Partial Algebras, Meaning Categories and Algebraization

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Abstract

Many approaches to natural language semantics are essentially model–theoretic, typically cast in type theoretic terms. Many linguists have adopted type theory or many–sorted algebras (see Hendriks 2001 and references therein). However, recently Hodges 2001 has offered an approach to compositionality using just partial algebras. An approach in terms of partial algebras seems at the outset more justified, since the typing is often just artificially superimposed on language (and makes many items look polymorphic). On the other hand, many–sorted algebras are easier to handle than partial algebras, and are therefore generally preferred. This paper investigates the dialectics between partial algebras and many–sorted algebras and tries to set the background for an approach in the spirit of Hodges 2001, which also incorporates insights from algebraic logic from Blok and Pigozzi 1990. The analytic methods that we shall develop here shall also be applied to combinatory algebras and algebraizations of predicate logic.

1 Introduction

The semiotic program by Montague is cast in algebraic terms (for a recent discussion see Hendriks 2001). The compositionality thesis, for example, makes reference only to expressions, their meanings, and functions that take expressions (and meanings, respectively) as their arguments. Compositionality comes down to the requirement that the meaning assignment is a homomorphism. This is the way it has been (re)constructed by Janssen. The formulations by Montague, Janssen and Hendriks all use many–sorted algebras. Yet, it does seem that there is no need for sorts in the first place if we are willing to admit partial functions. Elsewhere (see Kracht 2003) I have sketched a program that basically assumes no typing or sorts, but allows for partiality at the level of strings (or exponents), categories, and meanings. The reason for that is that often enough the partiality is purely arbitrary (certain morphological forms just don't exist, certain meaning combinations are 'not said', and so on). Thus, if we strip off the types we are left with partial algebras. However, the principal problem with this approach from a theoretical point of view is that the algebras are *partial*, and partial algebras are not the structures that algebraists are most comfortable with (see Macintyre 2003 for a complaint about the lack of interest of model–theorists even in many–sorted algebras). In fact, it seems that Hodges 2001 is one of the first attempts to provide a setting for compositionality within partial algebras.

Model–theoretic semantics has the problem of requiring us to fix the truth of each and every sentence in a model. This creates too much overload. Recently, Dresner 2002 has argued that algebraic semantics is actually more suitable for linguistic purposes than is model–theoretic semantics. For example, it is unclear in a model–theoretic setting how people can handle imperfect knowledge of meaning. (For example, there is no scenario how children can grow up learning the language.) Additionally, in the state of minimum knowledge, computation on meanings is unfeasible in a model–theoretic setting, while is trivial in an algebraic setting: there are no equations. On top of all, the functions we need to assume to compute the meaning of $(\forall x)\varphi$ from the meaning of φ are not computable — a point that makes the standard semantics noncompositional in the sense of Kracht 2003. (It is not immediately clear that the algebraic approach avoids this. However, see Kracht 2003 for radically different proposals for semantics which do.) While the arguments by Dresner are well–taken, an immediate question arises: can the alternative analyses (say, in terms of model–theory) be recast algebraically, and can conversely an approach in terms of algebras be remodeled into a type–theoretic one? In particular, what to do with the fact that we are not actually dealing with algebras but with partial algebras? These questions will receive an answer below. In particular, we shall demonstrate that first–order logic and type theory can be recast algebraically, although the algebraizations are not really insightful. Additionally, partial algebras and many–sorted algebras are not very far apart, so that eventually the choice between these approaches becomes a matter of parsimony (and taste).

There is an additional motive for going for partial algebras: type theory provides too much epistructure to worry about. Algebraic accounts turn out to be simpler, at least if we are willing to go partial. Also, often enough, the type–theoretic analysis is just made to fit the facts, which is to say that the typing introduced does not strictly follow from semantic principles. For example, the division into nouns and verbs in type–theoretic analyses is not grounded in semantics: both can denote the same kind of entities (eg trip denotes a set of events, just like (to) travel). In this paper we shall therefore discuss the problem of categorization arising from just one piece of datum: whether or not a particular function can be applied to a given argument. The advantage of the approach in terms of partial algebras is that once they are turned into many–sorted algebras, we can use tools of universal algebra rather than having to use λ–calculus.

2 Algebraic Preliminaries

For background reading in partial and many–sorted algebras see Burmeister 1986 and the somewhat more accessible Burmeister 2002. Where differences between his and our terminology arise, this will be pointed out. A *signature* is a pair $\langle F, \Omega\rangle$, where F is a set and $\Omega : F \to \mathbb{N}$ a function. To harmonize terminology with many–sorted algebras, we agree that if f is interpreted as an n–ary function, we put $\Omega(f) := n + 1$. A *partial Ω–algebra* is a pair $\mathfrak{A} = \langle A, \mathfrak{i}\rangle$, where A is a set and $\mathfrak{i}$ assigns to $f \in F$ a partial $(\Omega(f) - 1)$–ary function on A. An *assignment* is a function v from the variables into A. v defines a unique extension to the terms, also denoted by v. Notice, however, that the extension is in general a partial function. An *equation* is a pair of terms, written $s \approx t$. The equation $s \approx t$ is *weakly valid* in $\mathfrak{A}$ if for every v assigning elements of A to variables such that both $v(s)$ and $v(t)$ are defined, they are equal. The equation is *strongly valid*, in symbols $\mathfrak{A} \vDash^\star s \approx t$, if it is weakly valid and $v(s)$ is defined iff $v(t)$ is. Burmeister 1986 calls these also *Kleene–equations*; his approach is based on an even stronger notion, that of an *existence equation.* Write $s \stackrel{e}{\approx} t$ for such an equation; furthermore, $\mathfrak{A}, v \vDash s \stackrel{e}{\approx} t$ iff $v(s)$ and $v(t)$ exist and are equal. Many–sorted algebras can axiomatized using such equations (because of the use of sorted variables), but partial algebras do not seem to fit that notion easily. Note that

(1) $$\mathfrak{A}, v \vDash^\star s \approx t \quad\Leftrightarrow\quad \mathfrak{A}, v \vDash (s \stackrel{e}{\approx} s \to s \stackrel{e}{\approx} t) \wedge (t \stackrel{e}{\approx} t \to s \stackrel{e}{\approx} t)$$

and that

(2) $$\mathfrak{A}, v \vDash s \approx t \quad\Leftrightarrow\quad \mathfrak{A}, v \vDash (s \stackrel{e}{\approx} s \wedge t \stackrel{e}{\approx} t) \to (s \stackrel{e}{\approx} t)$$

Burmeister 2002 calls an *ECE–equation* a Horn clause of the form

(3) $$(\bigwedge_{i<n} s_i \stackrel{e}{\approx} s_i) \to t \stackrel{e}{\approx} t'$$

(So, the equality of t and t' depends only of the existence of certain terms.) Obviously, weak equations and Kleene–equations are ECE–equations. A *strong homomorphism* between partial algebras $\langle A, \mathfrak{i}\rangle$ and $\langle B, \mathfrak{j}\rangle$ is a total map $h : A \to B$ such that for all $f \in F$ and $\vec{a} \in A^{\Omega(f)-1}$: (a) $\mathfrak{j}(f)(h(\vec{a}))$ is defined iff $\mathfrak{i}(f)(\vec{a})$ is and (b) $\mathfrak{j}(f)(h(\vec{a})) = h(\mathfrak{i}(f)(\vec{a}))$. $\mathfrak{A} = \langle A, \mathfrak{i}\rangle$ is a *subalgebra* of $\mathfrak{B} = \langle B, \mathfrak{j}\rangle$ if $A \subseteq B$ and $\mathfrak{i}(f) = \mathfrak{j}(f) \cap A^{\Omega(f)}$. $\mathfrak{A}$ is a *strong subalgebra* if it is a subalgebra and the embedding is a strong homomorphism.

An equivalence relation Θ is a *weak congruence* if for every $f \in F$ and every $\vec{a}, \vec{c} \in A^{\Omega(f)-1}$: if $\vec{a} \; \Theta \; \vec{c}$ and $f(\vec{a})$, $f(\vec{c})$ are defined, $f(\vec{a}) \; \Theta \; f(\vec{c})$. (Here, $\vec{a} \; \Theta \; \vec{c}$ is short for $a_i \; \Theta \; c_i$ for every $i < \Omega(f)-1$.) The congruence classes are denoted by $[a]_\Theta := \{c : a \; \Theta \; c\}$. Θ is a *strong congruence* if it is a weak congruence and from $\vec{a} \; \Theta \; \vec{c}$ follows that $f(\vec{a})$ is defined iff $f(\vec{c})$ is (the latter property is called *closedness* in Burmeister 1986).

2.1 Total and Partial Algebras

Let $\langle A, \mathfrak{i} \rangle$ be a partial algebra, and $\star \notin A$. Then put $A^\star := A \cup \{\star\}$ and define

$$f^\star(\vec{x}) := \begin{cases} f(\vec{x}) & \text{if } f(\vec{x}) \text{ is defined, } \vec{x} \subseteq A, \\ \star & \text{if } f(\vec{x}) \text{ is undefined, } \vec{x} \subseteq A, \\ \star & \text{if } \vec{x} \text{ contains } \star. \end{cases} \tag{4}$$

Finally, $\mathfrak{i}^\star(f) := \mathfrak{i}(f)^\star$ and $\mathfrak{A}^\star := \langle A^\star, \mathfrak{i}^\star \rangle$. This is an algebra. So, every partial algebra has a completion.

Also, let $s \approx t$ be an equation. The *strong theory* of $\mathfrak{A}$ is defined by

$$\mathsf{Eq}\,\mathfrak{A} := \{\langle s, t \rangle : \mathfrak{A} \vDash^\star s \approx t\} \tag{5}$$

There are Birkhoff type results on partial algebras (see Burmeister 1986 and references therein). Recall from universal algebra the notion of a reduced product. Let I be a set, $\langle \mathfrak{A}_i : i \in I \rangle$ a family of algebras and F a filter on I. Define $\sim_F$ on $\prod_{i \in I} A_i$ by $\vec{a} \sim_F \vec{b}$ iff $\{i : a_i = b_i\} \in F$. Let $P := \prod_{i \in I} A_i / \sim_F$. Put

$$\mathfrak{p}(f)([\vec{a}_0]_{\sim_F}, \ldots, [\vec{a}_{\Omega(f)-2}]_{\sim_F}) := [f(\vec{a}_0, \ldots, \vec{a}_{\Omega(f)-2})]_{\sim_F} \tag{6}$$

Moreover, the left hand side is defined if it is defined for at least sequence of representatives. Then $\langle P, \mathfrak{p} \rangle$ is called a *reduced product* of the $\mathfrak{A}_i$. For a class $\mathcal{K}$ of partial algebras, let $\mathsf{H}_w(\mathcal{K})$ denote the class of weak homomorphic images of members of $\mathcal{K}$, $\mathsf{S}_s(\mathcal{K})$ the class of strong subalgebras of members of $\mathcal{K}$, and $\mathsf{P}_r(\mathcal{K})$ the closure of $\mathcal{K}$ under reduced products. For the proof of the following theorem see Burmeister 2002.

Theorem 1 *$\mathcal{K}$ is a class of partial algebras satisfying a set of ECE–equations iff $\mathcal{K} = \mathsf{H}_w\mathsf{S}_s\mathsf{P}_r(\mathcal{K})$.*

Theorem 2 *If $\mathfrak{A} \vDash^\star s \approx t$, then $\mathfrak{A}^\star \vDash s \approx t$.*

Proof. Let $v : V \to A^\star$. Two cases arise. (a) $v(s) \neq \star$. Then on the relevant variables, v is a map into A, so $v(s)$ is defined in $\mathfrak{A}$. By definition, so is $v(t)$. Then $v(s) = v(t)$ in $\mathfrak{A}$. (b) $v(s) = \star$. Then either v assigns $\star$ to one of the variables in s, or $v(s)$ is undefined in $\mathfrak{A}$. In the latter case, $v(t)$ is undefined, too, so that $v(t) = \star$ in $\mathfrak{A}^\star$. In the former case, we may by symmetry assume that v assigns $\star$ to one of the variables of t. But then also $v(t) = \star$. □

Now, add a new constant $\star$ and put

$$\mathsf{Part} := \{f(x_0, \ldots, x_{i-1}, \star, x_{i+1}, \ldots, x_{\Omega(f)-1}) = \star : i < \Omega(f), f \in F\} \tag{7}$$

$$\mathsf{Eq}(\mathfrak{A}^\star) := \mathsf{Eq}(\mathfrak{A}) \cup \mathsf{Part} \tag{8}$$

Theorem 3 *Let $\mathfrak{A}$ be a partial algebra. Then $\mathfrak{A}^\star$ is a total algebra. Moreover, the equational theory of $\mathfrak{A}^\star$ is the strong partial theory of $\mathfrak{A}$ plus the theory* Part.

This process can be reversed, showing that every partial algebra arises from a total algebra by eliminating an element. Let $\mathfrak{A} = \langle A, \mathfrak{i} \rangle$ be a total Ω–algebra, and $D \subseteq A$ a set closed under the functions. Then $A - D$ is a partial algebra with the functions

$$\mathfrak{i}_D(f) := \mathfrak{i}(f) \cap (A - D)^{\Omega(f)+1} \tag{9}$$

We denote this algebra by $\mathfrak{A}_D$.

(10) $$\mathfrak{A} \vDash s \approx t \quad \Rightarrow \quad \mathfrak{A}_D \vDash^\star s \approx t$$

Moreover, the map $d : D \to \{\star\} : x \mapsto \star$ is a homomorphism, as is easily calculated. Its image is a full algebra which is isomorphic to $(\mathfrak{A}_D)^\star$.

Using this we can reread a theorem from Hodges 2001. Hodges assumes that meanings are assigned to terms. However, he assumes that the meaning function μ is only partial; so it is only partially defined. Two terms s and t are *synonymous* if μ is defined on both s and t and $\mu(s) = \mu(t)$. μ is reinterpreted as a total map μ' from the domain of μ. The domain is therefore of the form Tm_D, where D is the complement of $\mathrm{dom}(\mu)$. While the term algebra Tm is total, Tm_D is partial. No particular assumptions on D are being made. However, if D is closed under the functions, the meaning function is compositional if synonymy is a strong congruence. This can easily be established on the basis of the theorems shown here.

2.2 From Partial to Many–Sorted

A *many–sorted algebra* is defined over a set $\mathcal{S}$ of sorts. A *sorted signature* is a pair $\Xi = \langle F, \Xi\rangle$, where $\Xi : F \to \mathcal{S}^+$ assigns to every function symbol a string over $\mathcal{S}^+$. An *algebra* over this signature (or a Ξ–*algebra*) is a pair $\langle\{A_\sigma : \sigma \in \mathcal{S}\}, \mathfrak{m}\rangle$ such that the A_σ (called *phyla*) are pairwise disjoint and if $\Xi(f) = \langle \sigma_i : i < n+1\rangle$ then $\mathfrak{m}(f) : A_{\sigma_0} \times \ldots A_{\sigma_{n-1}} \to A_{\sigma_n}$. For each $\sigma \in \mathcal{S}$, choose a denumerably infinite set V_σ of variables of sort σ. *Terms* of sort σ are defined by induction. A variable of sort σ is a term of sort σ; and if f has signature $\langle \sigma_i : i < n+1\rangle$, and if t_i, $i < n$, have sort σ_i, then $f(\vec{t})$ has sort σ_n. A *sorted valuation* is a family $\{h_\sigma : \sigma \in \mathcal{S}\}$ of functions $h_\sigma : V_\sigma \to A_\sigma$. This extends to a unique family of maps assigning to each term t of sort σ a value $h_\sigma(t)$. Since the sort is implicitly given, we also write $h(t)$ in place of $h_\sigma(t)$.

Write $\mathfrak{A} \vDash s \approx t$ if for every sorted valuation h, $h(s) = h(t)$. For a Ξ–algebra $\mathfrak{A}$, put

(11) $$\mathsf{Eq}\,\mathfrak{A} := \{\langle s, t\rangle : \mathfrak{A} \vDash s \approx t\}$$

It is clear that $s \approx t$ can only obtain if s and t have identical sorts. There is a Birkhoff type theorem for many–sorted algebras. Call a class of many–sorted algebras *primitive* if it is closed under reduced products, subalgebras and homomorphic images. (Notice that there is no distinction between weak and strong with respect to subalgebras and homomorphic images.)

Theorem 4 *A class of many–sorted algebras is equationally definable iff it is primitive.*

(See Burmeister 2002 for an example that closure under products is not enough — if the signature is infinite.) It follows that the theory of many–sorted algebras is more or less parallel to that of unsorted ($\cong$ single–sorted) algebras.

Evidently, a many–sorted algebra $\mathfrak{A}$ can be turned into a partial algebra $\mathfrak{A}_\circ := \langle A_\circ, \mathfrak{m}_\circ\rangle$, where $A_\circ := \bigcup_{\sigma\in\mathcal{S}} A_\sigma$ and $\mathfrak{m}_\circ(f) := \mathfrak{m}(f)$. If $h = \{h_\sigma : \sigma \in \mathcal{S}\}$ is a sorted homomorphism, $h_\circ := \bigcup_{\sigma\in\mathcal{S}} h_\sigma$ is a homomorphism of the unsorted partial algebras. Notice however that there are homomorphisms $\mathfrak{A}_\circ \to \mathfrak{B}_\circ$ that are *not* of this form. (For example, let $\mathcal{S} = \{\sigma, \tau\}$. $\mathfrak{A} = \mathfrak{B}$, $A_\sigma = \{a\}$, $A_\tau = \{b\}$. The signature is empty. The map $a \mapsto b$, $b \mapsto a$ is not of the form $h_\circ$.) So, if $\mathcal{V}$ is a variety of many–sorted algebras, $\mathcal{V}_\circ := \{\mathfrak{A}_\circ : \mathfrak{A} \in \mathcal{V}\}$ need not be a variety again. Moreover, if $s \approx t$ is a sorted equation, and if $\mathfrak{A} \vDash s \approx t$, then $\mathfrak{A}_\circ \vDash^\star s \approx t$ is not necessarily true. This is so since removing the sortal information from the variables allows for more valuations. Given the signature, Ξ, there is a final Ξ–algebra $\mathfrak{I}_\Xi = \langle\{I_\sigma : \sigma :\in \mathcal{S}\}, \mathfrak{m}\rangle$, where $I_\sigma = \{\sigma\}$, and $\mathfrak{m}(f) = \Xi(f)$. It turns out that every many–sorted algebra $\mathfrak{A}$ can be uniquely described as a homomorphism $h : \mathfrak{A}_\circ \to \mathfrak{I}_\Xi$ (see Burmeister 1986).

Theorem 5 *The category of Ξ–algebras is naturally equivalent to the comma category of strong partial algebras over $\mathfrak{I}_\Xi$.*

Now, let conversely $\mathfrak{A}$ be a partial algebra. Put $a \asymp_{\mathfrak{A}} c$ iff for all unary polynomials f: $f(a)$ is defined iff $f(c)$ is. The following is folklore, see Burmeister 1986.

Theorem 6 $\asymp := \asymp_{\mathfrak{A}}$ *is a strong congruence. Moreover, a weak congruence* Θ *is strong iff it is contained in* $\asymp$.

Proof. $\asymp$ is certainly an equivalence relation; further, if g is a unary polynomial and $a \asymp c$ then $g(a) \asymp g(c)$. For assume $f(g(a))$ is defined. Since $f(g(x))$ is unary and $a \asymp c$, $f(g(c))$ is defined, too. And conversely. So, $\asymp$ is a congruence. (We use the fact that Θ is a congruence iff for all unary polynomials g: if $a \ \Theta \ c$ then $g(a) \ \Theta \ g(c)$.) $\asymp$ is obviously strong. Now let Θ be a congruence. Then if $a \ \Theta \ c$ we must have that $f(a)$ defined iff $f(c)$ defined for every unary polynomial f, showing that $\Theta \subseteq \asymp$. On the other hand, if $\Theta \subseteq \asymp$ then this evidently holds. □

$\asymp$ is of some significance. For example, if $\mathfrak{A}$ is the algebra of meanings, then the equivalence classes of $\asymp$ are the meaning categories of Husserl, according to Hodges 2001. Given $\mathfrak{A}$, let $\mathfrak{S} := \{[x]_{\asymp} : x \in A\}$ be the set of congruence classes of $\asymp$. Then for each $\sigma \in \mathcal{S}$ put $A_\sigma := \sigma$. This means that every x is a representative of its $\asymp$–class. It is however problematic to define the functions. For each function symbol f of arity n will have to be split into up to $|\mathcal{S}|^n$ many function symbols of signature, one symbol for each n–tuple of equivalence classes. However, this is not a good approach. The next theorem spells out the condition under which f does not have to be split (so that we can use the old signature again).

Theorem 7 *A partial algebra* $\mathfrak{A} = \langle A, \mathfrak{i}\rangle$ *is of the form* $\mathfrak{B}_\circ$ *for some many–sorted algebra iff* ($\star$)*: for every* $f \in F$*, if* $\mathfrak{i}(f)(\vec{a})$ *and* $\mathfrak{i}(f)(\vec{c})$ *are both defined then* $\vec{a} \asymp_{\mathfrak{A}} \vec{c}$.

Proof. First of all, suppose that $\mathfrak{A} = \mathfrak{B}_\circ$. Let f be of signature $\langle \sigma_i : i < n+1\rangle$. Assume that $\mathfrak{i}(f)(\vec{a})$ and $\mathfrak{i}(f)(\vec{c})$ are both defined. Then $\vec{a}, \vec{c} \in \prod_{i<n} A_{\sigma_i}$. This means, however, that for every polynomial g, if g is defined on $\vec{a}$ then it has signature $\langle\langle \sigma_i : i < n\rangle, \tau\rangle$, and so it is defined on $\vec{c}$ as well. So, $\vec{a} \asymp \vec{c}$. Moreover, $\mathfrak{i}(f)(\vec{a}) \asymp \mathfrak{i}(f)(\vec{c})$, since both have sort σ_n. Conversely, assume that $\mathfrak{A}$ has the property ($\star$). Let the sorts be the equivalence classes of $\asymp$. Take $\vec{a}$ and b such that $f(\vec{a}) = b$. Then the signature of f is exactly $\langle\langle [a_i]_{\asymp} : i < n\rangle, [b]_{\asymp}\rangle$. By assumption, if f is defined on another n–tuple, it has the same sort, and the result has the same sort as b (since $\asymp$ is a congruence). □

This is reminiscent of the principle that Hodges 2001 attributes to Tarski:

> Tarski's Principle. For every nontrivial unary polynomial f: if $f(a)$ and $f(c)$ are defined, then $a \asymp c$.

In fact, a partial algebra satisfies Tarski's Principle iff it has the property ($\star$). See also below on partial combinatory algebras.

3 Polymorphism

In linguistic analysis, one often assumes that a particular symbol is polymorphic (for example, categorial grammar allows primitive symbols to have any (finite) number of categories). We can accommodate for this as follows. Say that a *generalized signature* is a pair $\langle F, \mathfrak{s}\rangle$ where $\mathfrak{s} : F \to \wp(\mathcal{S}^+)$ is such that if $\langle \sigma_i : i < n+1\rangle$ and $\langle \tau_i : i < n+1\rangle \in \mathfrak{s}(f)$ and $\sigma_i = \tau_i$ for all $i < n$ then $\sigma_n = \tau_n$. (So, $\mathfrak{s}$ can be seen as a function from $\mathcal{S}^*$ to $\mathcal{S}$.) Thus, a function symbol can take *any* set of strings over sorts as value. However, we shall generally look at cases where all the strings have the same length (so that they can be said to derive from the same unsorted function). A *generalized many–sorted algebra* is then defined in the obvious way. Notice that generalized many–sorted signatures are in some sense only notational variants of many–sorted algebras. Basically, the addition is that the generalized signature tells us which functions are to be looked as parts of one and the same global function. It turns out that the theory of generalized many–sorted algebras is largely equivalent to that of standard many–sorted algebras. Take a signature $\mathfrak{s}$. Now define the set $G := \{\langle f, \vec{\sigma}\rangle : f \in F, \vec{\sigma} \in \mathfrak{s}(f)\}$. Then put $\Xi(\langle f, \vec{\sigma}\rangle) := \vec{\sigma}$. Ξ is a many–sorted signature. Let $\mathfrak{A} = \langle \{A_\sigma : \sigma \in \mathcal{S}\}, \mathfrak{g}\rangle$ be an $\mathfrak{s}$–algebra. For $f \in F$ and $\vec{\sigma} = \langle \sigma_i : i < n+1\rangle \in \mathfrak{s}(f)$ put

(12) $$\mathfrak{m}(\langle f, \vec{\sigma}\rangle) := \mathfrak{g}(f) \cap \prod_{i<n+1} A_{\sigma_i}$$

Then $\mathfrak{A}^\heartsuit := \langle\{A_\sigma : \sigma \in \mathcal{S}\}, \mathfrak{i}\rangle$ is a Ξ–algebra. Given a Ξ–algebra $\mathfrak{B} = \langle\{B_\sigma : \sigma \in \mathcal{S}\}, \mathfrak{m}\rangle$, put

$$\mathfrak{g}(f) := \bigcup_{\vec{\sigma}\in\mathfrak{s}(f)} \mathfrak{i}(\langle f, \vec{\sigma}\rangle) \tag{13}$$

The pair $\mathfrak{B}_\heartsuit := \langle\{B_\sigma : \sigma \in \mathcal{S}\}, \mathfrak{g}\rangle$ is an $\mathfrak{s}$–algebra. It is easy to see that $(\mathfrak{A}^\heartsuit)_\heartsuit = \mathfrak{A}$ and $(\mathfrak{B}_\heartsuit)^\heartsuit = \mathfrak{B}$. (Identity, not just isomorphy!) Moreover, $h : \mathfrak{A} \to \mathfrak{C}$ is a homomorphism of $\mathfrak{s}$–algebras iff $h : \mathfrak{A}^\heartsuit \to \mathfrak{C}^\heartsuit$ is a homomorphism of Ξ–algebras.

Theorem 8 *The categories of $\mathfrak{s}$–algebras is isomorphic to the category of Ξ–algebras, where $\mathfrak{A}$ is mapped to $\mathfrak{A}^\heartsuit$ and $h : \mathfrak{A} \to \mathfrak{C}$ to $h : \mathfrak{A}^\heartsuit \to \mathfrak{C}^\heartsuit$.*

There is also a direct way to translate the equational theories. For a term t in the signature Ξ define

$$(x_\sigma)_\heartsuit := x_\sigma \tag{14}$$

$$(\langle f, \vec{\sigma}\rangle(\vec{s}))_\heartsuit := f(\vec{s}_\heartsuit) \tag{15}$$

Conversely, let s be an $\mathfrak{s}$–term. Notice first of all that every term s can be assigned a unique sort $Ⓡs$: a variable x_σ has sort σ, and if s_i has sort σ_i for $i < n$ and f is n–ary, then $f(\vec{s})$ has sort σ_n, where σ_n is the unique sort such that $\langle\sigma_i : i < n+1\rangle$.

$$(x_\sigma)^\heartsuit := x_\sigma \tag{16}$$

$$(f(s_0, \ldots, s_{n-1}))^\heartsuit := \langle f, \langle\langle Ⓡs_i : i < n\rangle, Ⓡf(\vec{s})\rangle\rangle(s_0^\heartsuit, \ldots, s_{n-1}^\heartsuit) \tag{17}$$

Notice that $(t_\heartsuit)^\heartsuit = t$ and $(s^\heartsuit)_\heartsuit = s$.

Theorem 9 *If T is an equational theory of Ξ–algebras axiomatizing $\mathcal{K}$, $T_\heartsuit$ is an equational theory of $\mathfrak{s}$–algebras axiomatizing $\mathcal{K}_\heartsuit$, and if U is an equational theory of $\mathfrak{s}$–algebras axiomatizing $\mathcal{L}$, $U^\heartsuit$ is an equational theory of Ξ–algebras axiomatizing $\mathcal{L}^\heartsuit$.*

Theorem 10 *Every partial algebra $\mathfrak{A}$ is of the form $\mathfrak{B}_\circ$ for some generalized many–sorted algebra.*

For a proof simply observe that we can take $\mathcal{S}$ to be just A (or the set of equivalence classes of $\asymp$). Then let

$$\mathfrak{s}(f) := \{\langle\vec{a}, f(\vec{a})\rangle : f(\vec{a}) \text{ is defined}\} \tag{18}$$

However, the categories of these kinds of algebras are not isomorphic. There are more homomorphisms between partial algebras than there are between (generalized) many–sorted algebras, since sortal restrictions apply.

We shall stress once again the linguistic significance of this notion. In linguistic theory one distinguishes a morpheme from a morph. The latter is but one manifestation of the morpheme. Typically, a morpheme is defined as a set of morphs having the same meaning (see Mel'čuk 2000). If a morph is a particular string function there is no connection between different morphs of a morpheme in a typed or many–sorted setting. Each morph is the manifestation of a different function. Generalized signatures allow to treat the morphs of a morpheme as the manifestation of a single abstract function. Similarly, in syntax it becomes possible to represent the polymorphism of a function directly, because the signature itself allows for polymorphic functions. This polymorphism is pervasive in categorial grammar. Even though Lambek–grammars introduce a systematic device to handle the categorial flexibility (and the meanings to go with the different categories), it does not actually eliminate the diversity of categories (and meanings) assigned to a given lexical head. It remains a fact that in categorial grammar heads are polymorphic: one category for each syntactic environment. Generalized signatures can bundle them into natural groups. Notice however that their power is potentially larger: a given symbol can even have an infinite signature, something which is normally excluded in categorial grammar. An exception to this, however, are the logical words `and`, `not` and `or`, etc. (See also below on the algebraization of predicate logic.) Finally, parallel polymorphism (different categories give rise to different meaning functions) is directly represented here, in fact is the norm. For example, `and` is massively polymorphous. Without having to assume a different symbol for each of them (and therefore assuming massive syncretism), we choose to give `and` the signature $\{\langle\sigma,\sigma,\sigma\rangle : \sigma \text{ conjoinable}\}$. This means that in the algebra of categories for each σ it is interpreted by a function that sends pairs of σ–categories to a σ–category; and for each σ it is interpreted in the algebra of meanings as a function from pairs of σ–meanings to σ–meanings.

Despite the usefulness of many–sorted and generalized many–sorted algebras, there are also reasons why they should not be used as the basic structures of analysis. One such reason is that the sorts restrict the maps between algebras and are therefore universal. If we compare different languages we often face the fact that classification systems of one language do not coincide with classification systems of another. A clear example are gender categories (see Corbett 1991). If we want to assume that languages can have different syntax and morphology but basically a semantics which is the same for all languages, the syntactic categories cannot always be mapped straightforwardly into semantic types whichever way they are chosen. In fact, we should perhaps not use any predefined semantic types.

4 Combinatory Algebras

An interesting case of an algebra for a generalized many–sorted signature are combinatory algebras. These are partial algebras with just one binary operation, denoted here by $\bullet$. Notice that most equations of the partial combinatory algebras hold only weakly. For example, the equation $k{\bullet}x{\bullet}y \approx x$ in a combinatory algebra, where k interprets the combinator K, is valid only in the weak sense: if both sides are defined, then equality holds. For if $\mathfrak{A} \vDash^\star k \bullet x \bullet y \approx x$, then for all a and b: $k \bullet a \bullet b = a$, for the right hand side is defined.

Introducing typing regimes removes this feature. The equation will be split into infinitely many equations, all of which are universally true. A *typed combinatory algebra* is a generalized many–sorted algebra such that the sorts are the sets of terms formed from a set B of so–called *basic types* using the type constructor $\rightarrow$, and $\mathfrak{s}(\bullet) = \{\langle\alpha \rightarrow \beta, \alpha, \beta\rangle : \alpha, \beta \in \mathcal{S}\}$. Thus, $a \bullet b$ is defined iff a has type $\alpha \rightarrow \beta$ for some α and β and b has type α; and then $a \bullet b$ has type β. Unfortunately, as is well known, not all combinatory algebras can be typed. The following characterizes those combinatory algebras which are derived from typed combinatory algebras.

Theorem 11 *A partial combinatory algebra $\mathfrak{A}$ is of the form $\mathfrak{B}_\circ$ for some typed combinatory algebra $\mathfrak{B}$ iff:*

① *(Tarski's Principle) For a and c: $a \asymp_{\mathfrak{A}} c$ iff if there is a single, nontrivial unary polynomial f such that $f(a)$ and $f(c)$ are defined.*

② *(Well–Foundedness) For every a there exists an n and b_i $(i < n)$ such that $(\ldots((a \bullet b_0) \bullet b_1)\ldots \bullet b_{n-1})$ is undefined.*

For a proof, let $\mathfrak{S} := \{[a]_{\asymp_{\mathfrak{A}}} : a \in A\}$ and

(19) $$\mathfrak{S}_1 := \{[a]_{\asymp_{\mathfrak{A}}} : \text{ there is no } b \text{ such that } a \bullet b \text{ is defined}\}$$

Let $A_\alpha := \alpha$ if $\alpha \in \mathfrak{S}_1$; furthermore, let $A_{\alpha\rightarrow\beta}$ be the set of a such that there exists $b \in A_\alpha$ such that $a \bullet b \in A_\beta$. We need to show that for every category α there is $a \in A$ such that $A_\alpha = [a]_{\asymp_{\mathfrak{A}}}$. By ②, every element is assigned a type. Uniqueness follows from Tarski's Pinciple. This is seen as follows. Call the least n for which ② is satisfied for a the *height*. By ①, the height is actually unique. For if a has height 1, it is in $\mathfrak{S}_1$. Namely, if $a \bullet b_0$ is undefined, $a \bullet c$ cannot be defined for any c, by ①. Now assume the claim holds for all a of height n. Assume that a has height $n+1$. Then if $a \bullet b$ and $a \bullet c$ are both defined, we get $b \asymp_{\mathfrak{A}} c$ (using ①). It follows that they have the same height and the same type.

5 The Leibniz Congruence

There is a congruence quite analogous to $\asymp_{\mathfrak{A}}$, namely the Leibniz congruence. It is construed on the basis of a set D of designated truth values, see Blok and Pigozzi 1990. Let $D \subseteq A$. Then put: $a\ \Omega_{\mathfrak{A}}(D)\ c$ iff for all unary polynomials f: $f(a) \in D$ iff $f(c) \in D$. This is called the *Leibniz congruence*. The intention is here that D is the set of trivially true sentences. They induce a synonymy on the elements of $\mathfrak{A}$ in virtue of Leibniz' Principle (a and b are synonymous iff they can be substituted for each other in every context *salva veritate*). Now, in the context of partial algebras we need to adjust the definition as follows. $a\ \Omega_{\mathfrak{A}}(D)\ c$ iff for all unary polynomials f: $f(a)$ is defined iff $f(c)$ is, and $f(a) \in D$ iff $f(c) \in D$. It is easy to see that $\Omega_{\mathfrak{A}}(D) \subseteq \asymp_{\mathfrak{A}}$. Compare this definition with the definition of the Husserl categories. The latter are identical in partial algebras with the equivalence classes of $\Omega_{\mathfrak{A}}(A)$! For the meaning categories are the classes of the Leibniz operator when there is no distinction between truth and definedness.

Next, let us move to the many–sorted algebras. For every σ, choose $D_\sigma \subseteq A_\sigma$ and put $\mathcal{D} := \{D_\sigma : \sigma \in \mathcal{S}\}$. Now, for $a, c \in A_\sigma$ put $a\ \Omega_{\mathfrak{A}}(\mathcal{D})\ c$ iff for all unary polynomials $f : A_\sigma \to A_\tau$: $f(a) \in D_\tau$ iff $f(c) \in D_\tau$. We mention an important particular case. Choose one sort $\gamma \in \mathcal{S}$ (the sort of 'propositions'). Then choose $D_\sigma = \varnothing$ if $\sigma \neq \gamma$, and $D_\gamma \subseteq A_\gamma$ is any subset. The rest is as above. This definition has been chosen in Simple Type Theory by Church 1940. One type has been distinguished. Furthermore, Simple Type Theory has a deductive calculus for the elements of type γ. This calculus effectively axiomatizes the set D. Write $a \triangleq c$ for two elements a and c of type σ if for every f of type $\sigma \to \gamma$: $f(a) \in D$ iff $f(c) \in D$, which can be rephrased as $f(a) \leftrightarrow f(c) \in D$. ($\gamma$ is the type of propositions, and $\leftrightarrow$ is definable.) The Henkin–completeness proof of Simple Type Theory simply defines a model by showing that $\triangleq$ is a congruence relation. It can be factored out, giving rise to a model of the theory.

6 First–Order Logic

Another application of the methods is the algebraization of predicate logic (FOL). Standardly it is assumed that all formulae are of the same type, and this has been the underlying assumption also with cylindric algebras. Unfortunately, algebraizations that have tried to maintain this did not succeed in characterizing the exact models of FOL and — even worse — the exact theory of FOL. Namely, it is not possible to recast the axioms of FOL in algebraic terms since the former make reference to free variables, which the latter cannot do. Hence, we must acknowledge that formulae are of different types. In fact, the types arise in a perfectly natural way. Suppose that we single out a special class Sent of formulae, the sentences. Then let $\mathcal{S}$ be the set of equivalence classes of $\Omega_{\mathrm{Tm}}(\mathrm{Sent})$, where Tm is the algebra of formulae. Two formulae are equivalent modulo $\Omega_{\mathrm{Tm}}(\mathrm{Sent})$ iff they have the same free variables. So, given that our set of variables is $V = \{x_i : i \in \omega\}$, the sorts can be identified by the finite sets of natural numbers. φ has sort H iff $\mathrm{fvar}(\varphi) = \{x_i : i \in H\}$. If the sentences are the only meaningful formulae (such as in ALGOL), then the equivalence classes are the Husserlian categories.

Once we have defined the classes, we need to define the signature. First, the variables are of the form x^i_H, where $H \in \wp_{fin}(\omega)$, and $i \in \omega$. Here, H is the sort of x^i_H. Again, if we follow the standard path to simply go many–sorted then a single function, say $\wedge$, splits into infinitely many functions. Using the generalized signatures we can unite them under one symbol again. So, the generalized signature of $\wedge$ and $\vee$ is

(20) $$\{\langle H, H', H \cup H'\rangle : H, H' \in \wp_{fin}(\omega)\}$$

The generalized signature of $\neg$ is

(21) $$\{\langle H, H\rangle : H \in \wp_{fin}(\omega)\}$$

In this way the standard symbols can be united. However, recall, that the algebraization yields an infinite set of quantifiers $\exists_i$, which represent the first–order quantifiers $(\exists x_i)$, with $\forall_i(\varphi)$ defined by $\neg\exists_i(\neg\varphi)$. Notice that $\exists_i$ has the signature

(22) $$\{\langle H, H - \{i\}\rangle : H \in \wp_{fin}(\omega)\}$$

No predicate letters are needed. If one wants to instantiate a special signature, choose for an n–ary predicate letter P a constant c^P of sort $\{0, 1, \ldots, \Omega(P) - 1\}$. (Think of c^P as representing $P(x_0, \ldots, x_{\Omega(P)-1})$.)

Cylindric algebras are known not to provide an exact characterization of the intended models. This is so because there is no way to tell which are the free variables of a formula. Recall that the problem in algebraization of FOL is that certain laws only hold modulo a restriction on free variables. An example is $(\forall x_i)\varphi \approx \varphi$, which holds only if x_i is not free in φ. The situation is remedied by the introduction of the sorts. The restriction is incorporated by taking only the following set of equations.

$$\{\forall_i x_H \approx x_H : i \notin H\} \tag{23}$$

In this way every equation gets a unique sort. Notice however that some equations of predicate logic cannot be written down any more.

$$x^0_{\{i\}} \vee \neg x^0_{\{i\}} \approx x^0_{\varnothing} \vee \neg x^0_{\varnothing} \tag{24}$$

Clearly, this can only arise when the left hand side has a different sort than the right hand side. This in turn means that some formula contains variables that play no role in it. We can take care of that as follows. Like in cylindric algebras assume elements d_H, $H \in \wp_{fin}(\omega)$, the 'diagonals'. Then add the following equations:

$$\mathsf{d}_H \wedge \mathsf{d}_J \approx \mathsf{d}_{H \cup J} \tag{25}$$

Notice that $\top \approx \mathsf{d}_{\varnothing}$ is valid, if $\top$ is a special propositional constant. We repair Equation (24) as follows:

$$x^0_{\{i\}} \vee \neg x^0_{\{i\}} \approx (x^0_{\varnothing} \vee \neg x^0_{\varnothing}) \wedge \mathsf{d}_{\{i\}} \tag{26}$$

This defines a complete set of equations for algebraic predicate logic. Notice that it is still conceivable that the models for this set of equations is not derived from a first–order structure. (For example, think of the canonical structure formed by formulae modulo equivalence.) However, the equational theory is faithful in the sense that $s \approx t$ is derivable iff $s \leftrightarrow t$ is a theorem of predicate logic for every interpretation which sends the variable of sort x_H to a formula φ such that $\mathrm{fvar}(\varphi) = \{x_i : i \in H\}$, which interprets d_H by $\bigwedge_{i \in H} x_i = x_i$, and $\exists_i$ by $\exists x_i$.

For the Leibniz Congruence, notice that D_H is the set of tautologies of sort H (which is the set of all s such that $s \approx \mathsf{d}_H$). Put $\mathcal{D} := \{D_H : H \in \wp_{fin}(\omega)\}$. Then $s\ \Omega_{Tm}(\mathcal{D})\ t$ iff $s \leftrightarrow t \in D_H$.

This strategy of turning a partial algebra into a generalized many–sorted algebra is completely general. The types can encode any property of the actual terms, so that any condition on the equations reflecting a property of the term can be encoded using types. This algebraization may not as inspiring as other ones (eg Manca and Salibra 1986, or Pigozzi and Salibra 1995). Nevertheless, it allows to use Birkhoff's theorems, and thus provides a canonical completeness proof. The models are quite unlike standard first–order models, but they capture the logic exactly. If one insists on having variables, they can also be added (though as constants). This further complicates the formalization, but the procedure itself is quite straightforward (see Kracht 2003).

7 Conclusion

From a foundational perspective partial algebras seem to be better motivated, since they do not force us to choose the sorts to begin with. On the other hand, partial algebras are not so well–behaved mathematically. Many–sorted algebras seem to be much more suited for the purpose. However, they are unnecessarily restrictive, since not every partial algebra arises from a many–sorted algebra. We have shown that there is a slight generalization of many–sorted algebras which allows to incorporate polymorphism in a rather direct way. An element can be given *one* generalized signature, which takes care of all of its different manifestations in language. Essentially, for any generalized signature $\mathfrak{s}$ the category of $\mathfrak{s}$–algebras is isomomorphic to the category of Ξ–algebras for some many–sorted signature. So, the notion of a homomorphism is not generalized. However, we also noted that partial algebras allow more homomorphisms than many–sorted algebras (and therefore even generalized many–sorted algebras), since there are no predefined sorts. While this complicates the algebraic theory somewhat, there are good reasons to believe that partial algebras are the fundamental structures of analysis.

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