Reducing Modal Consequence Relations

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Abstract

In this paper we will investigate the possibility of reducing derivability in a modal consequence relation to consistency in unimodal and polymodal **K** by means of so–called reduction functions. We will present new and easy methods to prove standard results on decidability, complexity, finite model property, interpolation and Halldén–completeness using only the reduction functions. Some new results on complexity of modal logics will be established. All proofs are in addition constructive.

1 Introduction

Proofs of the finite model property in modal logic typically proceed via the detour of the canonical models or the weak canonical models, where the weak canonical models are based on finitely many sentence letters only. The alternatives to this method, tableaux methods or normal forms, do not seem to be as flexible as the methods designed to produce finite models from the canonical model. Yet, canonical models too have disadvantages. They are highly abstract structures. Moreover, even when it is finite (for example when the logic determines a locally finite variety) it may be the case that the structure of a weak canonical model or the structure of a finitely generated free algebra cannot be determined if the logic is undecidable. So, this method of proof is highly inconstructive. This is not only a theoretical disadvantage. Typically, explicit solutions are easier to understand. Tableau systems can be used quite effectively for the standard systems. However, tableau calculi for many logics extending **K** are often rather ad–hoc. They use a mixture of closure rules (rules to be applied at a particular world) and step rules (rules to be applied when moving to a successor).

The present paper presents a combinatorial method that uses only the reduction of provability in a stronger system into provability of a weaker system. Though the proof of the success of the method typically involves model theoretic arguments, it is nevertheless finitistic. Its advantages are manifold. It is constructive, explicit, and yields proofs of the finite model property, decidability, complexity and interpolation, to name a few. Once the reduction of one logic M to a weaker logic L is shown, many properties of L are shown to transfer to M with no additional effort. The proof of interpolation, for example, is so simple that it reduces interpolation of the standard systems to a mere corollary of the method. (Compare this, for example, with the criterion developed in [14], which is difficult to state let alone apply.) This method is not new. It has been used, for example, by Balbiani and Herzig in [1] and De Giacomo in [5]. However, what is new is the systematic study of the method, and the observation that it can be used for much more than just the proof of finite model property.

I should say here that not everything presented here is new. In particular, most of the results of Sections 3, 5 and 6 have been anticipated in [7], though usually not in as general a form as they appear here. The results of Section 4 are, however, entirely new. Moreover, in an appendix we show how to give a completely constructive proof of a well–known theorem by Kit Fine on subframe logics. It is based on the ideas outlined here. Finally, I wish to thank Stefan Baier for his help with number theory and an anonymous referee for useful comments.

2 Preliminaries

We consider the language of modal logic with any number of unary modal operators. The vocabulary consists of a set of variables, $V := \{p_i : i \in \omega\}$, and the functors $\bot$, $\neg$, $\wedge$ and $\Box_i$, $i < \kappa$. Here κ is a cardinal number. If $\kappa = 1$ we will write $\Box$ rather than $\Box_0$. Formulae are built in the usual way; we use lower case Greek letters (φ, ψ, ...) to denote formulae, and upper case Greek letters (Δ, Γ, ...) to denote sets of formulae. We use the standard notation $\Delta; \varphi$ to denote $\Delta \cup \{\varphi\}$ and $\varphi; \psi$ for $\{\varphi\} \cup \{\psi\}$. The following symbols, also frequently used, are treated as abbreviations:

$$\begin{array}{lcl} \top & := & \neg\bot \\ \varphi \vee \psi & := & \neg(\neg\varphi \wedge \neg\psi) \\ \varphi \to \psi & := & \neg(\varphi \wedge \neg\psi) \\ \varphi \leftrightarrow \psi & := & (\varphi \to \psi) \wedge (\psi \to \varphi) \\ \Diamond_j\varphi & := & \neg\Box_j\neg\varphi \end{array}$$

$var(\varphi)$ denotes the set of variables occurring in φ and $sf(\varphi)$ the set of subformulae of φ. The notation $var[\Delta]$ and $sf[\Delta]$ are self–explanatory. (We remark here that if f is a function, and X a subset of the domain of f we write $f[X] := \{f(x) : x \in X\}$.) The modal depth, $dp(\varphi)$ is defined as follows.

$$\begin{array}{lcl} dp(p_i) & := & 0 \\ dp(\bot) & := & 0 \\ dp(\neg\varphi) & := & dp(\varphi) \\ dp(\varphi \wedge \psi) & := & max\{dp(\varphi), dp(\psi)\} \\ dp(\Box_j\varphi) & := & 1 + dp(\varphi) \end{array}$$

The set of all formulae is denoted by $\mathcal{F}_\kappa$. A substitution is a function $\sigma : V \to \mathcal{F}_\kappa$. The effect of applying σ to φ (Δ) is denoted by φ^σ (Δ^σ).

A *consequence relation* is a relation $\vdash \subseteq \wp(\mathfrak{F}_\kappa) \times \mathfrak{F}_\kappa$ such that

1. If $\varphi \in \Delta$ then $\Delta \vdash \varphi$.
2. If $\Delta \vdash \varphi$ and $\Delta \subseteq \Delta'$ then $\Delta' \vdash \varphi$.
3. If $\Delta \vdash \varphi$ for all $\varphi \in \Sigma$ and $\Sigma \vdash \psi$ then $\Delta \vdash \psi$.

$\vdash$ is *structural* if $\Delta \vdash \varphi$ implies $\Delta^\sigma \vdash \varphi^\sigma$ for every substitution σ. $\vdash$ is *finitary* if $\Delta \vdash \varphi$ implies that there exists a finite $\Delta_0 \subseteq \Delta$ such that $\Delta_0 \vdash \varphi$. In sequel, all consequence relations are finitary and structural. A *rule* is a pair $\rho = \langle \Delta, \varphi \rangle$, where $\Delta \subseteq \mathfrak{F}_\kappa$ and $\varphi \in \mathfrak{F}_\kappa$. ρ is *finitary* if Δ is finite. Examples of rules are MP := $\langle \{p_0, p_0 \to p_1\}, p_1 \rangle$ and $\mathrm{MN}_j := \langle \{p_0\}, \Box_j p_0 \rangle$. Let R be a set of finitary rules. We denote by $\vdash^R$ the smallest finitary structural consequence relation containing R. $\vdash^R$ can be described as follows. Call an *R–proof* of φ from Δ a finite sequence $\langle \delta_i : i < n \rangle$ such that

1. $\delta_{n-1} = \varphi$.
2. For all $i < n$, $\delta_i \in \Delta$ or there exists a subset $\Sigma \subseteq \{\delta_j : j < i\}$ such that $\langle \Sigma, \delta_i \rangle$ is a substitution instance of some element of R.

The following is stated without proof (see [7] for a complete proof).

Proposition 1 $\Delta \vdash^R \varphi$ *iff there exists an R–proof of* φ *from* Δ.

We define a modal logic as a set of formulae. We assume that A is a set of formulae which together with MP axiomatises classical propositional logic.

Definition 2 *A* ***normal*** κ***–modal logic*** *is a subset* L *of* $\mathfrak{F}_\kappa$ *which contains* A*, the formulae* $\Box_j(p_0 \to p_1) \to (\Box_j p_0 \to \Box_j p_1)$*, and is closed under substitution, and the rules MP and MN*$_j$*,* $j < \kappa$*. The smallest* κ*–normal modal logic is denoted by* $\boldsymbol{K}_\kappa$*.*

We shall denote modal logics in sequel by upper case Roman letters, for example L, M etc. Moreover, we will often speak of the rule MN, by which we denote the set of rules MN$_j$, $j < \kappa$. Now let L be a normal logic and Δ a set of formulae; the smallest normal logic containing L and Δ is denoted by $L \oplus \Delta$. (With particular axioms such as **4**, **T** etc. the notation $L.\Delta$ is also used.) The notions of frame and general frame are as usual. A general frame is a triple $\mathfrak{F} = \langle F, R, U\rangle$, where F is a set, R a function from κ into subsets of F^2, and $U \subseteq \wp(F)$ closed under relative complement, intersection and the operations $\tau_j(A) := \{y : \text{ if } y\ R(j)\ x \text{ then } x \in A\}$, $j < \kappa$. We shall write $\lhd_j$ or $\blacktriangleleft_j$ (with or without the index, when no confusion arises) to denote $R(j)$. If $U = \wp(F)$ we call $\mathfrak{F}$ a *Kripke–frame* and suppress mentioning U. A *valuation* into $\mathfrak{F}$ is a function $\beta : V \to U$. A triple $\mathfrak{M} = \langle \mathfrak{F}, \beta, x\rangle$ where $\mathfrak{F}$ is a frame, β a valuation into $\mathfrak{F}$ and $x \in F$ is called a **local model**, the pair $\mathfrak{N} = \langle \mathfrak{F}, \beta\rangle$ a *global model*. In case, where $\mathfrak{M} = \langle \mathfrak{F}, \beta, x\rangle$ and $\mathfrak{N} = \langle \mathfrak{F}, \beta\rangle$, $\mathfrak{M}$ is a *local expansion* of $\mathfrak{N}$. Given a local model $\mathfrak{M} = \langle \mathfrak{F}, \beta, x\rangle$ and a formula φ, we define $\mathfrak{M} \models \varphi$ as follows.

1. $\mathfrak{M} \not\models \bot$
2. $\mathfrak{M} \models \neg\varphi$ iff $\mathfrak{M} \not\models \varphi$.
3. $\mathfrak{M} \models \varphi \wedge \varphi'$ iff $\mathfrak{M} \models \varphi$ and $\mathfrak{M} \models \varphi'$.
4. $\mathfrak{M} \models \Box_j\varphi$ iff for all y such that $x\ R(j)\ y$: $\langle \mathfrak{F}, \beta, y\rangle \models \varphi$.

$\mathfrak{M}$ is a (local) L–model, if $\mathfrak{M} \models L$. We define $\mathfrak{N} \models \varphi$ iff $\mathfrak{M} \models \varphi$ for all local extensions $\mathfrak{M}$ of $\mathfrak{N}$.

Let $\vdash$ be a consequence relation. Then we put $\mathsf{Taut}(\vdash) := \{\varphi : \varnothing \vdash \varphi\}$. We call $\vdash$ a *normal modal consequence relation* if $\mathsf{Taut}(\vdash)$ is a normal modal logic. We associate with a normal logic L two special consequence relations, denoted by $\vdash_L$ and $\Vdash_L$. The first is called the **local consequence relation** of L and is defined as follows. $\Delta \vdash_L \varphi$ iff φ is derivable from $\Delta \cup L$ by means of MP. $\Vdash_L$ is called the **global consequence relation** of L and $\Delta \Vdash_L \varphi$ iff φ can be derived from $\Delta \cup L$ by means of MP and MN. (These consequence relations are not necessarily distinct. In fact, they are equal iff $p_0 \to \Box_j p_0 \in L$ for all $j < \kappa$.) A (global) *L–proof* of φ from Δ is a proof of φ from $\Delta \cup L$ using MP (MP and MN).

The following is a consequence of the general completeness theorem of modal logic (with respect to general frames). We shall not prove it here, since it will *not* be used in sequel.

Proposition 3 *Let L be a normal modal logic. Then the following holds.*

1. $\Delta \vdash_L \varphi$ *iff for all local L–models* $\mathfrak{M}$*: if* $\mathfrak{M} \models \Delta$ *then* $\mathfrak{M} \models \varphi$.

2. $\Delta \Vdash_L \varphi$ *iff for all global L–models* $\mathfrak{N}$*: if* $\mathfrak{N} \models \Delta$ *then* $\mathfrak{N} \models \varphi$.

We will also make use of the notion of a *compound modality*. A **compound modality** is a term in one variable, built up using only $\wedge$ and $\Box_j$. Examples are $\Box_0 p_0$, $p_0 \wedge \Box_1 p_0$, $\Box_1\Box_0 p_1$ etc. Compound modalities behave like unary modal operators. We use $\boxplus$ as a variable over compound modalities. Now let $\sigma \in \kappa^*$ be a finite sequence of elements of κ. We define inductively the symbol $\Box^\sigma$ as follows. (Here, ε is the empty sequence.)

$$\begin{array}{rcl} \Box^\varepsilon\varphi & := & \varphi \\ \Box^{j\cdot\sigma}\varphi & := & \Box_j\Box^\sigma\varphi \end{array}$$

Finally, for a set $S \subseteq \kappa^*$ we put

$$\Box^S\varphi := \{\Box^\sigma\varphi : \sigma \in S\}$$

Generally, we use the convention that a finite set of formulae Δ also denotes the conjunction $\bigwedge \Delta$. Hence, if S is finite we also have

$$\Box^S\varphi = \bigwedge\langle\Box^\sigma\varphi : \sigma \in S\rangle$$

The following is easy to verify.

Proposition 4 *Let* $\boxplus$ *be a compound modality of* $\mathcal{F}_\kappa$*. Then there exists a finite set* $S \subseteq \kappa^*$ *such that* $\boxplus p \leftrightarrow \Box^S p \in \boldsymbol{K}_\kappa$.

Fix a normal modal logic L. Let Δ be a set of formulae. We wish to describe the sets $\Delta^\vdash := \{\varphi : \Delta \vdash_L \varphi\}$, $\Delta^\Vdash := \{\varphi : \Delta \Vdash_L \varphi\}$ as well as the set $L \oplus \Delta$. In the first case we have to close under MP, in the second under MP and MN and in the third under MP, the rule MN and substitution. It turns out that these closures can be obtained in a canonical and simple way. Denote by Δ^s the closure of Δ under substitution, by Δ^p the closure of Δ under MP, and by Δ^n the closure of Δ under MN.

Proposition 5 *The following holds.*

1. $\Delta \vdash_L \varphi$ *iff* $\varphi \in (\Delta \cup L)^p$
2. $\Delta \Vdash_L \varphi$ *iff* $\varphi \in (\Delta^n \cup L)^p$
3. $\varphi \in L \oplus \Delta$ *iff* $\varphi \in ((\Delta^s)^n \cup L)^p$

Proof. The first is immediate from the definition. The second follows from the fact that applications of MN_j can be put before all applications of MP. Moreover, L is closed under MN. The third fact follows from the fact that substitutions commute with MN_j and MP (and that L is closed under MN_j and substitutions). We shall show this now. The proof is based on the notion of a proof tree (rather than a sequence).

Consider an application of MN_j which follows an application of MP as in the left hand side below. There is an alternative proof of $\Box_j\psi$ in which the order is reversed. This proof is shown to the right.

$$\dfrac{\dfrac{\varphi \quad \varphi \to \psi}{\psi}}{\Box_j\psi} \qquad\qquad \dfrac{\dfrac{\varphi}{\Box_j\varphi} \quad \dfrac{\dfrac{\varphi \to \psi}{\Box_j(\varphi \to \psi)} \quad \Box_j(\varphi \to \psi) \to \Box_j\varphi \to \Box_j\psi}{\Box_j\varphi. \to .\Box_j\psi}}{\Box_j\psi}$$

It is readily computed that the depth of the application of MN_j is reduced. A proper inductive argument will establish that each proof can be transformed into a proof where all applications of MN_j precede all applications of MP. It follows that we have $\Delta \vdash_L \varphi$ iff $\varphi \in ((\Delta \cup L)^n)^p = (\Delta^n \cup L^n)^p = (\Delta^n \cup L)^p$. For the rules MN are unary and so $(\Delta \cup L)^n = \Delta^n \cup L^n$. $L^n = L$, since L is normal.

Next we look at substitution. It is easy to show that each application of substitution can be moved up in the proof tree, so that the proof can be arranged in such a way that all applications of substitution precede all applications of MN, which in turn precede all applications of MP. Therefore, $\varphi \in L \oplus \Delta$ iff $\varphi \in ((\Delta^s)^n \cup (L^s)^n)^p = ((\Delta^s)^n \cup L)^p$, by similar arguments. $\Box$

3 Global Reduction and Local Reduction

Let L and M be normal modal logics and $L \subseteq M$. Then also $\Vdash_L \subseteq \Vdash_M$. Moreover, $M = L \oplus \Sigma$ for some set of formulae Σ. Then the following is easily shown.

Lemma 6 *Suppose that $M = L \oplus \Sigma$ and $\Delta \Vdash_M \varphi$. Then there exists a finite set $\Xi \subseteq \Sigma^s$ such that $\Delta; \Xi \Vdash_L \varphi$.*

Proof. We use Proposition 5. $M = (L \cup (\Sigma^s)^n)^p$. Now $\Delta \Vdash_M \varphi$ iff $\varphi \in (\Delta^n \cup M)^p$ iff $\varphi \in (\Delta^n \cup (\Sigma^s)^n \cup L)^p$ iff $\varphi \in ((\Delta \cup \Sigma^s)^n \cup L)^p$ iff $\Delta; \Xi \Vdash_L \varphi$ for some finite $\Xi \subseteq \Sigma^s$. $\Box$

Of course, Ξ depends on Δ and φ. Therefore, we regard Ξ as a function X mapping sets of formulae to sets of theorems of M. (More precisely, we should construct X as a function from pairs $\langle \Delta, \varphi \rangle$ to sets of formulae. However, in practice it is enough to assume X to be a function that takes the set $\Delta; \varphi$ rather than the pair $\langle \Delta, \varphi \rangle$ as its argument. Nothing of substance is lost.) We can also assume that the variables occurring in $X(\Delta)$ are also variables of Δ. For let σ be a substitution such that $\sigma(p) := p$ if p occurs in Δ or φ and $\sigma(p) := \bot$ else. Then if $\Delta; X(\Delta; \varphi) \Vdash_L \varphi$ then also $\Delta^\sigma; X(\Delta; \varphi)^\sigma \Vdash_L \varphi^\sigma$, which is $\Delta; X(\Delta; \varphi)^\sigma \Vdash_L \varphi$. Since $X(\Delta) \subseteq \Sigma^s$, also $X(\Delta; \varphi)^\sigma \subseteq \Sigma^s$.

Proposition 7 *Let $M = L \oplus \Sigma$. Then there exists a function $X : \wp(\mathcal{F}_\kappa) \to \Sigma^s$ such that*

1. *$X(\Delta; \varphi)$ is finite for any Δ.*
2. *$var[X(\Delta)] \subseteq var[\Delta]$*
3. *$\Delta \Vdash_M \varphi$ iff $\Delta; X(\Delta; \varphi) \Vdash_L \varphi$.*

Such a function is called a ***global reduction function*** *of M to L.*

Definition 8 *Let L be a normal modal logic. L is* ***globally decidable*** *if there is an algorithm computing the answer of the problem '$\Delta \Vdash_L \varphi$'. L has the* ***global finite model property*** *if $\Delta \Vdash_L \varphi$ iff for all finite global L–models $\mathfrak{N}$: if $\mathfrak{N} \models \Delta$ then $\mathfrak{N} \models \varphi$.*

The following is known as Harrop's Theorem.

Proposition 9 *Let L be a a finitely axiomatizable logic. If L has the (global) finite model property is also (globally) decidable.*

Proof. Since L is finitely axiomatizable, its set of theorems is recursively enumerable. Its set of nontheorems is enumerated as follows. Since L is finitely axiomatizable, there is a procedure to check whether a finite Kripke–frame is an L–frame.

Thus the set of L–frames is also recursively enumerable. It is now straightforward to see that the set of local models is r. e. and thus the set of nontheorems. □

For the purpose of the next definition a function from sets to finite sets is **computable** if its restriction to the set of finite sets is a computable function.

Definition 10 *Let M and L be normal modal logics. M is **globally constructively reducible** to L if there exists a **computable** global reduction function of M to L.*

Proposition 11 *Suppose that L is globally decidable and M is globally constructively reducible to L. Then M is globally decidable as well.*

Proof. Let Δ and φ be given. We describe an algorithm for deciding '$\Delta \Vdash_M \varphi$'. First, compute $X(\Delta;\varphi)$ and then compute the answer to the problem '$\Delta; X(\Delta;\varphi) \Vdash_L \varphi$'. This is decidable by assumption. The answer to the latter is the answer to the problem '$\Delta \Vdash_M \varphi$'. □

From this theorem the following can be deduced.

Proposition 12 *Suppose that L is globally decidable. Then there exists a computable global reduction function to any logic contained in L.*

Proof. Let $M \subseteq L$, Δ and φ be given. We have an algorithm deciding whether or not $\Delta \Vdash_L \varphi$. Suppose that $\Delta \Vdash_L \varphi$ does not hold. Then put $X(\Delta;\varphi) := \varnothing$. Then clearly $\Delta; X(\Delta;\varphi) \Vdash_M \varphi$ also does not hold. Suppose now that $\Delta \Vdash_L \varphi$ holds. Then start enumerating all global L–proofs starting with Δ. (This is possible since L is decidable. For a global L–proof is a sequence such that every element is either a member of Δ, which is decidable since Δ is finite, a member of L, which is decidable by the decidability of L, of follows from previous members of the sequence by application of MN or MP.) There is a proof Π for φ in this list. Let $X(\Delta;\varphi)$ be the set of all $\chi \in L$ occurring in Π. Then Π is a proof of φ from $\Delta; X(\Delta;\varphi)$ using MN and MP. Hence $\Delta; X(\Delta;\varphi) \Vdash_M \varphi$. □

Hence, the mere existence of computable global reduction functions is not an exciting fact. More interesting are the upper bounds for the size of such sets. Notice that in the previous theorem M could even be a globally undecidable logic.

Theorem 13 *Let M be an extension of L by means of finitely many variable free formulae. Then M is globally constructively reducible to L.*

Proof. By assumption, $M = L \oplus \Sigma$ for some finite set Σ of constant formulae. Notice that $M = (L \cup \Sigma)^{np} = (L \cup \Sigma^n)^p$, since Σ is closed under substitution. Put

$X(\Delta;\psi) := \Sigma$. Then we have $\Delta \Vdash_M \psi$ iff there is a global proof of ψ from $\Delta \cup M$ iff there is a global proof of ψ from $\Delta \cup L \cup \Sigma$ iff there is a global proof of ψ from $\Delta \cup X(\Delta;\psi) \cup L$ iff $\Delta; X(\Delta;\psi) \Vdash_L \psi$. $\Box$

Notice that the theorem is false if Σ is infinite. For there exists an infinite set C of independent constant formulae over **K**. Let Σ be a nonrecursive subset of C. Then $\mathbf{K} \oplus \Sigma$ is globally undecidable, as is easily seen.

Consider the following functions.

$$\begin{array}{lcl}
X_4(\Delta) & := & \{\Box\chi \to \Box\Box\chi : \Box\chi \in sf[\Delta]\} \\
X_T(\Delta) & := & \{\Box\chi \to \chi : \Box\chi \in sf[\Delta]\} \\
X_B(\Delta) & := & \{\neg\chi \to \Box\neg\Box\chi : \Box\chi \in sf[\Delta]\} \\
X_G(\Delta) & := & \{\neg\Box\chi \to \neg\Box(\chi \vee \neg\Box\chi) : \Box\chi \in sf[\Delta]\} \\
X_{Grz}(\Delta) & := & \{\neg\Box\chi \to \neg\Box(\chi \vee \neg\Box(\chi \to \Box\chi)) : \Box\chi \in sf[\Delta]\} \\
X_{alt1}(\Delta) & := & \{\neg\Box\chi \to \Box\neg\chi : \Box\chi \in sf[\Delta]\}
\end{array}$$

The reader may check that the formulae are indeed axioms. We give proofs of the fact that these are reduction functions to **K**.

Theorem 14 *Suppose that L has the global finite model property and that the class of finite L–frames is closed under replacement of $\lhd$ by its transitive closure. Then X_4 is a global reduction function of $L.4$ to L. In particular, X_4 is a global reduction function of* ***K4*** *to* ***K****.*

Proof. We have to show that

$$\Delta \Vdash_{L.4} \varphi \quad \Leftrightarrow \quad \Delta; X_4(\Delta;\varphi) \Vdash_L \varphi$$

From right to left is straightforward. From left to right, assume $\Delta; X_4(\Delta;\varphi) \Vdash_L \varphi$ is not the case. Then there exists a finite L–model $\langle\mathfrak{F},\beta,x\rangle$, $\mathfrak{F} = \langle F, \lhd\rangle$ such that $\langle\mathfrak{F},\beta\rangle \models \Delta; X_4(\Delta;\varphi)$ but $\langle\mathfrak{F},\beta,x\rangle \nvDash \varphi$. Let $\blacktriangleleft$ be the transitive closure of $\lhd$ and $\mathfrak{F}^4 := \langle\mathfrak{F}, \blacktriangleleft\rangle$. By assumption on L, $\mathfrak{F}^4$ is a L–frame. $\langle F, \blacktriangleleft\rangle$ is transitive; therefore $\mathfrak{F}^4$ is a L.**4**–frame. We show that for all subformulae χ of Δ or φ and all worlds y

$$(\dagger) \qquad \langle F, \blacktriangleleft, \beta, y\rangle \models \chi \quad \Leftrightarrow \quad \langle F, \lhd, \beta, y\rangle \models \chi$$

This then establishes $\langle F, \blacktriangleleft, \beta\rangle \models \Delta$ and $\langle F, \blacktriangleleft, \beta, x\rangle \models \neg\varphi$. We show $(\dagger)$ by induction on χ. For variables there is nothing to show. The steps for $\neg$ and $\wedge$ are straightforward. Now let $\chi = \Box\chi'$. Assume $\langle F, \blacktriangleleft, \beta, y\rangle \nvDash \Box\chi'$. Then there is a z such that $y \blacktriangleleft z$ and $\langle F, \blacktriangleleft, \beta, z\rangle \models \neg\chi'$. By induction hypothesis, $\langle F, \lhd, \beta, z\rangle \models \neg\chi'$. By definition of $\blacktriangleleft$ there is a chain $y = y_0 \lhd y_1 \lhd \ldots \lhd y_n = z$. Now $\langle F, \lhd, \beta, y_{n-1}\rangle \models$

$\neg\Box\chi'$. If $n-1>0$ then $\langle F,\lhd,\beta,y_{n-2}\rangle \models \neg\Box\Box\chi'$. Since $\Box\chi' \to \Box\Box\chi' \in X_4(\Delta;\varphi)$ and $\langle F,\lhd,\beta,y_{n-2}\rangle \models X_4(\Delta;\varphi)$ we must have $\langle F,\lhd,\beta,y_{n-2}\rangle \models \neg\Box\chi'$. Iterating this argument we get $\langle F,\lhd,\beta,y\rangle \models \neg\Box\chi'$. So, $\langle F,\lhd,\beta,y\rangle \not\models \Box\chi'$. Clearly, if $\langle F,\lhd,\beta,y\rangle \not\models \Box\chi'$ then $\langle F,\blacktriangleleft,\beta,y\rangle \not\models \Box\chi'$, since $\lhd\ \subseteq \blacktriangleleft$. □

Theorem 15 *Suppose that L has the global finite model property and that the class of finite L–frames is closed under replacement of $\lhd$ by its reflexive closure. Then X_T is a global reduction function of $L.\boldsymbol{T}$ to L. In particular, X_T is a global reduction function of $\boldsymbol{K.T}$ to $\boldsymbol{K}$.*

Proof. Suppose that we have an M–frame $\mathfrak{F} = \langle F,\lhd\rangle$ and $\langle\mathfrak{F},\beta\rangle \models X_T(\Delta;\varphi)$. Let $\mathfrak{F}^T$ be obtained by replacing $\lhd$ by its reflexive closure, $\blacktriangleleft$. By definition, $\mathfrak{F}^T \models M$ and so $\mathfrak{F}^T \models M.\mathbf{T}$. By induction on the set $sf[\Delta;\varphi]$ we show that for all w in the transit of x

$$\langle\mathfrak{F}^T,\beta,w\rangle \models \chi \qquad \Leftrightarrow \qquad \langle\mathfrak{F},\beta,w\rangle \models \chi$$

The only critical step is $\chi = \Box\tau$. From left to right this follows from the fact that if $x \lhd y$ then also $x \blacktriangleleft y$. For the other direction, assume we have $\langle\mathfrak{F}^T,\beta,w\rangle \not\models \Box\tau$. Then there is a v such that $w \blacktriangleleft v$ and $\langle\mathfrak{F}^T,\beta,v\rangle \models \neg\tau$. If $v \neq w$, we are done for then also $w \lhd v$. So assume the only choice for v is $v = w$ and that $w \not\lhd w$. Then we have $\langle\mathfrak{F},\beta,w\rangle \models \Box\tau$. But $\langle\mathfrak{F},\beta,w\rangle \models \Box\tau \to \tau$, by choice of the reduction function. Hence $\langle\mathfrak{F},\beta,w\rangle \models \tau$, and so by induction hypothesis $\langle\mathfrak{F}^T,\beta,w\rangle \models \tau$, which is a contradiction. So there always is a successor $v \neq w$. □

The proof for **B** is as in Theorem 20, so we will omit it here. Clearly, the reduction functions given above work also for polymodal logics under the conditions stated for the logic and for $\lhd$, where $\lhd$ is replaced by any of the operators. Furthermore, if X_2 is a reduction function from L_2 to L_1, and X_1 is a reduction function from L_1 to L_0 then $X_2 \circ X_1$ is a reduction function from L_2 to L_0. So, reductions may applied in succession. We apply this to the following theorem.

Definition 16 *A κ–modal logic L is called an* $\boldsymbol{RST}$*–logic if $L = \boldsymbol{K} \oplus A$ for some $A \subseteq \{p \to \Diamond_j p, p \to \Box_j\Diamond_j p, \Diamond_j\Diamond_j \to \Diamond_j p : j < \kappa\}$.*

Theorem 17 *Let L be a finitely axiomatizable RST–logic Then L has the global finite model property.*

Proof. Apply the constructive reduction in iteration. Since there are no interaction postulates for the operators, we may add the axioms for each operator independently. For a single operator the result follows from the following observations.

Let R be a relation. If R is reflexive, so is its symmetric closure and also its transitive closure. If R is reflexive and symmetric, so is its transitive closure. $\Box$

This result actually follow from the general transfer results of [8], but the proof offered here is much simpler. Now we turn to **G**, **Grz** and $\mathbf{alt}_1$. Both **G** and **Grz** are transitive logics. We will now show that the functions above establish a reduction from L.**G** to L.**K4** under certain conditions and a reduction from L.**Grz** to L.**S4** under certain analogous conditions. The first result is a generalization of a theorem by Balbiani and Herzig in [1]. To state it in full generality, we shall introduce the notion of a subframe logic. Let $\langle F, R, U\rangle$ be a κ–frame, and $G \in U$. Then put $S(j) := R(j) \cap F^2$, $V := \{A \subseteq F : A \subseteq U\}$. The pair $\langle G, S, V\rangle$ is called a *subframe* of $\mathfrak{F}$. A logic L is a *subframe logic* if its class of frames is closed unter taking subframes. By a well–known theorem of Fine [3], L has the finite model property (but see also the appendix for a proof using constructive reduction). The following is proved without this assumption, however.

Theorem 18 *Let $L \supseteq$ **K4** be a subframe logic with the finite model property. Assume that the class of finite L–Kripke frames is closed under replacement of a reflexive point by an irreflexive point. Then X_G is a global constructive reduction of L.**G** to L. In particular, it is a global constructive construction of **G** to **K4**.*

Proof. Notice that L is transitive, so we only need to consider reductions where the antecedent is identical to $\top$. From this follows the global finite model property of L. Put $\Box^+\varphi := \varphi \wedge \Box\varphi$. Now let $\mathfrak{F}$ be a finite transitive frame and

$$\langle \mathfrak{F}, \beta, w_0\rangle \models \varphi; \Box^+\{\neg\Box\chi \to \neg\Box(\chi \vee \neg\Box\chi) : \Box\chi \in sf(\varphi)\}\ .$$

Now, pick points from the frame as follows. Put $S_0 := \{w_0\}$. The sets S_n are now defined inductively. Let $x \in S_n$ and $\Box\chi \in sf(\varphi)$ such that $\langle \mathfrak{F}, \beta, x\rangle \not\models \Box\chi$, and no successor of x in $S_n - \{w_0\}$ exists such that $\langle \mathfrak{F}, \beta, y\rangle \models \neg\chi$. Then, by assumption on the reduction function, $\langle \mathfrak{F}, \beta, x\rangle \models \Diamond(\neg\chi \wedge \Box\chi)$. Hence there exists an $\widehat{x}$ such that $\widehat{x} \models \neg\chi; \Box\chi$. (Moreover, if $x = w_0$, then $\widehat{x} \neq w_0$. For $\widehat{x}$ is irreflexive, and so $w_0 \lhd \widehat{x}$ implies $w_0 \neq \widehat{x}$.) It follows that $\widehat{x}$ is irreflexive. Put $S_{n+1} := S_n \cup \{\widehat{x}\}$. The selection ends after some steps, since F is finite. Call the resulting set G. Let $\lhd^G := \lhd \cap (G \times G) - \{\langle w_0, w_0\rangle\}$. Then put $\mathfrak{G} := \langle G, \lhd^G\rangle$. (Alternatively, we might simply take $\mathfrak{G}$ to be the subframe consisting of w_0 and all irreflexive points from $\mathfrak{F}$, with the transition $w_0 \to w_0$ being removed.) $\mathfrak{G}$ is transitive and irreflexive, hence it is a frame for **G**. Since L is a subframe logic and closed under changing a reflexive point into an irreflexive point, $\mathfrak{G}$ is also a frame for L. Put $\gamma(p) := \beta(p) \cap G$. We now show that for every subformula ψ of φ and every point

$y \in G$, $\langle \mathfrak{G}, \gamma, y\rangle \models \psi$ iff $\langle \mathfrak{F}, \beta, y\rangle \models \psi$. This holds for variables by construction, and the steps $\neg$, $\wedge$ are straightforward. Now suppose $\langle \mathfrak{G}, \gamma, y\rangle \not\models \Box\chi$. Then also $\langle \mathfrak{F}, \beta, y\rangle \not\models \Box\chi$. Conversely, suppose that $\langle \mathfrak{F}, \beta, y\rangle \not\models \Box\chi$, for some $\Box\chi \in sf(\varphi)$. Then also $\langle \mathfrak{G}, \beta, y\rangle \not\models \Box\chi$, since a successor z for y has been chosen such that $\langle \mathfrak{F}, \beta, z\rangle \models \chi; \Box\neg\chi$. By induction hypothesis, $\langle \mathfrak{G}, \gamma, z\rangle \models \chi$. Moreover, $y \lhd^G z$. For if $y \neq w_0$ this holds by definition of $\lhd^G$. For $y = w_0$ observe that either $w_0 \lhd^F w_0$, and then $z \neq w_0$, since $z \not\lhd z$. From this follows $w_0 \lhd^G z$. Or else, $w_0 \not\lhd^F w_0$, in which case $w_0 \lhd^G z$ anyway. And so $\langle \mathfrak{G}, \gamma, y\rangle \not\models \Box\chi$, as required. $\Box$

Theorem 19 *Let L be a subframe logic containing* $\boldsymbol{S4}$. *Assume that L has the finite model property. Then X_{Grz} is a constructive global reduction function of $L.\boldsymbol{Grz}$ to L.*

Proof. L has the (global) finite model property. Let $\langle \mathfrak{F}, \beta, w_0\rangle$ a finite **S4**–model such that

$$\langle \mathfrak{F}, \beta, w_0\rangle \models \varphi; \Box^+\{\neg\Box\chi \to \neg\Box(\chi \vee \neg\Box(\chi \to \Box\chi)) : \Box\chi \in sf(\varphi)\} .$$

(Here, again $\Box^+\varphi := \varphi \wedge \Box\varphi$.) We select a subset G of F in the following way. We start with the set $S_0 := \{w_0\}$. S_{n+1} is defined inductively as follows. Suppose $x \in S_n$ and $\langle \mathfrak{F}, \beta, x\rangle \models \neg\Box\chi$, but no y exists in S_n such that $x \lhd y$ and $\langle \mathfrak{F}, \beta, y\rangle \models \neg\chi$. We choose a successor y of x such that $y \models \neg\chi; \Box(\chi \to \Box\chi)$ and put $S_{n+1} := S_n \cup \{y\}$. y exists by choice of the reduction function. Now the following holds. (Recall that in **S4**–frames, sets of the form $C(x) := \{y : x \lhd y \lhd x\}$ are called *clusters*. See also Section 8.) (a) The entire cluster $C(y)$ satisfies $\neg\chi$, (b) no point in a cluster succeeding $C(y)$ and different from $C(y)$ satisfies χ. This procedure comes to a halt after finitely many steps. The resulting set is called G, and the subframe based on it $\mathfrak{G}$. It is directly verified that G contains at most one point from each cluster. (Moreover, the selection procedure produces a model whose depth is bounded by the number of formulae in $sf(\varphi)$ of the form $\Box\chi$ as can easily be seen.) So all clusters have size 1. $\mathfrak{G}$ is reflexive and transitive, being a subframe of $\mathfrak{F}$. So, $\mathfrak{G}$ is a **Grz**–frame. Since L is a subframe logic, $\mathfrak{G}$ is a L–frame as well. Let $\gamma(p) := \beta(p) \cap G$. It is shown as in the previous proof that for every subformula χ of φ and every $x \in G$, $\langle \mathfrak{G}, \gamma, x\rangle \models \chi$ exactly when $\langle \mathfrak{F}, \beta, x\rangle \models \chi$. In particular, $\langle \mathfrak{G}, \gamma, w_0\rangle \models \varphi$. This concludes the proof. $\Box$

So far all the axioms have been unimodal. Here now is an example of a bimodal axiom. Axioms of this form are used to axiomatize tense logic. The axiom of symmetry is also of this form (except that there is only one modality, not two). If R is a binary relation, we shall write $R^{\smile}$ to denote the converse of R, i. e. the set $\{\langle y, x\rangle : x\, R\, y\}$.

Theorem 20 *Let L be a bimodal logic that has the global finite model property. Assume that the class of finite L–Kripke frames is closed under passing from $\langle F, \lhd_0, \lhd_1\rangle$ to $\langle F, \lhd_0, \lhd_1 \cup \breve{\lhd_0}\rangle$. Put*

$$X_{01}(\Delta) := \{\neg\chi \to \Box_0\neg\Box_1\chi : \Box_0\chi \in sf[\Delta]\}$$

Then X_{01} globally reduces $L \oplus p \to \Box_0\Diamond_1 p$ to L.

Proof. Let $L(01) := L \oplus p \to \Box_0\Diamond_1 p$. We show that

$$(\ddagger) \qquad \Delta \Vdash_{L(01)} \varphi \quad \Leftrightarrow \quad \varphi; X_{01}(\Delta;\psi) \Vdash_L \varphi$$

Let $\mathfrak{M} = \langle \mathfrak{F}, \beta, w_0\rangle$ be a local model where $\mathfrak{F} = \langle F, \lhd_0, \lhd_1\rangle$ is a finite $\mathbf{K}_2$–frame such that $\langle \mathfrak{F}, \beta\rangle \vDash \Delta; X_{01}(\Delta;\varphi)$ and $\langle \mathfrak{F}, \beta, w_0\rangle \vDash \neg\varphi$. Let $\blacktriangleleft_0 := \lhd_0$ and $\blacktriangleleft_1 := \lhd_1 \cup \breve{\lhd_0}$. Then $\langle F, \blacktriangleleft_0, \blacktriangleleft_1\rangle$ is a $L(01)$–frame, for it is a L–frame by assumption on L; and $\blacktriangleleft_1 \supseteq \breve{\blacktriangleleft_0}$. For all $\chi \in sf[\Delta;\varphi]$ we have

$$(\dagger) \qquad \langle F, \blacktriangleleft_0, \blacktriangleleft_1, \beta, y\rangle \vDash \chi \quad \Leftrightarrow \quad \langle F, \lhd_0, \lhd_1, \beta, y\rangle \vDash \chi\ .$$

This is clear for variables; the steps for $\neg$ and $\wedge$ are straightforward. Likewise the step for $\chi = \Box_0\tau$. Now let $\chi = \Box_1\tau$. From left to right is clear. Now right to left; assume $\langle F, \blacktriangleleft_0, \blacktriangleleft_1, \beta, y\rangle \nvDash \Box_1\tau$. Then there is a w such that $y \blacktriangleleft_1 w$ and $\langle F, \blacktriangleleft_0, \blacktriangleleft_1, \beta, w\rangle \vDash \neg\tau$. By induction hypothesis, $\langle F, \lhd_0, \lhd_1, \beta, w\rangle \vDash \neg\tau$. If $y \lhd_1 w$, we are done; for then $\langle F, \lhd_0, \lhd_1, \beta, y\rangle \nvDash \Box_0\tau\,(= \chi)$. Otherwise $w \lhd_0 y$. Now, $\langle F, \lhd_0, \lhd_1, \beta, w\rangle \vDash \Box_0\neg\Box_1\tau$, since $\langle F, \lhd_0, \lhd_1, \beta, w\rangle \vDash X_{01}(\Delta;\varphi)$. Thus $\langle F, \lhd_0, \lhd_1, y\rangle \vDash \neg\Box_1\tau$. So, $\langle F, \lhd_0, \lhd_1, \beta, y\rangle \nvDash \Box_1\tau$. $\Box$

Before we prove some more results in this vein, let us indicate something about the scope and the limits of this technique. Suppose that we have a postulate that is elementary on all finite frames, and that the condition on finite frames is a universal, positive restricted sentence. That is, it is of the form $\forall x.\alpha(x)$ where $\alpha(x)$ is made from statements $x \lhd_j y$, $j < \kappa$, and $x \doteq y$, using the connectives $\wedge$ and $\vee$ and

$$(\forall y \rhd_j x)\beta := (\forall y)(x \lhd_j y \to \beta)$$

Then α is a Sahlqvist condition and corresponds to a modal formula φ_α. More precisely, if $\alpha = Q\vec{x}.\Phi(\vec{x})$, where Φ is quantifier free, we can define for each disjunct δ of Φ a formula μ_δ such that

$$\varphi_\alpha = \sigma \to \bigvee \mu_\delta$$

Here, σ and the μ_δ are formulae made from of variables using $\wedge$, $\vee$ and $\Diamond_j$, $j < \kappa$. Let α not contain any occurrences of $x \doteq y$. Then we can view the elementary property α as a closure condition on the accessibility relations. If a certain tree can be mapped homomorphically into the frame, then some more relations must hold. The tree is actually defined by the variables of the restricted quantifiers plus the restrictions that apply to them. We call this the **carrier tree** of α. For example, transitivity is $(\forall x)(\forall y \rhd x)(\forall z \rhd y)(x \lhd z)$. So, if the tree $x \lhd y \lhd z$ can be mapped into the frame (not necessarily injectively) then $x \lhd z$ must obtain as well. Likewise for $\mathbf{alt}_1$, which is $(\forall x)(\forall y \rhd x)(\forall z \rhd x)(y \doteq z)$. If $\mathfrak{F}$ and $\mathfrak{G}$ are frame such that $F = G$ and $\lhd_j^{\mathfrak{F}} \subseteq \lhd_j^{\mathfrak{G}}$ for all $j < \kappa$ then we say that $\mathfrak{G}$ is an **arrow extension** of $\mathfrak{F}$. We may define for a frame $\mathfrak{F}$ the set $C^\alpha(\mathfrak{F})$ to be the set of all minimal arrow extensions $\mathfrak{G}$ of $\mathfrak{F}$ such that $\mathfrak{G} \models (\forall x)\alpha(x)$. Then if L has the finite model property and α is elementary on the finite frames, and it holds that for all finite L–frames $\mathfrak{F}$, $C^\alpha(\mathfrak{F})$ is a set of L–frames it seems that $L \oplus \varphi_\alpha$ has the finite model property, and that this can be shown by means of the global reduction sets.

We are not able to provide a proof for that claim, in fact we are not sure whether the claim holds in full generality. However, it might be worthwile to explain the idea of a proof that works quite well in many concrete cases. It is roughly as follows. Take Δ and φ such that φ does not follow globally from Δ in $L \oplus \varphi_\alpha$. Define $Y(\Delta)$ to be the set of those instances of φ_α, where conjunction of subformulae of Δ or their negations are substituted for the variables. This is a finite set, and $var[Y(\Delta)] \subseteq var[\Delta]$. We need to show that if there is a L–model $\mathfrak{M} = \langle \mathfrak{F}, \beta \rangle$ such that $\mathfrak{M} \models \Delta; Y(\varphi; \Delta)$ but $\mathfrak{M} \not\models \varphi$, then we define a closure $\mathfrak{G} \in C^\alpha(\mathfrak{F})$ in the following way. Suppose that the carrier tree of α is embeddable, say $x \mapsto w_x$. Then σ can be made true at the root of the tree. Substitute for the variable p_x the conjunction of all subformulae of $\Delta; \varphi$ that are true at w_x and the conjunction of all negations of such formulae which are false at w_x. (We call this the **atom** of w_x.) By force of $Y(\varphi; \Delta)$ there must be a disjunct μ_δ of φ_α, which is true. It corresponds to a disjunct δ of the matrix of α. Hence, we add those relations $w_x \lhd_j w_y$ for which $x \lhd_j y$ is a conjunct of δ. After having done so, we need to show that for this newly created model $\mathfrak{M}^1$ we have $\mathfrak{M}^1 \models \Delta; Y(\varphi; \Delta)$, but $\mathfrak{M}^1 \not\models \varphi$. (It is here that some more is needed to show this. We do not know how to supply the details in the general case, but for many specific cases it works.) If that is so, we continue the process so long as the α is still not satisfied. When we are done, however, the resulting frame is a member of $C^\alpha(\mathfrak{F})$ and so a model for $L \oplus \varphi_\alpha$, by assumption. Hence, it defines a global model form Δ in which φ is not satisfied.

A case in point are in addition to **4**, **T**, **X(01)** and **B** also the formulae $(\forall x)(\forall y \rhd_i x)(x \lhd_j y)$, corresponding to $\Diamond_i p \to \Diamond_j p$. Furthermore, we may add to a κ–modal logic a new modality whose relation includes a given set $S \subseteq \kappa$ of relations and that is either transitive (we call it an S–*master*) or an equivalence relation (then it is an S–universal modality, see [6]). Constructive reduction can be applied there as well. Notice that the commutation axioms $\Diamond_i\Diamond_j p \rightarrow \Diamond_j\Diamond_i p$ do not fall into this class since they do not correspond to a universal formula. The just mentioned axiom corresponds to $(\forall x)(\forall y \rhd_i y)(\forall z \rhd_j y)(\exists u \rhd_j x)(u \lhd_i z)$. Indeed, it is known that the logic of three commuting **S5**–modalities is undecidable (this follows from results in Gabbay and Shehtman [4]). Hence, since the logic $\mathbf{S5} \otimes \mathbf{S5} \otimes \mathbf{S5}$ is an RST–logic and therefore has the finite model property, it must be the commutation axioms that lead to undecidability.

Another technique, that works especially well with subframe axioms is that of dropping points from a model. In fact, we show in the Appending to this paper that such a procedure can be used to show that all subframe logics containing **K4** have the finite model property. This constitutes a constructive proof of the theorem by Fine ([3]). The proof is in fact quite involved, but not more complex than the original one. Likewise, it is possible to show that $\mathbf{alt}_n$, if added to a subframe logic with the (global) finite model property then the resulting logic has the finite model property again and is a subframe logic. It is well known that all subframe logics containing $\mathbf{K}.\mathbf{alt}_n$ have the local finite model property. Our result is a strengthening for some logics. However, these techniques need to be handled with care. We give the following negative example. Consider the monomodal logic corresponding to the following first–order properties:

$$(\forall x, y, z \rhd w)(x \doteq y \vee x \doteq z \vee y \doteq z), \qquad (\forall x, y \rhd w)(\forall z \rhd w)(z \rhd x \vee z \rhd y \vee x \doteq y)$$

This logic is an extension of $\mathbf{K}.\mathbf{alt}_2$. Its frame are such that any point has at most two direct successors and at most three two–step successors. By encoding a tiling problem, [15] has shown that this logic is globally undecidable. This shows that we cannot strengthen the methods to all subframe logics.

4 Complexity

In this section we will discuss upper bounds on complexity of modal logics that can be derived using the reduction functions. We will mainly deal with the global complexity, since this is the easiest point of attack. From the global complexity, one can also establish results on the local complexity (using the gl–reduction of next section), but they tend to be far worse than the known bounds. For the results on modal logic complexity we refer here generally to [15] and [2] and references therein, though some specific references are also given below.

Before we start, we have to discuss the notion of the length of a formula. Standardly, the length of a formula is the number of symbols occurring in it. This length is denoted here by $|\varphi|$. It is defined inductively as follows.

$$\begin{array}{lcl} |\bot| & := & 1 \\ |p_i| & := & 1 \\ |\neg\varphi| & := & 1 + |\varphi| \\ |\varphi \wedge \psi| & := & 1 + |\varphi| + |\psi| \\ |\Box_j\varphi| & := & 1 + |\varphi| \end{array}$$

(The variable p_i counts as one symbol, even though it would be more appropriate to code the index by a bit sequence. However, we shall ignore this detail here.) For a set Δ, put $|\Delta| := \sum_{\varphi\in\Delta} |\varphi|$. The symbol count is however not always the most economical way of writing down a formula, especially when Δ contains few subformulae. This is the case with the formula sets we are dealing with here. Therefore, we shall work with a different measure, in line with [2]. Namely, for a set Δ of formulae we put

$$\sharp(\Delta) := card(sf[\Delta])$$

To get acquainted with this measure, we shall note a few of its properties. The following is proved by induction.

Lemma 21 *Suppose that no formula occurs more than once in Δ. Then $|\Delta| = \sharp(\Delta)$.*

Proposition 22 $\log_2(|\Delta| + 1) \leq \sharp(\Delta) \leq |\Delta|$. *These bounds are sharp.*

Proof. Clearly, $\sharp(\Delta) \leq |\Delta|$. Now suppose that Δ is a set of formulae such that every variable and $\bot$ occur at most once. Then it is not hard to see that each formula occurs in Δ at most once, and in this case $|\Delta| = \sharp(\Delta)$ by the previous lemma. Now we turn to the other inequality. It is clear that the maximum depth of the formulae in Δ (in terms of their tree structure) provides a lower bound on the number of subformulae, since no two formulae of different depth can be equal. A formula of depth n can have up to $2^n - 1$ symbols, as is easily shown. So, $n \geq \log_2(|\Delta| - 1)$. Now define the following formulae.

$$\begin{array}{lcl} \chi_1 & := & \bot \\ \chi_{n+1} & := & \chi_n \wedge \chi_n \end{array}$$

Then $\sharp(\chi_n) = n$, but $|\chi_n| = 2^n - 1$. Thus, $\log_2(|\chi_n| + 1) = n = \sharp(\chi_n)$. ☐

Proposition 23 *The following holds.*

1. *If* $\varphi \wedge \varphi \in \Delta$ *then* $\sharp(\Delta; \varphi \wedge \psi) = \sharp(\Delta; \varphi; \psi)$.

2. *If* $*\varphi \in \Delta$, $* \in \{\Box_j : j < \kappa\} \cup \{\neg\}$, *then* $\sharp(\Delta; *\varphi) = \sharp(\Delta; \varphi)$.

3. $\sharp(\varphi \to \psi) = \sharp(\neg(\varphi \wedge \neg\psi)) = 3 + \sharp(\varphi; \psi)$.

4. $\sharp(\varphi \leftrightarrow \psi) = \sharp((\varphi \to \psi) \wedge (\psi \to \varphi)) = 7 + \sharp(\varphi; \psi)$.

5. $\sharp(\Delta; \Delta') = \sharp(\Delta) + \sharp(\Delta') - card(sf[\Delta] \cap sf[\Delta'])$.

These claims are easy to prove, and we shall make tacit use of them in sequel. We note now the following.

Proposition 24 *Let Y be any of the reduction functions of the previous section. Then for any set Δ, $\sharp(Y(\Delta)) \leq c_Y \sharp(\Delta)$, where $c_Y > 0$ is a constant depending only on Y.*

We verify this with X_4. Take Δ. For any $\Box\chi \in sf[\Delta]$, $X_4(\Delta)$ contains the additional formula $\Box\chi \to \Box\Box\chi$, which is in fact $\neg(\Box\chi \wedge \neg\Box\Box\chi)$. Now

$$sf(\Box\chi \to \Box\Box\chi) = sf(\chi) \cup \{\neg(\Box\chi \wedge \neg\Box\Box\chi), \Box\chi \wedge \neg\Box\Box\chi, \neg\Box\Box\chi, \Box\Box\chi, \Box\chi\}$$

It follows that for each subformula of Δ, $X(\Delta)$ adds 5 more formulae. Hence $\sharp(X_4(\Delta)) \leq 6\sharp(\Delta)$. (The reader may verify that $|X_4(\Delta)|$ is quadratic in $|\Delta|$.)

Definition 25 *Let X is a function from finite sets of formulae to finite sets of formulae. We say that X is* ***linear (polynomial, exponential)*** *if $\sharp(X(\Delta)) \leq f(\sharp(\Delta))$ for some linear (polynomial, exponential) function $f : \omega \to \omega$.*

Definition 26 *A logic L is* ***globally NP (PSPACE, EXPTIME)*** *if there exists a nondeterministic algorithm taking time polynomial in $\sharp(\Delta; \varphi)$ (a deterministic algorithm taking space polynomial/exponential in $\sharp(\Delta; \varphi)$) which computes the answer to the problem '$\Delta \Vdash_L \varphi$'.*

Similarly, L is **globally NP–hard** (globally **PSPACE–hard** etc.) if some problem that is NP–complete (PSPACE–complete etc.) can be polynomially reduced to a problem of the form '$\Delta \Vdash_L \varphi$'.

Now suppose that we have a logic L which is Q–hard, where Q is any of the three complexity classes. Let M be globally reducible to L by a linear reduction function X. Then the problem '$\Delta \Vdash_M \varphi$' is equivalent to the problem '$\Delta; X(\Delta; \varphi) \Vdash_L \varphi$'. The latter takes $O(\sharp(\Delta; \varphi; X(\Delta; \varphi))) = O(\Delta; \varphi)$ time (space). Hence, the space and time complexity does not rise.

Theorem 27 *Suppose that X is a linear global reduction function from L to M. Then if M is globally* ***NP (PSPACE, EXPTIME)****, so is L.*

This method only gives an upper bound. In general, L can have much lower complexity than M. A trivial example is the inconsistent logic. A more instructive example is the following. **S5** is globally linearly reducible to **K**, by the results of the preceding section. Yet, **K** is globally EXPTIME–complete while **S5** is NP–complete.

Corollary 28 *The systems* ***KB, KT, KBT, K4, G, S4, Grz*** *are globally in* ***EXPTIME****. Moreover, tense logic and RST–logics are in* ***EXPTIME****.*

This is for the transitive logics mentioned in the theorem not the best possible result. They are all PSPACE–complete. However, such a result would follow by the same techniques, if only it is established that **K4** is PSPACE–complete. This last statement, however, has to be established independently. There is a possibility to show this, namely using tableaux. It is known that **K** has a tableau calculus (for local consistency) in which the branches have a linear length. Using this tableau calculus one can establish the same property for **K4**. This means that validity can be checked in polynomial space. The reduction sets can be used to show this. We will not go into the details here. Suffice it to say that the length of branches in a tableau can be bounded from above by a polynomial function (see [14] for some tableau calculi for modal logics).

Now consider again the case where M is globally linearly reducible to L. Let L' be a logic such that $L \subseteq L' \subsetneq M$. Then M is globally linearly reducible to L' as well. Hence, the complexity class of M is also bounded from above by the complexity class of L'. It follows that the complexity class of M is the minimum over all complexity classes over the logics containing L and properly contained in M. Moreover, if M is C–hard for some complexity class C, then all logics in the interval $[L, M]$ are C–hard. This allows to establish lower bounds for entire intervals of logics, analogous to the result of Ladner ([10]) that all logics in $[\mathbf{K}, \mathbf{S4}]$ are locally PSPACE–hard. Unfortunately, using our methods we obtain the same lower bound for global complexity of these logics, since **S4** is globally PSPACE–complete.

In analogy to Chagrov and Zakharyaschev we define the global complexity function of a logic as follows.

Definition 29 *Let M be a modal logic with the global finite model property. Let f_M be the following function.* $f_M(n) = max\{\mu(\Delta; \varphi) : \sharp(\Delta; \varphi) \leq n\}$*, where* $\mu(\Delta; \varphi)$ *is*

0 *if* $\Delta \Vdash_M \varphi$ *and else it is the least number p such that there exists a global model* $\mathfrak{N}$ *based on p worlds such that* $\mathfrak{N} \models \Delta$ *and* $\mathfrak{N} \nvDash \varphi$.

The local complexity function is defined analogously, where the maximum is taken only over $\mu(\varnothing; \varphi)$. Chagrov and Zakharyaschev show that the local complexity function can be extremely complex, for example the k–fold iteration of the exponential function. However, here we are interested only in whether the complexity function is polynomial or exponential. The following is now clear.

Theorem 30 *Suppose that L has a polynomial (exponential) global complexity function and that M is globally reducible to L by means of a linear complexity function. Then M has a polynomial (exponential) global compexity function as well.*

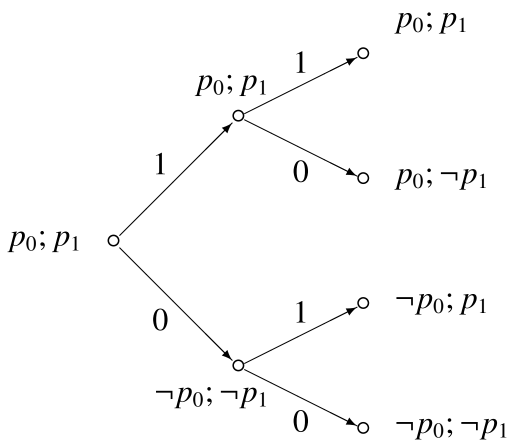
Proof. Let f_L be the global complexity function of L and X the reduction function of M to L. Let p be defined by $p(n) := max\{\sharp(X(\Delta)) : \sharp(\Delta) \leq n\}$. Suppose that $\Delta \nVdash_M \varphi$ and $\sharp(\Delta; \varphi) \leq n$. Then $\Delta; X(\Delta; \varphi) \nVdash_L \varphi$. By assumption there is a model of size $\leq f_L(\sharp(\Delta; \varphi; X(\Delta; \varphi)) \leq f_L(p(n) + 1)$. So, $f_M := f_L(p(n) + 1)$ is a global complexity function for M. If f_L and p are both polynomial (exponential), so is f_M. ☐

Again, these results establish only upper bounds. **K** does not have a polynomial local complexity function and hence also not a polynomial global complexity function as we shall show below. However, for **S5** the global complexity function is polynomial. Incidentally, we may use a result of Chagrov and Zakharyaschev to derive a lower bound for the global complexity function for **K**. Namely, it is shown that for the local complexity function for **K4**, f_4, it holds that $\log_2 f_4(n)$ is linear in the limit. Hence the local complexity function is exponential. It is an easy matter to show that the global complexity function, g_4, is then also exponential. For $\Delta \Vdash_{\mathbf{K4}} \varphi$ iff $\Delta; \Box\Delta \vdash_{\mathbf{K4}} \varphi$, where $g_4(n) \leq f_4(2n)$. Moreover, $f_4(n) \leq g_4(n)$. Now, since **K4** can be reduced globally to **K**, we obtain that the global complexity function for **K** cannot be less than exponential. We denote by $k_\sharp$ the global complexity function of $\mathbf{K}_\kappa$, and k_g the local complexity function.

Lemma 31 $k_\sharp(n) \leq 2^n$.

Proof. Let $\Delta \nVdash \varphi$, Δ finite. Then by Theorem 42 there exists a global model $\mathfrak{N} = \langle \mathfrak{F}, \beta \rangle$ based on a Kripke–frame $\mathfrak{F}$ such that $\mathfrak{N} \models \Delta$ but $\mathfrak{N} \nvDash \varphi$. Now filtrate this model with respect to the set $A := sf[\Delta; \varphi]$. That means the following. Write $x \approx_A y$ if $\langle \mathfrak{F}, \beta, x \rangle \models \chi$ is equivalent to $\langle \mathfrak{F}, \beta, y \rangle \models \chi$ for all $\chi \in A$. Now let $[x]_A := \{y : x \approx_A y\}$. Put $[x]_A \lhd_j [y]_A$ iff there exist $x' \approx_A x$ and $y' \approx_A y$ such that

Figure 1: The branching Tree



$x' \lhd_j y'$. This defines the frame $\mathfrak{F}/A$. Next we put $\beta/A(p) := \{[x]_A : x \in \beta(p)\}$. Then it is shown by induction that $\langle \mathfrak{F}/A, \beta/A, [x]_A\rangle \models \chi$ iff $\langle \mathfrak{F}, \beta, x\rangle \models \chi$ for all $\chi \in A$. Clearly, $\mathfrak{F}/A$ has at most 2^n elements, where $n = \sharp(\Delta; \varphi)$. $\square$

Theorem 32 *The logics $\boldsymbol{K}_\kappa$ have the global finite model property and are globally decidable. Furthermore, the following holds in the limit:*

1. $2^{n/16} < k_\sharp(n) \leq 2^n$.
2. $2^{\sqrt{n}} < k_g(n) \leq 2^n$.

Proof. Let m be a natural number. Consider the formulae

$$\varphi_i := \neg\Box^{m-i}\bot. \leftrightarrow .\neg\Box p_i \wedge \neg\Box\neg p_i$$

We claim that

$$(\dagger) \qquad \Box^{m+1}\bot; \{\varphi_i : i < m\} \nVdash \Box^m\bot \vee \Diamond^{m-1}\Box\bot$$

A countermodel is constructed as follows. Let F be the set of at most m–long sequences from $\{0, 1\}$. Put $\vec{x} \lhd \vec{y}$ iff $\vec{y} = \vec{x}a$ for $a \in \{0, 1\}$. This defines the frame $\mathfrak{F}$.

It is rooted at ε (the empty sequence) and has $2^{m+1} - 1$ elements. Furthermore, let $\beta(p_i)$ consist of those $\vec{x}$ such that either (a) $\vec{x}$ has length at least $i+1$ and $\vec{x} = \vec{u}1\vec{v}$ for some $\vec{u}$ of length i, or (b) $\vec{x} = \varepsilon$, or (c) $\vec{x} \neq \varepsilon$, $\vec{x}$ has length $\leq i$ and $\vec{x} = \vec{u}1$ for some $\vec{u}$. (Figure 1 illustrates the model for $m = 2$.) Then $\langle \mathfrak{F}, \beta, \varepsilon \rangle \nvDash \Diamond^{m-1}\Box\bot; \Box^m\bot$. However, $\langle \mathfrak{F}, \beta \rangle \vDash \Box^{m+1}\bot$. Further, if $\vec{x} \vDash \Diamond^{m-i}\top$, then $\vec{x}$ has length at most i. Hence $\vec{x}0 \vDash \neg p_i$ and $\vec{x}1 \vDash p_i$, from which follows $\vec{x} \vDash \Diamond p_i \wedge \Diamond\neg p_i$. If however $\vec{x} \nvDash \Diamond^{m-i+1}\top$ then $\vec{x}$ has length at least $i + 1$, and then either (a) $\vec{x} = \vec{u}0\vec{v}$ for some $\vec{u}$ of length i or (b) $\vec{x} = \vec{u}1\vec{v}$ for some $\vec{u}$ of length i. Then $\vec{x} \vDash \Box\neg p_i$ in Case (a) and $\vec{x} \vDash \Box p_i$ in Case (b), as is easily seen. Hence $\langle \mathfrak{F}, \beta \rangle \vDash \varphi_i$ for every $i < m$. This establishes (†).

Now we show that any model witnessing (†) must have at least $2^{m+1} - 1$ points. Suppose that $\langle \mathfrak{G}, \gamma \rangle$ is such a model. Then $\langle \mathfrak{G}, \gamma \rangle \vDash \Delta$, where $\Delta := \Box^{m+1}\bot; \{\varphi_i : < m\}$ and $\langle \mathfrak{G}, \gamma \rangle \nvDash \Box^m\bot \vee \Diamond^{m-1}\Box\bot$. Then there is an $x \in G$ satisfying $\Diamond^m\top \wedge \Box^{m-1}\Diamond\top$. Furthermore, $x \vDash \Box^{m+1}\bot$, so any maximal path from x has length m, and there exists a path of length m. Now, consider a point x of depth $m - i + 1$. Then $x \vDash \Diamond^{m-i}\top$. If p_i holds at x, it will hold at every successor of x; and if $\neg p_i$ holds, it holds at every successor of x. However, p_{i-1} holds at one successor and fails at another. So, x has at least two successors. It follows easily that G has least $1 + 2 + 2^2 + 2^n = 2^{m+1} - 1$ points.

This allows to prove the upper complexity bound. Let n be given. Then

$$\begin{array}{rcl} \sharp(\varphi_i) & = & \sharp(\neg\Box^{m-i}\top \leftrightarrow (\neg\Box p_i \wedge \neg\Box\neg p_i)) \\ & = & 7 + \sharp(\neg\Box^{m-i}\bot; \neg\Box p_i \wedge \neg\Box\neg p_i) \\ & = & 14 + \sharp(\neg\Box^{m-i}\bot) \\ & = & 16 + (m - i) \end{array}$$

Put $\Delta := \Box^{m+1}\bot; \{\varphi_i : i < n\}$. Then

$$\begin{array}{rcl} \sharp(\Delta; \neg(\Box^m\bot \wedge \Box^{m-1}\neg\Box\bot)) & = & 14m + \sharp(\Box^{m+1}\bot; \neg(\Box^m\bot \wedge \Box^{m-1}\neg\Box\bot)) - \sharp(\neg\Box\bot; \Box^m\bot) \\ & = & 14m + ((m + 2) + (2m + 1)) - (3 + m) \\ & = & 16m \end{array}$$

Hence, with $n := 16m$ we get the desired lower bound. For there exist Δ and φ such that $\sharp(\Delta; \varphi) = n$, $\Delta \nVdash \varphi$, and the smallest model witnessing this is of size $2^{m+1} - 1 > 2^{n/16}$.

Now derive the local complexity bounds we take a modified version of the previous example. We have

$$\Box^m\bot; \{\Box^j\varphi_i : i, j < m\} \nvdash \Box^m\top \vee \Diamond^{m-1}\Box\bot$$

The model constructed earlier serves here as well. Furthermore, a countermodel has at least $2^{m+1} - 1$ points, by the same argument. However, the number of subformulae is now $10m + m(m-1) = m^2 + 9m = (m + 9/2)^2 - (9/2)^2$. This gives the lower bound, assuming that $n = m^2 + 9m$. $\Box$

As we have remarked with the complexity class, these results can be used to derive bounds for the size of models for entire intervals of logics. We will give an example. The logic $\mathbf{K.D.alt}_1$ has a linear local complexity function, as is easily seen. However, the global complexity function is not even polynomial. For consider the following formulae

$$\gamma_n(p) := p \leftrightarrow \left(\Box^n p \wedge \bigwedge_{0<i<n} \Box^i \neg p\right)$$

$$\begin{array}{rcl} \sharp(\gamma_n(p)) & = & 7 + \sharp(p; \Box^n p \wedge \bigwedge_{0<i<n} \Box^i \neg p)) \\ & = & 7 + n + \sharp(\bigwedge_{0<i<n} \Box^i \neg p) \\ & = & 6 + 2n + \sharp(\{\Box^i \neg p : 0 < i < n\}) \\ & = & 6 + 2n + \sharp(\Box^{n-1} \neg p) \\ & = & 7 + 3n \end{array}$$

Now let q_i, $i < s$, be the first n prime numbers. Put $\Delta_n := \{\gamma_{q_i}(p_i) : i < n\} \cup \{\Box p_i \leftrightarrow \neg\Box\neg p_i : i < n\}$.

Lemma 33 *Let L be a modal logic in the interval $[\boldsymbol{K}, \boldsymbol{K.D.alt}_1]$. Then $\neg\Box\bot; \Delta_n$ is globally L–consistent. Moreover, any L–model witnessing this has size $\geq \prod_{i<n} q_i$.*

Proof. Suppose that $\langle \mathfrak{F}, \beta \rangle$ is a global L–model such that $\langle \mathfrak{F}, \beta \rangle \models \neg\Box\bot; \Delta_n$. Let $\langle w_i : i < \omega \rangle$ be a sequence of points such that $w_i \lhd w_{i+1}$ for all $i \in \omega$. Such sequence exists, by the fact that the model satisfies $\neg\Box\bot$. Then it follows by choice of the formulae $\gamma_{q_i}(p_i)$ that (1) $w_k \in \beta(q_i)$ iff $w_{k+q_i} \in \beta(q_i)$, (2) $w_k \in \beta(q_i)$ iff $w_{k+s} \notin \beta(q_i)$ for all $s < q_i$. It follows easily that there is a k such that $w_k \models p_i$ for all $i < n$. Then the smallest number $k' > k$ such that $w_{k'} \models p_i$ for all $i < n$ is $k + \prod_{i<n} q_i$. This is independent of the chosen sequence. It follows that all w_j for $k \leq j < k'$ must be distinct. So, there are at least $\prod_{i<n} q_i$ many points in this frame. It is easy to construct a $\mathbf{K.D.alt}_1$–model with exactly that many points. This shows the global consistency of this set. $\Box$

Let $g(n)$ be the product of all primes $\leq n$, and let $L(n) := \sharp(\Delta_n; \neg\Box\bot)$. Since the formulae in Δ_n use pairwise distinct variables, we have

$$\begin{array}{rcl} L(n) & = & \sum_{i<n} \sharp(\gamma_{q_i}(p_i)) + 3 + 7n \\ & = & 7n + \sum_{i<n}(8 + 3q_i) \\ & = & 15n + 3\sum_{i<n} q_i \end{array}$$

Then $f(n) := g(L^{-1}(n))$ is the function measuring the size of the models for these formulae in terms of their length. So, f is a lower bound on the global complexity of $\mathbf{K.D.alt}_1$. We prove that f grows faster than any polynomial. The proof uses some number theory.

To give a lower bound for f, we will establish lower bounds for L^{-1} and g. This is sufficient, since g is monotonously increasing. First we will deal with L^{-1}. Using the asymptotic formula $q_n \sim n \log n$ for the nth prime number, where log is the logarithm to the base e, we get that $\sum_{i<n} q_i$ is asymptotically equal to $\frac{n^2}{2 \log n}$. (See [9].) Namely, taking the integral $\int_1^{n/\log n} x \log x dx$ we get $x^2 \log x/2 - x/2\Big|_1^{n/\log n}$, which is asymptotically equal to $\frac{n^2}{2\log n}$. The latter is eventually $< \frac{n^2}{c}$, where c is any given positive real number. This allows to conclude that asymptotically $L(n) < n^2/c$ for any given $c > 0$. Putting $n = L^{-1}(m)$, we get $m < L^{-1}(m)^2/c$ from which $\sqrt{mc} < L^{-1}(m)$. Hence, changing m back to n, $L^{-1}(n) > \sqrt{cL(n)}$.

Now $\prod_{p \leq n} p \sim e^n$. (Namely, $\vartheta(n)$ is defined to be the sum of the $\log p$, where p is a prime number $\leq n$. It can be shown that $\vartheta(n) \sim n$. For [9] on Page 108, Theorem 5.16 gives $\pi(n) \sim \frac{\vartheta(n)}{\log n}$ and on Page 112, we have $\pi(n) \sim \frac{n}{\log n}$, showing $\frac{\vartheta(n)}{\log n} \sim \frac{n}{\log n}$.) Hence, for any $\varepsilon > 0$, $\prod_{p \leq n} p$ is eventually larger than $e^{(1-\varepsilon)n}$.

Now, given positive real numbers c and ε we get $f(n) = g(L^{-1}(n)) > e^{\sqrt{cL(n)}(1-\varepsilon)}$. Since we may choose $c^2/(1-\varepsilon)$ rather than c we have that eventually also $f(n) > e^{c\sqrt{n}}$. Now, from Lemma 33 we immediately the following result.

Theorem 34 *Let $\boldsymbol{K} \subseteq L \subseteq \boldsymbol{K.D.alt}_1$. Then the global complexity function of L is in the limit at least $2^{c\sqrt{n}}$ for any given c.*

Let U be a set of worlds in a frame. Let

$$\begin{array}{lcl} T(U) & := & U \cup \{y : x \lhd_j y, x \in U, j < \kappa\} \\ T^{n+1}(U) & := & T(T^n(U)) \end{array}$$

$T^n(U)$ is the set of all points reachable in at most n steps from a point in U. Now say that a frame $\mathfrak{F}$ is of **depth** δ if there is a point x such that $T^\delta(\{x\})$ is the entire set of worlds from $\mathfrak{F}$.

Definition 35 *Let M be a modal logic with the global finite model property. Let f_M be the following function. $f_M(n) = max\{\lambda(\Delta;\varphi) : \sharp(\Delta;\varphi) \leq n\}$, where $\lambda(\Delta;\varphi)$ is 0 if $\Delta \Vdash_M \varphi$ and else it is the least number δ such that there exists a global model $\mathfrak{N}$ of depth δ such that $\mathfrak{N} \models \Delta$ and $\mathfrak{N} \not\models \varphi$.*

The local complexity function is defined analogously, where the maximum is taken only over $\lambda(\varnothing;\varphi)$.

Theorem 36 *Let d_g and $d_\sharp$ be the global and local depth complexity functions of $\boldsymbol{K}_\kappa$. Then the following holds asymptotically, for all $c > 0$:*

1. $2^{c\sqrt{n}} < d_g(n) < 2^n$.
2. $d_\sharp(n) = n - 2$.

The local complexity bound is rather trivial to establish, using the fact that the depth of the models can be chosen to be at most the modal depth of the formulae, which in turn is at most half of the number of subformulae. The upper bound is reached by $\neg\Box^{n-1}\bot; \Box^n\bot$. This formula has length $n + 2$ and needs a model of depth exactly n. The global results follow from (the proof of) Theorem 34 and the fact that the depth complexity function never exceeds the global complexity function. It is clear that the theorem does not reveal mucht about transitive logics; here the depth complexity is ≤ 2. Finally, it is clear that there is an analogue of Theorem 30 with respect to the depth functions.

5 The Reduction from Global to Local

Now let $\vdash$ and $\vdash'$ be two consequence relations. We call X a **reduction function from $\vdash$ to $\vdash'$** if

1. $X(\Delta)$ is finite,
2. $var[X(\Delta)] \subseteq var[\Delta]$,
3. $X(\Delta) \subseteq \mathsf{Taut}(\vdash)$,
4. $\Delta \vdash \varphi$ iff $\Delta; X(\Delta; \varphi) \vdash' \varphi$.

In the previous section we have discussed the case where $\vdash$ and $\vdash'$ are both global consequence relations. Analogously we can reduce the local consequence relation of M to the local consequence relation of L.

Now notice that there is the following connection between local and global consequence relations.

Proposition 37 $\Delta \Vdash_L \varphi$ *iff for some compound modality $\boxplus$: $\boxplus\Delta \vdash_L \varphi$.*

Moreover, we can simplify the choices for $\boxplus$ somewhat. Define for finite κ:

$$\begin{array}{lcl} \Box^0\varphi & := & \varphi \\ \Box^1\varphi & := & \bigwedge_{j<\kappa}\varphi \\ \Box^{k+1}\varphi & := & \Box^1\Box^k\varphi \\ \Box^{\leq k}\varphi & := & \bigwedge_{i\leq k}\Box^i\varphi \end{array}$$

Definition 38 *Assume that κ is finite. Let f be a function from finite sets of formulae to ω. f is a **global–to–local reduction function** or **gl–reduction function** for L if for all finite sets Δ and φ*

$$\Delta \Vdash_L \varphi \qquad \Leftrightarrow \qquad \Box^{\leq k}\Delta \vdash_L \varphi$$

where $k := f(\Delta;\varphi)$.

The following theorem is easy to show.

Proposition 39 *Suppose that L is locally decidable and that there is a computable gl–reduction function for L. Then L is globally decidable.*

To put it negatively: if a logic is locally decidable but globally undecidable, then no computable gl–reduction function for L exists. [15] has proved the existence of such logics. A logic is weakly transitive if it has a theorem of the form $\Box^k p \to \Box^{k+1} p$. Clearly, weakly transitive logics have computable reduction functions: simply put $f(\Delta) := k$. Furthermore, observe the following.

Proposition 40 *Suppose that M has a (computable) global reduction function to L and that L has a (computable) gl–reduction function. Then M has a (computable) gl–reduction function.*

Proof. Let X be a reduction function from M to L, and f a gl–reduction function for L. Then

$$\begin{array}{rcl} \Delta \Vdash_M \varphi & \Rightarrow & \Delta; X(\Delta;\varphi) \Vdash_L \varphi \\ & \Rightarrow & \Box^{\leq k}\Delta; \Box^{\leq k}X(\Delta;\varphi) \vdash_L \varphi \\ & \Rightarrow & \Box^{\leq k}\Delta; \Box^{\leq k}X(\Delta;\varphi) \vdash_M \varphi \\ & \Rightarrow & \Box^{\leq k}\Delta \vdash_M \varphi \end{array}$$

where $k := f(\Delta;\varphi;X(\Delta;\varphi))$. Hence put $g(\Delta) := f(\Delta;X(\Delta))$. This is a gl–reduction function for M. If both f and X are computable, then so is g. □

A logic M is called **weakly transitive** if there exists a compound modality $\boxtimes$ such that $\boxtimes p \to p$, $\boxtimes p \to \boxtimes\boxtimes p$, $\boxtimes p \to \Box_j p$ and $\boxtimes p \to \Box_j \boxtimes p$ are theorems of M for all modal operators $\Box_j$. We call $\boxtimes$ a **master modality**. If M is weakly transitive with master modality $\boxtimes$ then

$$\Delta \Vdash_M \varphi \qquad \Leftrightarrow \qquad \boxtimes\Delta \vdash_M \varphi$$

Moreover, $\sharp(\boxtimes\Delta) \leq c \cdot \sharp(\Delta)$ for some constant c. Hence, the global derivability is linearly recodable as a local derivability problem. So we have the

Proposition 41 *Suppose that M is weakly transitive. Then there exists a linear gl–reduction function for M.*

It follows that in the weakly transitive case the local and the global complexity coincide. For we generally have

$$\Delta \vdash_M \varphi \qquad \Leftrightarrow \qquad \Vdash_M \bigwedge \Delta \rightarrow \varphi$$

Moreover, $\sharp(\bigwedge \Delta \rightarrow \varphi) \leq 2 + 2\sharp(\Delta; \varphi)$. So, the local derivability problem is linearly recodable into a global derivability problem.

We will discuss a special case to solve, and that is the gl–reduction of $\mathbf{K}_\kappa$. In [7] the following is proved. (We reproduce the proof here.)

Theorem 42 *Let Δ be a finite set of formulae and φ a formula. Put $k := 2^{\sharp(\Delta)+\sharp(\varphi)}$. Then*

$$\Delta \Vdash_{\mathbf{K}_\kappa} \varphi \qquad \Leftrightarrow \qquad \Box^{\leq k}\Delta \vdash_{\mathbf{K}_\kappa} \varphi$$

Proof. Assume $\Box^{\leq k}\varphi \nvdash_{\mathbf{K}_\kappa} \psi$. Then there exists a finite model $\langle \mathfrak{F}, \beta, w_0 \rangle \models \Box^{\leq k}\varphi; \neg\psi$. Moreover, we may assume that $\mathfrak{F}$ is cycle–free, and that between any pair of points there exists at most one path. Let $\Delta := sf(\varphi) \cup sf(\psi)$ and put $S(y) = \{\chi \in \Delta : \langle \mathfrak{F}, \beta, y \rangle \models \chi\}$. Let G be the set of all y in F such that along any path from w_0 to y there are no two distinct points v and w such that $S(v) = S(w)$. Then any path from w_0 to $y \in G$ has length $\leq k$, because there are at most k subsets of Δ. Now define $\blacktriangleleft_j$ on G as follows. $y \blacktriangleleft_j z$ iff (1) $y \lhd_j z$ or (2) for some $u \notin G$ we have $y \lhd_j u$ and $S(z) = S(u)$. Put $\gamma(p) := \beta(p) \cap G$. We will now show that for every $y \in G$ and $\chi \in \Delta$

$$\langle \mathfrak{G}, \gamma, y \rangle \models \chi \quad \Leftrightarrow \quad \langle \mathfrak{F}, \beta, y \rangle \models \chi \; .$$

This is true for variables by construction. The steps for negation and conjunction are clear. Now let $\chi = \Diamond_j\delta$. If $\langle \mathfrak{F}, \beta, y \rangle \models \Diamond_j\delta$ then for some z such that $y \lhd_j z$ we have $\langle \mathfrak{F}, \beta, z \rangle \models \delta$. There are two cases. **Case 1.** $z \in G$. Then by induction hypothesis, $\langle \mathfrak{G}, \gamma, z\rangle \models \delta$. From this we conclude $\langle \mathfrak{G}, \gamma, y\rangle \models \Diamond_j\delta$, since $y \blacktriangleleft_j z$. **Case 2.** $z \notin G$. Then there is a $u \in G$ such that $S(u) = S(z)$. Therefore, by construction of $\mathfrak{G}$, $y \blacktriangleleft_j u$. Furthermore, $\langle \mathfrak{F}, \beta, u\rangle \models \delta$ by definition of $S(-)$. So, $\langle \mathfrak{G}, \gamma, u\rangle \models \delta$ by induction hypothesis. From this follows $\langle \mathfrak{G}, \gamma, y\rangle \models \Diamond_j\delta$, since $y \blacktriangleleft_j u$. This exhausts the two cases. Now suppose $\langle \mathfrak{G}, \gamma, y\rangle \models \Diamond_j\delta$. Then $\langle \mathfrak{G}, \gamma, z\rangle \models \delta$ for some z such that $y \blacktriangleleft_j z$. By induction hypothesis, $\langle \mathfrak{F}, \beta, z\rangle \models \delta$. If $y \lhd_j z$, then also $\langle \mathfrak{F}, \beta, y\rangle \models \Diamond_j\delta$. If, however, $y \not\lhd_j z$, then there is a u such that $y \lhd_j u$ and $S(u) = S(z)$. By definition of $S(-)$, $\langle \mathfrak{F}, \beta, u\rangle \models \delta$, from which $\langle \mathfrak{F}, \beta, y\rangle \models \Diamond_j\delta$ as well. Now since from w_0 there is always a path of length $\leq 2^k$ to any point $y \in G$, we have $\langle \mathfrak{F}, \beta, y\rangle \models \varphi$ for all $y \in G$, and so $\langle \mathfrak{G}, \beta, y\rangle \models \varphi$ for all y. Consequently, $\langle \mathfrak{G}, \beta, w_0\rangle \models \Box^{\omega}\varphi; \neg\psi$, as required. □

6 Interpolation and Beth Theorems

Definition 43 *A modal logic L has* **local interpolation** *if for every pair φ and ψ of formulae with $\varphi \vdash_L \psi$ there is a χ such that $var(\chi) \subseteq var(\varphi) \cap var(\psi)$ and $\varphi \vdash_L \chi$ as well as $\chi \vdash_L \psi$. L has* **global interpolation** *if for every pair φ, ψ of formulae with $\varphi \Vdash_L \psi$ there is a χ such that $var(\chi) \subseteq var(\varphi) \cap var(\psi)$ and $\varphi \Vdash_L \chi$ as well as $\chi \Vdash_L \psi$.*

Since we have a deduction theorem for local deducibility, we can reformulate local interpolation in such a way that it depends only on the set of theorems. L has the **Craig Interpolation Property** if whenever $\varphi \to \psi \in L$ there exists a χ which is based on the common variables of φ and ψ such that $\varphi \to \chi; \chi \to \psi \in L$. A logic has the Craig Interpolation Property iff it has local interpolation.

Proposition 44 *If L has local interpolation it also has global interpolation.*

Proof. Suppose that L has local interpolation. Let $\varphi \Vdash_L \psi$. Then for some compound modality $\boxtimes$ we have $\boxtimes\varphi \vdash_L \psi$. Whence by local interpolation there is a χ with $var(\chi) \subseteq var(\varphi) \cap var(\psi)$ such that $\boxtimes\varphi \vdash_L \chi$ and $\chi \vdash_L \psi$. Hence $\varphi \Vdash_L \chi$ as well as $\chi \Vdash_L \psi$. □

The converse implication does not hold, as has been shown in [12]. Interpolation is closely connected with the so–called *Beth property*. It says, in intuitive terms, that if we have defined p implicitly, then there also is an explicit definition of p. An *explicit definition* is a statement of the form $\chi \leftrightarrow p$ where $p \notin var(\chi)$. An *implicit definition* is a formula $\psi(p, \vec{q})$, such that the value of p in a model is uniquely defined by the values of the variables $\vec{q}$. The latter can be reformulated syntactically. Given a consequence relation $\vdash$, we say that $\varphi(p, \vec{q})$ **implicitly defines** p (in $\vdash$) if $\varphi(p, \vec{q}); \varphi(r, \vec{q}) \vdash p \leftrightarrow r$. Given L, we may choose $\vdash$ to be either $\vdash_L$ or $\Vdash_L$. This gives rise to the notions of **local** and **global** implicit definitions.

Definition 45 *L is said to have the* ***local Beth Property*** *if the following holds. Suppose $\varphi(p, \vec{q})$ is a formula and*

$$\varphi(p, \vec{q}); \varphi(r, \vec{q}) \vdash_L p \leftrightarrow r.$$

Then there exists a formula $\chi(\vec{q})$ not containing p as a variable such that

$$\varphi(p, \vec{q}) \vdash_L p \leftrightarrow \chi(\vec{q}) \ .$$

Analogously, the ***global Beth property*** *is defined by replacing $\vdash_L$ by $\Vdash_L$.*

The lack of the deduction theorem for the global consequence makes the global Beth property somewhat more difficult to handle than the local equivalent. For the local Beth property we can actually prove that it is equivalent to the Craig Interpolation Property. The following two results were shown in [12].

Theorem 46 (Maksimova) *Let L be a modal logic. Then L has local interpolation iff it has the local Beth property.*

Theorem 47 (Maksimova) *A modal logic with local interpolation also has the global Beth–property.*

Proof. Assume that $\varphi(p, \vec{q}); \varphi(r, \vec{q}) \Vdash_L p \leftrightarrow r$. Then for some compound modality $\boxtimes$ we have

$$\boxtimes\varphi(p, \vec{q}); \boxtimes\varphi(r, \vec{q}) \vdash_L p \leftrightarrow r.$$

This can now be rearranged to

$$\boxtimes\varphi(p, \vec{q}); p \vdash_L \boxtimes\varphi(r, \vec{q}) \rightarrow r.$$

We get an interpolant $\chi(\vec{q})$ and so we have $\boxtimes\varphi(p, \vec{q}); p \vdash_L \chi(\vec{q})$, from which $\boxtimes\varphi(p, \vec{q}) \vdash_L p \rightarrow \chi(\vec{q})$. So $\varphi(p, \vec{q}) \vdash_L p \rightarrow \chi(\vec{q})$. And we have $\chi(\vec{q}) \vdash_L \boxtimes\varphi(r, \vec{q}) \rightarrow r$ from which we get $\boxtimes\varphi(r, \vec{q}) \vdash_L \chi(\vec{q}) \rightarrow r$, and so $\varphi(r, \vec{q}) \vdash_L \chi(\vec{q}) \rightarrow r$. Replacing r by p we get the desired result. □

The picture obtained thus far is the following.

$$\begin{tikzcd}
\text{local interpolation} \arrow[r] \arrow[d, shift left] & \text{global interpolation} \\
\text{local Beth Property} \arrow[r] \arrow[u, shift left] & \text{global Beth Property}
\end{tikzcd}$$

It can be shown that there exist logics without global interpolation while having the global Beth Property and that there exist logics with global interpolation without the global Beth Property. An example of the first kind is the logic **G.3**. (See [11].)

Definition 48 *Let L be a modal logic. L is* ***locally Halldén–complete*** *if whenever $\varphi \vdash_L \psi$ and $var(\varphi) \cap var(\psi) = \varnothing$ we have $\varphi \vdash_L \bot$ or $\vdash_L \psi$. L is* ***globally Halldén–complete*** *if whenever $\varphi \Vdash_L \psi$ and $var(\varphi) \cap var(\psi) = \varnothing$ we have $\varphi \Vdash_L \bot$ or $\Vdash_L \psi$.*

Global Halldén–completeness is called the **Pseudo Relevance Property** in [13]. In the literature, a logic L is called *Halldén–complete* if for φ and ψ disjoint in variables, if $\varphi \vee \psi \in L$ then also $\varphi \in L$ or $\psi \in L$. Clearly, this latter notion of Halldén–completeness coincides with local Halldén–completeness. This follows from the deduction theorem, since $\varphi \vdash_L \psi$ is equivalent to $\vdash_L \neg\varphi \vee \psi$. If a logic is locally Halldén–complete it is also globally Halldén–complete. For if $\varphi \Vdash_L \psi$ then $\boxplus\varphi \vdash_L \psi$. If φ and ψ have no variable in common, this holds for $\boxplus\varphi$ and ψ as well. Hence, $\varphi \vdash_L \bot$ or else $\vdash_L \psi$, from which we get $\varphi \Vdash_L \bot$ or $\Vdash_L \psi$. Further, if L logic is locally Halldén–complete then any constant formula must be equivalent to $\bot$ or $\top$. This is the case, for example, when all modal operators are reflexive.

Proposition 49 *Suppose that L is a logic such that every constant formula is locally equivalent to $\bot$ or $\top$. Then if L has local (global) interpolation, it is locally (globally) Halldén–complete.*

For a proof note that if $\varphi \vdash_L \psi$ with φ and ψ disjoint in variables, then there exists a constant formula χ such that $\varphi \vdash_L \chi$ and $\psi \vdash_L \psi$. By assumption on L, χ is equivalent to $\top$ or $\bot$. If the first holds then $\top \vdash_L \psi$ and if the second holds then $\varphi \vdash_L \psi$. Similarly for the global case.

Finally, we will establish some criteria for interpolation. Assume that we have logics L and L' with $L \subseteq L$ and global reduction sets for L' with respect to L. Let us say that the reduction sets **split** if there exists a reduction function X such that $X(\varphi; \psi) = X(\varphi \to \psi) = X(\varphi) \cup X(\psi)$.

Theorem 50 *Suppose that L' can be globally reduced to L with splitting reduction sets. Then L' has local (global) interpolation if L has local (global) interpolation. Moreover, L' is locally (globally) Halldén–complete if L is.*

Proof. Assume $\varphi \vdash_{L'} \psi$. Using the deduction theorem we get $\vdash_{L'} \varphi \to \psi$. Then $\Vdash_{L'} \varphi \to \psi$. By global reduction we get $X(\varphi \to \psi) \Vdash_L \varphi \to \psi$ and so for some compound modality $\boxtimes$

$$\boxplus X(\varphi \to \psi) \vdash_L \varphi \to \psi \ .$$

This is the same as

$$\boxplus X(\varphi); \boxplus X(\psi) \vdash_L \varphi \to \psi \ ,$$

by the assumption that the reduction sets split. We can rearrange this into

$$\boxplus X(\varphi); \varphi \vdash_L \boxplus X(\psi) \to \psi.$$

(We allow ourselves to write $\boxplus X(\psi)$ in place of $\bigwedge \boxplus X(\psi)$.) By assumption on X, $var[X(\varphi)] \subseteq var(\varphi)$ and $var[X(\psi)] \subseteq var(\psi)$. By local interpolation for L we obtain a τ in the common variables of φ and ψ such that

$$\varphi; \boxplus X(\varphi) \vdash_L \tau \vdash_L \boxplus X(\psi) \to \psi.$$

From this follows that $\varphi \vdash_{L'} \tau \vdash_{L'} \psi$, by the fact that the reduction sets only contain instances of theorems. Moreover, $var(\tau) \subseteq var(\varphi) \cap var(\psi)$. Pushing up global interpolation works essentially in the same way. Now, for Halldén–completeness, assume that $\varphi \vdash_{L'} \psi$ for φ and ψ disjoint in variables. Then

$$\varphi; \boxplus X(\varphi) \vdash_L \boxplus X(\psi) \to \psi.$$

The left hand side is disjoint in variables from the right hand side, and so either the left hand side is inconsistent or the right hand side is a theorem. In the first case, $\varphi \vdash_{L'} \bot$. In the second case $\vdash_{L'} \psi$, as required. The proof for global Halldén–completeness is analogous. □

We conclude from that the following theorem.

Corollary 51 *The monomodal logics* $\boldsymbol{K.alt}_1$, $\boldsymbol{K4}$, $\boldsymbol{K.B}$, $\boldsymbol{K.T}$, $\boldsymbol{K.BT}$, $\boldsymbol{S4}$, $\boldsymbol{S5}$, $\boldsymbol{G}$ *and* $\boldsymbol{Grz}$ *have local interpolation. Moreover,* $\boldsymbol{K.T}$, $\boldsymbol{K.BT}$, $\boldsymbol{S4}$, $\boldsymbol{S5}$ *and* $\boldsymbol{Grz}$ *are Halldén–complete.*

This is so since the reduction functions given earlier split, as an easy inspection reveals. The second claim follows from Proposition 49. The following has been observed first in [14]

Corollary 52 (Rautenberg) *Let L have local (global) interpolation and let Ψ be a set of constant formulae. Then $L \oplus \Psi$ has local (global) interpolation.*

By considerations analogous to Theorem 13, $X_\Psi(\Delta; \psi) := \Psi$ is a global reduction function. Obviously, X splits. An analogous theorem holds for local and global Halldén–completeness; however it is of no use. For if a logic is Halldén–complete, any constant formula is equivalent to either $\bot$ or $\top$.

We close with a remark on interpolation. Say that L has *constructible local interpolants* if for given φ and ψ such that $\varphi \vdash_L \psi$ we can construct a local interpolant χ. It can be shown that if M can be locally constructively reduced to L and L has constructible local interpolants, then this holds for M as well. Similarly for global interpolation. [1]

7 Conclusion

We have introduced the notion of a reduction function and shown how to use reduction functions to prove certain standard and some new results about decidability, finite model property, interpolation etc. of logical systems. All methods are constructive: the reduction functions are shown to be constructible if the logic is decidable. Moreover, if we are given reduction function, we assume that the base logic L has the finite model property and construct an M–model on the basis of a suitably defined L–model. The only handicap of the method is the fact that we use a global reduction function instead of a local one, even though we show also that the global consequence relation is reducible in the same way to the local reducibility. Yet, the bounds obtained in this way for the time and space complexity are sometimes far off the mark. However, our results are — so we think — only the beginning. It is conceivable that they can be improved to establish the bounds known from the literature.

8 Appendix: Subframe Logics containing K4

In this appendix, all frames and logics are assumed to be transitive. A subframe $\mathfrak{G}$ is **cofinal** in $\mathfrak{F}$ if every point in the subframe generated by $\mathfrak{G}$ in $\mathfrak{F}$ is covered by a point of $\mathfrak{G}$. That means, if $x \in G$ and $x \lhd y$ then either $y \in G$ or else $y \lhd z$ for some $z \in G$. A logic is called a **cofinal subframe logic** if its class of frames is closed

[1] I owe this observation to Oliver Kutz.

under taking cofinal subframes. Obviously, a subframe logic is a cofinal subframe logic; the converse need not hold, e. g. **S4.2**. Examples of subframe logics are **S4**, **S5**, **G**, **Grz**, **K4.3** and many more.

We will prove in this appendix the following theorem, proved in [3] for subframe logics and in [16] for cofinal subframe logics.

Theorem 53 (Fine, Zakharyaschev) *Every cofinal subframe logic containing* ***K4*** *has the finite model property.*

For subframe logics this is due to Kit Fine; the generalization to cofinal subframe logics (as well as Fine's result independently) has been obtained by Michael Zakharyaschev. Before we enter the proof let us introduce some useful terminology and draw important consequences. If L is a logic and $\mathfrak{F}$ rooted and finite then denote by $L/_{\mathrm{F}}\,\mathfrak{F}$ the smallest subframe logic containing L not having $\mathfrak{F}$ as a frame, and call $L/_{\mathrm{F}}\,\mathfrak{F}$ the **Fine–splitting** of L by $\mathfrak{F}$. It turns out that $L/_{\mathrm{F}}\,\mathfrak{F} = L \oplus C_{\mathfrak{F}}$ where $C_{\mathfrak{F}} = \Box^{+} SF(\mathfrak{F}). \to .\neg p_o$ with o a root

$$\begin{aligned} SF(\mathfrak{F}) := \quad & \bigwedge\langle p_x \to \neg p_y : x \neq y\rangle \\ & \wedge \bigwedge\langle p_x \to \Diamond p_y : x \lhd y\rangle \\ & \wedge \bigwedge\langle p_x \to \neg\Diamond p_y : x \ntriangleleft y\rangle \end{aligned}$$

Any subframe logic L is a Fine–splitting $\mathbf{K4}/_{\mathrm{F}}\, G$ with $G = \{\mathfrak{F} : \mathfrak{F} \notin Fr(L), \mathfrak{G} \text{ rooted}\}$. Analogously, a **Zakharyaschev–splitting** of L by $\mathfrak{F}$ is the least cofinal subframe logic containing L for which $\mathfrak{F}$ is not a frame. It is axiomatizable by $L \oplus \Box^{+} CSF(\mathfrak{F}). \to \neg p_o$. We write $L/_{\mathrm{Z}}\,\mathfrak{F}$.

$$\begin{aligned} CSF(\mathfrak{F}) := \quad & \bigwedge\langle p_x \to \neg p_y : x \neq y\rangle \\ & \wedge \bigwedge\langle p_x \to \Diamond p_y : x \lhd y\rangle \\ & \wedge \bigwedge\langle p_x \to \neg\Diamond p_y : x \ntriangleleft y\rangle \\ & \wedge \Box\Diamond \bigvee\langle p_x : x \in f\rangle \end{aligned}$$

Put $\mathfrak{F} \prec_F \mathfrak{G}$ if $\mathfrak{G}$ is a p–morphic image of a subframe of $\mathfrak{F}$, and $\mathfrak{F} \prec_Z \mathfrak{G}$ if $\mathfrak{G}$ is a p–morphic image of a cofinal subframe of $\mathfrak{F}$. Then $\prec_Z \subseteq \prec_F$. If $\mathfrak{F}$ is a **K4**–frame and $x, y \in f$ then y is called a **weak successor** of x, in symbols $x \trianglelefteq y$, if $x \lhd y$ or $x = y$. y is a **strong successor** of x, in symbols $x \vec{\lhd} y$, if $x \lhd x$ but $y \ntriangleleft x$. The cluster $C(x)$ of x is the set of weak successors which are not strong, that is the set of y such that $x \trianglelefteq y \trianglelefteq x$. The depth of a point x, $d(x)$, is the maximum number n such that there is a sequence $x = x_0 \vec{\lhd} x_1 \vec{\lhd} \ldots \vec{\lhd} x_{n-2} \vec{\lhd} x_n$. In the sequel we fix a formula φ, and let $S := sf(\varphi)$. Let $\mathfrak{M} = \langle \mathfrak{F}, \beta\rangle$ be a model. We make the crucial assumption from now on that the $\beta(p)$, $p \in var(\varphi)$, generate the algebra of sets of

$\mathfrak{F}$. We say in that case that the model is φ–**refined**. In a model, the **characteristic set** of a point x, $X(x)$, is the set of all formulae of $S = sf(\varphi)$ true at x. The **atom** or **characteristic formula** is

$$At(x) := \bigwedge_{\psi \in X(x)} \psi \wedge \bigwedge_{\psi \notin X(x)} \neg\psi$$

The set of all conjunctions of this type, i. e. the set of all characteristic formulae, is denoted by $A(\varphi)$ or simply A. Call $x \in F$ **maximal** (with respect to φ) in $\mathfrak{M}$ if no strong successor of x has the same characteristic set (or formula) as x. The subframe of all maximal points of $\mathfrak{M} = \langle \mathfrak{F}, \beta \rangle$ is denoted by $\mathfrak{F}^\mu$. In a finite model, every point x has a weak successor which is maximal for the atom of x. This weak successor is denoted by x^μ. Then $x = x^\mu$ iff x is maximal (by φ–refinement). Hence, $\mathfrak{F}^\mu$ is actually cofinal in $\mathfrak{F}$. It is useful to observe that if $x \lhd y^\mu$ and $x \models \Diamond At(y^\mu)$ then there is a weak successor x^μ such that $x^\mu \lhd y^\mu$. We now have the following

Proposition 54 *Let $\mathfrak{F}^\mu \subseteq \mathfrak{G} \subseteq \mathfrak{F}$ and $x \in G$. Then for all $\varphi \in S$:*

$$\langle \mathfrak{F}, \beta, x \rangle \models \varphi \qquad \Leftrightarrow \qquad \langle \mathfrak{G}, \beta, x \rangle \models \varphi$$

The proof of this theorem is straightforward. As a consequence, there is no distinction whether we choose maximal points in $\mathfrak{F}$ or in $\mathfrak{G}$. They will be the same set. And so x^μ does not depend on $\mathfrak{G}$ as long as $G \supseteq F^\mu$. Let $\mathfrak{R}$ be the refinement of the frame $\mathfrak{F}^\mu$, where the internal sets are those generated by $\beta(p_i)$, $p_i \in var(\varphi)$. (That is to say, we consider the general frame based on $\mathfrak{F}^\mu$, in which the algebra of sets is generated by the values of the variables p_i.) Write β for the valuation induced on $\mathfrak{R}$. Let $d^\mu(x)$ be the depth of x in $\mathfrak{R}$.

Fact 55 *For all $x \in G$, $d^\mu(x)$ is the maximum number n such that there is a chain $\langle x_i : i \in n \rangle$ with $x_0 = x$, $x_i \vec{\lhd} x_{i+1}$, and $x_{i+1} \models \neg At(x_i) \wedge \neg\Diamond At(x_i)$, $i \in n-1$. By consequence, $d^\mu(x) \leq 2^{\sharp S}$.*

Proof. This is seen by first noting that if there is a chain $\langle x_i : i \in n \rangle$ such that $x_{i+1} \models \neg At(x_i) \wedge \neg\Diamond At(x_i)$, then $x_{i+1} \ntriangleleft x_i$, and, starting with x_{n-1}, one can successively replace the x_i by a maximal weak successor x_i^μ so that $x_i^\mu \vec{\lhd} x_{i+1}^\mu$. Conversely, if there is a chain $\langle x_i : i \in n \rangle$ of maximal points such that $x_i \vec{\lhd} x_{i+1}$ then $x_{i+1} \models \neg At(x_i) \wedge \neg\Diamond At(x_i)$. So there is a chain of points with $x_{i+1} \models \neg At(x_i) \wedge \neg\Diamond At(x_i)$ iff there is a chain of maximal points of the same length satisfying $x_i \vec{\lhd} x_{i+1}$ iff $d^\mu(x_0) \geq n$. ☐

For $x \in F$ define

$$\begin{array}{lll} suc^+(x) & := & \{y \in F^\mu : x \vec{\lhd} y\}, \\ cl(x) & := & \{y \in F^\mu : x \lhd y \lhd x\}, \\ CL(x) & := & \{At(y) : y \in cl(x), y \text{ maximal}\} \end{array}$$

By induction on $d^\mu(x)$ we will now define formulae ε_x, λ_x. The formulae ε_x will encode the structure of the refined submodel of maximal points. The formulas λ_x define the layers of that model, that is, the set of all points y in the submodel of maximal points with $d^\mu(y) < d^\mu(x)$. The induction starts with $d^\mu(x) = -1$, where there is nothing to do, except to let $\lambda_{-1} := \bot$. Now let $d^\mu(x) := d + 1$ with $d \geq -1$. (We write $\Diamond^+\varphi$ for $\varphi \vee \Diamond\varphi$ and $\Box^+\varphi$ for $\varphi \wedge \Box\varphi$.)

$$\begin{array}{lll} \lambda_x & := & \zeta_d \\ \alpha_x & := & At(x) \wedge \neg\lambda_x \wedge \Box(\lambda_x \to \neg\Diamond^+ At(x)) \\ \beta_x & := & \begin{cases} \Box\neg At(x) & \text{if } cl(x) = \emptyset \\ \quad \bigwedge\langle \Diamond\alpha : \alpha \in CL(x)\rangle & \\ \wedge \bigwedge\langle \Box(\alpha_1 \to \Diamond\alpha_2) : \alpha_1, \alpha_2 \in CL(x)\rangle & \\ \wedge \bigwedge\langle \Box(\alpha \to \Diamond^+(\alpha \wedge \lambda_x)) : \alpha \in A(\varphi) - CL(x)\rangle & \text{otherwise} \end{cases} \\ \gamma_x & := & \begin{cases} \quad \bigwedge\langle \Diamond\varepsilon_y \wedge \Box(\Diamond^+ At(x) \to \Diamond\varepsilon_y) : y \in suc^+(x)\rangle \\ \wedge \bigwedge\langle \Box\neg\varepsilon_y : y \notin suc^+(x), d^\mu(y) \leq d\rangle \\ \wedge \bigwedge\langle \Box(\Box\neg\varepsilon_y \to \bigvee_{\alpha \in A(\varphi) - CL(x)} \Diamond^+(\alpha \wedge \lambda_x)) : y \in suc^+(x)\rangle \end{cases} \\ \varepsilon_x & := & \alpha_x \wedge \beta_x \wedge \gamma_x \\ \zeta_{d+1} := & := & \zeta_d \vee \bigvee\langle \varepsilon_x : d^\mu(x) = d + 1\rangle \end{array}$$

Define $SUC^+(x) := \{\varepsilon_y : y \in suc^+(x)\}$. Then if $\equiv$ denotes equivalence in **K** it is calculated that $\varepsilon_x \equiv \varepsilon_y$ iff either (α) $SUC^+(x) = SUC^+(y)$, or (β) $CL(x) = CL(y)$ and $At(x) = At(y)$. Define the frame $\mathfrak{Des} = \langle \delta, \lhd\rangle$ with $\delta = \{\varepsilon_x/\!\equiv\ : x \in F\}$ and $\varepsilon_x \lhd \varepsilon_y$ iff either $\varepsilon_y \in SUC^+(x)$ or $SUC^+(x) = SUC^+(y)$, $CL(x) = CL(y)$ and $At(y) \in CL(x)$. (Henceforth we will not distinguish between ε_x and its equivalence class $\varepsilon_x/\!\equiv$.) Our aim is to show that $\mathfrak{Des}$ is nothing but $\mathfrak{N}$, and that the natural valuation on $\mathfrak{Des}$ is β. The first lemma shows that the definition of the ε_s is *sound* for the maximal points:

Lemma 56 *Let $\mathfrak{F}^\mu \subseteq \mathfrak{G} \subseteq \mathfrak{F}$ and $x \in \mathfrak{G}^\mu$. Then $x \in f^\mu$ and $\langle \mathfrak{G}, \beta, s\rangle \models \varepsilon_x$.*

Proof. If $x \in \mathfrak{G}^\mu$, then its maximal successor x^μ is in $\mathfrak{G}$, since $\mathfrak{G} \supseteq \mathfrak{F}^\mu$. Hence, $x = x^\mu$, since x is maximal in $\mathfrak{G}$ and it follows that x is maximal in $\mathfrak{F}$ as well. By induction on $d := d^\mu(x)$ we show

(‡) $\quad \langle \mathfrak{G}, \beta, x\rangle \models \varepsilon_x; \quad$ moreover, if $\langle \mathfrak{G}, \beta, x\rangle \models \lambda_d$ then $d^\mu(x) \leq d$.

To begin with $d^\mu(x) = -1$, there is nothing to show. Thus let $d^\mu(x) = d + 1$. The proof is broken down into four parts:

(i) $x \models \alpha_x$

This is so because $y \models \lambda_x$ implies $y^\mu \models \lambda_x (= \zeta_d)$ by Proposition 54, from which $d^\mu(y^\mu) \leq d$. But no maximal successor of x of depth $\leq d$ can satisfy $At(x)$ or $\Diamond At(x)$.

(ii) $x \models \beta_x$

The case $CL(x) = \emptyset$ is straightforward. Let therefore $CL(x) \neq \emptyset$. By definition of $CL(x)$ and the fact that $G \supseteq F^\mu$ we get $x \models \Diamond\alpha_1; \Box(\alpha_1 \to \Diamond\alpha_2)$ for all $\alpha_1, \alpha_2 \in CL(x)$. Also $x \models \Box(\widehat{\alpha}. \to .\Diamond^+(\widehat{\alpha} \wedge \lambda_x))$ for $\widehat{\alpha} \notin CL(x)$, for if for a successor y: $y \models \widehat{\alpha}$, then $y^\mu \models \widehat{\alpha}$ and by induction hypothesis and the fact that $d^\mu(y^\mu) \leq d$, $y^\mu \models \lambda_x$. Thus if $y = y^\mu$ we have $y \models \lambda_x$ and if $y \lhd y^\mu$ we have $y^\mu \models \Diamond(\widehat{\alpha} \wedge \lambda_x)$.

(iii) $x \models \gamma_x$

$x \models \Diamond\varepsilon_y \wedge \Box(\Diamond^+ At(x). \to .\Diamond\varepsilon_y)$ for all $\varepsilon_y \in SUC^+(x)$ by the fact that x is $At(x)$–maximal and $x \models \Diamond\varepsilon_y$ for all $\varepsilon_y \in SUC^+(x)$. Furthermore, $x \models \neg\Diamond\varepsilon_y$ for all $y \notin suc^+(s)$ and $d^\mu(y) \leq d$. Finally, suppose for $x \lhd z$ that $z \models \neg\Diamond\varepsilon_y$ for some $y \in suc^+(x)$. By definition of depth, $d^\mu(z^\mu) \leq d$. Hence, by induction hypothesis $z^\mu \models \lambda_x$. If $z = z^\mu$ then $z \models \lambda_x \wedge \alpha$, if $z \lhd z^\mu$ then $z \models \Diamond(\alpha \wedge \lambda_x)$ for $\alpha = At(z) \in A$. And so

$$z \models \zeta := \bigvee \langle \Diamond^+(\alpha \wedge \lambda_x) : \alpha \in A\rangle$$

from which $x \models \Box(\Box\neg\varepsilon_y \to \zeta)$. This shows (iii). We have shown that $x \models \varepsilon_x$.

(iv) Now suppose $y \models \lambda_{d+1}$. If also $y \models \lambda_d$ then $d^\mu(y) \leq d$, by induction hypothesis. Hence let $y \models \neg\lambda_d$. Then $y \models \varepsilon_z$ for some maximal z with $d^\mu(z) = d+1$ and so $y \models \Diamond\varepsilon_x$ for some maximal x with $d^\mu(x) = d$. So, $d^\mu(y) > d$. But $y \models \beta_z$, implying that if $y \lhd v \models \Box^+\neg At(y)$, then $At(v) \notin CL(z)$ and so $v \models \Diamond^+(At(v) \wedge \lambda_d)$. If $v \models \lambda_d$, $d^\mu(v) \leq d$; but if $v \models \Diamond(At(v) \wedge \lambda_d)$ then $v^\mu \models \lambda_d$ and so $d^\mu(v) = d^\mu(v^\mu) \leq d$ as well. This proves $d^\mu(y) = d + 1$. $\Box$

Lemma 57 *For all* $x, y \in F$*:*

$$\begin{array}{llcl} (a) & \varepsilon_x \equiv \varepsilon_y & \Leftrightarrow & \vdash_{\mathbf{K4}} \varepsilon_x \leftrightarrow \varepsilon_y \\ (b) & \varepsilon_x \not\equiv \varepsilon_y & \Leftrightarrow & \vdash_{\mathbf{K4}} \varepsilon_x \to \neg\varepsilon_y \\ (c) & \varepsilon_x \lhd \varepsilon_y & \Leftrightarrow & \vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond\varepsilon_y \\ (d) & \varepsilon_x \not\lhd \varepsilon_y & \Leftrightarrow & \vdash_{\mathbf{K4}} \varepsilon_x \to \neg\Diamond\varepsilon_y \end{array}$$

Proof. It is enough to show only ($\Rightarrow$) in each case. (a) holds by definition of $\equiv$ and ε_x.

(b) If $SUC^+(x) \neq SUC^+(y)$, for example $\varepsilon_w \in SUC^+(x) - SUC^+(y)$, then $\vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond\varepsilon_w; \varepsilon_y \to \neg\Diamond\varepsilon_w$, whence $\vdash_{\mathbf{K4}} \varepsilon_x \to \neg\varepsilon_y$; likewise for $\varepsilon_w \in SUC^+(y) - SUC^+(x)$. Let us now suppose $SUC^+(x) = SUC^+(y)$. If $At(x) \neq At(y)$, the case is clear. Thus if $At(x) = At(y)$, we must have $CL(x) \neq CL(y)$. Without loss of generality we can assume that $\alpha \in CL(x) - CL(y)$. We have $\vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond\alpha; \varepsilon_x \to \Box(\Diamond^+ At(x) \to \neg\lambda_x)$ (by $\vdash_{\mathbf{K4}} \alpha_x \to \Box(\Diamond^+ At(x). \to .\neg\lambda_x)$ and $\vdash_{\mathbf{K4}} \varepsilon_x \to \alpha_x$). Furthermore, from $\vdash_{\mathbf{K4}} \varepsilon_x \to \Box(\alpha \to \Diamond At(x))$ we get $\vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond(\alpha \wedge \neg\lambda_x)$. Also, $\varepsilon_x \vdash_{\mathbf{K4}} \Box(\alpha \to \Diamond At(x))$ (this is a conjunct of β_x), and $\varepsilon \vdash_{\mathbf{K4}} \Box(\Diamond At(x) \to \neg\lambda_x)$ (this follows from the last conjunct of α_x). Together this gives $\varepsilon_x \vdash_{\mathbf{K4}} \Box(\alpha \to \neg\lambda_x)$. But

$$\vdash_{\mathbf{K4}} \varepsilon_y \to \Box(\alpha \to \Diamond^+(\alpha \wedge \lambda_y)) .$$

By definition of β_y and since $SUC^+(x) = SUC^+(y)$ we have $\lambda_x \equiv \lambda_y$. Consequently $\vdash_{\mathbf{K4}} \varepsilon_y \to \Box(\alpha \to \Diamond^+(\alpha \wedge \lambda_x))$. Now

$$\varepsilon_x \wedge \varepsilon_y \vdash_{\mathbf{K4}} \Diamond(\alpha \wedge \neg\lambda_x); \Box(\alpha \to \neg\lambda_x); \Box(\alpha \to \Diamond^+(\alpha \wedge \lambda_x)) \vdash_{\mathbf{K4}} \bot.$$

(The last two formulae give $\Box(\alpha \to \Diamond^+(\alpha \wedge \lambda_x \wedge \neg\lambda_x))$, by transitivity; therefore, $\Box(\alpha \to \Box^+\bot)$ is derivable, which gives $\Box\neg\alpha$. $\Diamond\alpha$ is derivable from the first formula. Hence, $\bot$ is derivable.) So, $\vdash_{\mathbf{K4}} \varepsilon_x \to \neg\varepsilon_y$.

(c) If $\varepsilon_y \in SUC^+(x)$, the case is trivial. So let us suppose the contrary. Then $x \lhd y \lhd x$ and so $cl(x) = cl(y)$ as well as $suc^+(x) = suc^+(y)$. Hence $SUC^+(x) = SUC^+(y)$ and $CL(x) = CL(y) \neq \emptyset$. Since ε_y is a conjunction of formulae ψ_i, $i < n$, in order to show $\vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond\varepsilon_y$ it is enough to show $\vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond\psi_0$ and $\vdash_{\mathbf{K4}} \varepsilon_x \to \Box(\psi_0 \to \psi_i)$, $0 < i < n$. We take $\psi_0 := At(y)$. Furthermore, $\beta_x = \beta_y$ and $\gamma_x = \gamma_y$, from which it easily follows with the help of $\vdash_{\mathbf{K4}} \varepsilon_x \to \Box(At(y) \to \Diamond^+ At(x))$ that $\vdash_{\mathbf{K4}} \varepsilon_x \to \Box(At(y). \to .\beta_y \wedge \gamma_y)$. Now

(i) $\vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond At(y)$, since $At(y) \in CL(x)$.

(ii) $\vdash_{\mathbf{K4}} \varepsilon_x \to \Box(At(y) \to \neg\lambda_y)$ follows from $\vdash_{\mathbf{K4}} \varepsilon_x \to \Box(At(y) \to \Diamond At(x))$, $\vdash_{\mathbf{K4}} \varepsilon_x \to \Box(\Diamond At(x) \to \neg\lambda_x)$ and $\lambda_x \equiv \lambda_y$.

(iii) $\vdash_{\mathbf{K4}} \varepsilon_x \to \Box(At(y) \to \Box(\lambda_y \to \neg\Diamond^+ At(y)))$. For $\vdash_{\mathbf{K4}} \varepsilon_x \to \Box(\lambda_x \to \neg\Diamond^+ At(x))$, whence $\vdash_{\mathbf{K4}} \varepsilon_x \to \Box(\lambda_y \to \Diamond^+ At(y))$. For $\lambda_x = \lambda_y$, and furthermore $\vdash_{\mathbf{K4}} \varepsilon_x \to \Box(At(y) \to \Diamond^+ At(x))$. The claim now easily follows.

(i), (ii) and (iii) together give $\vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond\alpha_y$. Thus $\vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond\varepsilon_y$, as required.

(d) **Case 1.** If $d^\mu(y) < d^\mu(x)$, then $\neg\Diamond\varepsilon_y$ is a conjunct of ε_x.

Case 2. $d^\mu(x) = d^\mu(y)$. Then $\lambda_x \equiv \lambda_y$. Suppose $\varepsilon_u \in SUC^+(x) - SUC^+(y)$ for some ε_u. If $\vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond\varepsilon_y$ then we have

$$\vdash_{\mathbf{K4}} \varepsilon_y. \to .\neg\Diamond\varepsilon_u \wedge \neg\lambda_x \wedge \bigwedge\langle \alpha \to \neg\Diamond^+(\alpha \wedge \lambda_x) : \alpha \in A(\varphi) - CL(x)\rangle$$

So we get

$$\vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond(\Box\neg\varepsilon_u. \wedge .\neg\lambda_x \wedge \bigwedge\langle \alpha \to \Diamond^+(\alpha \wedge \lambda_x) : \alpha \in A(\varphi) - CL(x)\rangle) .$$

But this is a contradiction to $\vdash_{\mathbf{K4}} \varepsilon_x \to \gamma_x$. Now suppose $\varepsilon_u \in SUC^+(y) - SUC^+(x)$. Then $\vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond\varepsilon_y$ yields $\vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond\varepsilon_u$ in contradiction to $\vdash_{\mathbf{K4}} \varepsilon_x \to \Box\neg\varepsilon_u$. Thus the case $SUC^+(x) = SUC^+(y)$ is left. Then we must have $CL(x) \neq CL(y)$ or $CL(x) = CL(y) = \emptyset$. The latter case is dealt with as follows. $\vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond\varepsilon_y$ implies $\vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond At(y)$; and since $\varepsilon_x \vdash_{\mathbf{K4}} \Box(At(y) \to \Diamond^+(At(y) \wedge \lambda_y))$ (for $At(y) \notin CL(x)$) we have $\vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond(\varepsilon_y \wedge \Diamond^+(At(y) \wedge \lambda_y))$, in contradiction to $\varepsilon_y \vdash_{\mathbf{K4}} \neg\lambda_y \wedge \Box(\lambda_y \to \neg At(y))$. Thus $CL(x) \neq CL(y)$. Now let $\alpha \in CL(y) - CL(x)$. Then $\vdash_{\mathbf{K4}} \varepsilon_x \to \Box(\alpha \to \Diamond^+(\alpha \wedge \lambda_x))$ and $\vdash_{\mathbf{K4}} \varepsilon_y \to \Box(\alpha \to \neg\lambda_x)$ and if $\vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond\varepsilon_y$ we get $\vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond(\alpha \wedge \lambda_x) \wedge \Box(\alpha \to \neg\lambda_x)$, again a contradiction. Assume finally $\gamma \in CL(x) - CL(y)$. Then $\vdash_{\mathbf{K4}} \varepsilon_x \to \Box(At(y) \to \Diamond\gamma)$ and $\vdash_{\mathbf{K4}} \varepsilon_y \to \Box(\gamma \to \Diamond^+(\gamma \wedge \lambda_x))$. We arrive at a contradiction with $\vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond\varepsilon_y$ because $\vdash_{\mathbf{K4}} \varepsilon_x \to \Box(\gamma \to \neg\lambda_x)$.

Case 3. $d^\mu(y) > d^\mu(x)$. If there is a z with $d^\mu(z) = d^\mu(x)$, $\varepsilon_y \lhd \varepsilon_z$ and $\varepsilon_z \not\lhd \varepsilon_x \not\lhd \varepsilon_z$, then $\vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond\varepsilon_y$ would imply $\vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond\varepsilon_z$, which is contradiction because of **Case 2**. But in the other case $\varepsilon_y \lhd \varepsilon_x$ and since $\vdash_{\mathbf{K4}} \varepsilon_y \to \Box(\lambda_y \to \Box\neg At(y))$ and $\vdash_{\mathbf{K4}} \varepsilon_x \to \lambda_y$ we get $\vdash_{\mathbf{K4}} \varepsilon_y \to \Box(\varepsilon_x \to \Box\neg At(y))$ showing $\vdash_{\mathbf{K4}} \varepsilon_x \to \neg\Diamond\varepsilon_y$. $\Box$

Since the depth of a point in $\mathfrak{R}$ is bounded we have a bounded number of formulae ε_x no matter what the frame $\mathfrak{F}$ is we started with. This number we denote by $c(\varphi)$. It can be computed from φ, but we will not do that here. Another consequence of Lemma 57 is

Proposition 58 *Let $\mathfrak{G}$ be a finite frame and $\mathfrak{F}^\mu$ be subreducible to $\mathfrak{G}$, that is, $\mathfrak{F}^\mu \supseteq \mathfrak{H} \overset{\pi}{\twoheadrightarrow} \mathfrak{G}$, and let $\chi_y := \bigvee\langle \varepsilon_w : w \in h, \pi(w) = y\rangle$, for $y \in G$. Then*

$$\vdash_{\mathbf{K4}} SF(\mathfrak{G})[\chi_y/p_y]$$

where $\varphi[\chi_y/p_y]$ is the result of replacing χ_y for all occurrences of p_y **for all** *y in φ.*

Proof. It is easily seen that $\vdash_{\mathbf{K4}} \chi_v \to \neg\chi_w$ if $v \neq w$, $\vdash_{\mathbf{K4}} \chi_v \to \Diamond\chi_w$ if $v \lhd w$ and $\vdash_{\mathbf{K4}} \chi_v \to \neg\Diamond\chi_w$ if $v \ntriangleleft w$. □

Proposition 59 *The map $\rho : \mathfrak{F}^\mu \twoheadrightarrow \mathfrak{R}$ given by $\rho : x \mapsto \varepsilon_x$ is a p–morphism admissible for β. Moreover, let $\mathbb{D}$ be the algebra of sets generated by the sets $\rho[\beta(p)] = \{\alpha \in \delta : \alpha \vdash_{\mathbf{K4}} p\}$. Then $\langle \mathfrak{Des}, \mathbb{D}\rangle$ is refined.*

Proof. Let $x \lhd y$. Then either $y \lhd x$ or $y \ntriangleleft x$. $y \ntriangleleft x$ implies $\varepsilon_x \lhd \varepsilon_y$ by definition, since $\Diamond\varepsilon_y$ is a conjunct of ε_x; if $y \lhd x$ then we have $SUC^+(x) = SUC^+(y)$, $CL(x) = CL(y)$ and $At(y) \in CL(x)$. Thus $\varepsilon_x \lhd \varepsilon_y$ as in the proof for Lemma 57(c). Hence $x \lhd y$ implies $\varepsilon_x \lhd \varepsilon_y$. Furthermore, if $\rho(x) \lhd \varepsilon_y$ then since $\langle \mathfrak{F}^\mu, \beta, x\rangle \models \varepsilon_x$ and $\vdash_{\mathbf{K4}} \varepsilon_x \to \Diamond\varepsilon_y$, $\langle \mathfrak{F}^\mu, \beta, z\rangle \models \varepsilon_y$ for some $x \lhd z$. Then $\varepsilon_y \equiv \varepsilon_z$, that is, $\rho(z) = \varepsilon_y$. This shows that ρ is a p–morphism. ρ is clearly admissible; let γ be the induced valuation on $\mathfrak{R}$. Because of $\langle \mathfrak{F}^\mu, \beta, z\rangle \models \varepsilon_x$, we have $\langle \mathfrak{R}, \gamma, \rho(z)\rangle \models \varepsilon_x$ but $\langle \mathfrak{R}, \gamma, \rho(z)\rangle \models \neg\varepsilon_y$ for $\varepsilon_x \neq \varepsilon_y$ and consequently $\langle \mathfrak{R}, \gamma\rangle$ is refined. □

We have now constructed formulas ε_x which completely describe the structure of the refined submodel of maximal points of any given finite model.

Theorem 60 *Let $L \supseteq$ **K4** be a subframe logic, $\mathfrak{F}$ a finite, rooted frame. If L has the finite model property, $L/_{\mathrm{F}}\mathfrak{F}$ has the finite model property as well. Moreover, if φ is consistent with $L/_{\mathrm{F}}\mathfrak{F}$ then it has a model of size $\leq c(\varphi)$.*

Proof. By constructive reduction. Suppose φ is consistent with $L/_{\mathrm{F}}\mathfrak{F}$. Then define $\varphi^\sharp$ to be the union of the sets $F(\langle \mathfrak{R}, \beta\rangle)$ where $\mathfrak{R}$ is of cardinality $\leq c(\varphi)$, β a valuation from $var(\varphi)$ into $\mathfrak{R}$ and

$$F(\langle \mathfrak{R}, \beta\rangle) := \{\Box^+ C_g[\varepsilon_{S(x)}/p_x] : S : G \to 2^r\}, \varepsilon_{S(x)} := \bigvee \langle \varepsilon_w : w \in S(x)\rangle$$

So $\varphi; \varphi^\sharp$ is $L_{\mathfrak{F}}$–consistent and a fortiori L–consistent. Therefore $\langle \mathfrak{Z}, \zeta, w\rangle \models \varphi; \varphi^\sharp$ for some L–model $\langle \mathfrak{Z}, \zeta, w\rangle$. Let $\mathfrak{R}$ be the reduced subframe of maximal points. We have $\langle \mathfrak{R}, \zeta, \varepsilon_w\rangle \models \varphi$, by Proposition 59. Now suppose $\mathfrak{R} \supseteq \mathfrak{H} \overset{\pi}{\twoheadrightarrow} \mathfrak{F}$. Then let $S : y \mapsto \pi^{-1}(y)$. If $v \in \pi^{-1}(x)$ with x generating $\mathfrak{F}$ then $\langle \mathfrak{Z}^\mu, \zeta, v\rangle \models \neg C_g[\varepsilon_{S(y)}/p_y]$ since $\langle \mathfrak{Z}^\mu, \zeta, v\rangle \models \varepsilon_{S(s)}$ and $\langle \mathfrak{Z}^\mu, \zeta\rangle \models SF(\mathfrak{F})[\varepsilon_{S(x)}/p_x]$, by Proposition 58. Thus $v \notin z$. Consequently, $\mathfrak{R}$ is not subreducible to $\mathfrak{F}$. $\langle \mathfrak{R}, \zeta, w\rangle$ is an $L_{\mathfrak{F}}$–model for φ and $card(\mathfrak{R}_w) \leq c(\varphi)$. □

Analogously the same theorem for cofinal subframe logics is proved. By induction one can now show that all Fine–splittings $\mathbf{K4}/_{\mathrm{F}}\, G$ (and all Zakharyaschev–splittings $\mathbf{K}/_{\mathrm{Z}}\, H$ for finite H) have the finite model property. But this is all we need to show the full theorem. Let φ be $\mathbf{K4}/_{\mathrm{F}}\, G$–consistent. Define $G^{\varphi} := \{\mathfrak{G} \in G : card(\mathfrak{G}) \leq c(\varphi)\}$. Then φ is $\mathbf{K4}/_{\mathrm{F}}\, G^{\varphi}$–consistent and has a finite model of size $\leq c(\varphi)$ by the preceding theorem. But this already is a $\mathbf{K4}/_{\mathrm{F}}\, G$–model. This completes the proof of Theorem 53

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