Invariant Logics

MARCUS KRACHT

Abstract. A modal logic Λ is called *invariant* if for all automorphisms α of $\mathrm{NExt}\,\mathbf{K}$, $\alpha(\Lambda) = \Lambda$. An invariant logic is therefore uniquely determined by its surrounding in the lattice. It will be established among other that all extensions of $\mathbf{K.alt}_1$, $\mathbf{S4.3}$ and $\mathbf{G.3}$ are invariant logics. Apart from the results that are being obtained, this work contributes to the understanding of the combinatorics of finite frames in general, something wich has not been done except for transitive frames. Certain useful concepts will be established, such as the notion of a *d–homogeneous frame*.

1. Introduction

In [2] we have investigated the groups of automorphisms of various lattices of modal logics. A full description of $\mathfrak{Aut}(\mathrm{NExt}\,\mathbf{S4.3})$ and $\mathfrak{Aut}(\mathrm{NExt}\,\mathbf{K.alt_1})$ was obtained. Moreover, we have shown that every automorphism of $\mathrm{NExt}\,\mathbf{S4}$ fixes all logics with the finite model property, and so also all logics of finite codimension. We raised the question about the group of automorphisms of $\mathrm{NExt}\,\mathbf{K}$. The motivation for such an endeavour can also be found in [2]. We should say here that apart from being interesting in its own right, this question provokes the development of new techniques of dealing with frames. Thus, the auxiliary results are at least as interesting as the main ones.

In this paper, we shall make a modest step towards the structure of the group of automorphisms of $\mathrm{NExt}\,\mathbf{K}$. We shall study logics which are invariant under all members of that group. Knowing which logics are invariant under any automorphism helps in this direction because it constrains the possible actions of an automorphism. It is our belief that all logics with the finite model property are invariant. This narrows down the choice of automorphisms substantially, though it is still conceivable that there are continuously many autormophisms.

2. Basic Terminology and Notation

We follow the notation and terminology of [3] for the basic terminology of modal logic and [2] for the notions relevant to automorphisms of lattices of modal logics. We will quickly review the basic instruments which we will need for our purposes. Recall that the normal logics form a distributive lattice (in fact a locale), which is denoted by $\mathrm{NExt}\,\mathbf{K}$. The lattice operations are denoted by $\sqcup$ and $\sqcap$ (and the infinitary operations by $\bigsqcup$ and $\bigsqcap$). Notice that $\Theta \sqcap \Lambda = \Theta \cap \Lambda$, although not necessarily $\Theta \sqcup \Lambda = \Theta \cup \Lambda$.

This paper is dedicated to Edward L. Keenan in gratitude.

Definition 2.1. *A modal logic Λ is said to be (**absolutely**) **invariant** if for all automorphisms α of NExt* $\boldsymbol{K}$ $\alpha(\Lambda) = \Lambda$.

A **Kripke–frame** is a pair $\mathfrak{F} = \langle F, \lhd \rangle$, where F is a set (possibly empty) and $\lhd \subseteq F^2$ a binary relation on F. If $G \subseteq F$ then $\mathfrak{F} \upharpoonright G := \langle G, \lhd \cap G^2 \rangle$ is the induced subframe on G. A frame identical to some $\mathfrak{F} \upharpoonright G$ is called a **subframe** of $\mathfrak{F}$. We write $\mathfrak{G} \subseteq \mathfrak{F}$ if $\mathfrak{G}$ is a subframe of $\mathfrak{F}$. Special subframes are the generated subframes, which are based on *generated subsets*. $S \subseteq F$ is a **generated subset** of $\mathfrak{F}$ if for all $x \in S$ and $x \lhd y$ also $y \in S$. We write $\mathfrak{F} \uparrow x$ for the generated subframe induced by x. It is based on the least generated subset containing x. $x \in F$ is a **root** of $\mathfrak{F}$ if the least generated subset containing x is F. $\mathfrak{F}$ is called **rooted** if it has a root. Given a Kripke–frame $\mathfrak{F}$, put $\mathsf{Th}\,\mathfrak{F} := \{\varphi : \mathfrak{F} \models \varphi\}$. Finally, let $\mathfrak{F} \leq \mathfrak{G}$ iff $\mathsf{Th}\,\mathfrak{F} \leq \mathsf{Th}\,\mathfrak{G}$. This is a converse well–quasi–ordering on the finite frames: it is transitive, and contains no infinite strictly upgoing chain. However, we can have $\mathfrak{F} \leq \mathfrak{G} \leq \mathfrak{F}$ and nevertheless $\mathfrak{F} \neq \mathfrak{G}$, even $\mathfrak{F} \ncong \mathfrak{G}$! One example is $\mathfrak{F} \oplus \mathfrak{F}$, the disjoint sum of $\mathfrak{F}$ with itself. However, an often used fact is the following folklore result (see [2]).

Lemma 2.2. *Let $\mathfrak{F}$ and $\mathfrak{G}$ be rooted and finite. Then $\mathsf{Th}\,\mathfrak{F} = \mathsf{Th}\,\mathfrak{G}$ iff $\mathfrak{F} \cong \mathfrak{G}$.*

By factoring out equivalence we get a converse well–partially ordered set, the set of all $\mathsf{Th}\,\mathfrak{F}$, $\mathfrak{F}$ a finite rooted Kripke–frame. We define $\mathbb{I}$ to be the set of all $\mathsf{Th}\,\mathfrak{F}$, $\mathfrak{F}$ rooted and finite. As is well–known (see [1]), $\mathbb{I}$ does not coincide with the set of intersectively irreducible logics of finite codimension. $\mathbb{I}$ is ordered by $\leq$. We shall often identify a logic of a frame with the frame itself, if that carries no risk of confusion. In that sense we may for example think of an automorphism of NExt $\mathbf{K}$ as inducing an automorphism of the set of all rooted finite Kripke–frames.

A logic Λ is **prime** in NExt $\mathbf{K}$ if whenever $\bigsqcap_{i \in I} \Theta_i \leq \Lambda$, there is an $i \in I$ such that $\Theta_i \leq \Lambda$. [1] has shown that Λ is prime in NExt $\mathbf{K}$ iff $\Lambda = \mathsf{Th}\,\mathfrak{F}$ for some finite, rooted, cycle–free frame. Let us denote the se of prime logics by $\mathbb{P}$. Then we have the following:

Lemma 2.3. *Every automorphism of NExt* $\boldsymbol{K}$ *fixes the set $\mathbb{P}$.*

It does *not* follow that each member of $\mathbb{P}$ is fixed by a given automorphism. In fact, establishing this for a small class of frames requires a lot of work. We shall briefly mention the following important fact.

Lemma 2.4. *Let $\Lambda \in \mathbb{P}$ and $\Theta \geq \Lambda$. Then Θ is tabular. Moreover, it is $\sqcap$–irreducible iff it is in $\mathbb{P}$.*

Further, notice the following:

Lemma 2.5. *$\mathbb{P}$ is an upper subset of $\langle \mathbb{I}, \leq \rangle$.*

This means that if $\Theta \in \mathbb{P}$ and $\Lambda \geq \Theta$ is a member of $\mathbb{I}$, then $\Lambda \in \mathbb{P}$ as well. It follows from Lemma 2.4 that given $\mathfrak{F} \in \mathbb{P}$, there is an upper bound on the number of logics containing $\mathsf{Th}\,\mathfrak{F}$, which can be uniformly determined on the basis of $|F|$.

Define $d_n := \Box^n \bot \wedge \Diamond^{n-1} \top$, $n > 0$. Say that a point x in a cycle–free frame $\mathfrak{F}$ is of **depth** n if $x \models d_n$. It is not hard to show that if $\mathfrak{F} \in \mathbb{P}$ then any point in $\mathfrak{F}$ has a depth and that this depth is unique. Further, if $\pi : \mathfrak{F} \to \mathfrak{G}$ is a p–morphism, then for $x \in F$ we have $x \models d_n$ iff $\pi(x) \models d_n$; whence the depth is invariant under p–morphisms. The depth of $\mathfrak{F}$ is the largest number n such that there exists a node of depth n.

In Rautenberg [5], Page 232, one finds a picture of the upper part of NExt $\mathbf{K}$. We shall be dealing only with tabular irreducible logics. First, there are two coatoms, by a theorem of Makinson ([4]). Then follow countably many logics of depth 2. These are among other the logics of the loops of prime order. Let $\mathfrak{Loop}_n := \langle \{0, 1, \dots, n-1\}, \lhd \rangle$, where $j \lhd j$ iff $j \equiv i+1 \pmod{n}$. The logic $\mathsf{Th}\,\mathfrak{Loop}_n$ has codimension 2 exactly when n is a prime number (see [3]). There are many more logics of depth 2, but this may suffice for an indication of the complexity of the lattice, even when we restrict the attention to tabular logics. Recall that an **antichain** in a poset $\langle P, \leq \rangle$ is a set A such that for all $x, y \in S$ such that $x \neq y$ also $x \nleq y$ and $y \nleq x$. By the above, $\langle \mathbb{I}, \leq \rangle$ possesses an infinite antichain, namely the set $\{\mathsf{Th}\,\mathfrak{Loop}_p : p \text{ prime}\}$. Further down we shall show that also $\langle \mathbb{P}, \leq \rangle$ has an infinite antichain.

In order to establish that some logic Λ is invariant under all automorphisms it is enough to find a formula $\varphi(x)$ in some logical language using the signature of lattices such that $\{\Lambda\} = \{\Theta : \text{NExt}\,\mathbf{K} \models \varphi(\Theta)\}$. In sequel it is enough to use monadic second order logic (MSO). The nonlogical symbols will be $\leq$, $\sqcap$ and $\sqcup$. Of course, some higher order language might also be used but is not necessary for our purposes. However, we shall make use of the following fact.

Lemma 2.6. *The set of invariant logics is closed under $\sqcap$, $\sqcup$, $\bigsqcap$ and $\bigsqcup$. In particular, it forms a sublocale of NExt $\boldsymbol{K}$.*

The supremum and infimum of infinite sets is defined as follows:

$$\begin{array}{lcl} \mathit{inf}(P,x) & := & (\forall y)(y \in P \to x \leq y) \wedge (\forall z)((\forall y)(y \in P \to y \leq z) \to z \leq x) \\ \mathit{sup}(P,x) & := & (\forall y)(y \in P \to x \geq y) \wedge (\forall z)((\forall y)(y \in P \to y \geq z) \to z \geq x) \end{array}$$

Definition 2.7. *Let $S \subseteq$ NExt $\boldsymbol{K}$. S is called **MSO–definable** if there is an MSO–formula φ such that $S = \{\Theta : \text{NExt}\,\boldsymbol{K} \models \varphi(\Theta)\}$. A logic $\Lambda \in$ NExt $\boldsymbol{K}$ is **MSO–definable** if $\{\Lambda\}$ is MSO–definable. Throughout this paper we shall use **definable** instead of **MSO–definable**.*

It is worthwile to establish the definability of some basic sets of logics.

Lemma 2.8. *The set of $\sqcap$–irreducible logics and the set of prime logics are definable.*

Proof. The defining formulae are

$$\begin{array}{lcl} \varphi_i(x) & := & (\forall yz)(x \doteq y \sqcap z \to (y \doteq x \vee z \doteq x)) \\ \varphi_p(x) & := & (\forall P)((\forall y)(\mathit{inf}(P,y) \to y \leq x) \to (\exists y)(y \in P \wedge y \leq x)) \end{array}$$

□

Lemma 2.9. *Suppose that S is definable. Then for every automorphism α of NExt $\boldsymbol{K}$, $\alpha[S] = S$. It follows that $\bigsqcap S$ and $\bigsqcup S$ are invariant. In particular, if Λ is definable, it is invariant.*

This proves once more the observation of Lemma 2.3. Notice that since there are only countably many formulae, not every subset is MSO–definable. Nevertheless, it may very well be that all subsets are invariant (namely when there exists only one automorphism), so the converse of the previous theorem is false. We shall note however that α is the identity already when it fixes every compact (= finitely axiomatizable) logic. Since the set of compact logics is definable, we can in principle determine the action of an automorphism α by its action on a countable definable subset of NExt $\mathbf{K}$.

3. Inherently 1–covered logics

We shall identify $\mathbb{P}$ in some canonical way with a set of finite rooted Kripke–frames, which we will also call $\mathbb{P}$. $\mathbb{P}$ is ordered by $\leq$, as defined above. It is an upper subset of $\langle \mathbb{I}, \leq \rangle$.

Definition 3.1. *Suppose that $\mathfrak{F} \in \mathbb{P}$. We say that $\mathfrak{F}$ is **n–covered** if it has at most n covers in $\mathbb{I}$ (iff it has at most n covers in $\mathbb{P}$, by Lemma 2.5). $\mathfrak{F}$ is **inherently n–covered** if every $\mathfrak{G} \geq \mathfrak{F}$ is n–covered. We denote by $\mathbb{C}^n$ the set of all inherently n–covered $\mathbb{P}$–frames (and the logics thereof). Finally, let*

$$\Gamma^n := \mathsf{Th}\,\mathbb{C}^n$$

Clearly, these notions are definable. From Lemma 2.9 we infer

Lemma 3.2. *Let $\alpha \in \mathfrak{Aut}(NExt\,\boldsymbol{K})$. Then $\alpha(\Gamma^n) = \Gamma^n$.*

$\mathbb{C}^0$ consists of the empty frame, and therefore $\Gamma^0 := \mathsf{K} \oplus \bot$, the inconsistent logic. The case $n = 0$ is therefore trivial. Of particular interest in the study are the logics in $\mathbb{C}^1$. Here is a useful way to identify $\mathbb{C}^1$–logics.

Lemma 3.3. *Let Λ be prime. Λ is in $\mathbb{C}^1$ iff NExt Λ is finite and linear.*

It does not follow, of course, that the generating frame of Λ is linear; neither does it follow that NExt Γ^1 is linear. To the contrary, this is quite a complex lattice, whose structure we will unravel to some extent. We remark here the following fact, which immediately shows that nontrivial invariant logics exist.

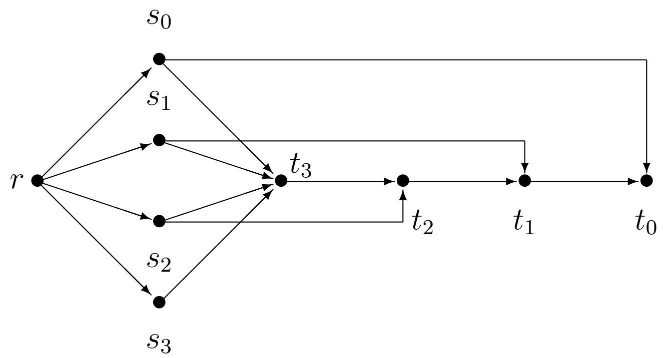
Proposition 3.4. *$\boldsymbol{K} = \prod_{n \in \omega} \Gamma^n$. Furthermore, $\Gamma^{n+1} < \Gamma^n$ for all $n \in \omega$.*

Proof. First, for any finite Kripke–frame there is an n such that it is inherently n–covered. So, the first claim follows from the fact that $\mathbf{K}$ has the finite model property. For the second claim, consider the following frame $\mathfrak{L}_n := \langle L_n, \lhd_n \rangle$ defined by

$$L_n \quad := \quad \{r\} \cup \{s_i : i < n\} \cup \{t_i : i < n\}$$

$$\lhd_n \quad := \quad \begin{cases} & \{\langle r, s_i \rangle : i < n\} \\ \cup & \{\langle t_j, t_{j-1} \rangle : 0 < j < n\} \\ \cup & \{\langle s_i, t_{n-1} \rangle : i < n\} \\ \cup & \{\langle s_i, t_i \rangle : i < n\} \end{cases}$$

The picture below shows the frame $\mathfrak{L}_4$.



It is easily established that $\mathfrak{L}_n$ has no contraction images other than $\mathfrak{L}_n$. (A contraction must be depth preserving, and so it can only collapse some s–points. But this is impossible, as they have different successors.) The subframes generated by the s_i are therefore the only immediate covers of $\mathfrak{L}_n$. They are all non–isomorphic, showing that $\mathfrak{L}_n$ has n covers (using Lemma 2.2). □

Before we consider the structure of $\mathbb{C}^1$–frames, let us specialize a little further.

Definition 3.5. *Let $\mathfrak{F} \in \mathbb{P}$. Then $\mathfrak{F}$ is called **linear** if for any $n \in \omega$ there is at most one point of depth n in $\mathfrak{F}$.*

(For a general notion of linearity, see Section 6.) So far we have not established any tool to identify logics of linear frames. However, a useful fact to note is the following:

Lemma 3.6. *Let $\mathfrak{F}$ be linear and in $\mathbb{P}$. Then any p–morphic image of $\mathfrak{F}$ is isomorphic to $\mathfrak{F}$. Furthermore, the codimension of $\mathsf{Th}\,\mathfrak{F}$ equals the number of points of $\mathfrak{F}$.*

Proof. The first claim is easy to verify. Now suppose that $\mathfrak{G} \geq \mathfrak{F}$. Then $\mathfrak{G}$ is isomorphic to a generated subframe of $\mathfrak{F}$. Now suppose that the root of $\mathfrak{F}$ has depth n and let y be of depth $n-1$. Then y generates a frame $\mathfrak{G}$ of depth $n-1$. By linearity, $|G| = |F| - 1$. This shows that NExt $\mathsf{Th}\,\mathfrak{F}$ has $n+1$ points, and so $\mathsf{Th}\,\mathfrak{F}$ has codimension n. □

Definition 3.7. *Suppose that $\mathfrak{F}$ is cycle–free. $\mathfrak{F}$ is called **d–homogeneous** if for any pair x, y of points of equal depth the following holds: if $x \lhd u$ and u has depth k, then there is a v of depth k such that $y \lhd v$.*

Obviously, linear $\mathbb{P}$–frames are d–homogeneous.

Lemma 3.8. *Suppose that $\mathfrak{F} \in \mathbb{P}$ is d–homogeneous. Then there exists a (uniquely determined) linear frame $\mathfrak{G}$ and a p–morphism $\pi : \mathfrak{F} \twoheadrightarrow \mathfrak{G}$. Furthermore, if $\mathfrak{F}$ is contractible to a linear $\mathbb{P}$–frame, $\mathfrak{F}$ is d–homogeneous.*

Proof. Put $G := \{n : \text{ exists } x \in F : x \models d_n\}$, and let $m \blacktriangleleft n$ iff there exists x of depth m and y of depth n such that $x \lhd y$. $\mathfrak{G} := \langle G, \blacktriangleleft \rangle$. Now put $\pi(x) := n$ if x has depth n. It is easy to check that this is a p–morphism. The uniqueness follows from the fact that if x is of depth n in $\mathfrak{F}$, it must be of depth n in $\mathfrak{G}$. So, $\mathfrak{G}$ is the smallest frame onto which a p–morphism exists, and it is easily seen that there exist no surjective p–morphism between nonisomorphic linear $\mathbb{P}$–frames. Hence, $\mathfrak{G}$ exists and is unique. Now assume that $\mathfrak{F}$ is contractible to (that is, can be mapped onto) a linear $\mathbb{P}$–frame, say $\pi : \mathfrak{F} \twoheadrightarrow \mathfrak{G}$, $\mathfrak{G}$ linear. Then x and y have the same depth iff $\pi(x) = \pi(y)$. So, assume that $x \lhd u$ and let y be of same depth as x. Then $\pi(x) = \pi(y)$ and therefore $\pi(y) \blacktriangleleft \pi(u)$. It follows that there is a v such that $y \lhd v$ and $\pi(v) = \pi(u)$. So, v has the same depth as u. Hence $\mathfrak{F}$ is d–homogeneous. □

So, $\mathfrak{F}$ is d–homogeneous exactly if it is contractible to a linear $\mathbb{P}$–frame. For $\mathbb{P}$–frames in general we shall define the concept of a *depth profile*.

Definition 3.9. *Let $\mathfrak{F}$ be a $\mathbb{P}$–frame. Then the* **depth profile** *of $\mathfrak{F}$, $\wp(\mathfrak{F})$, is the set of pairs of numbers $\langle m, n\rangle$ such that there exists a point of depth m seing a point of depth n. For a point x, the* **depth profile** *of x, $\wp(x)$, is the set of all numbers n such that there exists a successor of x of depth n.*

Clearly, a linear frame is uniquely characterized by its depth profile. Moreover, we can count the number of linear frames of given depth in the following way.

Theorem 3.10. *Let $\mathfrak{F}$ be a frame of depth n. Then $\wp(\mathfrak{F})$ is a subset of $\{1, \ldots, n\}^2$ with the following properties.*

(1) *$\langle m, m-1\rangle \in \wp(\mathfrak{F})$ for all $n \geq m > 1$.*
(2) *If $\langle m, m'\rangle \in \wp(\mathfrak{F})$ then $m > m'$.*

If $P \subseteq \{1, \ldots, n\}^2$ is any set satisfying these two properties, there is a linear frame $\mathfrak{L}_P$ such that $\wp(\mathfrak{L}_P) = P$. Hence there are $2^{\binom{n-1}{2}} = 2^{(n-1)(n-2)/2}$ many linear $\mathbb{P}$–frames of depth n.

We may also note the following, which is immediate from the definitions.

Lemma 3.11. *Let $\mathfrak{F}$ be a d–homogeneous $\mathbb{P}$–frame, x of depth m and $\langle m, n\rangle \in \wp(\mathfrak{F})$. Then there exists a y of depth n such that $x \lhd y$.*

Lemma 3.12. *Let $\pi : \mathfrak{F} \twoheadrightarrow \mathfrak{G}$ be a p–morphism. Then*

(1) *$\mathfrak{F}$ is d–homogeneous iff $\mathfrak{G}$ is.*
(2) *$\wp(\mathfrak{F}) = \wp(\mathfrak{G})$.*

Proof. The first claim follows from Lemma 3.8. For the second notice that a p–morphism is depth preserving. □

Write $\wp(\mathfrak{F}) \restriction n := \wp(\mathfrak{F}) \cap \{1, \ldots, n\}^2$.

Lemma 3.13. *Let $\mathfrak{F}$ be a d–homogeneous $\mathbb{P}$–frame and $\mathfrak{G} \geq \mathfrak{F}$ of depth n. Then $\wp(\mathfrak{G}) = \wp(\mathfrak{F}) \restriction n$. Further, if $\mathfrak{L}$ is linear and $\wp(\mathfrak{L}) = \wp(\mathfrak{F}) \restriction n$, then $\mathfrak{L} \geq \mathfrak{F}$.*

Proof. If $\mathfrak{G}$ is a p–morphic image of $\mathfrak{F}$, this follows from the previous theorem. Suppose then that $\mathfrak{G}$ is a p–morphic image of $\mathfrak{F}$, say $\pi : \mathfrak{F} \twoheadrightarrow \mathfrak{G}$. π is depth preserving. So, let $\langle i, j\rangle \in \wp(\mathfrak{F})$. Then there are x of depth i and y of depth j such that $x \lhd^F y$. Then $\pi(x)$ is of depth i, $\pi(y)$ of depth j and $\pi(x) \lhd^G \pi(y)$. Hence $\langle i, j\rangle \in \wp(\mathfrak{G})$. Conversely, suppose that $\langle i, j\rangle \in \wp(\mathfrak{G})$. Then there are x' of depth i

and y' of depth j such that $x' \lhd^G y'$. There is $x \in F$ such that $\pi(x) = x'$. π is a p–morphism, so there exists a $y \in F$ such that $x \lhd^F y$ and $\pi(y) = y'$. x' has depth i and y' has depth j. Therefore, $\langle i, j\rangle \in \wp(\mathfrak{F})$. Now for the last claim. Suppose that $\mathfrak{L}$ is linear, and that $\wp(\mathfrak{L}) = \wp(\mathfrak{F}) \upharpoonright n$. Consider the contraction of $\mathfrak{F}$ onto a linear frame $\mathfrak{L}'$. We have $\wp(\mathfrak{L}') = \wp(\mathfrak{F})$, by Lemma 3.12. Let $\mathfrak{L}''$ be the subframe of depth n of $\mathfrak{L}'$. Then $\wp(\mathfrak{L}'') = \wp(\mathfrak{L})$, so $\mathfrak{L}'' \cong \mathfrak{L}$, from which the claim follows, since $\mathfrak{L}'' \geq \mathfrak{F}$. ☐

Using this result one can show that $\langle \mathbb{P}, \leq\rangle$ possesses an infinite antichain. Namely, let $\mathfrak{Q}_n$ be the unique linear frame in $\mathbb{P}$ such that $\wp(\mathfrak{Q}_n) = \{\langle j, j-1\rangle : 1 < j \leq n\} \cup \{\langle n, n-2\rangle\}$. Clearly, for $n \leq m$ we do have $\wp(\mathfrak{Q}_m) \upharpoonright n \neq \wp(\mathfrak{Q}_n)$, from which follows $\mathfrak{Q}_m \not\leq \mathfrak{Q}_n$ (and anyway $\mathfrak{Q}_n \not\leq \mathfrak{Q}_m$). So, $\{\mathfrak{Q}_n : n \in \omega\}$ is an infinite antichain of $\mathbb{P}$–frames.

We define for a given $\mathfrak{F}$ the **order depth** the length of a longest ascending chain of $\sqcap$–irreducible logics. This is by definition a lattice invariant. (It is *not* the same as the codimension of that logic, see [2].)

Lemma 3.14. *Let $\mathfrak{F} \in \mathbb{P}$ be d–homogeneous. Then the order depth of $\mathfrak{F}$ coincides with $|F|$.*

Proof. We have shown this already for linear $\mathbb{P}$–frames. So, assume that $\mathfrak{F}$ is not linear. Then there are x and y, $x \neq y$, of same depth such that x and y see the same set of points. Such a pair exists. For let m be the least number such that there is more than one point of depth m. Since $\mathfrak{F}$ is not linear, m exists. Choose x and y of depth m. Then x and y see the same set of points, as is easy to see. (For otherwise x and y generate the nonisomorphic linear frames $\mathfrak{F} \uparrow x$ and $\mathfrak{F} \uparrow y$.) Now collapse x and y into the same point and call the resulting frame $\mathfrak{G}$. Then $\mathfrak{G} \in \mathbb{P}$ and $|G| = |F| - 1$. Now an easy induction on $|F|$ establishes the claim. ☐

As it turns out, $\mathbb{C}^1$–frames are d–homogeneous:

Lemma 3.15. *Let $\mathfrak{F}$ be a $\mathbb{C}^1$–frame. Then $\mathfrak{F}$ is d–homogeneous.*

Proof. Suppose the claim is false. Then there exists a smallest $\mathbb{C}^1$–frame $\mathfrak{F}$ which is not d–homogeneous (recall that $\mathbb{C}^1$ consists of finite frames). This means that there exist points x and y of identical depth, say n, such that x has a successor of depth $m < n$, but y does not. Consider the subframes $\mathfrak{F} \uparrow x$ and $\mathfrak{F} \uparrow y$ generated by x and y. By choice of $\mathfrak{F}$ they are d–homogeneous and therefore possess a linear contraction image $\mathfrak{G}_x$ and $\mathfrak{G}_y$, respectively. By choice of x and y, $\mathfrak{G}_x$ and $\mathfrak{G}_y$ have the same depth but are not isomorphic. Hence we have neither $\mathfrak{G}_x \geq \mathfrak{G}_y$ nor $\mathfrak{G}_x \leq \mathfrak{G}_y$. But this is impossible by Lemma 3.3. ☐

As a consequence we note that, by Lemma 3.14, the order depth of a $\mathbb{C}^1$–frame $\mathfrak{F}$ is exactly $|F|$. So we can reconstruct the number of elements in such a frame just from the lattice of its extensions. A cycle–free frame is of depth n if no point has depth $> n$ and there is at least one point of depth n. Now, rooted frames of depth 2 have a rather simple structure. They consist of a single root and a number of depth 1 successors. Let $\mathfrak{F}$ be of depth 2 and let $\mathfrak{F}$ have $n+1$ elements; then we denote $\mathfrak{F}$ by $\mathfrak{B}_n$.

Lemma 3.16. *Let $n > 1$. $\mathfrak{B}_n$ has exactly one $\mathbb{C}^1$–cocover, namely $\mathfrak{B}_{n+1}$.*

Proof. Suppose that $\mathfrak{F}$ is a cocover of $\mathfrak{B}_n$ and assume that $\mathfrak{F} \in \mathbb{C}^1$. Then $\mathfrak{F}$ cannot be of depth 3, since then it has several cocovers (namely, there is a generated subframe of depth 2 isomorphic to $\mathfrak{B}_n$ and a contraction image, which has $n-1$ points of depth 1). So $\mathfrak{F}$ is of depth 2, and hence isomorphic to $\mathfrak{B}_m$ for some m. It is easy to see that $m = n+1$. This shows that $\mathfrak{B}_n$ has exactly one cocover in $\mathbb{C}^1$. ☐

Now, there are three non–isomorphic rooted, cycle–free frames of cardinality 3, two of which are linear. The third is $\mathfrak{B}_2$. $\mathfrak{B}_2$ is definable, since it has less $\mathbb{C}^1$–cocovers that the other two. For it follows from the next lemma that the linear frames with three elements have no less than 4 $\mathbb{C}^1$–cocovers.

Lemma 3.17. *Let $\mathfrak{F}$ be rooted and linear and let $|F| = n$. Then $\mathfrak{F}$ has at exactly 2^{n-1} linear cocovers.*

Proof. We may assume that $F \subsetneq G$ with $G = F \cup \{y\}$. The new point y is the new root. Its set of successors is any set containing at least the root of $\mathfrak{F}$. There are 2^{n-1} such sets. Since $\mathfrak{F}$ is linear, all these frames linear as well and in addition non–isomorphic since they have different depth profile. ☐

Notice that there are $2^{(n-1)(n-2)/2}$ linear $\mathbb{P}$–frames of depth n, each possessing 2^{n-1} different linear $\mathbb{C}^1$–cocovers. Thus we have $2^{n-1} \times 2^{(n-1)(n-2)/2} = 2^{(2(n-1)+(n-1)(n-2))/2} = 2^{n(n-1)/2}$ linear $\mathbb{P}$–frames of depth $n+1$. Thus a simple counting argument would have sufficed for the linear frames.

Proposition 3.18. *$\mathfrak{B}_n$ is definable for every $n > 0$.*

Proof. $\mathfrak{B}_1$ is definable by the property expressing that is a prime frame of codimension 2. $\mathfrak{B}_2$ is definable by the formula expressing that it is a prime frame of codimension 3 and has exactly one $\mathbb{C}^1$–cocover (by Lemma 3.14 the first condition defines frames of cardinality 3, and by Lemma 3.17 the second then defines $\mathfrak{B}_2$ in conjunction with the first). $\mathfrak{B}_n$ is definable by the formula expressing that it is a prime frame of codimension n, and below $\mathfrak{B}_2$ (for $n > 2$). ☐

As we have seen, not all $\mathbb{C}^1$–frames are actually linear. But they are not far from being linear. We first establish a weak version of a theorem that will reveal the structure of $\mathbb{C}^1$–frames.

Lemma 3.19. *Let $\mathfrak{F}$ be a $\mathbb{C}^1$–frame of depth n and not linear. Then the root has two immediate successors. Furthermore, if $\mathfrak{G} \leq \mathfrak{F}$ is a $\mathbb{C}^1$–frame, $\mathfrak{G}$ has the same depth as $\mathfrak{F}$.*

Proof. We show the first claim. Suppose that $\mathfrak{F}$ does not have 2 points at depth $n-1$. Then there are two covers. (1) The frame obtained by collapsing two points into one, (2) the generated subframe of depth $n-1$. They are both generated and nonisomorphic. The second claim follows in this way. Suppose that $\mathfrak{G}$ is a cocover of $\mathfrak{F}$. Then it is obtained by adding a single point. This point cannot be a new root, otherwise we would violate the first property. ☐

In order to make progress in distinguishing various frames of identical size we shall develop a formula that allows us to count the number of cocovers. The idea is as follows. We have already established how many $\mathbb{C}^1$–frames exist of which $\mathfrak{F}$ is a maximal generated subframe, now we shall establish how many $\mathbb{C}^1$–frames exist that cover $\mathfrak{F}$ and can be mapped onto it. By previous theorems, the fact that $\mathfrak{G}$ cocovers $\mathfrak{F}$ means that $\mathfrak{G}$ has one more point than $\mathfrak{F}$. So let $\pi : G \to F$ be a contraction (= a surjective p–morphism) and x and y different points such that $\pi(x) = \pi(y)$. (By our assumptions, $\{x, y\}$ is unique.) Then the set of successors of x and y in $\mathfrak{G}$ coincide, by the p–morphism conditions. Furthermore, the set P of predecessors of $\pi(x)$ in $\mathfrak{F}$ is the union of two sets $\pi[X]$ and $\pi[Y]$, where X is the set of predecessors of x in $\mathfrak{G}$, and Y the set of predecessors of y in $\mathfrak{G}$. Both X and Y are nonempty, since $\mathfrak{G}$ is rooted. Conversely, if in $\mathfrak{G}$ we find two points x and y such that their sets of successors are identical, then (and only then) is the collapsing of x and y a p–morphism, and the set P of predecessors of $\pi(x)$ is the union of (the π-image of) the set of predecessors of x and that of y. Thus, any contraction cocover is obtained in this way. Obviously, an upper bound on the number of cocovers of which $\mathfrak{F}$ is a p–morphic image is obtained by taking all possible points x distinct from the root, and all possible representations of $P := \{y : y \lhd x\}$ as unions of nonempty subsets. This bound can be very high. Notice also that it is only an upper bound, since some cocovers may turn out to be isomorphic.

Now let $\mathfrak{F}$ be a $\mathbb{C}^1$–frame. We have counted the number of linear cocovers. Now we count the number of nonlinear $\mathbb{C}^1$–cocovers.

Lemma 3.20. *Let $\mathfrak{L}$ be a linear $\mathbb{P}$–frame and x its root. Then there are card($\wp(x)$) many nonlinear $\mathbb{C}^1$–cocovers of $\mathfrak{L}$, which are all inherently 1–covered.*

Proof. Assume that $\mathfrak{L}$ has depth d with root x. Let $\mathfrak{F}$ be a d–homogeneous $\mathbb{C}^1$–cocover of $\mathfrak{L}$. By Lemma 3.14, $\mathfrak{F}$ has one point more than $\mathfrak{L}$ itself. Further, $\mathfrak{F}$ has the same depth profile as $\mathfrak{L}$, by Lemma 3.12. This means that we can consider $\mathfrak{L}$ to be a subframe of $\mathfrak{F}$. So, we now assume that $L \subseteq F$. By Lemma 3.14, $F - L = \{y\}$ for some y. Let y be of depth k. It is easy to see that we must have $v \lhd y$ iff $v = x$. (For suppose otherwise. Let u be a successor of x of depth $d-1$. y is not of depth $d-1$ (otherwise we have an immediate contradiction, since $x \lhd^F y$). So, u is unique and different from y. Then $\mathfrak{F} \uparrow u$ is a cover of $\mathfrak{F}$ not isomorphic to $\mathfrak{L}$ (since it contains y). But $\mathfrak{F}$ is an $\mathbb{C}^1$–frame. Contradiction.) Consequently, as $x \lhd^F y$, we have $k \in \wp(x)$. Now let w be of depth k, $w \neq y$. w and y have the same set of predecessors, namely only x. The contraction collapsing w and y is an admissible p–morphism from $\mathfrak{F}$ onto $\mathfrak{L}$ (the only one in fact). So we must have $\wp(y) = \wp(w)$, which is to say that w and y have the same successors. So, $\mathfrak{F}$ is uniquely determined by the number k. On the other hand, for each $k \in \wp(\mathfrak{L})$, such a frame exists. □

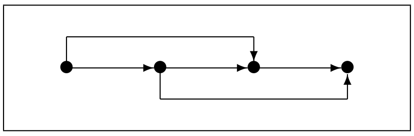
There is an immediate application. Let $\mathfrak{Ch}_n$ be defined by

$$\begin{array}{rcl}\mathfrak{Ch}_n & := & \langle\{0, 1, \ldots, n-1\}, \lhd_n\rangle \\ \lhd_n & := & \{\langle i, i-1\rangle : 0 < i < n\}\end{array}$$

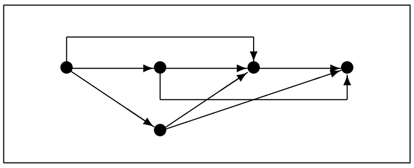
This chain has, by the above theorems, $2^{n-1} + 1$ $\mathbb{C}^1$–cocovers. For either $\mathfrak{Ch}_n \rightarrowtail \mathfrak{F}$, in which case $\mathfrak{F}$ is linear and there are 2^{n-1} choices up to isomorphism. But if

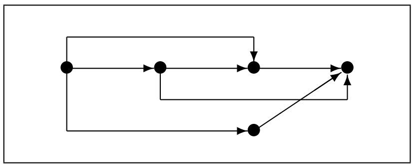
$\mathfrak{G} \twoheadrightarrow \mathfrak{Ch}_n$, then $\mathfrak{G}$ is obtained by adding a point of depth $< n$. For each given depth d there exists such a cocover iff $\langle n, d\rangle \in \wp(\mathfrak{Ch}_n)$ iff $d = n - 1$. Hence there is only 1 nonisomorphic choice.

We denote the frames defined in the proof by $\mathfrak{L}_k$ (see also below for the notation). For example, the following frame



has these two nonlinear $\mathbb{C}^1$–cocovers:





In order to discriminate the nonlinear cocovers from the linear cocovers, we shall investigate the structure of $\mathbb{C}^1$–frames in more detail.

Definition 3.21. *Let $\mathfrak{F}$ be a $\mathbb{P}$–frame. We say that $\mathfrak{F}$ is of d–**width** m iff there are exactly m points of depth d. The **width** of $\mathfrak{F}$ is the maximum of all d–widths of $\mathfrak{F}$, $d \in \omega$.*

Lemma 3.22. *Let $\mathfrak{F}$ be a $\mathbb{C}^1$–frame. Suppose that $\mathfrak{F}$ is of d–width 4. Then for all $d' < d$, $\mathfrak{F}$ is of d'–width 1.*

Proof. We let $\mathfrak{F}$ be a minimal counterexample to this claim. Let u_i, $i < 4$, be the points of depth d. Assume that there are two points of depth d', where $d' < d$. Then there is no $d'' \neq d'$ and $d'' < d$ such that $\mathfrak{F}$ has d''–width > 1. (Otherwise there are some admissible p–morphisms contracting $\mathfrak{F}$ to a smaller counterexample.) Moreover, we can assume that $\mathfrak{F}$ has d'–width 2. (Again, we can apply an admissible p–morphism otherwise.) Call the points of depth d' x and y. (Case 1.) $\langle d, d'\rangle \notin \wp(\mathfrak{F})$. Then the contraction of u_0 and u_1 into one point is a p–morphism, and so is the contraction of x and y into one point. So, $\mathfrak{F}$ has two covers. Hence this case does not arise. (Case 2.) $\langle d, d'\rangle \in \wp(\mathfrak{F})$. Then each u_i ($i < 4$) sees at least x or y by Lemma 3.15. As $\{x, y\}$ has only three nonempty subsets, there are i and j such that $i < j$ and u_i and u_j have the same set of successors of depth d'. It follows that they have the same set of successors of any given depth $< d$. Hence, contracting u_i and u_j to a single point is a p–morphism, and so $\mathfrak{F}$ has two nonisomorphic covers. So this case does not arise either. We have reached a contradiction, and the claim is proved. ☐

Lemma 3.23. *Suppose that $\mathfrak{F}$ is a nonlinear $\mathbb{C}^1$–frame. Then there is no infinite, homogeneously 2–branching tree of $\mathbb{C}^1$–frames rooted at $\mathfrak{F}$ in NExt $\boldsymbol{K}$.*

Proof. If $\mathfrak{F}$ is nonlinear and $\mathfrak{G}$ a $\mathbb{C}^1$–cocover of $\mathfrak{F}$, then $\mathfrak{G}$ has the same depth as $\mathfrak{F}$ by Lemma 3.19. Let it be d. Consider the number of frames which have the order depth κ, where $\kappa > 3d-2$. These frames cannot be of width < 4. Hence there is some d' such that they are of d'–width ≥ 4. By Lemma 3.22, $\mathfrak{F}$ is of d''–width < 4 for all $d'' \neq d'$. The number of such frames is bounded by some number just depending on d. This shows that no homogeneously 2–branching tree of $\mathbb{C}^1$–frames rooted at $\mathfrak{F}$ can exist. □

Lemma 3.24. *Let $\mathfrak{L} \in \mathbb{P}$ be linear and of depth > 1. Then there is an infinite, homogeneously 2–branching tree of $\mathbb{C}^1$–frames rooted at $\mathfrak{F}$.*

This follows basically from Lemma 3.17 and the fact that we are dealing with $\mathbb{C}^1$–frames.

Definition 3.25. *$\mathbb{L}$ denotes the set of linear $\mathbb{P}$–frames, $\mathbb{D}$ the set of d–homogeneous $\mathbb{P}$–frames.*

Lemma 3.26. *The class $\mathbb{L}$ is definable.*

Proof. There is an MSO–formula $t(x)$ which is true of Θ iff Θ is the root of some homogeneously 2–branching tree of $\mathbb{C}^1$–frames. Now consider the following formula:

$$\ell(x) := \mathbb{C}^1(x) \wedge (\forall y)(y \geq x \rightarrow t(x)) \ .$$

We claim that Θ is the logic of a linear frame iff $\ell(\Theta)$. But this is easily shown by induction on the order depth of Θ. □

Here is now a first result:

Lemma 3.27. *A linear frame is isomorphic to some $\mathfrak{Ch}_n$ iff its unique nonlinear $\mathbb{C}^1$–cocover has exactly one $\mathbb{C}^1$–cocover.*

It follows that the set of logics $\{\mathsf{Th}\,\mathfrak{Ch}_n : n \in \omega\}$ is definable. Recall that $\mathbf{K.alt}_n$ is the logic of all frames with the property that a given point has at most n immediate successors. Then we get the following result.

Theorem 3.28. *$\boldsymbol{K.alt}_1$ is invariant.*

Lemma 3.29. *$\mathfrak{F}$ is rooted and d–homogeneous iff for any two linear frames $\mathfrak{G}_1, \mathfrak{G}_2 \geq \mathfrak{F}$ we have $\mathfrak{G}_1 \geq \mathfrak{G}_2$ or $\mathfrak{G}_2 \geq \mathfrak{G}_1$.*

Proof. Suppose first that $\mathfrak{F}$ is d–homogeneous. Let $\mathfrak{L}_1, \mathfrak{L}_2 \geq \mathfrak{F}$ be linear $\mathbb{P}$–frames of depth n_1 and n_2, repsectively. Without loss of generality, $n_1 \leq n_2$. We have $\wp(\mathfrak{L}_1) = \wp(\mathfrak{F}) \upharpoonright n_1$ and $\wp(\mathfrak{L}_2) = \wp(\mathfrak{L}_2) \upharpoonright n_2$, from which we get that $\wp(\mathfrak{L}_1) = \wp(\mathfrak{L}_2) \upharpoonright n_1$. It follows from Lemma 3.13 that $\mathfrak{L}_1 \geq \mathfrak{L}_2$. Now suppose that $\mathfrak{F}$ is not d–homogeneous. Then there are $x, y \in F$ or identical depth and a u of depth k such that $x \lhd u$ but no v of depth k exists such that $y \lhd v$. We choose x of minimal depth with this property. Consider $\mathfrak{F} \uparrow x$ and $\mathfrak{F} \uparrow y$. We claim that these frames are d–homogeneous. For example, if $u, v \in \mathfrak{F} \uparrow x$ and the depth of the two points is less than the depth of x, by minimality of x, u has a successor of depth n iff v has. However, there is exactly one point of depth equal to the depth of x. So, the claim is proved. It follows that $\mathfrak{F} \uparrow x$ and $\mathfrak{F} \uparrow y$ can be mapped onto some linear frames $\mathfrak{L}_1$ and $\mathfrak{L}_2$. By choice of x and y, $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are not isomorphic, but of the same depth. It follows that neither $\mathfrak{L}_1 \leq \mathfrak{L}_2$ nor $\mathfrak{L}_2 \leq \mathfrak{L}_1$ holds. □

Theorem 3.30. *The following sets are definable. (a) $\mathbb{L}$, (b) $\mathbb{D}$, (c) the set of all frames from $\mathbb{D}$ (or $\mathbb{L}$) which have n points.*

Proof. From Lemma 3.26, Lemma 3.29 and Lemma 3.14. □

4. Linear $\mathbb{P}$–Frames

In this section we shall show that in fact every logic of a linear $\mathbb{P}$–frames is definable, from which we can deduce that any logic of linear $\mathbb{P}$–frames is invariant. To achieve this goal, we must first define special classes of frames.

Definition 4.1. *A frame $\mathfrak{P}$ is called **prelinear** if it is a d–homogeneous $\mathbb{C}^1$–frame whose cover is linear.*

Obviously, the class of prelinear frames is definable. Recall the notation $\mathfrak{L}_k$ from the previous section. Lemma 4.6 will establish that the prelinear frames are the frames of the form $\mathfrak{L}_k$, where $\mathfrak{L}$ is linear.

Definition 4.2. *Let $\mathfrak{F}$ be a $\mathbb{P}$–frame. Call x **n–branching** if there exist y_i, $i < n$, pairwise different, such that $x \lhd y_i$ for all $i < n$, and if the generated frame of y_i contains y_j, then $j = i$. x is **branching** if it is at least 2–branching. Call x **same depth n–branching** if it is not $n+1$–branching and there exist y_i, $i < n$, pairwise different of same depth (except for one point whose depth is one less than the depth of x), such that $x \lhd y_i$ for all $i < n$, and if the generated frame of y_i contains y_j then $j = i$. Say that a frame is **same depth branching** if for each node x there is n such that it is same depth n–branching. $\mathbb{R}$ denotes the class of $\mathbb{P}$–frames which are d–homogeneous and such that only the root x is branching.*

Lemma 4.3. $\mathbb{R}$ *is definable.*

Proof. Let α be the following property:

(a) $\mathfrak{F}$ is d–homogeneous and
(b) for all $\mathfrak{L}, \mathfrak{P} \geq \mathfrak{F}$: if $\mathfrak{L}$ is linear and $\mathfrak{P}$ is prelinear then $\mathfrak{P} \leq \mathfrak{L}$.

This is obviously an MSO–statement. Now, we shall show that for a d–homogeneous frame $\mathfrak{F}$, $\mathfrak{F}$ has α iff it is in $\mathbb{R}$. ($\Rightarrow$) Suppose that $\mathfrak{F}$ is not in $\mathbb{R}$. Then there is a node y which is not the root and 2–branching. Let $\mathfrak{P}$ be a prelinear contraction image of $\mathfrak{F} \uparrow y$, and let $\mathfrak{L}$ be the (unique) linear contraction image of $\mathfrak{F}$. Then $\mathfrak{P}$ must have lesser depth than $\mathfrak{L}$, whence $\mathfrak{L} \not\geq \mathfrak{P}$. ($\Leftarrow$) Suppose that $\mathfrak{F}$ is in $\mathbb{R}$. Consider a prelinear frame $\mathfrak{P} \geq \mathfrak{F}$. Then $\mathfrak{P}$ must have the same depth as $\mathfrak{F}$. Then if $\mathfrak{L} \geq \mathfrak{F}$ is linear, $\mathfrak{P} \leq \mathfrak{L}$, since both have the same depth profile. This completes the proof. □

Let $\pi : \mathfrak{F} \twoheadrightarrow \mathfrak{G}$ be a p–morphism. π is called **minimal** if for any p–morphisms ρ, σ such that $\pi = \rho \circ \sigma$, either ρ or σ are bijective. π is **2–collapsing**, if there are two points, x and y, $x \neq y$, such that: (a) $\pi(x) = \pi(y)$ and (b) for all z, z' such that $\{z, z'\} \neq \{x, y\}$: if $\pi(z) = \pi(z')$ then $z = z'$. The pair $\{x, y\}$ is called the **critical pair of** π. Obviously, a 2–collapsing p–morphism is minimal. If $\mathfrak{F}$ is d–homogeneous and $\mathfrak{G}$ a p–morphic image of $\mathfrak{F}$, then by the proof of Lemma 3.14 there exists a series of 2–collapsing p–morpisms mapping $\mathfrak{F}$ onto $\mathfrak{G}$. We distinguish two types of p–morphisms:

Type 1: The distinguished pair is $\{x, y\}$ and there is a z such that $z \lhd x, y$.
Type 2: The distinguished pair is $\{x, y\}$ and no z exists such that $z \lhd x, y$.

Obviously, for $\mathfrak{F} \in \mathbb{R}$ if π is of Type 1 and $z \lhd x, y$, z must be the root. It is easy to see the following:

Lemma 4.4. *Let $\mathfrak{F}$ be a $\mathbb{R}$–frame and $\pi : \mathfrak{F} \twoheadrightarrow \mathfrak{G}$ and $\rho : \mathfrak{G} \twoheadrightarrow \mathfrak{H}$ be 2–collapsing p–morphisms of Type 1. Then there exists $\mathfrak{G}'$ and $\rho' : \mathfrak{F} \twoheadrightarrow \mathfrak{G}'$ and $\pi' : \mathfrak{G}' \twoheadrightarrow \mathfrak{H}$ 2–collapsing of Type 1 such that $\rho \circ \pi = \pi' \circ \rho'$ and the critical pair of π' is the ρ'–image of the critical pair of π.*

Recall that, in general, for a 2–collapsing p–morphism all we need to know to fix $\mathfrak{F}$ up to isomorphism is: (a) $\mathfrak{G}$, (b) the image point of the collapsed pair, and (c) two sets A and B such that $A \cup B$ is the set of predecessors of z. If $A \cup B$ consists of one point only, we have a Type 1 p–morphism. For the sake of brevity, let us introduce the following notation.

Definition 4.5. *Let $\mathfrak{G}$ be a d–homogeneous frame, z a point and A and B sets such that $A \cup B$ is the set of predecessors of z. Then $W_z^{A,B}(\mathfrak{G})$ denotes the frame (unique up to isomorphism) such that there is a 2–collapsing p–morphism $\pi : W_z^{A,B}(\mathfrak{G}) \twoheadrightarrow \mathfrak{G}$ whose critical pair $\{x, y\}$ is mapped onto z, and A is the set of predecessors of x, B the set of predecessors of y. If A, B or z are clear from the context, they are omitted.*

Now let $\mathfrak{F}$ be an $\mathbb{R}$–frame and $\pi : \mathfrak{F} \twoheadrightarrow \mathfrak{G}$ be of Type 1. We claim that $\mathfrak{F}$ is uniquely determined (up to isomorphism) by the following data: (a) $\mathfrak{G}$, (b) the number k such that the for the critical pair $\{x, y\}$, both x and y are of depth k. (It is clear that x and y must be of the same depth.) Namely, we obviously have $\mathfrak{F} \upharpoonright (F - \{x, y\}) \cong \mathfrak{G} \upharpoonright (G - \{\pi(x)\})$. We may actually consider these frames to be identical rather than isomorphic. Let $z := \pi(x)$. Let $u \neq x, y$. Then $u \lhd^F x$ iff $u \lhd^F y$ iff $u \lhd^G z$; $x \lhd^F u$ iff $y \lhd^F u$ iff $z \lhd^G u$. This establishes the claim. We denote $\mathfrak{F}$ by $\mathfrak{G}_k$. By the Lemma 4.4 it is established that $(\mathfrak{G}_m)_n \cong (\mathfrak{G}_n)_m$.

Let us take an $\mathbb{R}$–frame $\mathfrak{F}$. Then there exists a linear frame $\mathfrak{L}$ and a chain of 2–collapsing p–morphisms mapping $\mathfrak{F}$ onto some linear frame $\mathfrak{L}$. $\mathfrak{L}$ is uniquely determined, the chain of p–morphisms is not. As remarked above, the members of the chain are either of Type 1 or of Type 2. Clearly, if $\mathfrak{F}$ is a cocover of $\mathfrak{L}$, the chain consists of a single 2–collapsing p–morphism $\pi : \mathfrak{F} \twoheadrightarrow \mathfrak{L}$. This p–morphism is of Type 1.

Lemma 4.6. *Let $\mathfrak{L}$ be a linear frame and let $\mathfrak{F}$ be an $\mathbb{R}$–cocover of $\mathfrak{L}$. If $\mathfrak{F}$ is not linear, then $\mathfrak{F} \cong \mathfrak{L}_k$ for some k. Moreover, k is such that there exists a point of depth k in $\mathfrak{L}$ which is seen by the root.*

So, for every k such that the root sees a point of depth k, $\mathfrak{L}_k$ exists; and the frames of the form $\mathfrak{L}_k$ are the only $\mathbb{R}$–cocovers of $\mathfrak{L}$ which are not linear. It remains to establish a method such that, given a linear frame $\mathfrak{L}$ and a nonlinear $\mathbb{R}$–cocover $\mathfrak{F}$, we can establish the number k such that $\mathfrak{F} \cong \mathfrak{L}_k$. To that end, let us look at $\mathbb{R}$–frames which are in addition inherently 1–covered.

Definition 4.7. *A frame is called an $\mathbb{R}^n$–**frame** if it is an $\mathbb{R}$–frame and also a $\mathbb{C}^n$–frame.*

Notice that $\mathbb{C}^1 \subseteq \mathbb{R}^1$ and so $\mathbb{C}^1 = \mathbb{R}^1$. It is useful to keep this in mind.

Take the frame $\mathfrak{L}_k$ and an $\mathbb{R}^1$–cocover $\mathfrak{F}$. Then there exists a 2–collapsing p–morphism $\pi : \mathfrak{F} \twoheadrightarrow \mathfrak{L}_k$. Suppose first that π is of Type 1. Then $\mathfrak{F} \cong \mathfrak{L}_{k,m}$ for some m. By Lemma 4.4 we must have $m = k$. So, $\mathfrak{F} \cong \mathfrak{L}_{k,k}$. It is easy to see that we have an infinite downgoing chain of $\mathbb{R}^1$–frames:

$$\mathfrak{L} > \mathfrak{L}_k > \mathfrak{L}_{k,k} > \mathfrak{L}_{k,k,k} > \dots$$

We write $\mathfrak{L}_{k^n}$ for the n–fold iteration of the map $\mathfrak{M} \mapsto \mathfrak{M}_k$, starting with $\mathfrak{L}$. Now let us consider the case when π is of Type 2. If $\rho : \mathfrak{L}_k \twoheadrightarrow \mathfrak{L}$ has distinguished pair $\{x, y\}$, then $\{x, y\}$ is the set of predecessors of the point z which is doubled. If π is of Type 2, several choices appear for A and B (up to renaming): (a) $A = \{x, y\}$ and $B = \{x, y\}$, (b) $A = \{x\}$ and $B = \{y\}$. (c) $A = \{x\}$, $B = \{x, y\}$. However, since $\mathfrak{F}$ is an $\mathbb{R}$–frame, A and B can each only contain one point. (Otherwise, assume that A contains 2 points. Contract $\mathfrak{F}$ such that everything after x and y is linear. Then the map contracting A to a single point is a p–morphism. Likewise, contraction of x and y. So, $\mathfrak{F}$ is not inherently 1–covered.) Therefore, we are in situation (b). So, A and B are uniquely fixed and therefore omitted, and we have $\mathfrak{F} \cong W_z(\mathfrak{L}_k)$.

Now assume that $\mathfrak{G}$ is an $\mathbb{R}^1$–cocover of $\mathfrak{F}$. If $\mathfrak{G} \cong \mathfrak{F}_m$ for some m, then — as is easy to verify — $\mathfrak{G}$ is not inherently 1–covered, since the maps $\mathfrak{G} \twoheadrightarrow \mathfrak{F} \twoheadrightarrow \mathfrak{L}_k$ 'commute'. This is asserted by the following lemma.

Lemma 4.8. *Let $\mathfrak{F}$ be an $\mathbb{R}$–frame and $\pi : \mathfrak{F} \twoheadrightarrow \mathfrak{G}$ and $\rho : \mathfrak{G} \twoheadrightarrow \mathfrak{H}$ 2–collapsing p–morphisms. Then if π is of Type 1 and ρ is of Type 2, there exists $\mathfrak{G}'$ and 2–collapsing p–morphisms $\rho' : \mathfrak{F} \twoheadrightarrow \mathfrak{G}'$ and $\pi' : \mathfrak{G}' \twoheadrightarrow \mathfrak{H}$ such that $\pi' \circ \rho' = \rho \circ \pi$ and π' is of Type 1 and ρ' is of Type 2.*

So, if we have $A = B$, then we never get a $\mathbb{C}^1$–frame. It follows from this first of all the following.

Lemma 4.9. *Suppose that $\mathfrak{F}$ is an $\mathbb{R}^1$–frame, and $u, z \in F$. If and $W_z(\mathfrak{F})$ and $W_u(\mathfrak{F})$ are $\mathbb{R}^1$–frames, then $u = z$.*

Proof. Verify by induction that there is exactly one node having two immediate predecessors. Clearly, if $W_z(\mathfrak{F})$ is defined, z must have two immediate predecessors. This proves the claim. □

This allows to drop mentioning the world z. Now we conclude the following.

Lemma 4.10. *Let $\mathfrak{L}$ be linear and $k > 0$. $\mathfrak{L}_k$ has exactly two $\mathbb{R}^1$–cocovers: $\mathfrak{L}_{k,k}$ and $W(\mathfrak{L}_k)$. If $k = 0$, $\mathfrak{L}_k$ has exactly one $\mathbb{R}^1$–cocover, $\mathfrak{L}_{k,k}$.*

Similarly, the following is established.

Lemma 4.11. *Let $k > 0$, $m > 0$. $W^m(\mathfrak{L}_k)$ has at most one $\mathbb{R}^1$–cocover. If $m = k$, it has no $\mathbb{R}^1$–cocover. If $m < k$, $W^{m+1}(\mathfrak{L}_k)$ is the unique $\mathbb{R}^1$–cocover of $W^m(\mathfrak{L}_k)$.*

Let us put this together. We are given a linear frame $\mathfrak{L}$. We want to establish its structure. First, we can determine its depth. Let it be d. We want to know, given two numbers $m, n \leq d$, whether $\langle m, n\rangle \in \wp(\mathfrak{L})$. To that end we establish first the generated subframe of depth m. (This is easily done; just go $d - m$ steps up in $\langle \mathbb{I}, \leq\rangle$.) For simplicity, we therefore assume $m = d$. Now, take an $\mathbb{R}^1$–cocover $\mathfrak{M}$ of $\mathfrak{L}$. If it has exactly one $\mathbb{R}^1$–cocover, then $\mathfrak{M} \cong \mathfrak{L}_0$, and the root sees a point of depth 0. Otherwise, $\mathfrak{M}$ has two $\mathbb{R}^1$–cocovers, $\mathfrak{N}_1$ and $\mathfrak{N}_2$. One of them, say $\mathfrak{N}_1$, has an infinite downgoing chain of $\mathbb{R}^1$–frames below it. Now take the number p of $\mathbb{R}^1$–frames strictly below $\mathfrak{N}_2$. Then $\mathfrak{M} \cong \mathfrak{L}_{p+1}$.

Lemma 4.12. *Let i, j be natural numbers and $i > j$. The class of all linear frames $\mathfrak{L}$ such that $\langle i, j\rangle \in \wp(\mathfrak{L})$ is definable.*

Proof. We first define the class of such frames which are of depth i. From this the result follows easily. (Take the property of being below a frame $\mathfrak{L}$ of depth i with $\langle i, j\rangle \in \wp(\mathfrak{L})$.) The property in question is:

> $\mathfrak{L}$ has codimension i and has a $\mathbb{C}^1$–cocover $\mathfrak{F}$ such that there are only j many $\mathbb{C}^1$–frames below $\mathfrak{F}$.

Clearly, $\mathfrak{F} \leq \mathfrak{L}$ has finitely many $\mathbb{R}^1$–frames below it only if it has the form $W^p(\mathfrak{L}_q)$ for some $p > 0$ and $q \geq p$. Since there are j many such frames, we must have $q = j$. This means however that $\langle i, j\rangle \in \wp(\mathfrak{L})$. ☐

Theorem 4.13. *Let Θ be the logic of a linear frame. Then Θ is definable.*

Proof. Let $\mathfrak{L}$ be linear such that $\Theta = \mathsf{Th}\,\mathfrak{L}$. Let $\mathfrak{L}$ have depth d. For a frame $\mathfrak{M}$: $\mathfrak{M} \cong \mathfrak{L}$ iff (a) $\mathfrak{M}$ is a $\mathbb{P}$–frame of depth d, (b) for all $0 < i, j \leq d$: $\langle i, j\rangle \in \wp(\mathfrak{L})$ iff $\langle i, j\rangle \in \wp(\mathfrak{M})$. This is a finite conjunction of definable properties, and so definable. ☐

Corollary 4.14. *Let Θ be the logic of a given set of linear frames. Then Θ is invariant. In particular, **K.alt**$_1$ and every extension of **G.3** is invariant.*

Let us cash out a little bit further on the results established so far. Notice that the logics $\mathsf{Th}\,\mathfrak{L}_{k^n}$ are all definable. This opens the way for the following result.

Definition 4.15. *Let $\mathfrak{F}$ be a $\mathbb{P}$–frame and x a point of depth d. We say that x is **directly** n–**branching** if x has exactly n successors of depth $d - 1$.*

Theorem 4.16. *The class of d–homogeneous frames of depth $\geq \delta$ which have an at least (at most, exactly) directly n–branching point of depth d is definable.*

Proof. Let $\mathfrak{F} \leq \mathfrak{L}_{k^{n-1}}$ for some linear frame $\mathfrak{L}$ of depth $\leq \delta$. Then it is immediately verified that $\mathfrak{F}$ contains a point of depth $\leq \delta$ which is immediately $\geq n$–branching. Conversely, suppose that $\mathfrak{F}$ contains an immediately at least n–branching point x of depth $d \leq \delta$. Take $\mathfrak{G} := \mathfrak{F} \uparrow x$. For all $k < d - 1$, contract all points of depth k onto a single point. The resulting frame is isomorphic to a frame $\mathfrak{L}_{k^p}$ for some $\mathfrak{L}$ of depth d and some $p \geq n$. This frame is contractible to a frame $\mathfrak{L}_{k^n}$. ☐

Lemma 4.17. *Let $\mathfrak{F}$ be a d–homogeneous frame of width $\geq n$. Then there exists a contraction image $\mathfrak{G}$ which has width exactly n. There is a unique largest frame with this property.*

Theorem 4.18. *The class of d–homogeneous frames of depth $\leq \delta$ which have at least (at most, exactly) n points of depth d is definable.*

Proof. The proof is by induction on d. Suppose the claim is true of all $k < d$. Then take the frame $\mathfrak{G} \geq \mathfrak{F}$ of largest size such that it has exactly one point of depth k for all $k < d$. Since the number of points of depth $k < d$ can be established by induction hypothesis, we can also construct $\mathfrak{G}$, since it is uniquely defined by its description, by Lemma 4.17. It is the result of reducing $\mathfrak{F}$ to a linear frame at depth $< d$. This frame exists, again by Lemma 4.17. (The reader is asked to note the special case $d = 0$, where is fact nothing needs to be proved, and $\mathfrak{G} \cong \mathfrak{F}$.) Let $\mathbb{H}_{\delta,d}$ the set of all $\mathbb{R}$–frames of depth $\leq \delta$, width 2, which have exactly one point of depth k for all $k < d$. This set is finite, and it is definable. Now, we claim that for all d–homogeneous $\mathfrak{F}$: $\mathfrak{F}$ has only one point of depth d iff $\mathfrak{F} \not\leq \mathfrak{H}$ for all $\mathfrak{H} \in \mathbb{H}_{\delta,d}$. For let $\mathfrak{F} \leq \mathfrak{H}$ for some $\mathfrak{H} \in \mathbb{H}_{\delta,d}$. Then clearly $\mathfrak{F}$ must have two points of depth d. Now assume that $\mathfrak{F}$ has two points of depth d. We contract $\mathfrak{F}$ to a frame of width 2, and of width 1 at all depths $< d$ and call the result $\mathfrak{F}^1$. Let x and y be the points of depth d. Let z be the point of least depth such that $\mathfrak{F}^1 \uparrow z$ contains x and y. Then $\mathfrak{F}^1$ is an $\mathbb{R}$–frame; it is of width 2 and has width 1 at all depths $< d$. Further, $\mathfrak{F}^1 \uparrow z$ has depth $\leq \delta$. So, it is in $\mathbb{H}_{\delta,d}$. This shows that $\mathfrak{F} \leq \mathfrak{H}$ for some $\mathfrak{H} \in \mathbb{H}_{\delta,d}$, as promised. Finally, we need to find the number of points of depth d. In order to do this, let us return to $\mathfrak{F}$ and $\mathfrak{G}$ as constructed above. Let $\mathfrak{H}$ be the largest frame such that $\mathfrak{H} \geq \mathfrak{G}$ and such that $\mathfrak{H}$ has exactly one point at depth d. This frame is uniquely defined. Let m be the size of the longest chain of frames between $\mathfrak{G}$ and $\mathfrak{H}$ (including $\mathfrak{G}$ and $\mathfrak{H}$). Then m is the number of points at depth d. For the contraction of all points of depth d to a single point can be split into a series of $d-1$ 2–collapsing p–morphisms, giving rise to a chain of length d. On the other hand, in between $\mathfrak{G}$ and $\mathfrak{H}$ no other frames exist, that is to say, $\mathfrak{H}$ has the same number of points at all depths $> d$ as $\mathfrak{G}$. This finally concludes the proof. □

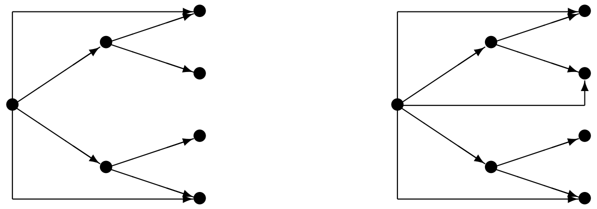
This allows us to define another set of very important frames, namely homogeneously branching trees.

Definition 4.19. *A $\mathbb{P}$–frame $\mathfrak{F}$ is a* ***tree*** *if for all x and y we either have $\mathfrak{F} \uparrow x \subseteq \mathfrak{F} \uparrow y$ or $\mathfrak{F} \uparrow y \subseteq \mathfrak{F} \uparrow x$ or $\mathfrak{F} \uparrow x \cap \mathfrak{F} \uparrow y = \varnothing$. $\mathfrak{F}$ is* ***homogeneously n–branching*** *if every point of depth > 1 is directly n–branching.*

Theorem 4.20. *Let $\mathbb{T}$ be the class of d–homogeneous, homogeneously n–branching trees of depth δ. Then $\mathsf{Th}\,\mathbb{T}$ is invariant.*

Proof. $\mathfrak{F} \in \mathbb{T}$ iff it is of depth δ, is d–homogeneous, n–branching at each depth and has exactly $n^{\delta-h}$ points of depth h. This set is definable by Theorems 3.30, 4.16 and 4.18. □

Unless the depth profile consists only of the pairs $\langle n, n-1\rangle$, this does not fix the trees up to isomorphism. The following trees are d–homogeneous, homogeneously 2–branching, and have the same depth profile.



It requires considerable effort to establish the definability of each individual frame of this kind. We will not go into the details here and concentrate on the linear frames instead.

If the depth profile is minimal, the trees can be fixed up to isomorphism. The logic of homogeneously n–branching trees with minimal depth profile is the following logic.

$$\mathbf{K.T}^n := \mathbf{K.alt}_n.\{\Diamond^{n+1}\top \to \Box^n\Diamond\top : n \in \omega\}$$

This logic is therefore invariant, as any extension determined by homogeneously n–branching trees (which is not to say that every extension is of this kind).

5. Logics of Depth 1

The results of the previous section can be pushed up remarkably by the following technique. Let P be a set of logics, and let Θ be a logic. Then a subset $Q \subseteq P$ is called a P–**subreduction** of Θ if $\bigsqcap Q \leq \Theta$. A set H of logics is P–**stable** if it is a set of logics having the same set of P–subreductions. Θ is called P–stable if $\{\Theta\}$ is P–stable. Then the following is immediate from Lemma 2.6.

Lemma 5.1. *Assume that P is a set of definable logics. If Θ is P–stable, it is invariant.*

The content of the concept of P–stability is the following. Although we have immediate hopes only to define logics of cycle–free frames, we can extend this to other logics as well. Take a frame $\mathfrak{F}$, not cycle–free. Then, by unravelling, there is a sequence $\mathfrak{P}_n$ of cycle–free frames such that $\bigsqcap \mathsf{Th}\,\mathfrak{P}_n \leq \mathsf{Th}\,\mathfrak{F}$. Therefore, $\{\mathsf{Th}\,\mathfrak{P}_n : n \in \omega\}$ forms a $\mathbb{P}$–subreduction of $\mathsf{Th}\,\mathfrak{F}$. If $\mathfrak{F}$ and $\mathfrak{G}$ have different unravellings, then no automorphism can map $\mathfrak{F}$ onto $\mathfrak{G}$. In particular, if no logic has the same unravellings as $\mathfrak{F}$, $\mathfrak{F}$ is invariant.

We shall precisify the notion of unravelling as follows. Let $\mathfrak{F}$ be a Kripke–frame, and $x \in F$. Then denote by $T^n(x, \mathfrak{F})$ the set of points that can be reached in at most n steps from x. This set is called the n–**transit** of x in $\mathfrak{F}$ (see [3]). The subframe $\mathfrak{T}^n(x, \mathfrak{F}) := \mathfrak{F} \upharpoonright T^n(x, \mathfrak{F})$ is likewise called the n–**transit** of x in $\mathfrak{F}$. We omit $\mathfrak{F}$ whenever it is clear which frame is meant. The following is easily shown by induction on n.

Lemma 5.2. *Suppose that $\mathfrak{F}$ and $\mathfrak{G}$ are frames, $x \in F$ and $y \in G$ points and $\mathfrak{T}^n(x, \mathfrak{F}) \cong \mathfrak{T}^n(y, \mathfrak{G})$. Then for every formula φ of modal degree $\leq n$:*

$$\langle \mathfrak{F}, x\rangle \models \varphi \quad \Leftrightarrow \quad \langle \mathfrak{G}, y\rangle \models \varphi$$

Definition 5.3. *An* ***unravelling of*** $\mathfrak{F}$ *is an infinite series* $\underline{G} = \langle \mathfrak{G}_n : n \in \omega \rangle$ *of frames such that the following holds.*

(1) *For all* $n \in \omega$: $\mathfrak{G}_n \subseteq \mathfrak{G}_{n+1}$.
(2) *For all* $n \in \omega$: $\mathfrak{G}_n \not\leq \mathfrak{F}$.
(3) *For each* $m, n \in \omega$ *and* $x \in G_n$ *there exists a* p *such that for all* $q \geq p$:
$$\mathfrak{T}^m(x, \mathfrak{G}_q) = \mathfrak{T}^m(x, \mathfrak{G}_p) \ .$$
(4) $\lim \underline{G} \leq \mathfrak{F}$.
(5) *There exists a* $k \in \omega$ *such that for all* $q \in \omega$ *and* $x \in G_{q+k} - G_k$ *there exists a* $y \in G_q$ *with*
$$\mathfrak{G}_{q+k} \upharpoonright x \cong \mathfrak{G}_q \upharpoonright y \ .$$

The last condition is called the ***cyclicity condition****.* $\underline{G}$ *is called* ***finite*** *if all* $\mathfrak{G}_k$ *are finite.*

Here, $\lim \underline{G}$ is the frame $\langle \bigcup_n G_n, \bigcup_n \lhd_n \rangle$. Obviously, unravellings are some kind of subreductions. However, the existence of some subreduction is not enough for us. What we must establish for stability (and invariance) is which frames a given sequence of frames subreduces. This is what unravellings let us do quite easily.

The following is an easy consequence of cyclicity.

Lemma 5.4. *Let* $\underline{G}$ *be an unravelling and* k *as in Condition 5. Suppose that* $\langle \mathfrak{G}_q, x \rangle \models \varphi$*. Then there exists a* $p \leq q$ *and a* $y \in G_k$ *such that* $\langle \mathfrak{G}_p, y \rangle \models \varphi$.

Proof. Suppose $\langle \mathfrak{G}_q, x \rangle \models \varphi$. If $x \in G_k$ we are done. Otherwise, we have $q \geq k+1$ and there exists a $x' \in G_{q-k}$ such that $\mathfrak{G}_{q-k} \upharpoonright x' \cong \mathfrak{G}_q \upharpoonright x$. It follows that for this x', $\langle \mathfrak{G}_{q-k}, x' \rangle \models \varphi$. Now reason with x' instead of x and $q-k$ instead of q. □

Lemma 5.5. *Let* $\underline{G}$ *be an unravelling of* $\mathfrak{F}$*. Then, given* $x \in G_n$ *and some formula* φ *we either have* $\langle \mathfrak{G}_q, x \rangle \models \varphi$ *for finitely many* q*, or for almost all* q*.*

Proof. Let φ be given. Suppose φ has modal depth m. Then there exists a p such that for all $q \geq p$: $\mathfrak{T}^m(x, \mathfrak{G}_q) = \mathfrak{T}^m(x, \mathfrak{G}_p)$. Whence by Lemma 5.2
$$\langle \mathfrak{G}_q, x \rangle \models \varphi \quad \Leftrightarrow \quad \langle \mathfrak{G}_p, x \rangle \models \varphi$$
□

Lemma 5.6. *Let* $\underline{G}$ *be an unravelling of* $\mathfrak{F}$ *and* $x \in \mathfrak{G}_n$*. Then* $\langle \lim \underline{G}, x \rangle \models \varphi$ *iff* $\langle \mathfrak{G}_q, x \rangle \models \varphi$ *for almost all* q*.*

Proof. Notice that $\mathfrak{T}^m(x, \lim \underline{G}) = \mathfrak{T}^m(x, \mathfrak{G}_q)$ for almost all q. □

Consider the formulae $\varphi_n := p_y \wedge \Box^n \delta(\mathfrak{G})$, where $\delta(\mathfrak{G})$ is the diagram of $\mathfrak{G}$. This is defined as follows.
$$\begin{array}{rrl} \delta(\mathfrak{G}) := & & \bigvee \langle p_y : y \in G \rangle \\ & \wedge & \bigwedge \langle p_x \to \neg p_y : x \neq y \rangle \\ & \wedge & \bigwedge \langle p_x \to \Diamond p_y : x \lhd y \rangle \\ & \wedge & \bigwedge \langle p_x \to \neg \Diamond p_y : x \not\lhd y \rangle \end{array}$$

Definition 5.7. *Let* $\mathfrak{F}$ *and* $\mathfrak{G}$ *be frames,* $x \in F$*, and* $h : F \to G$ *a map.* h *is* n*–****localic with respect to*** x *if the following holds.*

(1) *If* $y \lhd^F z$ *for* $y, z \in \mathfrak{T}^n(x, \mathfrak{F})$ *then* $h(y) \lhd^G h(z)$.
(2) *If* $y \in \mathfrak{T}^{n-1}(x, \mathfrak{F})$ *and* $h(y) \lhd^G u$ *then there exists a* $z \in F$ *such that* $y \lhd^F z$ *and* $h(z) = u$.

Lemma 5.8. $p_y \wedge \Box^n \delta(\mathfrak{G})$ *can be satisfied at* x *in* $\mathfrak{F}$ *iff there exists an* n*–localic map from* $\mathfrak{F} \uparrow y$ *onto* $\mathfrak{G} \uparrow x$.

The following can be found in Kracht [3].

Theorem 5.9. *Let* $\mathfrak{G}$ *be a finite frame.* $\Delta := \{p_y \wedge \Box^n \delta(\mathfrak{G}) : n \in \omega\}$ *is satisfiable in a frame* $\mathfrak{F}$ *iff* $\mathfrak{G}$ *is a generated subframe of some contraction of an ultraproduct of* $\mathfrak{F}$ *iff* $\mathfrak{G} \geq \mathfrak{F}$. *If* $\mathfrak{F}$ *is finite, then* $\mathfrak{G}$ *is the contraction image of some generated subframe of* $\mathfrak{F}$.

Lemma 5.10. *Let* $\underline{G}$ *be a finite unravelling of* $\mathfrak{F}$ *and* $\mathfrak{K}$ *a finite frame. Then there exists a point* x *and a valuation* β *such that* $\langle \lim \underline{G}, \beta, x\rangle \models \{p_y \wedge \Box^n \delta(\mathfrak{K}) : n \in \omega\}$ *iff for all* n, $p_y \wedge \Box^n \delta(\mathfrak{K})$ *is satisfiable in almost all* $\mathfrak{G}_m$.

Proof. ($\Rightarrow$) Suppose that $\langle \lim \underline{G}, \beta, x\rangle \models \{p_y \wedge \Box^n \delta(\mathfrak{K}) : n \in \omega\}$ for some x and some β. Let n be given. Then there exists a q such that $\mathfrak{T}^{n+1}(x, \mathfrak{G}_q) \cong \mathfrak{T}^{n+1}(x, \lim \underline{G})$. Hence, for the restriction β_q of β to $\mathfrak{G}_q$ we have $\langle \mathfrak{G}_q, \beta_q, x\rangle \models p_y \wedge \Box^n \delta(\mathfrak{K})$. So, for each n, $p_y \wedge \Box^m \delta(\mathfrak{K})$ is satisfiable in at least some member of the unravelling. Now, by Condition 3 of unravellings, $\mathfrak{T}^{n+1}(x, \mathfrak{G}_q) \cong \mathfrak{T}^{n+1}(x, \mathfrak{G}_p)$ for almost all p. Hence, $p_y \wedge \Box^n \delta(\mathfrak{K})$ is satisfiable in almost all $\mathfrak{G}_m$. ($\Leftarrow$) Suppose that for all n, $p_y \wedge \Box^n \delta(\mathfrak{K})$ is satisfiable in almost all $\mathfrak{G}_m$. By Lemma 5.4, we may assume that $p_y \wedge \Box^n \delta(\mathfrak{K})$ is satisfiable in some $\mathfrak{G}_p$ at a point in G_k. Since G_k is finite, there must be a point $x \in G_k$ such that x satisfies for each m the formula $p_y \wedge \Box^m \delta(\mathfrak{K})$ in $\mathfrak{G}_{p_m}$ for some p_m (under some valuation β_m). It is not necessarily the case that $\beta_{m+1}(p_x) \cap G_{p_m} = \beta_m(p_x)$ for all $x \in K$. However, by choosing an appropriate subsequence we get (for infinitely many and so) for each m a model

$$\langle \mathfrak{G}_{p_m}, \beta_m, x\rangle \models p_y \wedge \Box^m \delta(\mathfrak{K})$$

and $\beta_n(p_x) \cap G_m = \beta_m(p_x)$ for all $x \in K$ and all m, n such that $n \geq m$. Put $\beta(p_x) := \bigcup \langle \beta_m(p_x) : m \in \omega\rangle$. This is a valuation into $\lim \underline{G}$. Choosing this valuation we now have

$$\langle \lim \underline{G}, \beta, x\rangle \models \{p_y \wedge \Box^m \delta(\mathfrak{K}) : m \in \omega\} \;.$$

$\Box$

Theorem 5.11. *Let* $\underline{G} = \langle \mathfrak{G}_n : n \in \omega\rangle$ *be a finite unravelling of* $\mathfrak{F}$. *Then the following holds for any finite rooted frame* $\mathfrak{H}$*: either (a)* $\mathfrak{H} \geq \mathfrak{G}_n$ *for some* n*, or (b)* $\mathfrak{H} \geq \lim \underline{G}$*, or (c)* $\mathfrak{H}$ *is not subreduced by* $\underline{G}$.

Proof. Consider $\Delta := \{p_y \wedge \Box^m \delta(\mathfrak{H}) : m \in \omega\}$, where y is the root of $\mathfrak{H}$. (Case 1) Δ is simultaneously satisfiable in some $\mathfrak{G}_n$. Then, since $\mathfrak{G}_n$ is finite, some generated subframe of $\mathfrak{G}_n$ can be mapped onto $\mathfrak{H}$, and $\mathfrak{H} \leq \mathfrak{G}_n$. (Case 2) Δ is not simultaneously satisfiable in some $\mathfrak{G}_n$, but every member of Δ is satisfiable somewhere. It follows by Lemma 5.10 that every member is almost always satisfiable, and so there exists a point $x \in \lim \underline{G}$ such that Δ is satisfiable at x in $\lim \underline{G}$. So, there exists a series $\pi^m : \lim \underline{G} \twoheadrightarrow \mathfrak{H}$ of maps such that π^m is m–localic with respect to x for each m. Now, for each m there exists an infinite subset $N_m \subseteq \omega$ such that $\pi^p \upharpoonright T^m(x, \lim \underline{G}) = \pi^q \upharpoonright T^m(x, \lim \underline{G})$ for all $p, q \in N_m$. We may choose N_m inductively such that $N_m \supseteq N_{m+1}$. Define $\rho : \lim \underline{G} \twoheadrightarrow \mathfrak{H}$ as follows. If $y \in T^m(x, \lim \underline{G})$ then $\rho(y) := \pi^p(y)$, where $p \in N_m$ is arbitrary. By choice of the sets N_m, this does not depend on the number p. One easily shows that ρ is m–localic with respect to x for each m, which is to say that it is a p–morphism. (Case 3) Almost all members of Δ are nowhere satisfiable in any of $\mathfrak{G}_n$. Then $\mathfrak{H}$ is not subreduced by $\underline{G}$, by the remarks above. □

Corollary 5.12. *Let $\underline{G}$ be a finite unravelling. Suppose that all $\mathfrak{G}_m$ are invariant. Then $\lim \underline{G}$ is invariant as well.*

Proof. $\lim \underline{G}$ is an intersection of logics with finite model property, and therefore has the finite model property as well. From Theorem 5.11 it follows that $\mathsf{Th} \lim \underline{G} \geq \bigcap \langle \mathsf{Th}\, \mathfrak{G}_m : m \in \omega \rangle$. The latter is invariant, by Lemma 2.6. Now, $\mathsf{Th} \lim \underline{G}$ is the intersection of the logics of all finite frames which are above the intersection of all $\mathsf{Th}\, \mathfrak{G}_m$, but not above any $\mathfrak{G}_m$. This is again invariant. □

We shall apply this technique to cyclic frames.

Definition 5.13. *A Kripke–frame $\mathfrak{F}$ is called* ***cyclic*** *if it is not the singleton irreflexive point and $\mathfrak{F}$ possesses at most two nonisomorphic generated subframes: the empty frame, and $\mathfrak{F}$ itself.*

The following theorem shows why the name cyclic is appropriate.

Lemma 5.14. *Let $\mathfrak{F}$ be a finite cyclic Kripke–frame. Then $F = \{c_i : i < n\}$ and $\lhd \supseteq \{\langle c_i, c_j \rangle : j \equiv i + 1 \pmod{n}\}$.*

Let $\mathfrak{F}$ be a finite, cyclic Kripke–frame. Then put $\mathfrak{U}_\omega(\mathfrak{F}) := \langle \omega, \blacktriangleleft \rangle$, where $p \blacktriangleleft q$ iff (a) $q = p + 1$ or (b) $p = \gamma n + j$, $q = \gamma n + k$ for some $\gamma \in \omega$ and $j < k < n$ such that $c_j \lhd c_k$ or (c) $p = \gamma n + j$, and $q = (\gamma + 1)n + k$ for some $\gamma \in \omega$ and $k \leq j < n$ such that $c_j \lhd c_k$. (Actually, (a) is a subcase of (b).) Set $\mathfrak{U}_p(\mathfrak{F}) := \mathfrak{U}_\omega(\mathfrak{F}) \upharpoonright \{0, 1, \ldots, p - 1\}$.

Lemma 5.15. *$\{\mathfrak{U}_p(\mathfrak{F}) : p \in \omega\}$ is an unravelling and an $\mathbb{L}$–subreduction of $\mathfrak{F}$.*

Proof. The proof is a longish verification of the details. We shall only do part of the work. Clearly, $\mathfrak{U}_p(\mathfrak{F})$ is linear for every $p \in \omega$, so we need to show only that we have an unravelling. Condition 1 is obviously satisfied. Condition 2 is also easy. The unravelling frames are cycle free. For Condition 3 $p \geq m + n$ is clearly sufficient. For Condition 4 one only needs to show that there exists a p–morphism from $\mathfrak{U}_\omega(\mathfrak{F})$ onto $\mathfrak{F}$. But this is easy. Finally, Condition 5. Put $k := n$, where n is the size of the cycle. Consider the point j of $\mathfrak{U}_{q+n}(\mathfrak{F})$ Suppose that $j \geq n$, then $\mathfrak{U}_{q+n}(\mathfrak{F}) \upharpoonright j \cong \mathfrak{U}_q(\mathfrak{F}) \upharpoonright j - n$, as is easily verified. (The isomorphism is given by $i \mapsto i - n$.) □

We shall start with the logics of the frames $\mathfrak{Loop}_n$. We have $\mathfrak{U}_\omega(\mathfrak{Loop}_n) \cong \langle \omega, \blacktriangleleft \rangle$ regardless of n, where $i \blacktriangleleft j$ iff $j = i + 1$. So, using $\mathfrak{U}_\omega$ we cannot discriminate between any of these loops. Fortunately, we can revise our definition as follows. Let

$\mathfrak{U}^1_\omega(\mathfrak{F})$ differ from $\mathfrak{U}_\omega(\mathfrak{F})$ in that for each i, j such that $i \blacktriangleleft j$, the arc $\langle i, i+j+n\rangle$ is added. Let $\mathfrak{U}^1_q(\mathfrak{F})$ be the subframe of $\mathfrak{U}^1_\omega(\mathfrak{F})$ of the first q points. It is shown as above that $\{\mathfrak{U}^1_q(\mathfrak{F}) : q \in \omega\}$ is an $\mathbb{L}$–subreduction of $\mathfrak{F}$ if $\mathfrak{F}$ is cyclic. Now consider a p–morphism $\pi : \mathfrak{U}^1_\omega(\mathfrak{Loop}_n) \twoheadrightarrow \mathfrak{Loop}_m$. Take a point j. It has two successors, $j+1$ and $j+1+n$. However, $\pi(j)$ has only one successor. So, $\pi(j+1+n) = \pi(j+1)$, for all j. So we conclude that $m \leq n$. (Since π factors through the natural projection, $\rho_n : \mathfrak{U}^1_\omega(\mathfrak{Loop}_n) \twoheadrightarrow \mathfrak{Loop}_n$ we even have $m|n$.) So, let $Q_n := \{\mathfrak{U}^1_q(\mathfrak{Loop}_n) : q \in \omega\}$. Then Q_n is not a subreduction of any $\mathfrak{Loop}_m$ where $m > n$. Thus, we have shown that all loops have different subreduction sequences. A last detail is missing. Let us note that if a frame has the subreduction sequence $\{\mathfrak{Ch}_q : q \in \omega\}$ iff it is an $\mathbf{K.alt}_1$–frame, we see that each cyclic $\mathbf{K.alt}_1$–frame is uniquely characterized by its $\mathbb{L}$–subreduction sequence.

Theorem 5.16. *Every extension of* $\boldsymbol{K.alt}_1$ *is invariant.*

Proof. We may restrict our attention to $\bigsqcap$–irreducible logics. These are the logics of the chains, or of the frames $\mathfrak{Loop}_{p,q} := \langle\{0, 1, \ldots, p+q\}, \lhd\rangle$, where $i \lhd j$ iff $j = i+1$ or $i = p+q$ and $j = p$. In our previous notation, $\mathfrak{Loop}_n = \mathfrak{Loop}_{0,n}$. We have established that $\mathfrak{Loop}_{0,n}$ is invariant for all n. Now observe that $\mathfrak{Loop}_{p,n}$ is characterized by the following: (A) its logic is $\sqcap$–irreducible, (B) the largest loop $\geq \mathfrak{Loop}_{p,n}$ is $\mathfrak{Loop}_{0,n}$, (C) there are $p+1$ many irreducible logics between $\mathfrak{Loop}_{p,n}$ and $\mathfrak{Loop}_{0,p}$, namely all logics of the form $\mathfrak{Loop}_{j,n}$. Now it is not hard to see that also $\mathfrak{Loop}_{p,n}$ is invariant for all p and n. □

This does not yet establish that all logics of depth 1 are invariant. But this will follow from the results of the next section, which are even far more general.

6. Linear Logics

Finally, let us extend the notion of linearity to all finite Kripke–frames in the following way.

Definition 6.1. *Let $\mathfrak{F}$ be a finite Kripke–frame and x and y points of $\mathfrak{F}$. x and y are called **g–equivalent** if $\mathfrak{F} \uparrow x = \mathfrak{F} \uparrow y$. x is of **depth 1** if every $y \in \mathfrak{F} \uparrow x$ is g–equivalent with x. x is of **depth n + 1** iff for all $y \in \mathfrak{F} \uparrow x$: either y is of depth n or y is g–equivalent to x. $\mathfrak{F}$ is of **depth n** if it has a point of depth n but no point of depth $n+1$.*

Definition 6.2. *Let $\mathfrak{F}$ be a finite Kripke–frame. $\mathfrak{F}$ is **linear** if for all m and n such that $m \leq n$: if x is of depth n and y of depth m, then $y \in \mathfrak{F} \uparrow x$.*

Now we are going to prove that every logic of linear frames is invariant. The proof will be done by induction on the number of cycles.

Definition 6.3. *A **cycle** is a maximal set of g–equivalent points which is not identical to a set containing a single irreflexive point. The number of cycles of a given frame $\mathfrak{F}$ is denoted by $cyc(\mathfrak{F})$.*

Consider the frame $\mathfrak{F}$. Suppose first that it is generated by x, and that x is contained in a cycle C. (This assumption shall be retracted later.) We may assume that $C = \{c_i : i < n\}$, where $\lhd \supseteq \{\langle c_i, c_j\rangle : i \equiv i+1 \pmod{n}\}$. Without loss of generality, $x = c_0$. We construct the following frames.

$$\begin{array}{rcl} U(\mathfrak{F}) & := & \langle U(F), \lhd^U\rangle \\ V(\mathfrak{F}) & := & \langle U(F), \lhd^V\rangle \end{array}$$

Here,

$$U(F) := n \times \{0\} \cup C \times \omega \times \{1\} \cup (F - C) \times \{2\}$$

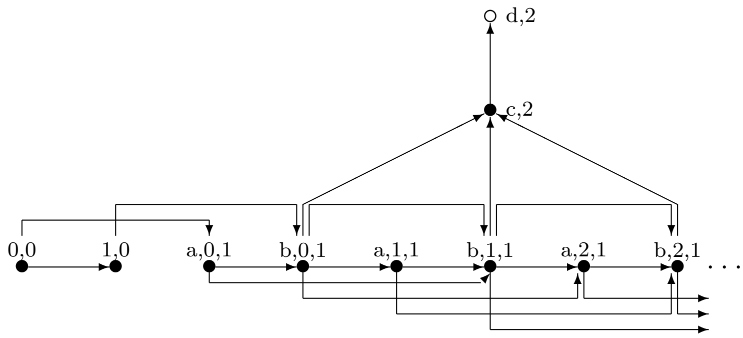
The relations are defined as follows.

$$\lhd^U := \left\{ \begin{array}{ll} & \{\langle\langle i, 0\rangle, \langle i+1, 0\rangle\rangle : i+1 < n\} \\ \cup & \{\langle\langle i, 0\rangle, \langle c_i, 0, 1\rangle\rangle : i < n\} \\ \cup & \{\langle\langle c_i, j, 1\rangle, \langle c_k, j, 1\rangle\rangle : i < k, c_i \lhd c_k\} \\ \cup & \{\langle\langle c_i, j, 1\rangle, \langle c_k, j+1, 1\rangle\rangle : c_i \lhd c_k\} \\ \cup & \{\langle\langle c_i, j, 1\rangle, \langle x, 2\rangle\rangle : c_i \lhd x\} \\ \cup & \{\langle\langle x, 2\rangle, \langle y, 2\rangle\rangle : x \lhd y\} \end{array} \right.$$

$$\lhd^V := \lhd^U \cup \{\langle\langle i, 0\rangle, \langle c_i, j, 1\rangle\rangle : i < n, j \in \omega\}$$

To give an example, consider the following frame. (Here, $\bullet$ denotes an irreflexive and $\circ$ a reflexive point.)

Then $U(\mathfrak{F})$ has the following form (where $c_0 = a$, $c_1 = b$).



The frames $U^p(\mathfrak{F})$ and $V^p(\mathfrak{F})$ are obtained from the frames $U(\mathfrak{F})$ and $V(\mathfrak{F})$ by restricting the set of worlds to the set $n \times \{0\} \cup C \times p \times \{1\} \cup (F - C) \times \{2\}$. Further, we let $U^{-k}(\mathfrak{F})$ (and $V^{-k}(\mathfrak{F})$) denote the subframe obtained by removing the points $\langle i, 0\rangle$ for all $i < k$. It is clear from the results of the previous section that $U^p(\mathfrak{F})$ subreduces $U(\mathfrak{F})$ and $V^p(\mathfrak{F})$, since they are unravellings. Moreover, for a finite rooted $\mathfrak{G}$, by Theorem 5.11, $\mathfrak{G}$ is subreduced by $\{U^p(\mathfrak{F}) : p \in \omega\}$ if (1) $\mathfrak{G} \geq U^p(\mathfrak{F})$ for no $p \in \omega$, and (2) $\mathfrak{G} \geq U(\mathfrak{F})$.

Notice that $cyc(U^p(\mathfrak{F})) = cyc(\mathfrak{F}) - 1$. We shall investigate the logics of finite frames that are subreduced by $U^p(\mathfrak{F})$ and $V^p(\mathfrak{F})$.

Definition 6.4. *Let $\mathbb{S}(\mathfrak{F})$ be the set of finite $\mathbb{I}$–frames subreduced both by $\{U^p(\mathfrak{F}) : p \in \omega\}$ and $\{V^p(\mathfrak{F}) : p \in \omega\}$.*

$\mathbb{S}(\mathfrak{F})$ is a convex subset of $\langle \mathbb{I}, \leq \rangle$.

Lemma 6.5. *Suppose that $\mathfrak{G} \in \mathbb{S}(\mathfrak{F})$ is generated by a cycle. Then $\mathfrak{G}$ has infinite dimension in $\mathbb{S}(\mathfrak{F})$.*

Proof. First, notice that $\mathfrak{G}$ is not isomorphic to a proper subframe of $\mathfrak{F}$. Otherwise we have $\mathfrak{G} \geq U^p(\mathfrak{F}), V^p(\mathfrak{F})$ for all p even, and so neither $\{U^p(\mathfrak{F}) : p \in \omega\}$ not $\{V^p(\mathfrak{F}) : p \in \omega\}$ is a subreduction of $\mathfrak{G}$. Consider the following frames. $T^k(\mathfrak{F}) := \langle T^k(F), \lhd^k \rangle$, defined by

$$T^k(\mathfrak{F}) := C \times k \cup (F - C) \times \{k\}$$

$$\lhd^k := \begin{cases} & \{\langle \langle c_i, j\rangle, \langle c_{i'}, j+1\rangle\rangle : c_i \lhd c_{i'}, j+1 < k\} \\ \cup & \{\langle \langle c_i, k-1\rangle, \langle c_{i'}, 0\rangle\rangle : c_i \lhd c_{i'}\} \\ \cup & \{\langle \langle c_i, j\rangle, \langle c_{i'}, j\rangle\rangle : c_i \lhd c_{i'}, i < i'\} \\ \cup & \{\langle \langle c_i, j\rangle, \langle x, k\rangle\rangle : j < k, c_i \lhd x\} \\ \cup & \{\langle \langle x, k\rangle, \langle y, k\rangle\rangle : x \lhd y\} \end{cases}$$

Take the generated subframe $U^{-n}(\mathfrak{F})$ of $U(\mathfrak{F})$ consisting of all points with last component 1 or 2. This frame is isomorphic to the generated subframe $V^{-k}(\mathfrak{F})$ of $V(\mathfrak{F})$. This frame can be mapped p–morphically onto $T^k(\mathfrak{F})$. (Namely, put $\pi(\langle x, 2\rangle) := \langle x, k\rangle$, and $\pi(\langle c_i, \lambda k + q, 1\rangle) := \langle c_i, q\rangle$, $q < k$.) Hence, $T^k(\mathfrak{F})$ is subreduced by both sets, and is a member of $\mathbb{S}(\mathfrak{F})$. Now, $T^k(\mathfrak{F}) \leq T^{k'}(\mathfrak{F})$ if k is a multiple of k'. (Again, there is a p–morphism from the latter onto the former.) This shows the claim. □

Lemma 6.6. *Suppose that $\mathfrak{G}$ is a minimal member of $\mathbb{S}(\mathfrak{F})$. Then $\mathfrak{G}$ is a p–morphic image of $U(\mathfrak{F})$ (and $V(\mathfrak{F})$), and $\mathfrak{F}$ a generated subframe of $\mathfrak{G}$.*

Proof. Now, first of all, $\mathfrak{G}$ is not generated by a cycle, by the previous lemma. Moreover, jus as before, $\mathfrak{G}$ contains a full copy of the frame $\mathfrak{F} \restriction (F - C)$. We have $\mathfrak{G} \geq U(\mathfrak{F})$ as well as $\mathfrak{G} \geq V(\mathfrak{F})$. So we must have p–morphisms $\pi : U^{-k}(\mathfrak{F}) \twoheadrightarrow \mathfrak{G}$ and $\rho : V^{-k}(\mathfrak{F}) \twoheadrightarrow \mathfrak{G}$ for some $k < n$, by these assumptions. Clearly, by minimality of $\mathfrak{G}$, $k = 0$. Consider now the point $\langle k, 0\rangle$. In the first frame it sees at most two points, in the latter infinitely many. It follows that $\rho(\langle c_k, \lambda, 1\rangle) = \rho(\langle c_k, \mu, 1\rangle)$ for all $\lambda, \mu \in \omega$ and for all $k < n$. Hence, there is a cycle isomorphic to C. This cycle generates a frame isomorphic to $\mathfrak{F}$, since we have assumed that C generates $\mathfrak{F}$. □

Unfortunately, $\mathfrak{G}$ is not isomorphic to $\mathfrak{F}$. Therefore, we must perform some complex reasoning. Let $lc(\mathfrak{F})$ be the size of the cycle of largest depth in $\mathfrak{F}$ (if that exists; otherwise, this number is zero). We show by induction on $cyc(\mathfrak{F})$ and $lc(\mathfrak{F})$ that $\mathfrak{F}$ is invariant.

Theorem 6.7. *Let $\mathfrak{F}$ be a linear frame. Then $\mathfrak{F}$ is invariant.*

Proof. The proof proceeds by induction on $cyc(\mathfrak{F})$ and $lc(\mathfrak{F})$. The claim proved inductively is the following.

(‡) Let p and q be natural numbers. $\mathsf{Th}\,\mathfrak{F}$ is invariant for all frames $\mathfrak{F}$ such that

(1) $cyc(\mathfrak{F}) < p$ or
(2) $cyc(\mathfrak{F}) = p$ and $lc(\mathfrak{F}) < q$

Put $\langle p, q\rangle \ll \langle p', q'\rangle$ if (i) $p < p'$ or (ii) $p = p'$ and $q < q'$. We show that the claim holds for $\langle p^*, q^*\rangle$ provided that it holds for all $\langle p, q\rangle \ll \langle p^*, q^*\rangle$. The induction starts therefore with $\langle 0, 0\rangle$. By invoking Theorem 4.13 we know that (‡) holds for $\langle 1, 0\rangle$. Notice that if (‡) holds for all $\langle p, q\rangle$ such that $p < p^*$, then it holds for $\langle p^*, q^*\rangle$, by virtue of its definition. Hence, we only need to prove the step from $\langle p, q\rangle$ to $\langle p, q+1\rangle$.

Let $\mathfrak{F}$ be given, with $cyc(\mathfrak{F}) = p$ and $lc(\mathfrak{F}) = q$. We first consider the case where $\mathfrak{F}$ is generated by a cycle. Consider the frame $\mathfrak{G}$ which is minimal in $\mathbb{S}(\mathfrak{F})$. By Lemma 6.6, we know that $\mathfrak{F}$ is a generated subframe of $\mathfrak{G}$. Let C be the leading cycle of $\mathfrak{F}$, $\mathfrak{F}^\circ := \mathfrak{F} \restriction (F - C)$. By induction hypothesis, $\mathfrak{F}^\circ$ is invariant. Let $\mathfrak{H}$ be the least frame $\geq \mathfrak{G}$ that has infinite dimension in $\mathbb{S}(\mathfrak{F})$. We know that $\mathfrak{H} < \mathfrak{F}^\circ$ and that $\mathfrak{H} > \mathfrak{G}$. Hence, as is easy to see, $\mathfrak{H} \cong \mathfrak{F}$. So, $\mathfrak{F}$ is invariant.

Now we consider the case where $\mathfrak{F}$ is not generated by a cycle. Let C be the cycle of largest depth, D the largest generated subset of $\mathfrak{F}$ not containing C, and $A := F - (C \cup D)$. Then define the following unravellings. Put

$$\begin{array}{rcl} X(\mathfrak{F}) & := & \langle U(F), \lhd^X\rangle \\ Y(\mathfrak{F}) & := & \langle U(F), \lhd^Y\rangle \end{array}$$

$$\lhd^X := \begin{cases} & \{\langle\langle y, 0\rangle, \langle c_i, 0, 1\rangle\rangle : y \lhd c_i, y \in A\} \\ \cup & \{\langle\langle c_i, j, 1\rangle, \langle c_{i'}, j+1, 1\rangle\rangle : c_i \lhd c_{i'}\} \\ \cup & \{\langle\langle c_i, j, 1\rangle, \langle c_{i'}, j, 1\rangle\rangle : c_i \lhd c_{i'}, i < i'\} \\ \cup & \{\langle\langle c_i, j, 1\rangle, \langle y, 0\rangle\rangle : c_i \lhd y, y \in D\} \\ \cup & \{\langle\langle x, 0\rangle, \langle\langle y, 0\rangle\rangle : x \lhd y, x, y \in A \cup D\} \end{cases}$$

Further, let

$$\lhd^Y := \lhd^X \cup \{\langle\langle y, 0\rangle, \langle c_i, j, 1\rangle\rangle : y \lhd c_i, j \in \omega\}$$

These define unravellings in the natural way, by restricting to finite parts of the unravelled cycle. These frames have one cycle less than $\mathfrak{F}$, and therefore the induction hypothesis applies to them. This means that they are invariant. Consider the set of frames subreduced by both sets. Denote this set by $\mathbb{T}(\mathfrak{F})$. By analogous reasoning we find that $\mathbb{T}(\mathfrak{F})$ has no elements of infinite dimension. Consider a minimal element $\mathfrak{M}$ in $\mathbb{T}(\mathfrak{F})$. M must contain the set A. So it is a p–morphic image of $X(\mathfrak{F})$ and $Y(\mathfrak{F})$. Hence, it is obtained from these frames by squashing the set $C \times \omega \times \{1\}$. It is now easy to see that $\mathfrak{M} \cong \mathfrak{F}$. □

Corollary 6.8. *Every logic determined by finite linear frames is invariant.*

Since every extension of **S4.3** has the finite model property, we have the following result.

Corollary 6.9. *Every extension of* ***S4.3*** *is invariant.*

7. Conclusion

Continuing the research of [2], where it was established that all extensions of **S4.3** are invariant under the automorphisms of the lattice NExt **S4**, we have shown here that they are invariant under all automorphisms of NExt **K**. In contrast to that, the lattice NExt $\mathbf{K.alt}_1$ was shown in [2] to have $2^{\aleph_0}$ automorphisms, but here we have established that only the identity extends to an automorphism of NExt $\mathbf{K}$, since all extensions of $\mathbf{K.alt}_1$ are invariant. Although this is just a very modest beginning, the present paper shows that completely new techniques must be developed in order to obtain the results. We believe that it can be established with the present methods that all d–homogeneous frames are invariant, and that the unravelling technique allows to extend this result even further. With respect to $\mathbb{P}$–frames in general our intuitions are not so well developed yet.

References

[1] Wim Blok. On the degree of incompleteness in modal logics and the covering relation in the lattice of modal logics. Technical Report 78 – 07, Department of Mathematics, University of Amsterdam, 1980.

[2] Marcus Kracht. Lattices of Modal Logics and Their Groups of Automorphisms. *Journal of Pure and Applied Logic*, 100:99 – 139, 1999.

[3] Marcus Kracht. *Tools and Techniques in Modal Logic.* Number 142 in Studies in Logic. Elsevier, Amsterdam, 1999.

[4] David C. Makinson. Some embedding theorems for modal logic. *Notre Dame Journal of Formal Logic*, 12:252 – 254, 1971.

[5] Wolfgang Rautenberg. *Klassische und nichtklassische Aussagenlogik.* Vieweg, Braunschweig/Wiesbaden, 1979.

II. Math. Institut, *FU Berlin*, *Arnimallee 3*, *D – 14195 Berlin*, `kracht@math.fu-berlin.de`