap f_{sim} | a \in \mathbf{F}\} \\ \mathfrak{F}_{sim} & := & \langle f_{sim}, \lhd, \blacktriangleleft, \mathbf{F}_{sim}\rangle \end{array}$$

This is immediately verified to be a 2–frame. This is well–defined even if $\mathfrak{F}$ is empty. In that case the unsimulation is also empty. This case is exceptional insofar as the simulation of $\mathfrak{F}_{sim}$ is not isomorphic to $\mathfrak{F}$. So, that case has to be taken care of independently. Next, if we have a 1–morphism $\phi : \mathfrak{F} \to \mathfrak{G}$, where $\mathfrak{F}$ and $\mathfrak{G}$ are **Sim**–frames, we put $\phi_{sim} := \phi \upharpoonright f_{sim}$. This is a 2–morphism. For let $x \lhd y$. Then $x \prec y$, from which $\phi(x) \prec \phi(y)$, since ϕ is a 1–morphism. Hence, by definition of ϕ_{sim}, $\phi_{sim}(x) \lhd \phi_{sim}(y)$. Likewise for $x \blacktriangleleft y$. Now assume that $\phi_{sim}(x) \lhd u$. Then, as $\phi(x) \prec u$, we know that there exists a $y \in f$ such that $x \prec y$ and $\phi(y) = u$. It is clear that y must be in f^w, since u is in g^w. Hence $x \lhd y$, and $\phi_{sim}(y) = u$, as required. Similarly for $\blacktriangleleft$.

Proposition 32 *The unsimulation map $(-)_{sim}$ is a covariant functor from the category of nonempty refined **Sim**–frames onto the category of refined 2–frames. Moreover, for every 2–frame we have $(\mathfrak{F}^{sim})_{sim} \cong \mathfrak{F}$.* $\dashv$

Proposition 33 *The category of nonempty simulation frames and the category of 2–frames are isomorphic.* $\dashv$

This last proposition is not a direct consequence of Proposition 32; we can only show that these categories are equivalent (which would be sufficient for our purposes). But the construction of $\mathfrak{F}^{sim}$ in set theoretic terms can be done in such a way that $(\mathfrak{F}^{sim})_{sim}$ is always the same as $\mathfrak{F}$.

In addition, if $\mathfrak{F}$ is a simulation frame and $\iota : \mathfrak{G} \rightarrowtail \mathfrak{F}$, then $\mathfrak{G}$ is a simulation frame and there exists a 2–morphism ϕ such that $\iota = \phi^{sim}$; and if $\iota : \mathfrak{F} \twoheadrightarrow \mathfrak{G}$ then $\mathfrak{G}$ is a simulation frame and there exists a 2–morphism ϕ such that $\iota = \phi^{sim}$. (We employ the convention that $\rightarrowtail$ denotes an injective and $\twoheadrightarrow$ a surjective map.)

In a similar fashion the unsimulation is defined for algebras. If $\mathfrak{A} = \langle A, 1, -, \cap, \boxminus \rangle$ is a **Sim**–algebra, then let $A_{sim} := \{a | a \leq \mathsf{w}\}$. Then $\mathfrak{A}_{sim} := \langle A_{sim}, 1, -', \cap', \Box, \blacksquare \rangle$, where $a \cap' b := a \cap b$, $-'a := \mathsf{w} \cap (-a)$ and

$$\begin{array}{rcl} \Box a & := & \boxminus(\mathsf{w} \to a) \\ \blacksquare a & := & \boxminus(\mathsf{b} \to \boxminus(\mathsf{b} \to \boxminus(\mathsf{w} \to a))) \end{array}$$

It is not hard to see that for a **Sim**–algebra $\mathfrak{A}$, $(\mathfrak{A}_{sim})^+ \cong (\mathfrak{A}^+)_{sim}$, unless $\mathfrak{A} \cong \mathbf{1}$. It is also not hard to see that $\mathbf{1}_{sim} \cong \mathbf{1}$, while $\mathbf{1}^{sim}$ is the two–element algebra of subsets of $\boxed{\bullet}$. We denote this algebra by $\mathbf{2}^{\bullet}$. Using the duality between descriptive frames and modal algebras we can prove analogous theorems for algebras. One consequence is worth noting. Let $Sub(\mathfrak{A})$ denote the lattice of subalgebras of $\mathfrak{A}$, and $Con(\mathfrak{A})$ the lattice of congruences of $\mathfrak{A}$. In $Con(\mathfrak{A})$ the element $\Delta := \{\langle a, a \rangle | a \in A\}$ is the bottom element and $\nabla := A \times A$ the top element. $\mathfrak{A}$ is called *subdirectly irreducible* if in $Con(\mathfrak{A})$ the set of congruences different from Δ has a lowest element. This congruence is called the *monolith* of $\mathfrak{A}$. Given a lattice $\mathfrak{L}$ we denote by $\mathfrak{L} + 1$ the lattice obtained from $\mathfrak{L}$ by adjoining a new element as a top element.

Theorem 34 *Let $\mathfrak{A}$ be a simulation algebra. Then $Sub(\mathfrak{A}_{sim}) \cong Sub(\mathfrak{A})$ and $Con(\mathfrak{A}_{sim}) + 1 \cong Con(\mathfrak{A})$. In particular, $\mathfrak{A}$ is subdirectly irreducible iff $\mathfrak{A}_{sim}$ is.*

Proof. Consider a homomorphism $h : \mathfrak{B} \rightarrowtail \mathfrak{A}$ based on the inclusion map $h : B \subseteq A$. Then by definition of h_{sim}, $h_{sim} : \mathfrak{B}_{sim} \rightarrowtail \mathfrak{A}_{sim}$ is the natural inclusion. Conversely, assume that $i : \mathfrak{C} \rightarrowtail \mathfrak{A}_{sim}$ is an inclusion map. We show that $\mathfrak{C}$ is a simulation algebra. For that we have to show that $\mathtt{t}_{\mathfrak{C}}$ is an atom of $\mathfrak{C}$. We certainly have $i(\mathtt{t}_{\mathfrak{C}}) = \mathtt{t}_{\mathfrak{A}_{sim}}$ because $\mathtt{t}$ is constant and $\mathtt{t}_{\mathfrak{C}}$ and $\mathtt{t}_{\mathfrak{A}_{sim}}$ are the respective values of $\mathtt{t}$ in the algebras. Now, since $\mathtt{t}_{\mathfrak{A}_{sim}}$ is an atom, $\mathtt{t}_{\mathfrak{C}}$ must be nonzero, and an atom too, since i is injective. So, $\mathfrak{C}$ is a simulation algebra. There exists a $\mathfrak{B}$ such that $\mathfrak{C} = \mathfrak{B}^{sim}$. Furthermore, $i_{sim} : \mathfrak{C}_{sim} \rightarrowtail (\mathfrak{A}^{sim})_{sim}$. Let $g : (\mathfrak{A}^{sim})_{sim} \to \mathfrak{A} : x^w \mapsto x$. This is an isomorphism, and $k := g \restriction (B^{sim})_{sim} : (\mathfrak{B}^{sim})_{sim} \to \mathfrak{B}$ is an isomorphism as well. Then $g \circ i_{sim} \circ k^{-1} : \mathfrak{C}_{sim} \rightarrowtail \mathfrak{A}$ is an inclusion map. This shows that $\mathfrak{B} \mapsto \mathfrak{B}_{sim}$ induces a bijection between subalgebras of $\mathfrak{A}$ and subalgebras of $\mathfrak{A}_{sim}$. For congruences it is actually better to switch to the dual category of descriptive frames. Consider $\mathfrak{F} := \mathfrak{A}^+$. $\mathfrak{F}$ is a descriptive **Sim**–frame. Now let $\phi : \mathfrak{G} \rightarrowtail \mathfrak{F}$. Then either $\mathfrak{G}$ is empty (and so not a simulation frame), or it contains f^t (since the latter consists of a single point). Then $\phi_{sim} : \mathfrak{G}_{sim} \rightarrowtail \mathfrak{F}_{sim}$. Conversely, let $\psi : \mathfrak{H} \rightarrowtail \mathfrak{F}_{sim}$. Then $\psi^{sim} : \mathfrak{H}^{sim} \rightarrowtail (\mathfrak{F}_{sim})^{sim}$. Fix an isomorphism $\iota : (\mathfrak{F}_{sim})^{sim} \to \mathfrak{F}$. Then $\iota \circ \psi^{sim} : \mathfrak{H}^{sim} \rightarrowtail \mathfrak{F}$. Hence $\phi \mapsto \phi_{sim}$ induces an inclusion preserving bijection between nonempty generated subframes of $\mathfrak{F}$ and generated subframes of $\mathfrak{F}_{sim}$. By duality (identifying $(\mathfrak{F}_{sim})_+$ with $(\mathfrak{F}_+)_{sim}$ in a natural way) there exists an inclusion preserving isomorphism from $Con(\mathfrak{A}) - \{\nabla\}$ to $Con(\mathfrak{A}_{sim})$. ⊣

The next important step is to extend the simulation and unsimulation maps to varieties. For a class $\mathcal{K}$ of 2–algebras put $\mathcal{K}^{sim} := \{\mathfrak{A}^{sim} | \mathfrak{A} \in \mathcal{K}\}$ and for a class $\mathcal{L}$ of **Sim**–algebras put $\mathcal{L}_{sim} := \{\mathfrak{A}_{sim} | \mathfrak{A} \in \mathcal{L}\}$.

Corollary 35 *For a class $\mathcal{L}$ of 1–algebras, $(H\mathcal{L})_{sim} = H(\mathcal{L}_{sim})$ and $(SS\,\mathcal{L})_{sim} = SS\,(\mathcal{L}_{sim})$. For a class $\mathcal{K}$ of 2–algebras, $H(\mathcal{K}^{sim}) = \{\mathbf{1}\} \cup (H\mathcal{K})^{sim}$ and $SS(\mathcal{K}^{sim}) = (SS\mathcal{K})^{sim}$.* ⊣

This immediately follows from the proof of the previous theorem, by duality. Now, let $\mathcal{V}$ be a variety of 2–algebras. Then let $\mathcal{V}^v$ be the variety of 1–algebras generated by $\mathcal{V}^{sim}$. By a theorem of Birkhoff, $\mathcal{V}^v = \mathsf{HSP}(\mathcal{V}^{sim})$. Likewise, for a variety $\mathcal{W}$ of **Sim**–algebras we let $\mathcal{W}_v := \mathsf{HSP}(\mathcal{W}_{sim})$. We want to show that the maps $(-)^v$ and $(-)_v$ are inverses of each other. Since they are both order preserving with respect to inclusion it follows from this fact that the lattice of varieties of 2–algebras is isomorphic to the lattice of nontrivial varieties of **Sim**–algebras. Here, a variety $\mathcal{V}$ is **nontrivial** if $\mathcal{V}$ contains an algebra different from $\mathbf{1}$. As it turns out, this is equivalent to requiring that $\mathbf{2}^\bullet$ is contained in $\mathcal{V}$.

Proposition 36 *Let $\mathcal{W}$ be a class of 1–algebras. Then $\mathcal{W}_{sim}$ is a variety if $\mathcal{W}$ is.*

Proof. We have to show that $\mathcal{W}_{sim}$ is closed under H, S and P. By Corollary 35 $\mathcal{W}_{sim}$ is closed under H and S. Closure under products is also relatively straightforward. If $\mathfrak{B}^i$, $i \in I$, is of the form $(\mathfrak{A}^i)_{sim}$ for some $\mathfrak{A}^i$, then $\prod_{i\in I}\mathfrak{B}^i$ is isomorphic to $(\prod_{i\in I}\mathfrak{A}^i)_{sim}$, since unsimulation is basically a projection. ⊣

So, if $\mathcal{W}$ is a variety then $\mathcal{W}_v = \mathcal{W}_{sim}$. The case of simulation is not as simple. However, by a theorem of Birkhoff, two varieties are equal iff they contain the same subdirectly irreducible algebras. Now denote by $si(\mathcal{V})$ the class of subdirectly irreducible members of $\mathcal{V}$. The variety of n–modal algebras is congruence distributive. Consequently, by Jónsson's Lemma, for a class $\mathcal{K}$ of n–modal algebras, $si(\mathcal{K}) \subseteq \mathsf{HSUp}(\mathcal{K})$. It is not hard to verify that for $\mathfrak{A}_i$, $i \in I$, 2–algebras and F an ultrafilter on I we have

$$(\prod_F \mathfrak{A}_i)^{sim} \cong \prod_F (\mathfrak{A}_i)^{sim}$$

Hence, $\mathsf{Up}(\mathcal{K})^{sim} = \mathsf{Up}(\mathcal{K}^{sim})$. So, if $\mathcal{V}$ is a variety,

$$\mathsf{HSUp}(\mathcal{V}^{sim}) = \{\mathbf{1}\} \cup (\mathsf{HSUp}(\mathcal{V}))^{sim} = \{\mathbf{1}\} \cup \mathcal{V}^{sim}$$

Hence

$$\begin{array}{rclcl} si(\mathcal{V}^v) & = & si(\mathsf{HSUp}(\mathcal{V}^{sim})) & = & \\ & = & si(\{\mathbf{1}\} \cup \mathcal{V}^{sim}) & = & \\ & = & si(\mathcal{V}^{sim}) & = & (si(\mathcal{V}))^{sim} \end{array}$$

Lemma 37 *Let $\mathcal{W}$ be a variety of 1–algebras, and $\mathcal{V}$ be a variety of 2–algebras. Then $si((\mathcal{V}^v)_v) = si(\mathcal{V})$ and $si((\mathcal{W}_v)^v) = si(\mathcal{W})$.*

Proof. We have seen that $si(\mathcal{V}^v) = (si(\mathcal{V}))^{sim}$. Furthermore, $si(\mathcal{W}_v) = si(\mathcal{W}_{sim})$ by Proposition 36 and $si(\mathcal{W}_{sim}) = (si(\mathcal{W}))_{sim}$, by Theorem 34.

$$\begin{array}{rcrcr} si((\mathcal{W}_v)^v) & = & (si(\mathcal{W}_v))^{sim} & = & (si(\mathcal{W}_{sim}))^{sim} \\ & = & ((si(\mathcal{W}))_{sim})^{sim} & = & si(\mathcal{W}) \end{array}$$

$$\begin{array}{rcrcr} si((\mathcal{V}^v)_v) & = & si((\mathcal{V}^v)_{sim}) & = & (si(\mathcal{V}^v))_{sim} \\ & = & ((si(\mathcal{V})^{sim})_{sim} & = & si(\mathcal{V}). \end{array}$$ ⊣

The following can now be concluded.

Theorem 38 *The map $(-)^v$ is a bijection between the varieties of 2–algebras and the varieties of **Sim**–algebras containing the algebra $\boldsymbol{2}^\bullet$.* ⊣

Given a bimodal logic Λ we will write Λ^{sim} for the logic $\mathsf{Th}\,(\mathsf{Alg}\,\Lambda)^{sim}$, and given a monomodal logic Θ we write Θ_{sim} for the logic $\mathsf{Th}\,(\mathsf{Alg}\,\Theta)_{sim}$. We call the map $\Lambda \mapsto \Lambda^{sim}$ the *Thomason–Simulation.*

Theorem 39 *The Thomason–Simulation is an isomorphism from the lattice $\mathcal{E}\boldsymbol{K}_2$ onto the interval $[\boldsymbol{Sim}, \mathsf{Th}\,\boldsymbol{2}^\bullet]$ in the lattice $\mathcal{E}\boldsymbol{K}_1$.* ⊣

A lot of conclusions can be drawn. For example, if Λ is a bimodal logic, then $\mathcal{E}\,\Lambda^{sim} \cong \mathcal{E}\,\Lambda + 1$. Moreover, by the previous theorems, if Λ is complete with respect to a class $\mathfrak{X}$ of refined 2–frames then so is Λ^{sim} with respect to $\mathfrak{X}^{sim}$. And conversely, if Λ^{sim} is complete with respect to a class $\mathfrak{Y}$ of refined 1–frames then Λ is complete with respect to $\mathfrak{Y}_{sim}$. In the next two sections we will improve these results drastically.

The simulation can be generalized in a straightforward way to obtain an isomorphism from the lattice $\mathcal{E}\,\mathbf{K}_n$ onto the interval $[\mathbf{Sim}(n,1), \mathsf{Th}\,\mathfrak{ch}_{n-1}]$ in the lattice $\mathcal{E}\,\mathbf{K}_1$, where $\mathbf{Sim}(n,1)$ is some monomodal logic and $\mathfrak{ch}_n = \langle\{0,1,\ldots,n-1\},>\rangle$ denotes the transitive, irreflexive n–element chain. The details will only be sketched. Namely, given $\mathfrak{f} = \langle f, \langle \lhd_i | i < n\rangle\rangle$, $\mathfrak{f}^{sim}$ will be defined thus. Each point x will be multiplied into distinct copies x^i, one per basic operator; a copy of $\mathfrak{ch}_n$ is added. The x^i are kept distinct by the points in the chains that they can see. Moreover, we let $x^i \prec x^j$ iff $i \neq j$. The sets f^i are then definable by constant formulae, as are the individual points of $\mathfrak{ch}_n$. Once this is done, we can encode $\lhd_i$ into the set f^i. This construction is lifted to general frames, and the theorems proved so far are proved in the same way for this general construction. One can also generalize the Thomason–Simulation to a map from $\mathcal{E}\,\mathbf{K}_{m+n}$, $m,n > 0$, onto an interval $[\Theta, \mathsf{Th}\boxed{\bullet}]$ in $\mathcal{E}\,\mathbf{K}_m$, where $\boxed{\bullet}$ is the one–point m–frame in which all relations are empty.

§ 10. Properties of the Simulation.

The simulation has been established on an abstract level. We have shown that for a bimodal logic Λ there exists a simulation Λ^{sim} and for every monomodal logic Θ there exists an unsimulation Θ_{sim}. In this section and the next we will show how to axiomatize the simulations and unsimulations, respectively, and determine what properties are transferred back and forth under this simulation. We start with the case of simulation. An axiomatization of Λ^{sim} can be given on the basis of an axiomatization of Λ. Define the following translation.

$$\begin{array}{lcl}
p^\tau & := & p \\
(\neg P)^\tau & := & \neg P^\tau \\
(P \wedge Q)^\tau & := & P^\tau \wedge Q^\tau \\
(\Box P)^\tau & := & \boxminus_w P^\tau \\
(\blacksquare P)^\tau & := & \boxminus_b \boxminus_b \boxminus_w P^\tau
\end{array}$$

Lemma 40 *Let $\mathfrak{F}$ be a 2–frame, and P a bimodal formula. Let $\langle \mathfrak{F}, \beta, x\rangle \models P$. Assume that γ is a valuation on $\mathfrak{F}^{sim}$ such that $\gamma(p) \cap f^w = \beta(p)^w$. Then*

$$\langle \mathfrak{F}, \beta, x\rangle \models P \qquad \Leftrightarrow \qquad \langle \mathfrak{F}^{sim}, \gamma, x^w\rangle \models P^\tau$$

Proof. By induction on the complexity of P. ⊣

Let $\mathfrak{F}$ be a bimodal frame, β a valuation on $\mathfrak{F}$. Then put $\beta^{sim}(p) := \beta(p)^w$. This defines a valuation on $\mathfrak{F}^{sim}$. The following now holds for any set X of bimodal formulae. (Here, $\mathsf{w} \rightarrow X^\tau$ denotes the set $\{\mathsf{w} \rightarrow P^\tau | P \in X\}$.)

$$\begin{array}{rcr}
\langle \mathfrak{F}, \beta, x\rangle \models X & \Leftrightarrow & \langle \mathfrak{F}^{sim}, \beta^{sim}, x^w\rangle \models X^\tau \\
\langle \mathfrak{F}, \beta\rangle \models X & \Leftrightarrow & \langle \mathfrak{F}^{sim}, \beta^{sim}\rangle \models \mathsf{w} \rightarrow X^\tau \\
\mathfrak{F} \models X & \Leftrightarrow & \mathfrak{F}^{sim} \models \mathsf{w} \rightarrow X^\tau
\end{array}$$

The first is the content of the previous lemma. The second easily follows from the first. For the third equivalence note that if γ_1 and γ_2 are valuations on a 1–frame $\mathfrak{G} = \mathfrak{F}^{sim}$ such that $\gamma_1(p) \cap f^w = \gamma_2(p) \cap f^w$ for all p then $\langle \mathfrak{G}, \gamma_1, x\rangle \models \mathsf{w} \wedge P^\tau$ iff $\langle \mathfrak{G}, \gamma_2, x\rangle \models \mathsf{w} \wedge P^\tau$, so that if there exists a model for $\mathsf{w} \wedge P^\tau$ it can always be based on a valuation of the form β^{sim} for some β on $\mathfrak{F}$. The first equivalence asserts that local satisfiability in a frame is equivalent to local satisfiability of X^τ in $\mathfrak{F}^{sim}$ in the simulating frame at 'white' points, the second asserts that global satisfiability of X^τ is equivalent to global satisfiability of $\mathsf{w} \rightarrow X^\tau$ in the simulating frame. The third asserts that validity of X in a frame is equivalent to validity of $\mathsf{w} \rightarrow X^\tau$ in its simulation.

Corollary 41 *Let $\Lambda = \boldsymbol{K}_2 \oplus X$. Then $\Lambda^{sim} = \boldsymbol{Sim} \oplus \{w \rightarrow P^\tau | P \in X\}$. In particular, if Λ is n–axiomatizable, so is Λ^{sim}.*

Proof. Put $\Theta := \mathbf{Sim} \oplus \{\mathsf{w} \rightarrow P^\tau | P \in X\}$. We show that the rooted refined frames validating Θ coincide with the rooted refined frames validating Λ^{sim}. This establishes the first part of the theorem. The second follows immediately. Now let $\mathfrak{F}$ be a rooted frame which is not a frame for Λ^{sim}. Let x be a root of $\mathfrak{F}$. Then there exists a bimodal rooted frame $\mathfrak{G}$ such that $\mathfrak{G}^{sim}$ is isomorphic to $\mathfrak{F}$. We then have that $\mathfrak{G}$ is not a frame for Λ, and thus there exists a $P \in X$ such that $\mathfrak{G}\ \mathbf{2}\ P$. By the lemma above, $\mathfrak{F}\ \mathbf{2}\ \mathsf{w} \rightarrow P^\tau$. Hence $\mathfrak{F}$ is not a frame for Θ. The argument can be run backwards as well. Assume that $\mathfrak{F}$ is a rooted refined frame which is not a frame for Θ. Then $\mathfrak{F}\ \mathbf{2}\ \mathsf{w} \rightarrow P^\tau$ for some $P \in X$. $\mathfrak{F}$ is a simulation frame and so we may assume that for some $\mathfrak{G}$, $\mathfrak{F} = \mathfrak{G}^{sim}$. By the previous lemma, $\langle \mathfrak{G}, \beta, y\rangle\ \mathbf{2}\ P$. So $\mathfrak{G}$ is not a frame for Λ. By definition of Λ^{sim}, $\mathfrak{F}$ (which is $\mathfrak{G}^{sim}$) is not a frame for Λ^{sim}. $\dashv$

The next theorem provides one half of the simulation theorem. Before we can state it, we have to provide some more definitions. An n–modal logic Λ is **elementary** (**Δ–elementary**) if the class of kripke–frames of Λ is elementary (Δ–elementary). Here, the language to speak about n–frames is first–order predicate logic with equality ($\doteq$) and binary relation symbols $\lhd_i$, $i < n$, which get interpreted by the corresponding relations in the frame. The following so–called **restricted quantifiers** are defined.

$$\begin{array}{rcl}
(\forall y \rhd_j x)\beta & := & (\forall y)(x \lhd_j y. \rightarrow .\beta) \\
(\exists y \rhd_j x)\beta & := & (\exists y)(x \lhd_j y. \wedge .\beta)
\end{array}$$

For the elementary language, too, we can define a translation on formulae.

$$\begin{array}{lcl}
t(x) & := & (\forall y \succ x)\neg(y \doteq y) \\
w(x) & := & (\exists y \succ x)t(x) \\
b(x) & := & \neg t(x) \wedge (\forall y \succ x)\neg t(x)
\end{array}$$

These formulae define the sets f^b, f^w and f^t in a **Sim**–frame. Now put

$$\alpha^e := \bigwedge_{x \in fvar(\alpha)} w(x). \rightarrow .\alpha^f$$

$$\begin{array}{lcl}
(x \doteq y)^f & := & x \doteq y \\
(x \lhd y)^f & := & x \prec y \\
(x \blacktriangleleft y)^f & := & (\exists v \succ x)(\exists w \succ y)(b(v) \wedge b(w) \wedge v \prec w) \\
(\alpha_1 \wedge \alpha_2)^f & := & \alpha_1^f \wedge \alpha_2^f \\
(\neg\alpha)^f & := & \neg\alpha^f \\
((\exists x)\alpha)^f & := & (\exists x)(w(x) \wedge \alpha^f)
\end{array}$$

It is not difficult to show that for a sentence α,

$$\mathfrak{F} \models \alpha \quad \Leftrightarrow \quad \mathfrak{F}^{sim} \models \alpha^e$$

It is not hard to see that the class of **Sim**–kripke frames is elementary. This also follows from the fact that axioms for **Sim** are Sahlqvist.

An n–modal formula is **Sahlqvist** if it is of the form $\boxtimes(P \rightarrow Q)$ where $\boxtimes$ is a prefix of box–like operators, P is strongly positive and Q is positive. Here a formula is **positive** if it is built from variable free formulae and variables with the help of $\wedge$, $\vee$ and the operators $\Box_i$, $\Diamond_i$, $i < n$. A formula is **strongly positive** if it is positive and no formula of the form $R_1 \vee R_2$, $\Diamond_i R_1$, $i < n$, which contains a variable is contained in the scope of a $\Box_j$, $j < n$. By a well–known theorem of Sahlqvist, logics axiomatizable by a Sahlqvist formula are locally $\mathfrak{D}$–persistent and elementary. Let us call a logic **Sahlqvist** if it is axiomatizable by a set of Sahlqvist formulae. As is well–known, a Sahlqvist logic is $\mathfrak{D}$–persistent and Δ–elementary. By a theorem of van Benthem [2], if a modal axiom P determines an elementary condition δ on kripke–frames then δ is equivalent to a formula of the form $\forall x.\alpha(x)$, where $\alpha(x)$ is obtained from (positive) atomic formulae $x \doteq y$, $x \lhd_i y$, $i < n$, using $\wedge$, $\vee$, and the restricted quantifiers. In [2] the following class of formulae is defined. Call an n–modal formula P a **Sahlqvist–van Benthem** formula if it is composed from variables and constants using $\neg$, $\wedge$, $\vee$, $\Box_j$ and $\Diamond_j$ ($j < n$) such that for all variables p either (i) no positive occurrence of p is in a subformula of P of the form $Q \wedge R$ or $\Box_j Q$, $j < n$, if that formula is in the scope of a $\Diamond_k$, $k < n$, or (ii) no negative occurrence of p in P is in a subformula of the form $Q \wedge R$ or $\Box_j Q$, $i < n$, if that occurrence is in the scope of some $\Diamond_k$, $k < n$. This describes a wider class of formulae. However, for every Sahlqvist–van Benthem formula P there exists a Sahlqvist–formula Q such that $\mathbf{K} \oplus P = \mathbf{K} \oplus Q$. (See [19].)

Definition 42 *An n–modal logic Λ has **local interpolation** if whenever $P \vdash_\Lambda Q$ there exists a formula R such that $var(R) \subseteq var(P) \cap var(Q)$ and $P \vdash_\Lambda R \vdash_\Lambda Q$. Λ has **global interpolation** if whenever $P \Vdash_\Lambda Q$ there exists a R such that $var(R) \subseteq var(P) \cap var(Q)$ and $P \Vdash_\Lambda R \Vdash_\Lambda Q$.*

Theorem 43 *The simulation map preserves the following properties of logics.*

- *n–axiomatizability, finite axiomatizability, recursive axiomatizability,*
- *$\mathfrak{G}$–persistence, $\mathfrak{R}$–persistence, $\mathfrak{D}$–persistence,*
- *being Sahlqvist,*
- *elementarity, Δ–elementarity.*

The simulation reflects the following properties of logics.

- *local/ global completeness with respect to kripke frames (finite kripke frames),*
- *local/global interpolation.*

Proof. The first set of properties is clear. For persistence only $\mathfrak{G}$–persistence is not a direct consequence. However, by a theorem of van Benthem [2], a logic is $\mathfrak{G}$–persistent iff it is 0–axiomatizable. So the claim follows from the first set. For Sahlqvist logics, it is enough if we show that if P is a Sahlqvist–van Benthem formula, so is $\mathsf{w} \to P^\tau$ (viewed as a formula in the language with $\boxminus$, $\diamondminus$). Now, it is checked that the translation is such that occurrences of variables in P are in one–to–one correspondence with occurrences of variables in $\mathsf{w} \to P^\tau$. Moreover, if p occurs positively (negatively) in P, its related occurrence in $\mathsf{w} \to P^\tau$ is also positive (negative). Also, if p occurs in a subformula $Q \wedge R$ ($\Box Q$, $\blacksquare Q$) within a formula $\Diamond S$ ($\blacklozenge S$) then its related occurrence in $\mathsf{w} \to P$ is in $Q^\tau \wedge R^\tau$ ($\boxminus T$ for $T = \mathsf{w} \to Q^\tau$, $\boxminus T$ for $T = \mathsf{b} \to \boxminus_b \boxminus_w Q^\tau$) within the subformula $\diamondminus U$, where $U = \mathsf{w} \wedge S^\tau$ (within the subformula $\diamondminus U$, where $U = \mathsf{b} \wedge \diamondminus_b \diamondminus_w S^\tau$). Hence $\mathsf{w} \to P^\tau$ is a Sahlqvist–van Benthem formula. Next assume that Λ is (Δ–)elementary. Then the class of kripke frames is determined by some condition α (set Γ of conditions). Then the class of kripke frames for Λ^{sim} is determined by some sentence characterizing the **Sim**–kripke frames plus α^e (Γ^e). Now we turn to interpolation. We make use of the following criteria.

Definition 44 *A variety $\mathcal{V}$ of n–modal algebras has the* ***amalgamation property*** *if for any three algebras $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2 \in \mathcal{V}$ and embeddings $\iota_1 : \mathfrak{A}_0 \rightarrowtail \mathfrak{A}_1$ and $\iota_2 : \mathfrak{A}_0 \rightarrowtail \mathfrak{A}_2$ there exists an algebra $\mathfrak{A}_3$ in $\mathcal{V}$ and embeddings $\epsilon_1 : \mathfrak{A}_1 \rightarrowtail \mathfrak{A}_3$, $\epsilon_2 : \mathfrak{A}_2 \rightarrowtail \mathfrak{A}_3$ such that $\epsilon_1 \circ \iota_1 = \epsilon_2 \circ \iota_2$. $\mathcal{V}$ has the* ***superamalgamation property*** *if under the same conditions the ϵ_i can be required to have the property that whenever $\epsilon_1(a_1) \leq \epsilon_2(a_2)$ for $a_1 \in A_1$ and $a_2 \in A_2$ then there exists an $a_0 \in A_0$ such that $a_1 \leq \iota_1(a_0)$ and $\iota_2(a_0) \leq a_2$.*

Theorem 45 (Maksimova) *Λ has local interpolation iff it has the superamalgamation property. Λ has global interpolation iff it has the amalgamation property.*

The proof is an immediate generalization of [21]. Now assume that Λ^{sim} has global interpolation. We have to show that Λ has global interpolation as well. It is enough to show that $\mathsf{Alg}\,\Lambda$ has the amalgamation property. Suppose that $\mathfrak{B}_0$, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are in the variety of Λ–algebras, and $\iota_1 : \mathfrak{B}_0 \rightarrowtail \mathfrak{B}_1$, $\iota_2 : \mathfrak{B}_0 \rightarrowtail \mathfrak{B}_2$. Then $(\iota_i)^{sim} : (\mathfrak{B}_0)^{sim} \rightarrowtail (\mathfrak{B}_1)^{sim}$, $i = 1,2$. By assumption on Λ^{sim}, there exists a $\mathfrak{A}_3$ and maps $\kappa_1 : (\mathfrak{B}_1)^{sim} \rightarrowtail \mathfrak{A}_3$, $\kappa_2 : (\mathfrak{B}_2)^{sim} \rightarrowtail \mathfrak{A}_3$ such that $\kappa_1 \circ (\iota_1)^{sim} = \kappa_2 \circ (\iota_2)^{sim}$. Put $\mathfrak{B}_3 := (\mathfrak{A}_3)_{sim}$ and $\epsilon_i := (\kappa_i)_{sim}$, $i = 1,2$. Then $\epsilon_i : \mathfrak{B}_i \rightarrowtail \mathfrak{B}_3$ and $\epsilon_1 \circ \iota_1 = (\kappa_1)_{sim} \circ ((\iota_1)^{sim})_{sim} = (\kappa_1 \circ (\iota_1)^{sim})_{sim} = (\kappa_2 \circ (\iota_2)^{sim})_{sim} = (\kappa_2)_{sim} \circ ((\iota_1)^{sim})_{sim} = \epsilon_2 \circ \iota_2$. Thus the variety of Λ–algebras has the amalgamation property. Now assume that the variety of Λ^{sim}–algebras has the superamalgamation property. Then in addition we can have $\mathfrak{A}_3$ and the embeddings in such a way that if $\kappa_1(a_1) \leq \kappa_2(a_2)$ then there exists a $a_0 \in A_0$ such that $a_1 \leq (\iota_1)^{sim}(a_0)$ and $(\iota_2)^{sim}(a_0) \leq a_2$. Now let $b_1 \in B_1$, $b_2 \in B_2$ be such that $\epsilon_1(b_1) \leq \epsilon_2(b_2)$. Then put $a_1 := (b_1)^w$ and $a_2 := (b_2)^w$. Then $\kappa_1(a_1) = (\epsilon_1(b_1))^w \leq (\epsilon_2(b_2))^w = \kappa_2(a_2)$ and so there exists $a_0 \in A_0$ such that $a_1 \leq (\iota_1)^{sim}(a_0)$ and $(\iota_2)^{sim}(a_0) \leq a_2$. Put $b_0 := \mathsf{w} \cap a_0$. Then $b_1 = \mathsf{w} \cap a_1 \leq \mathsf{w} \cap (\iota_1)^{sim}(a_0) = \mathsf{w} \cap (\iota_1(b_0))^{sim} = \iota_1(b_0)$, and likewise $\iota_2(b_0) \leq b_2$ is proved. $\dashv$

§ 11. Properties of the Unsimulation.

Now we will show how to axiomatize the unsimulation of a logic on the basis of an axiomatization for it. The proof is rather longwinded. Before we enter it, we need some terminology. Let P be a formula and Q a subformula of P. Fix an occurrence of Q in P. A **modal cover** of that occurrence of Q is a minimal subformula R of modal degree greater than Q containing that occurrence of Q. We also say that that particular occurrence of R **modally covers** Q. If Q has a modal cover, it is unique and a formula beginning with a modal operator. (We will often speak of formulae rather than occurrences of formulae, whenever the context allows this.) Now let P be a formula of the language with operators $\boxminus_\alpha$, $\diamondminus_\alpha$, $\alpha \in \{w,b,t\}$. Let us agree to say that an occurrence of a formula Q in P is α**–covered** if it modally covered by a formula of the form $\boxminus_\alpha R$ or $\diamondminus_\alpha R$. Call P and Q **white–equivalent** if $\mathsf{w} \vdash_{\mathbf{Sim}} P \leftrightarrow Q$ and **black–equivalent** if $\mathsf{b} \vdash_{\mathbf{Sim}} P \leftrightarrow Q$. Given a formula, we say that a subformula occurs **white** if it is not in the scope of a modal operator or else is w–covered. A subformula occurs **black** if it is b–covered. If P occurs white (black) in R, and P is white–equivalent (black–equivalent) to Q, then that occurrence of P may be replaced in R by Q preserving white–equivalence. By axioms (a) of § 8, $\boxminus_b M$ is white–equivalent to $\diamondminus_b M$ and $\boxminus_w M$ is black–equivalent to $\diamondminus_w M$.

Lemma 46 *Let P be a formula in the language with $\boxminus_\alpha$ and $\diamondminus_\alpha$, $\alpha \in \{w,b,t\}$. There exists a finite number n and formulae Q_i and R_i, $i < n$, such that Q_i is nonmodal for all $i < n$, and R_i is in the language with $\boxminus_w, \boxminus_b$, $\diamondminus_w$ and $\diamondminus_b$ for all $i < n$, and P is white–equivalent to the formula*

$$\bigvee_{i<n} (\diamondminus_t Q_i) \wedge R_i$$

Proof. Let us consider now a maximal subformula R of P of the form $\boxminus_t Q$. Clearly, by some **Sim**–equivalences, $\boxminus_t Q$ can be transformed into a subformula $R' = \boxminus_t Q'$ where Q' is nonmodal. So, let us assume that P contains only subformulae $\boxminus_t Q$, where Q is nonmodal. Now let $S \subseteq var(P)$. Put

$$\chi(S) := \bigwedge_{p \in S} p \wedge \bigwedge_{p \in var(P)-S} \neg p$$

Then

$$(\ddagger) \qquad \mathsf{w} \vdash_{\mathbf{sim}} P \quad \leftrightarrow \quad \bigvee_{S \subseteq var(P)} P \wedge \diamondminus_t \chi(S)$$

Consider a particular disjunct $P \wedge \diamondminus_t \chi(S)$ of the formula to the right in ($\ddagger$), corresponding to S. Consider a subformula $R = \boxminus_t Q$ in P. We are allowed to replace Q by $\chi(S) \wedge Q$. The latter reduces via some **Sim**–equivalences to either $\chi(S)$ or to $\bot$. (This is clear if one argues semantically.) So we may assume that $R = \boxminus_t \bot$ or $R = \boxminus_t \chi(S)$. Several cases need to be distinguished now. (a) R is w–covered. Then R can be replaced by $\bot$ when $R = \boxminus_t \bot$ and by $\top$ otherwise. (b) R is b–covered. Then it can be replaced by $\top$. (c) R is not in the scope of an operator. Then it can be replaced by $\bot$ when $R = \boxminus_t \bot$ and by $\top$ otherwise. All these replacements are white **Sim**–equivalences. This shows the lemma. ⊣

Definition 47 *A monomodal formula P is called **simulation transparent** if it is of the form p, $\neg p$, $\diamondminus_b p$, $\neg \diamondminus_b p$, $\diamondminus_t p$, $\neg \diamondminus_t p$, p a variable, or of the form $Q \wedge R$, $Q \vee R$, $\diamondminus_w Q$, $\boxminus_w Q$, $\diamondminus_b \diamondminus_b \diamondminus_w Q$ or $\boxminus_b \boxminus_b \boxminus_w Q$ where Q and R are simulation transparent.*

Definition 48 *Call a formula P **white based** if there do not exist occurrences of subformulae Q, R, S and T such that Q b–covers R, R b–covers S, and S b–covers T.*

Lemma 49 *For every formula P there exists a formula Q which is white–based and white–equivalent to P.*

Proof. Suppose that there is a quadruple $\langle Q, R, S, T \rangle$ of occurrences of subformulae such that Q b–covers R, R b–covers S, and S b–covers T. Then there exists such a quadruple in which Q occurs white. Now replace the occurrence of T by $\boxminus_w \boxminus_b T$. Since T is black equivalent with $\boxminus_w \boxminus_b T$, this replacement yields a formula P' which is white equivalent to P. Now repeat this procedure with P'. It is not hard to see that this process terminates with a white based formula. (For example, count the number of occurrences of quadruples $\langle Q, R, S, T \rangle$ such that Q b–covers R, R b–covers S, and S b–covers T. It decreases by at least one in passing from P to P'. If it is zero, the formula is white based.) ⊣

Lemma 50 *Let P be a monomodal formula. Then there exists a simulation transparent formula S such that*

$$\mathsf{w} \vdash_{\mathbf{Sim}} P \leftrightarrow S$$

Proof. First we simplify the problem somewhat. Namely, by some standard manipulations we can achieve it that no operator occurs in the scope of negation. We call a formula in such a form **basic**. So, let us assume P to be basic. By Lemma 46 we can assume P to be a disjunction of formulae of the form $U \wedge R$, where $U = \Diamond_t S$, for a nonmodal S, and R contains only $\boxminus_w$, $\Diamond_w$, $\boxminus_b$ and $\Diamond_b$. In general, if the claim holds for P_1 and P_2, then it also holds for $P_1 \vee P_2$ and $P_1 \wedge P_2$. Therefore, we have two cases to consider: (i) P contains no occurrences of $\Diamond_w$, $\boxminus_w$, $\Diamond_b$ or $\boxminus_b$, or (ii) P contains no occurrences of $\Diamond_t$ and $\boxminus_t$. In case (i), we know that $\Diamond_t$ distributes over $\vee$ and $\wedge$, so that we can reduce P (modulo white–equivalence) to the form $\Diamond_t p$, and $\Diamond_t \neg p$. Now we have

$$\mathsf{w} \vdash_{\mathbf{Sim}} \Diamond_t \neg p \leftrightarrow \neg \Diamond_t p$$

So in Case (i) P is white–equivalent to a simulation transparent formula. From now on we can assume to be in Case (ii). Furthermore, by Lemma 49, we can assume that P is white based, and (inspecting the proof of that lemma) that P is built from variables and negated variables, using $\wedge$, $\vee$, and the modal operators $\Diamond_w$, $\Diamond_b$, $\boxminus_w$ and $\boxminus_b$.

Let $\mu_b(P)$ denote the maximum of nested black operators ($\Diamond_b$, $\boxminus_b$) in P. Call P **thinner** than Q if either $\mu_b(P) < \mu_b(Q)$ or P is a subformula of Q. We will show that for given white based basic P there exists a simulation transparent formula Q which is white–equivalent to P on the condition that this holds already for all white based basic formulae P' thinner than P.

If $P = p$ we are done; for P is simulation transparent. Likewise, if $P = \neg p$. Suppose $P = P_1 \wedge P_2$. P_1 and P_2 are thinner than P. Therefore there exist simulation transparent formulae Q_1 and Q_2 such that Q_i is white–equivalent to P_i, $i \in \{1, 2\}$. Then $Q_1 \wedge Q_2$ is white–equivalent to $P_1 \wedge P_2$. Similarly for $P = P_1 \vee P_2$. If $P = \Diamond_w P_1$ there exists a simulation transparent Q which is white–equivalent to P_1. So $\mathsf{w} \to P_1 \dashv\vdash_{\mathbf{Sim}} \mathsf{w} \to Q$, and therefore $\Diamond_w P_1 \dashv\vdash_{\mathbf{Sim}} \Diamond_w Q$; it follows that $\Diamond_w Q$ is white–equivalent to P. Similarly for $P = \boxminus_w P_1$. We are left with the case that P is either $\Diamond_b R$ or $\boxminus_b R$. By inductive hypothesis, for every basic white based Q such that $\mu_b(Q) < \mu_b(P)$ there is a simulation transparent S such that S is white–equivalent to Q. As P occurs white, we can distribute $\boxminus_b$ and $\Diamond_b$ over $\wedge$ and $\vee$, and so reduce R to the form p, $\neg p$ or $\boxminus_\alpha N$ or $\Diamond_\alpha N$, with $\alpha \in \{w, b\}$ and N basic. This reduction does not alter $\mu_b(P)$. **Case 1.** $R = p$. Then P is simulation transparent. **Case 2.** $R = \neg p$. Observe that $\Diamond_b \neg p$ is white equivalent to $\neg \Diamond_b p$. So we are done. **Case 3.** $R = \boxminus_w N$ or $R = \Diamond_w N$. Then P is white–equivalent to N. The claim follows by induction hypothesis for N. **Case 4.** $R = \boxminus_b N$ or $R = \Diamond_b N$, N basic. Now let us look at N. N occurs black. Furthermore, N is the result of applying a lattice polynomial to formulae of the form p, $\neg p$, $\boxminus_b A$, $\Diamond_b B$, $\boxminus_w C$, $\Diamond_w D$. (Here, a lattice polynomial is an expression formed from variables and constants using only $\wedge$ and $\vee$, but no other functions. It turns out that $\top$ and $\bot$ can be eliminated from this polynomial as long as it contains at least one variable. If it does not contain a variable, it is equivalent to either $\top$ or $\bot$. Finally, $\top$ may be replaced by $p \vee \neg p$ and $\bot$ by $p \wedge \neg p$, so we may assume that the polynomial does not contain any occurrences of $\top$ and $\bot$.) However, as P is white based, formulae of the form $\diamondminus_b A$ or $\boxminus_b B$ do not occur. Furthermore, if $\mu_b(N) > 0$, we replace the occurrences of p by $\boxminus_w \boxminus_b p$, and the occurrences of $\neg p$ by $\boxminus_w \boxminus_w \neg p$. This replacement does not change $\mu_b(N)$. (The case $\mu_b(N) = 0$ needs some attention. Here we replace N by $\boxminus_w \boxminus_b N$. Then P is white equivalent to either $\boxminus_b \boxminus_b \boxminus_w \boxminus_b N$ or $\diamondminus_b \diamondminus_b \diamondminus_w \boxminus_b N$, N nonmodal. Now we are down to the case of a formula of the form $\boxminus_b N$. $\boxminus_b$ commutes with $\wedge$ and $\vee$, which leaves the cases $\boxminus_b p$ and $\boxminus_b \neg p$ to consider. These are immediate.) Now, after this replacement, N is a lattice polynomial over formulae of the form $\boxminus_w C$, $\diamondminus_w D$. The latter occur black, so $\boxminus_w C$ is intersubstitutable with $\diamondminus_w C$ and $\diamondminus_w D$ is intersubstitutable with $\boxminus_w D$. Finally, $\diamondminus_w \top$ is intersubstitutable (modulo equivalence) with $\top$. So N is without loss of generality of the form $f(\langle \boxminus_w E_i | i < n\rangle)$ for some lattice polynomial f. Thus, N can be substituted by the formula $\boxminus_w f(\langle E_i | i < n\rangle)$. Therefore, N can be reduced to the form $\boxminus_w E$ for some basic E. E is white based. Therefore, by induction hypothesis, E is white–equivalent to a simulation transparent formula G. R has the form $\boxminus_b \boxminus_w E$ (**Case A**) or $\diamondminus_b \boxminus_w E$ (**Case B**), so P is white–equivalent to either $Q_1 := \boxminus_b \boxminus_b \boxminus_w G$ (Case A) or $\boxminus_b \diamondminus_b \boxminus_w G$ (Case B). The latter is white–equivalent to $Q_2 := \diamondminus_b \diamondminus_b \diamondminus_w G$. Both Q_1 and Q_2 are simulation transparent, by assumption on G. $\dashv$

Lemma 51 *Let P be a monomodal formula in the variables $\{p_i | i < n\}$. There exists a bimodal formula Q in the variables $\{p_i^\circ, p_i^\bullet, p_i^* | i < n\}$ such that*

$$(\ddagger) : \qquad \mathsf{w} \vdash_{\mathbf{Sim}} P \leftrightarrow Q^\tau[p_i/p_i^\circ, \diamondminus_b p_i/p_i^\bullet, \diamondminus_t p_i/p_i^* | i < n]$$

*Q is called an **unsimulation** of P. Moreover, if P is a Sahlqvist–van Benthem formula, then Q can be chosen to be a Sahlqvist–van Benthem formula as well.*

Proof. The first part of the proof is straightforward. There exists a simulation transparent S which is white–equivalent to P. Q is obtained from S by applying the following translation outside in.

$(Q_1 \wedge Q_2)_\tau$	$:=$	$(Q_1)_\tau \wedge (Q_2)_\tau$	$(Q_1 \vee Q_2)_\tau$	$:=$	$(Q_1)_\tau \vee (Q_2)_\tau$
$(\diamondminus_w Q)_\tau$	$:=$	$\Diamond Q_\tau$	$(\boxminus_w Q)_\tau$	$:=$	$\Box Q_\tau$
$(\diamondminus_b \diamondminus_b \diamondminus_w Q)_\tau$	$:=$	$\blacklozenge Q_\tau$	$(\boxminus_b \boxminus_b \boxminus_w Q)_\tau$	$:=$	$\blacksquare Q_\tau$
$(\diamondminus_t p)_\tau$	$:=$	p^*	$(\neg \diamondminus_t p)_\tau$	$:=$	$\neg p^*$
$(\diamondminus_b p)_\tau$	$:=$	$p^\bullet$	$(\neg \diamondminus_b p)_\tau$	$:=$	$\neg p^\bullet$
p_τ	$:=$	p°	$(\neg p)^\tau$	$:=$	$\neg p^\circ$

The formula S is of course not uniquely determined by P, but is unique only up to equivalence. The proof of Lemma 50 is actually a construction of S, and so let us denote by P_τ the particular formula that is obtained by performing that construction.

For the second claim, assume that P is a Sahlqvist–van Benthem formula. Then there exists a simulation transparent P^δ which is white equivalent to P.

Inspection of the actual construction shows that the transformation of P to P^δ preserves positive and negative occurrences of variables. Moreover, suppose that $(P^\delta)_\tau$ is *not* Sahlqvist–van Benthem. Then it contains a positive occurrence of a variable p in a subformula of the form $Q \wedge R$, $\Box Q$ or $\blacksquare Q$ in the scope of a $\Diamond$ or $\blacklozenge$, and likewise a negative occurrence of that same variable in a subformula of such kind. It is not hard to see that the corresponding occurrences of p in P^δ are in a similar configuration, and that — finally — there are corresponding occurrences in P which are also in such a configuration. Thus P is not Sahlqvist–van Benthem. (A remark. This last step is not straightforward to prove, the details are cumbersome, since a lot of elementary transformations are being made to pass from P to P^δ. The interested reader may simply note that each of these transformations takes a Sahlqvist–van Benthem formula into a Sahlqvist–van Benthem formula (and back). Spelling out these details is rather unrevealing.) $\dashv$

Now take a set X of monomodal formulae; put $X_\tau := \{P_\tau | P \in X\}$. Assume that $\mathfrak{F}$ is a simulation frame and $\langle \mathfrak{F}, \beta, x\rangle \models \mathsf{w}; X$. Then we have

$$\langle \mathfrak{F}, \beta, x\rangle \models \mathsf{w}; \sigma((X_\tau)^\tau),$$

where σ is a substitution satisfying $\sigma(p^\circ) = p$, $\sigma(p^\bullet) = \Diamond_b p$, $\sigma(p^*) = \Diamond_t p$. Now define a valuation $\beta^\diamond$ of the set $\{p^\circ, p^\bullet, p^* | p \in var(X)\}$ by

$$\beta^\diamond(p^\circ) := \beta(p) \cap f^w, \quad \beta^\diamond(p^\bullet) := \Diamond_b \beta(p) \cap f^w, \quad \beta^\diamond(p^*) := \Diamond_t \beta(p) \cap f^w$$

By definition of $\beta^\diamond$,

$$\langle \mathfrak{F}, \beta, x\rangle \models \mathsf{w}; \sigma(Q) \qquad \Leftrightarrow \qquad \langle \mathfrak{F}, \beta^\diamond, x\rangle \models \mathsf{w}; Q$$

Thus we conclude

$$\langle \mathfrak{F}, \beta^\diamond, x\rangle \models \mathsf{w}; (X_\tau)^\tau$$

Define a valuation β_{sim} on $\mathfrak{F}_{sim}$ by $\beta_{sim}(q) := \beta^\diamond(q)$. x is of the form y^w for some $y \in f_{sim}$; in fact, by construction, $y^w = x$. By the previous results,

$$\langle \mathfrak{F}, \beta^\diamond, x\rangle \models \mathsf{w}; (X_\tau)^\tau \qquad \Leftrightarrow \qquad \langle \mathfrak{F}_{sim}, \beta_{sim}, x\rangle \models X_\tau.$$

It therefore turns out that the satisfaction of X in a simulation frame at a white point is equivalent to the satisfaction of X_τ in the unsimulation of the frame. The satisfaction of X at a black point is equivalent to the satisfaction of $\boxminus_b X := \{\boxminus_b P | P \in X\}$ at a white point. The satisfaction of X at f^t is likewise reducible to satisfaction of $\boxminus_t X$, which is defined analogously.

Now let Λ be a monomodal logic contained in the interval $[\mathbf{Sim}, \mathsf{Th}\boxed{\bullet}]$. We will show that it can be axiomatized by formulae of the form $\mathsf{w} \to Q$. To that end, let $\Lambda = \mathbf{K} \oplus X$ and let $P \in X$. For simplicity we may actually assume that $X = \{P\}$. Then $\Lambda = \mathbf{K} \oplus \{\mathsf{t} \to P, \mathsf{w} \to P, \mathsf{b} \to P\}$. Since $\boxed{\bullet}$ is a frame for Λ, $\mathsf{t} \to P$ can only be a theorem if it becomes a boolean tautology after substituting $\top$ for maximal subformulae of the form $\boxminus Q$ (and $\bot$ for maximal subformulae of the form $\Diamond Q$). Hence, $\mathsf{t} \to P$ is in Λ only if P is an instance of a boolean tautology. However, in that case P is a theorem of **K**. $\mathsf{b} \to P$ is **Sim**–equivalent to $\mathsf{w} \to \boxminus_b P$. So, $\Lambda = \mathbf{K} \oplus \{\mathsf{w} \to P, \mathsf{w} \to \boxminus_b P\}$, as promised. So we can always assume that an axiom is of the form $Q := \mathsf{w} \to P$ for some P. (This follows independently from the surjectivity of the simulation map and the fact that an axiomatization of this form for simulation logics has been given above.) Now Q is rejected in a model based on $\mathfrak{F}$ iff P is rejected at a white point of $\mathfrak{F}$ iff P_τ is rejected in a model based on $\mathfrak{F}_{sim}$.

We summarize our findings as follows. Given a monomodal rooted **Sim**–frame $\mathfrak{G}$, a set X, a valuation β, we define β_{sim} by $\beta_{sim}(q) := \beta^\diamond(q)$, where q is a variable of the form p°, $p^\bullet$ or p^*.

$$\begin{array}{rcr} \langle \mathfrak{G}, \beta, x\rangle \models \mathsf{w} \wedge X & \Leftrightarrow & \langle \mathfrak{G}_{sim}, \beta_{sim}, x\rangle \models X_\tau \\ \langle \mathfrak{G}, \beta\rangle \models X & \Leftrightarrow & \langle \mathfrak{G}_{sim}, \beta_{sim}\rangle \models X_\tau; (\boxminus_b X)_\tau; (\boxminus_t X)_\tau \\ \mathfrak{G} \models X & \Leftrightarrow & \mathfrak{G}_{sim} \models X_\tau; (\boxminus_b X)_\tau; (\boxminus_t X)_\tau \end{array}$$

Two cases may arise. Suppose that $\mathfrak{G}$ contains only one point. Then $\mathfrak{G}_{sim}$ is empty, and no formula is satisfiable in it. This case has to be dealt with separately. Else, let $\mathfrak{G}$ have more than point, and be rooted. Then satisfiability of X in $\mathfrak{G}$ is reducible to satisfiability of either $\mathsf{w}; X_\tau$ or $\mathsf{w}; \boxminus_t X_\tau$ or $\mathsf{w}; \boxminus_b X_\tau$. All problems are reducible to satisfiability of a set Y in $\mathfrak{G}_{sim}$. Global satisfiability of X in $\langle \mathfrak{G}, \beta\rangle$ is equivalent to the global satisfiability of $\{\mathsf{w} \to P | P \in X^+\}$, where $X^+ = X; \boxminus_b X; \boxminus_t X$.

Theorem 52 *Let Λ be a monomodal logic in the interval $[\boldsymbol{Sim}, \mathsf{Th}\boxed{\bullet}]$. Then Λ is axiomatizable as $\Lambda := \boldsymbol{Sim} \oplus \{\mathsf{w} \to P | P \in X\}$ for some X. Furthermore, $\Lambda_{sim} = \boldsymbol{K}_2 \oplus X_\tau$.* $\dashv$

The case of unsimulating elementary properties is likewise more complex than the simulating part. Take a formula ϕ in the first–order language for 1–modal frames. We may assume (to save some notation) that the formula does not contain $\forall$. Furthermore, we may assume that the formula is a sentence, that is, contains no free variables. However, we do not assume that structures are nonempty. We introduce new quantifiers $\exists^\alpha$, $\alpha \in \{b, w, t\}$, which are defined by

$$(\exists^\alpha x)\phi(x) := (\exists x)(\alpha(x) \wedge \phi(x))$$

Furthermore, for each variable x we introduce three new variables, x^α, $\alpha \in \{b, w, t\}$. Now define a translation $(-)^\dagger$ as follows.

$$\begin{array}{lcl} (\phi \wedge \psi)^\dagger & := & \phi^\dagger \wedge \psi^\dagger \\ (\phi \vee \psi)^\dagger & := & \phi^\dagger \vee \psi^\dagger \\ (\neg\phi)^\dagger & := & \neg\phi^\dagger \\ ((\exists x)\phi(x))^\dagger & := & (\exists^w x^w)\phi(x^w)^\dagger \vee (\exists^b x^b)\phi(x^b)^\dagger \vee (\exists^t x^t)\phi(x^t)^\dagger \end{array}$$

It is clear that ϕ and $\phi^\dagger$ are deductively equivalent. In a next step replace $(\exists^b x^b)\phi(x^b)$ by

$$(\exists^w x^w)(\exists^b x^b \succ x^w)\phi(x^b)$$

and $(\exists^t x^t)\phi(x^t)$ by
$$(\exists^w x^w)(\exists^t x^t \succ x^w)\phi(x^t)$$
Call ψ^δ the result of applying this replacement to ψ. It turns out that (in the first–order logic of simulation frames)
$$(\exists x)w(x) \vdash \psi^\delta \leftrightarrow \psi$$
That means, the two are equivalent on all frames $\mathfrak{G}^{sim}$ where $\mathfrak{G}$ is not empty. In ψ^δ the variables x^b and x^t are bound by a restricted quantifier with restrictor x^w; x^w in turn is bound by $\exists^w$. To see whether such formulae are valid in a frame we may restrict ourselves to assignments h of the variables in which x^α is in the α–region for each α and each x, and furthermore $h(x^w) \succ h(x^b)$. In a final step, translate as follows
$$\begin{array}{lll}
(\phi \wedge \psi)^\ddagger & := & \phi^\ddagger \wedge \psi^\ddagger \\
(\phi \vee \psi)^\ddagger & := & \phi^\ddagger \vee \psi^\ddagger \\
(\neg\phi)^\ddagger & := & \neg\phi^\ddagger \\
((\exists^w x^w)\phi(x^w))^\ddagger & := & (\exists x)\phi(x^w)^\ddagger \\
((\exists^b x^b \succ x^w)\phi(x^b))^\ddagger & := & \phi(x^b)^\ddagger \\
((\exists^t x^t \succ x^w)\phi(x^t))^\ddagger & := & \phi(x^t)^\ddagger
\end{array}$$
For atomic formulae, $\phi^\ddagger$ is computed as follows:

$x^\alpha \prec y^\beta$		$\beta \rightarrow$ w	b	t
α	w	$x \lhd y$	$x \doteq y$	$\top$
$\downarrow$	b	$x \doteq y$	$x \blacktriangleleft y$	$\bot$
	t	$\bot$	$\bot$	$\bot$

$x^\alpha \doteq y^\beta$		$\beta \rightarrow$ w	b	t
α	w	$x \doteq y$	$\bot$	$\bot$
$\downarrow$	b	$\bot$	$x \doteq y$	$\bot$
	t	$\bot$	$\bot$	$\top$

Let ψ be a subformula of $\phi^{\dagger\delta}$ for some sentence ϕ. Let $\mathfrak{G}$ be a 1–modal simulation frame. Then $\mathfrak{G}$ is isomorphic to $\mathfrak{F}^{sim}$ for some bimodal frame $\mathfrak{F}$ (for example, $\mathfrak{F} := \mathfrak{G}_{sim}$) and therefore $\mathfrak{G}$ has exactly one point in the t–region. Suppose $\mathfrak{G} \models \psi[h]$. Then we may assume that $h(x^\alpha)$ is in the α–region for each α, and that $h(x^b) \prec h(x^w)$. Now put $k(x) := h(x^w)$. It is verified by induction on ψ that $\mathfrak{G} \models \psi[h]$ iff $\mathfrak{G}_{sim} \models \psi^\ddagger[k]$. On the other hand, if k is given, define h as follows: $h(x^w) := k(x)$, $h(x^b)$ is the unique world u in the b–region such that $k(x) \succ u$, and $h(x^t)$ is the unique world in the t–region. h is uniquely determined by k, and again it is verified that $\mathfrak{G} \models \psi[h]$ iff $\mathfrak{G}_{sim} \models \psi^\ddagger[k]$. In particular, for $\psi = \phi^{\dagger\delta}$ we get $\mathfrak{G} \models \psi$ iff $\mathfrak{G}_{sim} \models \psi^\ddagger$. Now we return to ϕ. We have $\phi \equiv \phi \wedge (\forall x)t(x) \vee \phi \wedge (\exists x)\neg t(x)$. The first formula is either equivalent to $\bot$ (**Case 1**) or to $(\forall x)t(x)$ (**Case 2**). **Case 1.** Put $\phi_e := (\phi^{\dagger\delta})^\ddagger$. Then $\mathfrak{G} \models \phi$ iff $\mathfrak{G}_{sim} \models \phi_e$. **Case 2.** Put $\psi_e := ((\forall x)\neg(x \doteq x)) \vee (\phi^{\dagger\delta})^\ddagger$.

Proposition 53 *Let $\mathfrak{X}$ be a class of simulation frames. If $\mathfrak{X}$ is elementary (Δ–elementary) so is $\mathfrak{X}_{sim}$.* $\dashv$

Now say that a property $\mathfrak{P}$ **transfers under simulation** if Λ has $\mathfrak{P}$ iff Λ^{sim} has $\mathfrak{P}$.

Theorem 54 (Simulation Theorem) *The following properties transfer under simulation.*

- 0*–axiomatizability, finite axiomatizability, recursive axiomatizability,*
- $\mathfrak{G}$*–persistence,* $\mathfrak{R}$*–persistence,* $\mathfrak{D}$*–persistence,*
- *being Sahlqvist,*
- *elementarity,* Δ*–elementarity,*
- *local (global) completeness with respect to kripke frames (finite frames),*
- *local (global) interpolation.*

Proof. In view of the results of the preceding section we only need to prove one direction in each of the cases. Finite and recursive axiomatizability are clear. Likewise $\mathfrak{R}$–persistence and $\mathfrak{D}$–persistence. For $\mathfrak{G}$–persistence note that it is equivalent with 0–axiomatizability. For the property of being Sahlqvist, we have established that if $\mathsf{w} \to P$ is a Sahlqvist–van Benthem formula, so is P_τ. Now suppose that Λ^{sim} is $(\Delta-)$elementary and let $\mathfrak{X}$ be the class of kripke frames for it. Then the class of simulation frames in $\mathfrak{X}$ is elementary (since such a frame is a simulation frame iff it satisfies $(\forall xy)(t(x) \wedge t(y) \to x \doteq y)$). Then $\mathfrak{X}_{sim}$ is $(\Delta-)$elementary as well by Proposition 53, and it is the class of Λ frames. Hence Λ is $(\Delta-)$elementary. The case of completeness is clear. Now for the proof of interpolation. Assume that Λ has local interpolation. Put $\vdash := \vdash_{\Lambda^{sim}}$. Assume $P \vdash Q$. Then we have (a) $\mathsf{w} \to P \vdash \mathsf{w} \to Q$. Moreover, we also have (b) $\mathsf{b} \to P \vdash \mathsf{b} \to Q$ and (c) $\mathsf{t} \to P \vdash \mathsf{t} \to Q$. (b) can be reformulated into (d) $\mathsf{w} \to \Diamond_b P \vdash \mathsf{w} \to \Diamond_b Q$. The cases (a) and (d) are now similar. Take (a). We have $P_\tau \vdash_\Lambda Q_\tau$. There exists by assumption on Λ an R such that $var(R) \subseteq var(P_\tau) \cap var(Q_\tau)$ and $P_\tau \vdash_\Lambda R \vdash_\Lambda R_\tau$. Then $\mathsf{w} \to (P_\tau)^\tau \vdash \mathsf{w} \to R^\tau \vdash \mathsf{w} \to (P_\tau)^\tau$. Now take the substitution σ as defined above. Then

$$\mathsf{w} \to \sigma((P_\tau)^\tau) \vdash \mathsf{w} \to \sigma(R^\tau) \vdash \mathsf{w} \to \sigma((Q_\tau)^\tau).$$

while by Lemma 51

$$\mathsf{w} \to P \dashv\vdash \mathsf{w} \to \sigma((P_\tau)^\tau)$$

as well as

$$\mathsf{w} \to Q \dashv\vdash \mathsf{w} \to \sigma((Q_\tau)^\tau)$$

Put $A := \sigma(R^\tau)$. Then we have $var(\mathsf{w} \to A) \subseteq var(P) \cap var(Q)$ and

$$\mathsf{w} \to P \vdash \mathsf{w} \to A \vdash \mathsf{w} \to Q$$

Similarly, we find a formula B such that

$$\mathsf{w} \to \Diamond_b P \vdash \mathsf{w} \to B \vdash \mathsf{w} \to \Diamond_b Q$$

In case (c) we can appeal to the fact that classical logic has interpolation to find a C such that

$$\mathsf{w} \to \diamondsuit_t P \vdash \mathsf{w} \to \diamondsuit_t C \vdash \mathsf{w} \to \diamondsuit_t Q$$

(For by Lemma 46, in the scope of $\diamondsuit_t$, P and Q reduce to nonmodal formulae.) Then put

$$N := (\mathsf{w} \to A) \wedge (\mathsf{b} \to \diamondsuit_w B) \wedge (\mathsf{t} \to C)$$

It follows that $P \vdash N \vdash Q$. Thus Λ^{sim} has interpolation. Exactly the same proof can be used for global interpolation (no use of the deduction theorem has been made). $\dashv$

Not all properties are transferred under simulation. A case in point is Halldén–completeness. Recall that a logic Λ is **Halldén–complete** if from $P \vee Q \in \Lambda$ and $var(P) \cap var(Q) = \emptyset$ we can infer that $P \in \Lambda$ or $Q \in \Lambda$. For a monomodal logic to be Halldén–complete it must be the case that either $\boxminus\bot \in \Lambda$ or $\diamondsuit\top \in \Lambda$. It follows that no logic Λ^{sim} is Halldén–complete except if Λ is inconsistent. For since $\boxminus\bot \vee \neg \boxminus \bot$ is a theorem of Λ^{sim}, Halldén–completeness of Λ^{sim} implies that either $\boxminus\bot$ or $\neg \boxminus \bot$ is a theorem. But the latter cannot hold. Hence $\boxminus\bot \in \Lambda^{sim}$, and so $\Lambda = \mathbf{K}_2 \oplus \bot$. On the other hand, there are monomodal logics which are both complete and Halldén–complete (e. g. **S4**). Denote by $\Lambda \otimes \Theta$ the bimodal logic whose first operator satisfies the postulates of Λ and whose second operator satisfies the postulates of Θ. This is called the **independent join** or **fusion** of Λ and Θ. Now let Λ be both Halldén–complete and complete. Then, by a theorem of [20], $\Lambda \otimes \Lambda$ is Halldén–complete as well, while $(\Lambda \otimes \Lambda)^{sim}$ is not.

§12. Applications to Decidability Problems.

In [7] it is shown that a great many properties of logics are undecidable even for logics containing **K4**. The results of that paper are proved using a technique which is rather involved. Here we will present some results which can easily be derived from known ones using the special simulations we have developed here. The results are based on the fact that it is possible to encode word–problems in modal logic. Let us review the basic facts. Given an alphabet A, A^* denotes the set of finite strings over A, a member of which we denote by vector arrow, e. g. $\vec{u}$. The empty string is denoted by ϵ the concatenation of $\vec{u}$ and $\vec{v}$ by $\vec{u} \cdot \vec{v}$. A *Thue–process* is a finite set of equations $\vec{v} \approx \vec{w}$, where $\vec{v}, \vec{w} \in A^*$. A Thue–process $\mathfrak{T}$ can be viewed as a finite presentation of a (finitely generated) semigroup. Given an alphabet over two letters, Thue processes thus are presentations of 2–generated semigroups. The *word problem* of $\mathfrak{T}$ is the problem to decide whether in the semigroup presented by $\mathfrak{T}$ the equation $\vec{x} \approx \vec{y}$ holds for arbitrary given $\vec{x}, \vec{y} \in A^*$. The following is known.

Theorem 55 *Let A be a two–letter alphabet.*

1. *([23]) There exist Thue–processes over A with undecidable word problem.*
2. *([22]) It is undecidable whether a Thue–process over A has a decidable word problem.*
3. *It is undecidable whether two Thue–processes over A present the same semigroup.*
4. *([24]) It is undecidable whether a Thue–process over A is trivial, that is, whether it presents the one–element semigroup.*
5. *It is undecidable whether a Thue–process over A presents a finite semigroup.*

The first actually follows from the second, and the third from the fourth assertion. The fifth can be seen as follows. Suppose we can decide whether $\mathfrak{T}$ presents a finite semigroup. Then we can decide whether $\mathfrak{T}$ presents the one–element semigroup in the following way. With $\mathfrak{T}$ given, decide whether it presents an infinite semigroup. If yes, then that semigroup is not the one–element semigroup. If not, $\mathfrak{T}$ is decidable. For the set of equations derivable from $\mathfrak{T}$ is recursively enumerable. Its complement is now also recursively enumerable, since an equation fails in $\mathfrak{T}$ iff it fails in a finite semigroup validating all equations of $\mathfrak{T}$.

Let us note that the above theorem would be false if A had only one symbol. In that case, every Thue process is decidable. With a Thue–process over $\{w, b\}$ we associate a modal logic as follows. A word is translated as a sequence of modalities.

$$
\begin{array}{lcllcl}
[\epsilon]P & := & P & \langle\epsilon\rangle P & := & P \\
[b]P & := & \blacksquare P & \langle b\rangle P & := & \blacklozenge P \\
[w]P & := & \Box P & \langle w\rangle P & := & \Diamond P \\
[\vec{u}\cdot\vec{v}]P & := & [\vec{u}]([\vec{v}]P) & \langle\vec{u}\cdot\vec{v}\rangle P & := & \langle\vec{u}\rangle(\langle\vec{v}\rangle P)
\end{array}
$$

We translate an equation $E = \vec{x} \approx \vec{y}$ by the axiom

$$E^m := \langle\vec{x}\rangle p. \to .[\vec{y}]p$$

This axiom is first–order and says that for every point s every successor t related by an $\vec{x}$–path from s and every successor u related by a $\vec{y}$–path from s we have $t = u$. Here, an $\vec{x}$–path is a path formed by $\vec{x}$ under the identification of b (= *black*) with $\blacktriangleleft$ and of w (= *white*) with $\lhd$. Now let $\mathfrak{T}$ be given. With $\mathfrak{T}$ we associate the following two logics.

$$
\begin{array}{rcl}
\Sigma(\mathfrak{T}) & := & \mathbf{K.Alt}_1 \otimes \mathbf{K.Alt}_1 \oplus \{E^m | E \in \mathfrak{T}\} \\
\Lambda(\mathfrak{T}) & := & \Sigma(\mathfrak{T}) \oplus \Diamond\top \oplus \blacklozenge\top
\end{array}
$$

Let $\mathfrak{F}$, $\mathfrak{G}$ be n–modal frames. $\mathfrak{G}$ is a **subframe** of $\mathfrak{F}$ if $g \in \mathbf{F}$, $\lhd_i^g = \lhd_i^f \cap (g \times g)$ and $\mathbf{G} = \{a \cap g : a \in \mathbf{F}\} = \{a \in \mathbf{F} : a \leq g\}$. In other words, subframes are relational reducts to internal subsets of a frame. For the purpose of this definition, kripke–frames are identified with the corresponding full frames. A **suframe logic** is a logic whose class of frames is closed under taking subframes (see [32]). The axioms of $\Lambda(\mathfrak{T})$ are elementary, and the corresponding 1^{st}–order formulae can be written using only restricted universal quantifiers. Thus the logics $\Lambda(\mathfrak{T})$ are subframe logics (see [32] or [18]), which is Claim (4.) of the theorem below.

Proposition 56 *Let $\mathfrak{T}$ be a Thue–process over $A = \{w, b\}$ and $\Sigma := \Sigma(\mathfrak{T})$ and $\Lambda := \Lambda(\mathfrak{T})$ be as above. The following holds.*

1. *Σ and Λ are finitely axiomatizable by one–letter axioms.*
2. *Σ and Λ are complete.*
3. *$\vdash_\Lambda P$ iff $\Diamond\top; \blacklozenge\top \Vdash_\Sigma P$.*
4. *Σ is a subframe logic.*
5. *Σ has the local finite model property and is locally decidable.*

The first claim is straightforward since $\mathbf{alt}_1$ can be axiomatized by $\Diamond p \to \Box p$, and the second follows from a theorem of [32], which states that any extension of polymodal $\mathbf{K.Alt}_1$ is complete; this generalizes a theorem of Bellissima [1]. The third claim is straightforward. A proof can be found in [18]. The fourth follows by inspection on the elementary condition imposed by the axioms of Σ. The axioms $\mathbf{Alt}_1$ are preserved when passing to a subframe, and so are the axioms E^m. The last claim now follows easily. Assume that $P \notin \Sigma$. Then there exists a model $\langle \mathfrak{f}, \beta, x\rangle \models \neg P$, since Σ is complete. Now let d be the maximum nesting of modal operators in P, and let g be the subset of all points reachable from x in at most d steps following the relations $\lhd$ or $\blacktriangleleft$. Then g is finite, containing at most $2^{d+1} - 1$ points. The subframe based on g, $\mathfrak{g}$, is a frame for Σ. Let $\gamma(p) := \beta(p) \cap g$. Then $\langle \mathfrak{g}, \gamma, x\rangle \models \neg P$. The local decidability of Σ is an immediate consequence.

Now take a Thue–process $\mathfrak{T}$. Denote by $\mathfrak{Fr}_{SG}(A)/\mathfrak{T}$ the semigroup presented by $\mathfrak{T}$. Then this semigroup can be turned into a kripke frame as follows. Write $\vec{x} \approx_\mathfrak{T} \vec{y}$ if the equation $\vec{x} \approx \vec{y}$ is derivable in $\mathfrak{T}$. Furthermore, let $[\vec{x}] := \{\vec{y} : \vec{y} \approx_\mathfrak{T} \vec{x}\}$. Put $[\vec{x}] \lhd [\vec{y}]$ iff $\vec{x} \cdot w \in [\vec{y}]$ (iff $\forall \vec{v} \in [\vec{x}] : \vec{v} \cdot w \in [\vec{y}]$), and $[\vec{x}] \blacktriangleleft [\vec{y}]$ iff $\vec{x} \cdot b \in [\vec{y}]$ (iff $\forall \vec{v} \in [\vec{x}] : \vec{v} \cdot b \in [\vec{y}]$). We denote by $\mathfrak{Sgr}(\mathfrak{T})$ the frame $\langle \{[\vec{x}] | \vec{x} \in A^*\}, \lhd, \blacktriangleleft\rangle$. Then the following holds (see [16] and [18]).

Proposition 57 (Grefe) *(1.)* $\Lambda(\mathfrak{T}) = \mathsf{Th}\, \mathfrak{Sgr}(\mathfrak{T}) = \mathsf{Th}\, \langle \mathfrak{Sgr}(\mathfrak{T}), [\epsilon]\rangle$. *(2.) $\Lambda(\mathfrak{T})$ is decidable iff $\mathfrak{T}$ is decidable. (3.) $\Lambda(\mathfrak{T})$ is tabular iff $\mathfrak{T}$ presents a finite semigroup.*

Proof. (1.) First of all, $\langle \mathfrak{Sgr}(\mathfrak{T}), [\epsilon]\rangle \models \Lambda(\mathfrak{T})$, and so $\Lambda(\mathfrak{T}) \subseteq \mathsf{Th}\, \langle \mathfrak{Sgr}(\mathfrak{T}), [\epsilon]\rangle$, as is easily checked. Now, take a rooted kripke frame $\langle \mathfrak{f}, x\rangle$ such that $\langle \mathfrak{f}, x\rangle \models$

$\Lambda(\mathfrak{T})$. We claim that there is a p–morphism $\pi : \mathfrak{Sgr}(\mathfrak{T}) \twoheadrightarrow \mathfrak{f}$ such that $\pi([\epsilon]) = x$. Namely, put $\pi([\epsilon]) := x$. Furthermore, assume π is defined on $[\vec{u}]$ and $\pi([\vec{u}]) = y$. If $y \lhd z$ put $\pi([\vec{u} \cdot w]) := z$. If $y \blacktriangleleft z$ put $\pi([\vec{u} \cdot b]) := z$. In this way, π is defined on all strings and is onto. Now we show that it is a p–morphism. Let $[\vec{u}] \lhd [\vec{v}]$. Then $[\vec{v}] = [\vec{u} \cdot w]$. Furthermore, $\pi([\vec{u} \cdot w]) = z$ for the unique z such that $\pi([\vec{u}]) \lhd z$. Hence $\pi([\vec{u}]) \lhd \pi([\vec{v}])$. Similarly if $[\vec{u}] \blacktriangleleft [\vec{v}]$. Next assume that $y = \pi([\vec{u}])$ and that $y \lhd z$. Then put $\vec{v} := \vec{u} \cdot w$. With this choice we get $[\vec{u}] \lhd [\vec{v}]$ and $\pi([\vec{u}]) \lhd \pi([\vec{v}])$. Likewise for $\blacktriangleleft$. This shows that π is a p–morphism. Moreover, $\pi([\epsilon]) = x$, by definition of π. By Proposition 56, $\Lambda(\mathfrak{T})$ is complete. Hence, if $P \notin \Lambda(\mathfrak{T})$ there exists a rooted kripke model $\langle \mathfrak{f}, \beta, x\rangle$ such that $\langle \mathfrak{f}, x\rangle \models \Lambda(\mathfrak{T})$ and $\langle \mathfrak{f}, \beta, x\rangle$ **2** P. Then let $\pi : \mathfrak{Sgr}(\mathfrak{T}) \twoheadrightarrow \mathfrak{f}$ be a p–morphism such that $\pi([\epsilon]) = x$. Put $\gamma(p) := \{[\vec{u}] : \pi([\vec{u}]) \in \beta(p)\}$. With this valuation, $\langle \mathfrak{Sgr}(\mathfrak{T}), \gamma, [\epsilon]\rangle$ **2** P. The first claim is now proved, since we also note that $\langle \mathfrak{Sgr}(\mathfrak{T}), x\rangle \models \Lambda(\mathfrak{T})$. (2.) By (1.), $\langle \vec{x}\rangle p \leftrightarrow \langle \vec{y}\rangle p \in \Lambda(\mathfrak{T})$ iff $\langle \mathfrak{Sgr}(\mathfrak{T}), [\epsilon]\rangle \models \langle \vec{x}\rangle p \leftrightarrow \langle \vec{y}\rangle p$ iff $[\vec{x}] = [\vec{y}]$ iff $\vec{x} \approx_{\mathfrak{T}} \vec{y}$. Therefore, if $\mathfrak{T}$ is undecidable, so is $\Lambda(\mathfrak{T})$. Now let $\mathfrak{T}$ be decidable. Let us call a formula P **straight** if it is of the form $\langle \vec{x}\rangle C$, where C is a variable or a negated variable. It is not hard to show that in $\mathbf{K.Alt.D} \otimes \mathbf{K.Alt.D}$, any formula is deductively equivalent to a disjunction of conjunctions of straight formulae. (Moreover, there is an effective procedure which transforms a given formula into a deductively equivalent formula of such a form.) Therefore, it is enough to show that for such formulae, satisfiability is decidable in $\Lambda(\mathfrak{T})$. For that it is in turn enough to show that the satisfiability of a conjunction of straight formulae is decidable. Let $Q = \bigwedge_{i<k} \langle \vec{x}_i\rangle C_i$ be such a formula. By (1.), Q is satisfiable iff $\langle \mathfrak{Sgr}(\mathfrak{T}), [\epsilon]\rangle$ **2** $\neg Q$ iff there exists a valuation β such that $\langle \mathfrak{Sgr}(\mathfrak{T}), \beta, [\epsilon]\rangle \models Q$. $\langle \mathfrak{Sgr}(\mathfrak{T}), \beta, [\epsilon]\rangle \models Q$ is equivalent to (†).

(†) For all $i < k$, if $C_i = p$ then $[\vec{x}_i] \in \beta(p)$
and if $C_i = \neg p$ then $[\vec{x}_i] \notin \beta(p)$.

β exists iff (†) is noncontradictory iff for no i, j there is a variable p such that $C_i = p$ and $C_j = \neg p$ and $[\vec{x}_i] = [\vec{x}_j]$. Since $[\vec{x}_i] = [\vec{x}_j]$ iff $\vec{x}_i \approx_{\mathfrak{T}} \vec{x}_j$, it is decidable (by decidability of $\mathfrak{T}$) whether (†) is consistent, hence whether Q is satisfiable. So, $\Lambda(\mathfrak{T})$ is decidable. (3.) Is straightforward. $\dashv$

For the statement of the next theorem, recall that an element x in a modular lattice $\mathcal{L}$ with top element $\top$ has codimension n if there exists a sequence $Y = \langle y_i : i < n+1\rangle$ such that $y_0 = x$, $y_n = \top$, and for all $i < n$, $y_i < y_{i+1}$ and there is no z such that $y_n < z < y_{i+1}$. The condition of modularity guarantees that n —- if it exists — is independent of the sequence Y. The lattices $\mathcal{E}\,\mathbf{K}_n$ are distributive, and therefore modular.

Theorem 58 *The following properties of logics are undecidable for finitely axiomatizable monomodal logics on the basis of a finite axiomatization*

1. *local decidability,*
2. *local finite model property,*
3. *global finite model property,*

4. *tabularity,*

5. *being of finite codimension in $\mathcal{E}\,\boldsymbol{K}$.*

Proof. By the Simulation Theorem and the fact that these properties are undecidable for bimodal logics. That they are undecidable for bimodal logic we will show now. Local decidability is undecidable by Theorem 55 and Proposition 57, likewise tabularity. For logics of the form $\Lambda(\mathfrak{T})$, tabularity and finite codimension coincide. The undecidability of the finite model property is proved thus. Suppose that the finite model property is decidable. Then a decision procedure can be given that decides whether $\mathfrak{T}$ presents a one–element semigroup. Namely, take $\mathfrak{T}$. If $\Lambda(\mathfrak{T})$ fails to have the finite model property, it is anyway not tabular. If $\Lambda(\mathfrak{T})$ has the finite model property, however, it is decidable, because not only the theorems are recursively enumerable, but also the nontheorems. (Namely, $\Lambda(\mathfrak{T})$ is finitely axiomatizable, and so the finite frames are enumerable.) Now, by Proposition 57, $\mathfrak{T}$ is trivial exactly when $\Diamond p \leftrightarrow p \in \Lambda(\mathfrak{T})$ as well as $\blacklozenge p \leftrightarrow p \in \Lambda(\mathfrak{T})$. Since both problems are decidable, it can be decided whether or not $\mathfrak{T}$ is trivial. Contradiction. Therefore, the finite model property is not decidable. The same argument can be given for the global finite model property of $\Sigma(\mathfrak{T})$. Since $\Sigma(\mathfrak{T})$ has the local finite model property, it follows that the global finite model property is not decidable. $\dashv$

We remark that what we have shown in fact is that it is not even decidable whether a logic has the global finite model property when we know that the logic has the local finite model property. That tabularity is undecidable has independently been proved by ALEXANDER CHAGROV in [5].

Theorem 59 *The following properties are undecidable for finitely axiomatizable bimodal logics on the basis of a finite axiomatization:*

1. *independent axiomatizability,*

2. *consistency,*

3. *containment of a given tabular logic.*

Proof. (1.) Take the logics $\Lambda(\mathfrak{T})$. Suppose $\Lambda(\mathfrak{T})$ is independently axiomatizable. Then it is decidable, for it is the fusion of monomodal extensions of $\mathbf{K.Alt}_1$, which are all decidable. Hence if independent axiomatizability is decidable, so is the problem whether $\Lambda(\mathfrak{T})$ presents a one–element semigroup. Contradiction. (2.) Let $\boxed{\bullet|\bullet}$ be the one–element kripke frame with both relations empty. Consider the logic $\Sigma(\mathfrak{T}) \sqcup \mathsf{Th}\boxed{\bullet|\bullet}$. This logic is consistent iff $\mathfrak{Sgr}(\mathfrak{T})$ contains a point which is irreflexive in both relations. This is the case exactly when neither $[\epsilon] = [w]$ nor $[\epsilon] = [b]$ iff neither $w \approx_{\mathfrak{T}} \epsilon$ nor $b \approx_{\mathfrak{T}} \epsilon$. The latter problem, however, is undecidable. Suppose otherwise. Then the problem '$w \approx_{\mathfrak{T}} \epsilon$ or $b \approx_{\mathfrak{T}} \epsilon$' is also decidable. This gives a decision procedure for the problem whether $\mathfrak{T}$ is trivial as follows. Let $\mathfrak{T}$ be given. Decide '$w \approx_{\mathfrak{T}} \epsilon$ or

$b \approx_{\mathfrak{T}} \epsilon$'. If it is false, $\mathfrak{T}$ is not trivial. If it is true, however, $\Lambda(\mathfrak{T})$ is the fusion of two monomodal logics $\Theta_1 = \Lambda(\mathfrak{U}_1)$ and $\Theta_2 = \Lambda(\mathfrak{U}_2)$ where $\mathfrak{U}_1$ and $\mathfrak{U}_2$ are one–letter Thue processes. (One of $\mathfrak{U}_1$ and $\mathfrak{U}_2$ is actually trivial.) Since Θ_1 and Θ_2 are decidable and complete (this is shown in [28]) their fusion, $\Lambda(\mathfrak{T})$, is decidable, by a theorem of [20]. $\mathfrak{T}$ is trivial if $p \to [b]p \in \Lambda(\mathfrak{T})$ and $p \to [w]p \in \Lambda(\mathfrak{T})$, which is now decidable. Hence it follows from our assumption that we can decide whether $\mathfrak{T}$ is trivial. But we cannot. So the supposition fails. (3.) Assume that containment of Θ is decidable, and Θ is tabular, $\Theta = \mathsf{Th}\,\mathfrak{f}$ for a finite kripke frame. We show that it follows that consistency is decidable. To that end, take a bimodal logic Λ. Test first whether it contains Θ. If not, it is consistent. If yes, test all generated subframes of $\mathfrak{f}$ whether they are frames for Λ. If none of them is, Λ is inconsistent, otherwise consistent. $\dashv$

Corollary 60 *For a monomodal logic it is undecidable whether it is a subframe logic.*

Proof. This claim follows from the fact that for an extension Θ of **Sim**, Θ is a subframe logic iff it is inconsistent or $\Lambda = \mathsf{Th}\,\mathbf{2}^{\bullet}$. (For let $\mathfrak{F}$ be a Θ–frame. If Θ is a subframe logic, the subframe $\mathfrak{G}$ based on $f^w \cup f^b$ is a frame for Θ. But $\mathfrak{G} \models \Diamond\top$. Hence $f^w \cup f^b = \emptyset$. So, the only Θ–frames are the empty frame and the one–point frame.) It follows that for a bimodal logic Λ, Λ is consistent iff Λ^{sim} is a subframe logic. Since the first is undecidable, the latter is undecidable as well. $\dashv$

The last two parts of Theorem 59 have also been shown by Alexander Chagrov in [5]. With these results in our hands we can use the technique of fusion to obtain a number of cardinality and undecidability results. The method has already been established with respect to decidability in [31]. Suppose that $\mathfrak{P}$ is a property of logics such that the inconsistent n–modal logics have $\mathfrak{P}$, and which is reflected under fusion in the following sense. If $\Lambda \otimes \Theta$ has $\mathfrak{P}$, then both Λ and Θ have $\mathfrak{P}$. Now take a logic Λ that fails to have $\mathfrak{P}$. Then $\Lambda \otimes \Theta$ fails to have $\mathfrak{P}$ iff Θ is consistent. It follows that to have $\mathfrak{P}$ is undecidable, and that there exist $2^{\aleph_0}$ logics without $\mathfrak{P}$. Such properties are $\mathfrak{G}$–persistence, $\mathfrak{R}$–persistence, $\mathfrak{D}$–persistence, completeness, the finite model property, elementarity, Sahlqvist, Halldén–completeness, interpolation and many more. (To show this, one needs to establish that these properties are reflected under fusion in each of these cases. It is enough to take Λ a monomodal and Θ a bimodal logic. Reflection of $\mathfrak{P}$ is straightforward for the listed properties, see for example [20]. The preservation of $\mathfrak{P}$ is in many cases much harder to show, but not needed here.) However, this argument does not allow to deduce that there are $2^{\aleph_0}$ logics with $\mathfrak{P}$. In the case of tabularity this is false. This scheme is therefore asymmetrical in this respect. We will not spell out the results in detail; once the way to obtain them is known, the results become of lesser importance.

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