Simple Games:

Representation and Solutions

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Chapter 1

Representation

By a representation of a simple game $\boldsymbol{v}$ we understand a pair $(\boldsymbol{m}, \alpha)$ such that $\boldsymbol{v}$ appears as the majority game (in the most general sense) resulting form a distribution of votes according to $\boldsymbol{m}$ and a majority level α. That is to say, we have $\boldsymbol{v} = \boldsymbol{v}_\alpha^{\boldsymbol{m}}$ with

$$\boldsymbol{v}_\alpha^{\boldsymbol{m}}(S) = \left\{ \begin{array}{lll} 1 & \text{if} & \boldsymbol{m}(S) \geq \alpha \\ 0 & \text{if} & \boldsymbol{m}(S) < \alpha \quad . \end{array} \right\}$$

Of course representations do not necessaily exist nor, if they exist, do they have to be unique. The first example for a non-representable simple game is due to Von Neumann – Morgenstern and appears in [4]. One needs 5 players to construct a general simple game of this type and 6 players to construct a zero-sum simple game.

Within this chapter we will provide some basic properties of games that admit of a representation. We discuss also tests for representability. Finally we present the apportionment problem though this is generally not seen as a "representation peoblem". Nevertheless, apportionment (which is of wide interest in the political context) is an approach to generate a representation (the parlamentary body) of a game with smaller weights then the original version (the representation within the voting population). This attempt fails frequently, thus the game played in parlament may not be the one of the voting populace – that is, many apportionment procedures distort the game – a disadvantage that is mostly disregarded in the framework of political discussion...

1 The Apportionment Problem

Consider a parlament or a committee which is to be established via some voting procedure. There are $n \in \mathbb{N}$ political parties acting in the population and after counting the polls it turns out that the distribution of votes is given by the (integer) vector

$$\boldsymbol{g} \;=\; (g_1, \ldots, g_n)\ .$$

Assuming an issue would be voted about in the population via simple majority we would (in Game theoretical terms) have to construct the majority level which is

$$\lambda \;:=\; \left[\frac{\boldsymbol{g}(I)+1}{2}\right],$$

and then consider the weighted majority game $\boldsymbol{v}_\lambda^{\boldsymbol{g}}$ given by

$$\boldsymbol{v}_\lambda^{\boldsymbol{g}}(S) \;=\; \left\{ \begin{array}{lll} 1 & \text{if} & \boldsymbol{g}(S) \geq \lambda \\ 0 & \text{if} & \boldsymbol{g}(S) < \lambda \end{array} \right\}\ .$$

Now let the number of seats in the parliament be $h \in \mathbb{N}$. In order to allot the seats to the parties in the contest one is tempted to use the distribution of votes suggested my the vector

$$\boldsymbol{x} \;:=\; \frac{h}{\boldsymbol{g}(\boldsymbol{I})}\boldsymbol{g}$$

which is proportional to $\boldsymbol{g}$ and has the right size. However, $\boldsymbol{x}$ is in general not an integer and hence the problem of allotting the seats, the ***apportionment problem*** is on our hands.

We have to develop procedures following certain principles or axioms such that, for any apportionment problem, the result of casting votes in the population is transferred into a distribution of votes in the parlament such that it is acceptable for the parties, the population, the courts, ...

Let us start out with some more precise formulations. These will have to be augmented sometimes (mainly because its inevitable that sometimes a lottery is involved in case of certain ties), nevertheless the follwing is an attempt to provide exact definitions.

Definition 1.1. *1. An **apportionment** is a mapping*

$$\begin{aligned} G : \mathbb{N}_0^n \times \mathbb{N} &\longrightarrow \mathbb{N}_0^n \\ (\boldsymbol{g}, h) &\to \boldsymbol{m}\ . \end{aligned} \tag{1}$$

2. *An apportionment* ***preserves the house*** *if it satisfies*

$$G(\boldsymbol{g}, h)(I) \;=\; h \quad ((\boldsymbol{g}, h) \in \mathbb{N}_0^n \times \mathbb{N}). \tag{2}$$

3. *An apportionment is* ***house–monotone*** *if it satisfies*

$$h \geq h' \Longrightarrow G(\boldsymbol{g}, h) \geq G(\boldsymbol{g}, h') \quad (\boldsymbol{g} \in \mathbb{N}_0^n, h, h' \in \mathbb{N}) \tag{3}$$

(coordinatewise or seen as a set function).

4. *An apportionment* ***preserves the coalitional function*** *if it satisfies*

$$\boldsymbol{v}_\alpha^{G(\boldsymbol{g},h)} \;=\; v_\lambda^{\boldsymbol{g}} \tag{4}$$

with

$$\lambda \;:=\; \left[\frac{\boldsymbol{g}(I)+1}{2}\right], \qquad \alpha \;:=\; \left[\frac{G(\boldsymbol{g},h)(I)+1}{2}\right]$$

.

5. *An apportionment is* ***close to the proportional solution*** *if it satisfies*

$$\begin{aligned} &\left| G(\boldsymbol{g}, h) - \frac{h}{\boldsymbol{g}(I)}\boldsymbol{g} \right| \\ &= \min\left\{ \left| \boldsymbol{m} - \frac{h}{\boldsymbol{g}(I)}\boldsymbol{g} \right| \;\middle|\; \boldsymbol{m} \in \mathbb{N}_0^n, \boldsymbol{m}(I) = h \right\}. \end{aligned} \tag{5}$$

Here is a simple example:

Example 1.2. Choose an integer $c \in \mathbb{N}$ and define an apportionment by

$$S^c(\boldsymbol{g}, h) \;:=\; \left(\left[\frac{g_1}{c}\right], \ldots, \left[\frac{g_n}{c}\right]\right) \quad ((\boldsymbol{g}, h) \in \mathbb{N}_0^n \times \mathbb{N}).$$

the mapping S^c is called a ***leveling procedure***.Intuitively, a party obtains for every c votes in the population exactly one vote or one seat in the parlament. However, as a quotient $\frac{g_i}{c}$ is not necessarily an integer, some votes remaining after the assignment of $\left[\frac{g_1}{c}\right]$ seats are not distributed. Thus, in general a leveling procedure does not preserve the house.

The first candidate for the level is given by $c \;=\; \left[\frac{\boldsymbol{g}(\boldsymbol{I})}{h}\right]$ for, if it so happens that $\frac{\boldsymbol{g}(\boldsymbol{I})}{h}$ is an integer, then

$$\left[\frac{g_i}{c}\right] \;=\; \left[\frac{g_i}{\frac{\boldsymbol{g}(\boldsymbol{I})}{h}}\right] \;=\; \left[\frac{g_i h}{\boldsymbol{g}(\boldsymbol{I})}\right]$$

which resembles (though may well be different from) the proportionality $\boldsymbol{x} = \frac{h}{\boldsymbol{g}(\boldsymbol{I})}\boldsymbol{g}$.

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Thus, after some inspection a leveling procedure has several disadvantages, this first and possibly naive approach cannot be supported by the presence of some desirable properties.

We are going to discuss two further prodedures in the following sections.

2 The Hare Procedure

Within this section we attempt the description of a very much debated concept, the Hare procedure (which has various names in different countries historical context). It constitutes one of the most widely used procedures and has some great advantages. Nevertheless, it is not satidfying all the conditions that are set doen in the previous section.

Definition 2.1. *Let* $\boldsymbol{g} \in \mathbb{N}_0^n$ *and* $h \in \mathbb{N}$. *Define*

$$\boldsymbol{x} \ := \ \frac{h}{\boldsymbol{g}(I)}\boldsymbol{g}$$

to be the proportional solution. Then $\boldsymbol{m} \in \mathbb{N}_0^n$ *is said to be consistent with the* ***Hare procedure*** *if the following holds true.*

1. $[\boldsymbol{x}_i] \leq \boldsymbol{m}_i \leq [\boldsymbol{x}_i] + 1 \quad (i \in I)$,
2. $\boldsymbol{x}_i - [\boldsymbol{x}_i] < \boldsymbol{x}_j - [\boldsymbol{x}_j] \Longrightarrow \boldsymbol{m}_i - [\boldsymbol{x}_i] \leq \boldsymbol{m}_j - [\boldsymbol{x}_j] \quad (i, j \in I)$
3. $\boldsymbol{m}(I) = h$.

Verbally, each party or player is first alloted the largest integer below his proportional share, thereafter the remaining votes are distributed according to the size of the remainders $\boldsymbol{x}_i - [\boldsymbol{x}_i]$.

The Hare procedure can be seen as an apportionment, however, the assignment $(\boldsymbol{g}, h) \to \boldsymbol{m}$ as described by the above definition is not unique.

Remark 2.2. Given $\boldsymbol{g}$ and h (and hence $\boldsymbol{x}$), we denote the remainders by

$$\boldsymbol{q}_i \ := \ \boldsymbol{x}_i - [\boldsymbol{x}_i] \quad (i \in I),$$

again $\boldsymbol{q} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n)$ can be viewed as a vector or as a set function. Now, if $\boldsymbol{x}$ is not an integer (a case prevailing most frequently), then we have

$$0 < \boldsymbol{q}(I) < n \ .$$

But because of

$$\boldsymbol{q}(I) = \boldsymbol{x}(I) - \sum_{i \in I} [\boldsymbol{x}_i] = h - \sum_{i \in I} [\boldsymbol{x}_i]$$

we observe that $\boldsymbol{q}(I)$ is an integer, hence we know that

$$1 \leq \boldsymbol{q}(I) \leq n - 1 \tag{1}$$

holds true. Therefore, if $\boldsymbol{x}$ is not an integer, then at least one and at most $n-1$ votes are to be allotted beyond the quantities $[\boldsymbol{x}_i]$.

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Within the following presentation we employ the l^1-norm in $\mathbb{R}^n$, i.e., e write

$$|z| \ := \ \sum_{i\in I} |z_i| \quad (z \in \mathbb{R}^n).$$

Theorem 2.3. *Let $(\boldsymbol{g}, h) \in \mathbb{N}_0^n \times \mathbb{N}$. If an integer distribution $\boldsymbol{m} \in \mathbb{N}_0^n$ is obtained by the Hare procedure, then $\boldsymbol{m}$ is close to the proportional solution $\boldsymbol{x} \ = \ \frac{h}{\boldsymbol{g}(I)}\boldsymbol{g}$, i.e.,*

$$|\boldsymbol{m} - \boldsymbol{x}| = \min\left\{ \left| \boldsymbol{m}' - \boldsymbol{x} \right| \ \middle|\ \boldsymbol{m}' \in \mathbb{N}_0^n, \boldsymbol{m}'(I) = h \right\}. \tag{2}$$

On the other hand, any minimizer in (2) *is obtained from the Hare procedure.*

Proof: Throughout this proof we may assume that $\boldsymbol{x}$ is *not* integer valued, for otherwise nothing has to be proved.

We start out with the second claim, so let $\boldsymbol{m}$ be an integer distribution minimizing the l^1-distance to the proportional solution $\boldsymbol{x}$.

1stSTEP : Assume that

$$\boldsymbol{m}_i \geq [\boldsymbol{x}_i] + 2 \tag{3}$$

holds true for some $i \in I$. Then, necessarily there has to be some $j \in I$ satisfying

$$\boldsymbol{m}_j \leq [\boldsymbol{x}_j], \tag{4}$$

for otherwise we would have

$$\boldsymbol{m}(I) > \sum_{i\in I}([\boldsymbol{x}_i] + 1) > \sum_{i\in I} \boldsymbol{x}_i = h,$$

contradicting the definition of $\boldsymbol{m}$. Now, allot an aditional vote to player (party) j and deduct it from i, i.e., consider the distribution

$$\boldsymbol{m}' \ := \ \boldsymbol{m} - \boldsymbol{e}^i + \boldsymbol{e}^j.$$

Clearly, we have

$$|\,\boldsymbol{m}'_i - \boldsymbol{x}_i\,| = |\,\boldsymbol{m}_i - \boldsymbol{x}_i\,| - 1, \tag{5}$$

that is, with respect to coordinate i, the distance has been decreased by a unit.

On the other hand, with respect to coordinate j, two cases may be distinguished:

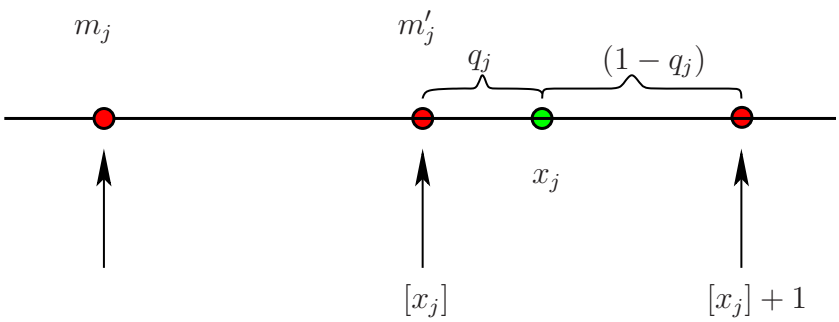

Figure 2.1: Case $\boldsymbol{A}$

-)
  Case $\boldsymbol{A}$: $\boldsymbol{m}_j < [\boldsymbol{x}_j]$. Then obviously

  $$|\,\boldsymbol{m}'_j - \boldsymbol{x}_j\,| = |\,\boldsymbol{m}_j - \boldsymbol{x}_j\,| - 1$$

  is true.

-) Case $\boldsymbol{B}$: $\boldsymbol{m}_j = [\boldsymbol{x}_j]$. Then we obtain

  $$\begin{aligned} |\,\boldsymbol{m}_j - \boldsymbol{x}_j\,| &= \boldsymbol{q}_j, \\ |\,\boldsymbol{m}'_j - \boldsymbol{x}_j\,| &= 1 - \boldsymbol{q}_j. \end{aligned}$$

Combining the results for both the coordinates that have been changed we obtain

$$|\,\boldsymbol{m}' - \boldsymbol{x}\,| = |\,\boldsymbol{m} - \boldsymbol{x}\,| + \begin{cases} -2 & \text{in Case } \boldsymbol{A} \\ -2\boldsymbol{q}_j & \text{in Case } \boldsymbol{B}, \end{cases} \tag{6}$$

which shows that $\boldsymbol{m}'$ has a smaller distance towards the proportional solution compared to $\boldsymbol{m}$. Hence, (3) cannot be the case, we must have

$$\boldsymbol{m}_i \le [\boldsymbol{x}_i] + 1 \quad (i \in I).$$

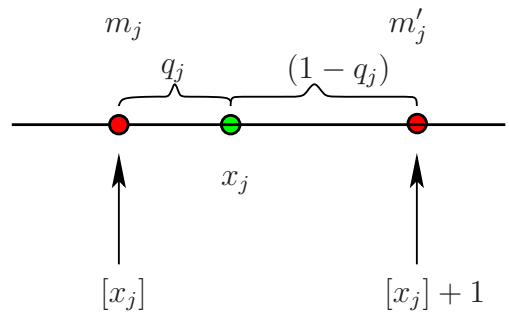

Figure 2.2: Case $\boldsymbol{B}$

2ndSTEP : By a similar procedure we establish the second inequality $[\boldsymbol{x}_i] \leq \boldsymbol{m}_i \;\; (i \in I)$, in other words we know that

$$[\boldsymbol{x}_i] \leq \boldsymbol{m}_i \leq [\boldsymbol{x}_i] + 1 \;\; (i \in I) \tag{7}$$

is satisfied.

3rdSTEP : It remains to show the following:

If $\boldsymbol{q}_i > \boldsymbol{q}_j$ then $\boldsymbol{m}_i = [\boldsymbol{x}_i], \;\; \boldsymbol{m}_j = [\boldsymbol{x}_j] + 1$ is *not* true.

However, if the above would be the case, then again we change from $\boldsymbol{m}$ to dm' via $\boldsymbol{m}' = \boldsymbol{m} + \varepsilon^i - \boldsymbol{e}^j$. Comparing the distance coordinate-wise we obtain

$$\begin{aligned} |\,\boldsymbol{m}_i - \boldsymbol{x}_i| &= \boldsymbol{q}_i\ , \qquad |\,\boldsymbol{m}_j - \boldsymbol{x}_j| = 1 - \boldsymbol{q}_j \\ |\,\boldsymbol{m}'_i - \boldsymbol{x}_i| &= 1 - \boldsymbol{q}_i\ , \qquad |\,\boldsymbol{m}'_j - \boldsymbol{x}_j| = \boldsymbol{q}_j. \end{aligned} \tag{8}$$

Now, as

$$1 - \boldsymbol{q}_i + \boldsymbol{q}_j < \boldsymbol{q}_i + 1 - \boldsymbol{q}_j$$

follows from

$$2\boldsymbol{q}_j < 2\boldsymbol{q}_i,$$

we observe that again $\boldsymbol{m}'$ would have a smaller distance towards $\boldsymbol{x}$, which contradicts the minimal-distance property of $\boldsymbol{m}$.

This shows that anx minimizer of the distance towards $\boldsymbol{x}$ satisfies the Hare procedure.

4thSTEP :

The converse direction we will treat somewhat superficially, for the intuition sems to be rather obvious. Any result of the Hare procedure is indeed minimizing the distance towards $\boldsymbol{x}$.

Indeed, inspect the typical distribution of votes exceding $[\boldsymbol{x}_i]$ according to the size of the remainders $\boldsymbol{q}_i$. For simplicity we assume in the following sketch that the remainders decrease according to the natural order.

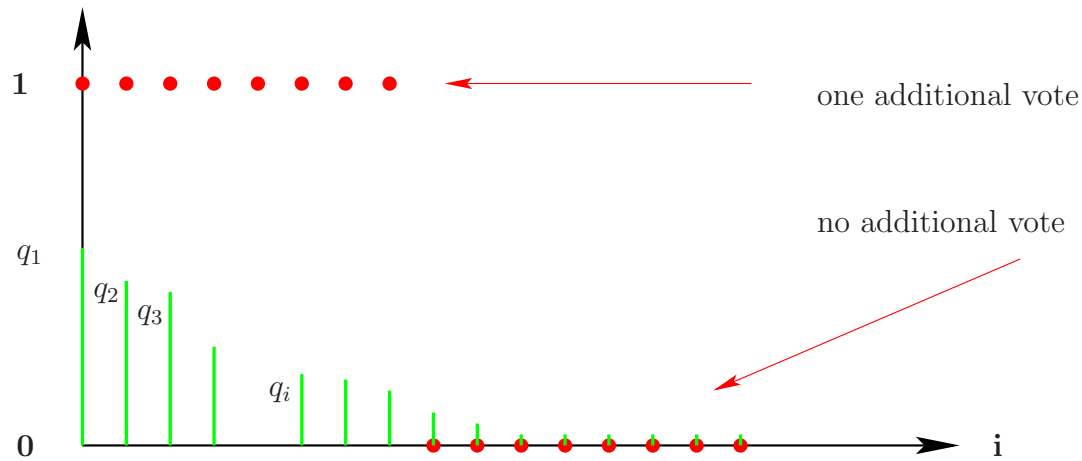

Figure 2.3: Additional votes in the natural ordering

It is seen that any two results of the Hare procedure, say $\boldsymbol{m}$, $\boldsymbol{m}'$, differ only at most on a set of the type $\{i|\boldsymbol{x}_i = const\}$. From this it follows at once that $|\boldsymbol{m} - \boldsymbol{x}| = |\boldsymbol{m}' - \boldsymbol{x}|$ holds true. But as we have seen in the first 3 STEPs, all minimizers are among the Hare-generated distributions, hence the minimizers and the results of the Hare procedure coincide,

q.e.d.

Remark 2.4. Let

$$\overline{\underline{\mathbf{X}}}^h := \left\{ \boldsymbol{z} \in \{\mathbb{R}^n_+ \,\middle|\, \sum_{i=1}^n z_i = h\} \right\}$$

denote the h–multiple of the unit simplex in $\mathbb{R}^n$. For $n = 3$, the Hare procedure can nicely be viewed in $\overline{\underline{\mathbf{X}}}^3$ as follows. Assuming $0 < \delta(\boldsymbol{I}) < 3$ we know that $\boldsymbol{q}(\boldsymbol{I}) \in \{1, 2\}$ and we distinguish zwo cases accordingly.

Note that these two cases correspond to two situations in $\overline{\underline{\mathbf{X}}}$ represented by the accompanying sketches as follows. The case $\boldsymbol{q}(\boldsymbol{I}) = 1$ reflects the situation in which the proportionality $\boldsymbol{x}$ is located in a subtriangle of $\overline{\underline{\mathbf{X}}}$ with edges parallel to those of $\overline{\underline{\mathbf{X}}}$.

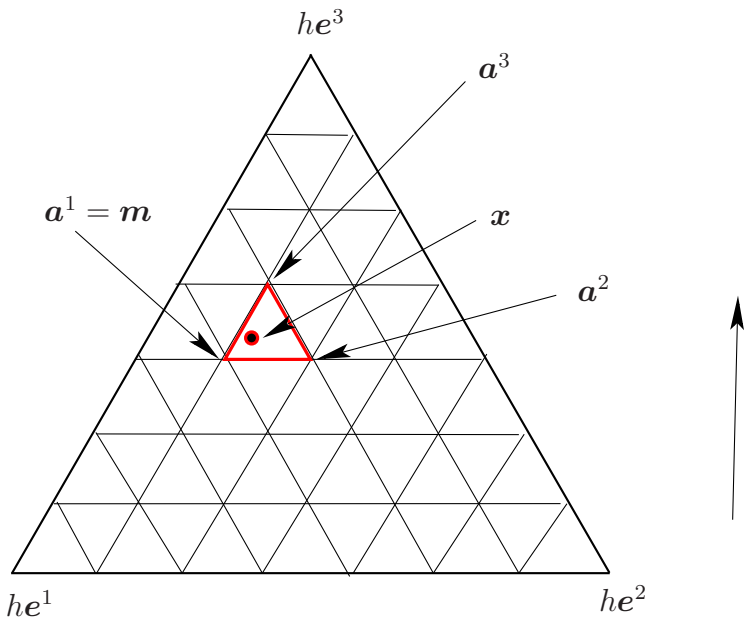

Figure 2.4: $1^{st} STEP : \boldsymbol{q}(\boldsymbol{I}) = 2$

1stSTEP :

More formally, assume now tht $\boldsymbol{q}(\boldsymbol{I}) = 1$ holds true. Writing $[\boldsymbol{z}] = ([z_1], \ldots, [z_n]$ we obtain

$$\boldsymbol{x} = [\boldsymbol{x}] + \boldsymbol{q} \ .$$

Introducing furthermore

$$\boldsymbol{a}^i := [\boldsymbol{x}] + \boldsymbol{e}^i \in \underline{\overline{\mathbf{X}}}^3 \ (i = 1, 2, 3) \ ,$$

we obtain

$$\boldsymbol{x} = [\boldsymbol{x}] + \boldsymbol{q} = \sum_{i=1}^{n} q_i([\boldsymbol{x}] + \boldsymbol{e}^i) = \sum_{i=1}^{n} q_i \boldsymbol{a}^i$$

as $\sum_{i=1}^{n} q_i = 1$. Thus, the remainders, i.e., the coordinates of $\boldsymbol{q}$ yield exactly the barycentric coordinates of $\boldsymbol{x}$ within the simplex spanned by the $\boldsymbol{a}^i$ $(i \in \boldsymbol{I})$. The integer vector chosen by the Hare procedure is obviously the one $\boldsymbol{a}^i$ which is closest to $\boldsymbol{x}$ as it is obtained by alloting one additional unit to the i with largest q_i.

2ndSTEP : Consider now the case that $\boldsymbol{q}(\boldsymbol{I}) = 2$ holds true. Then we define

$$\boldsymbol{b}^k := [\boldsymbol{x}] + (1, 1, 1) - \boldsymbol{e}^k = [\boldsymbol{x}] + (1, \underbrace{0}_{k}, 1) \ \ (k = 1, 2, 3)$$

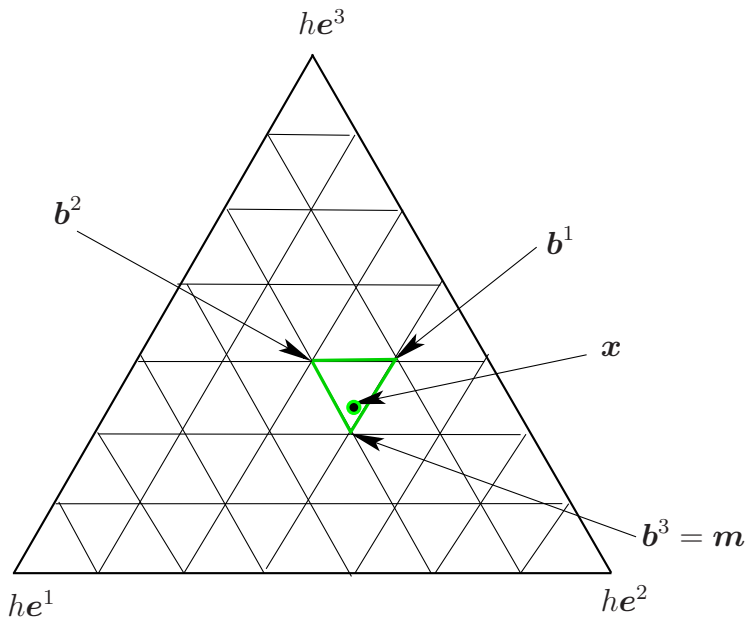

Figure 2.5: $2^{nd} STEP : \boldsymbol{q}(\boldsymbol{I}) = 1$

and observe

$$\sum_{k=1}^{3}(1-q_k) = 3 - \sum_{k=1}^{3} q_k = 1 \ .$$

Now we obtain

$$\begin{aligned}
\boldsymbol{x} &= [\boldsymbol{x}] + (1,1,1) - \sum_{k=1}^{3}(1-q_k)\boldsymbol{e}^k \\
&= \sum_{k=1}^{3}(1-q_k)\left([x] + (1,1,1) - \boldsymbol{e}^k\right) \\
&= \sum_{k=1}^{3}(1-q_k)\boldsymbol{b}^k \quad .
\end{aligned}$$

Thus, in this case the weights $(1-q_k)$ are the barycentric coordinates of $\boldsymbol{x}$ with respect to the simplex spanned by the vectors $\boldsymbol{b}^k \quad (k = 1,2,3)$. Now, if q_k is minimal, then $1-q_k$ is maximal hence $\boldsymbol{b}^k$ is the vector nearest to $\boldsymbol{x}$. However, $\boldsymbol{b}^k$ is characterized by the fact that *two additional* votes are allotted to the two indices $i \neq k$. Observe that the simplex spanned by the vectors $\boldsymbol{b}^k$ in this case has edges that as well are parallel to those of $\underline{\overline{\mathbf{X}}}^h$ – but $\boldsymbol{b}^1$ eg. is maximally distant from $\boldsymbol{e}^1$ and similar for $\boldsymbol{b}^2, \boldsymbol{b}^3$.

The reader is obliged to ponder about the analogue geometric representation of the Hare procedure in higher dimensions. There are only two types of triangles in the case $n = 3$ we have considered, but with growing dimension there will be more and more possible arrangements of extremal vctors of the subsimplices generated by vectors with integer coordinates. According to the size of $\boldsymbol{q}(\boldsymbol{I})$ one can single out such a subsimplex and look at the appropriate representation of $\boldsymbol{x}$ in $\underline{\overline{\mathbf{X}}}^h$ by means of barycentric coordinates. this will show that the Hare procedure chooses the nearest point to the proportionality as in the lower dimensional case....

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3 The d'Hondt Procedure

The next procedure we consider is widely used in the political systems of Western Europe: the d'Hondt procedure (Victor d'Hondt, Professor iuris University of Gent, Belgium, 1941 – 1901). As prefiously, the procedure does not define a function as its result is not nesessarily unique.

Definition 3.1. *Let* $\boldsymbol{g} \in \mathbb{N}_0^n$ *and* $h \in \mathbb{N}$. *Define*

$$\boldsymbol{Q} \;=\; (q_{ij})_{i\in\boldsymbol{I},j\in\mathbb{R}} \;:=\; \left(\frac{g_i}{j}\right)_{i\in\boldsymbol{I},j\in\mathbb{N}}$$

and put, for every $(i,j) \in \boldsymbol{I} \times \mathbb{N}$,

$$N_{ij} \;:=\; \Big|\, \{q_{lm} \in \boldsymbol{I} \times \mathbb{N} \,|\, q_{lm} \geq q_{ij}\} \,\Big| \;=\; \sum_{l=1}^{N} \max\left\{ m \in \mathbb{N} \,\middle|\, \frac{g_l}{m} \geq q_{ij} \right\} .$$

We shall say that $\boldsymbol{m} \in \mathbb{N}$ *is consistent with the* ***d'Hondt procedure*** *if there is a unique pair* $(i_0, j_0) \in \boldsymbol{I} \times \mathbb{N}$ *such that*

$$N_{i_0 j_0} \;=\; h \text{ and } m_{i_0} \;=\; |\, \{j \,|\, q_{ij} \geq q_{i_0 j_0}\} \,| \qquad (i \in \boldsymbol{I})$$

holds true.

Intuitively we list the ratios $\left(\frac{g_i}{j}\right)_{i\in\boldsymbol{I},j\in\mathbb{N}}$ and order them according to size. We take the first h in size of these quotients. Whenever a quotient $\frac{g_i}{j}$ is among these largest h, then party $i \in \mathbb{N}$ obtains a seat in the parlament.

Or else, consider the (infinite) matrix $\boldsymbol{Q}$ in the sketch of Figure 3.1. For every $(i,j) \in \boldsymbol{I} \times \mathbb{N}$, the entries of this matrix are partitioned into those greater than or equal to $\frac{g_i}{j}$ and those strictly smaller. The ("left side" of the) i^{th} row of $\boldsymbol{Q}$ then shows exactly the seats allotted to party i.

We emphasize again that a pair $(i_0; j_0)$ satisfying $N_{i_0 j_0} = h$ is not necessarily unique. If there are several pairs with this property, then the votes remaining after all larger quotients have been served will have to be distributed by a chance mechanism. However, to incorporate this into a precise model with stochastic background is not worthwile; hence we will allways tacitly assume that $(i_0; j_0)$ is unique.

Remark 3.2. 1. If it so happens that $N_{i_0 j_0} = h$, then player/party i_0 receives exactly $m_{i_0} = j_0$ votes/seats. Intuitively (inspect Figure 3.2)

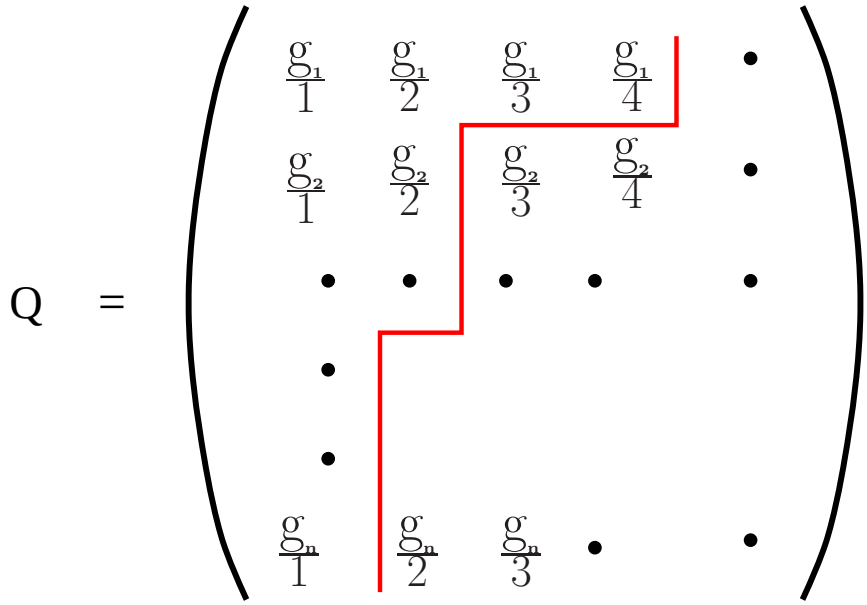

Figure 3.1: Partitioning the ratios

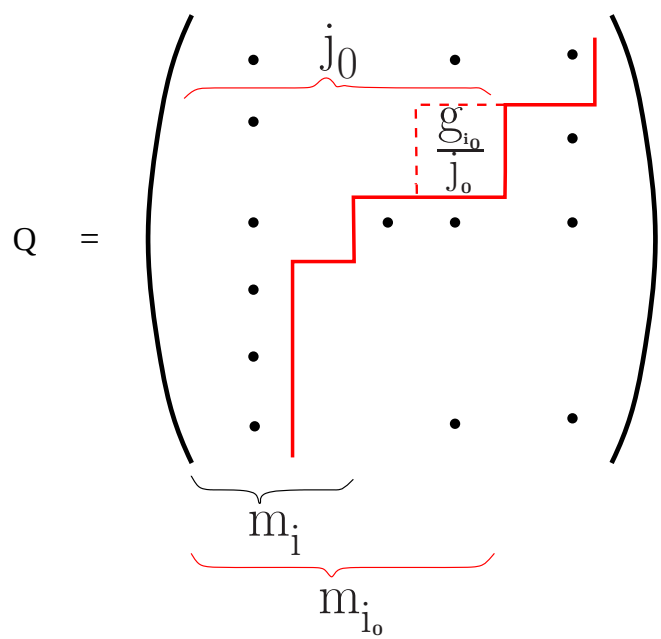

Figure 3.2: Party i_0 obtaining j_0 seats

player i_0 is the last one to receive a vote as his ratio $\frac{g_{i_0}}{j_0}$ is the smallest that is taken into consideration. This ratio appears as the last entry of the matrix $\boldsymbol{Q}$ in the intersection of row i_0 with the left half of the partition.

2. For player $i \neq i_0$ the number m_i of his votes is precisely determined by

$$\frac{g_i}{m_i} \geq \frac{g_{i_0}}{j_0} > \frac{g_i}{m_i + 1}$$

i.e.,

$$\frac{g_i}{g_{i_0}} j_0 - 1 < m_i \leq \frac{g_i}{g_{i_0}} j_0$$

from which we deduce

$$m_i = \left[\frac{g_i}{g_{i_0}} j_0 \right] \tag{1}$$

3. Thus we obtain for two players i, l

$$\frac{m_i}{m_l} = \frac{\left[\frac{g_i}{g_{i_0}} j_0 \right]}{\left[\frac{g_i}{g_{i_0}} j_0 \right]} \quad ,$$

which can be seen as the "version of proportionality" underlying the d'Hondt procedure.

4. An algorithm to determine the distribution of seats according to the d'Hondt procedure is shortly described as follows:

   (a) Determine (i_0, j_0) such that

   $$N_{i_0 j_0} = h \quad .$$

   (b) For each party $i \in \boldsymbol{I}$, determine the number of seats m_i by

   $$m_i = \left[\frac{g_i}{g_{i_0}} j_0 \right] \quad .$$

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Lemma 3.3. *Let*

$$\underline{\underline{\mathbf{Q}}} \;:=\; \left\{ \frac{g_i}{k} \,\middle|\, i \in \boldsymbol{I}, k \in \mathbb{N} \right\}$$

be the set of all ratios and let

$$p \;:=\; \max \left\{ q \in \underline{\underline{\mathbf{Q}}} \,\middle|\, \sum_{i \in \boldsymbol{I}} \left[\frac{g_i}{q} \right] \;=\; h \right\} \quad . \tag{2}$$

Then the levelling procedure S^p (see Example 1.2) results in the D'Hondt procedure. In other words, there is one seat per p votes allotted.

Proof: Let (i_0, j_0) be such that $N_{i_0 j_0} \;=\; h$ as previously and put $\bar{q} = \frac{g_{i_0}}{j_0}$. Then $\bar{q} \in \underline{\underline{\mathbf{Q}}}$. For any $i \in \boldsymbol{I}$ we have $m_i = \left[\frac{g_i}{\bar{q}} \right]$ and, as $\sum_{i \in \boldsymbol{I}} m_i = h$, we see that $\bar{q}$ is an element of the set used to define p in formula (2).

Hence $p \geq \bar{q}$.

On the other hand, assuming $p > \bar{q}$ we would obtain

$$\left[\frac{g_i}{p} \right] \leq \left[\frac{g_i}{\bar{q}} \right] \qquad (i \in \boldsymbol{I})$$

and for $i = i_0$ we have "$<$" since

$$\left[\frac{g_{i_0}}{\bar{q}} \right] = \left[\frac{g_{i_0}}{\frac{g_{i_0}}{j_0}} \right] = j_0 .$$

Hence we would come up with

$$\sum_{i \in \boldsymbol{I}} \left[\frac{g_i}{p} \right] < \sum_{i \in \boldsymbol{I}} \left[\frac{g_i}{\bar{q}} \right] \;<\; h$$

contradicting the definition of p. Hence $p = \bar{q}$.

q.e.d.

Chapter 2

Homogeneous Games

The term "homogenous" for a certain simple zero sum game is due to von Neumann–Morgenstern. It refers to a game with a very "homogenous" class of minimal winning coalitions. The notion is easily extended to general simple games. In this context, the property rather resembles some kind of fine divisibilty of the votes considered as a measure over the players; one can with some justification consider the property to yield a finite analogue to a nonatomic measure. We will discuss the definitions and the historical development provided in [4] within the first section and explain the strong implications in the general case within the further sections.

1 Definitions, The Zero-Sum-Case

The following definition is the general version valid for all simple games. However, this section will essentially treat the zero sum case.

Definition 1.1. *1. Let $\boldsymbol{m} \in \mathbb{A}_+$ be a measure and let $\alpha \in \mathbb{R}_+$ be a real number. We shall say that $\boldsymbol{m}$ is **homogeneous** with respect to α if, for every $T \in \underline{\underline{\mathbf{P}}}$ satisfying $\boldsymbol{m}(T) > \alpha$, there exists $S \subseteq T$ satisfying $\boldsymbol{m}(T) = \alpha$. If the definition prevails, then we write $\boldsymbol{m}$ hom α*

*2. A simple game $\boldsymbol{v}$ is called **homogeneous** if there exists a pair $(\boldsymbol{m}, \alpha) \in \mathbb{A}_+ \times \mathbb{R}_+$ such that*

(a) $\boldsymbol{v} = \boldsymbol{v}_\alpha^{\boldsymbol{m}}$,

(b) $\boldsymbol{m}$ hom α

holds true.

A coalition is ***minimal winning*** if every proper subcoalition is losing. The system of minimal winning coalitions is denoted by $\underline{\underline{\mathbf{W}}}^{min}$. Using this notation, a game is homogenous if there exists a representation $(\boldsymbol{m}, \alpha)$ such that

$$\underline{\underline{\mathbf{W}}}^{min} = \left\{ S \in \underline{\underline{\mathbf{P}}} \,\middle|\, \boldsymbol{m}(S) = \alpha \right\} \tag{1}$$

holds true.

There is a second property of simple games (or rather of representations of such games) which sometimes is appropriately used as well to imitate nonatomic measures or distributions of votes in the finite context.

Definition 1.2. *1. Let $\boldsymbol{m} \in \mathbb{A}_+$ be a measure and let $\alpha \in \mathbb{R}_+$ be a real number. Then the system*

$$\underline{\underline{\mathbf{Q}}}_\alpha := \left\{ S \in \underline{\underline{\mathbf{P}}} \,\middle|\, \boldsymbol{m}(S) = \alpha \right\} \tag{2}$$

*is said to be the **defining** system of $\boldsymbol{m}$ (w.r.t. α).*

*2. We shall say that $\boldsymbol{m}$ is **nondegenerate** with respect to α (written $\boldsymbol{m}$ nd. α) if $\boldsymbol{m}$ is uniquely defined by the system $\underline{\underline{\mathbf{Q}}}_\alpha$, more precisely, if the linear system of equations in variables $x_1, \ldots, x_n$ given by*

$$\sum_{i \in S} x_i = \alpha \quad (S \in \underline{\underline{\mathbf{Q}}}_\alpha) \tag{3}$$

has the unique solution $\boldsymbol{x} = \boldsymbol{m}$.

Note that, for a homogeneous simple game, we have immediately a close relation between the system of minimal winning coalitions and the defining system, obviously

$$\underline{\underline{\mathbf{W}}}^{min} \subseteq \underline{\underline{\mathbf{Q}}}_{\alpha} \tag{4}$$

holds true. Any coalition $S \in \underline{\underline{\mathbf{Q}}}_{\alpha}$ contains a unique minimal winning coalition which is obtained by throwing out the dummies. If no dummies are present, then both systems obviously coincide.

In passing let us prove that non–degeneracy implies rationality of the quantities involved.

Lemma 1.3. *Let $(\boldsymbol{m}, \alpha) \in \mathbb{A}_+ \times \mathbb{R}_+$ such that $\boldsymbol{m}$ is normalized to $\boldsymbol{m}(\boldsymbol{I}) = \alpha$. If $\boldsymbol{m}$ nd. α holds true, then $\boldsymbol{m}$ α are rational.*

Proof: According to Cramers rule we can obtain $\boldsymbol{m}$ as the unique solution of the linear system of equations given by (3) via

$$m_i = \frac{|\, \boldsymbol{D}^i \,|}{|\, \boldsymbol{D} \,|} \quad ; \tag{5}$$

here dD is the coefficient matrix of the system (3) and Δ^i is the matrix obtained from $\boldsymbol{D}$ by replacing the ith column of $\boldsymbol{D}$ by the column $(\alpha, \ldots, \alpha)$. $|\bullet|$ denotes the determinant. Therefore, $\boldsymbol{D}^i = \alpha \boldsymbol{D}^e$ where the $\boldsymbol{D}^e$ has a column of 1 in position i. Hence, both $\boldsymbol{D}^e$ and $\boldsymbol{D}$ have entries 0 and 1 only. Therefore, both their determinants are rationals. From this and (5) it follows that

$$m_i = \alpha\, r_i \quad (i \in \boldsymbol{I}) \tag{6}$$

with certain rationals r_i $(i \in \boldsymbol{I})$. Because of the normalization we find

$$1 = \boldsymbol{m}(\boldsymbol{I}) = \alpha \sum_{i \in \boldsymbol{I}} r_i, \quad \text{i.e. } , \alpha = \frac{1}{\sum_{i \in \boldsymbol{I}} r_i}, \tag{7}$$

thus α is rational. Now (6) implies that all the m_i $(i \in \boldsymbol{I})$ are rational.

q.e.d.

Let us now discuss the traditional version of a uniqueness theorem for homogeneous games due to von Neumann–Morgenstern.

Theorem 1.4 (von Neumann–Morgenstern)**.** *Let $\boldsymbol{v}$ be a simple game which is superadditive, constant sum, and homogeneous. Then there is a unique homogeneous representation which assigns zero to the dummies and one to the grand coalition.*

Proof:

1st STEP : Given a homogeneous representation $(\boldsymbol{m}, \alpha)$, we can at once assume that dummies have zero voting power (i.e., $\boldsymbol{m}_i = 0 \ \ (i \in \Delta)$) and that it is normalized to yield $\boldsymbol{m}(\boldsymbol{I}) = 1$ and $\alpha \leq 1$. Suppose there is a further representation with these properties, say $(\boldsymbol{m}', \beta)$. We may w.l.g. assume that $\beta \geq \alpha$ and hence the convex compact polyhedron

$$\mathcal{X} \ := \ \left\{ \boldsymbol{x} \in \mathbb{R}^n_+ \,\middle|\, \boldsymbol{x}(\boldsymbol{I}) = 1, \ \boldsymbol{x}(\Delta) = 0, \ \boldsymbol{x}(S) \geq \alpha \ (S \in \underline{\underline{\mathbf{W}}}) \right\} \tag{8}$$

has at least one extremepoint differing from $\boldsymbol{m}$, say $\bar{\boldsymbol{x}} \in \mathcal{X}$.

2nd STEP : We are going to show that $\bar{\boldsymbol{x}}_k = 0$ holds true for at least one $k \in \boldsymbol{I} - \Delta$.

To this end, assume *per absurdum* that $\bar{\boldsymbol{x}}_i > 0$ $(i \in I - \Delta)$ is true. We claim that, for sufficiently small $\varepsilon > 0$ the vectors/measures

$$\boldsymbol{x}^{\pm\varepsilon} \ := \ (1 \pm \varepsilon)\bar{\boldsymbol{x}} \mp \varepsilon \boldsymbol{m} \ \in \mathcal{X} \tag{9}$$

are elements of $\mathcal{X}$. Indeed, they are linear combinations of $\bar{\boldsymbol{x}}$ and $\boldsymbol{m}$, hence satisfy all the equations appearing in (8). Also, the dummy coordinates are obviously 0 and the non–dummy coordinates are positive for small $\varepsilon > 0$. In order to establish our claim we shall, therefore, show that

$$\boldsymbol{m}(S) > \alpha \Longrightarrow \bar{\boldsymbol{x}}(S) > \alpha \tag{10}$$

is true. Now, if S is such that $\boldsymbol{m}(S) > \alpha$ holds true, then pick $T \subseteq S$ with $\boldsymbol{m}(T) = \alpha$ by means of homogeneity. Clearly, $\boldsymbol{m}(S - T) > 0$ implies that there are nondummies contained in $S - T$ (all dummies receive 0 at $\boldsymbol{m}$ as well as at $\bar{\boldsymbol{x}}$). Hence, by the assumption of this step, $\bar{\boldsymbol{x}}(S - T) > 0$ is also true and we have $\bar{\boldsymbol{x}}(S) > \bar{\boldsymbol{x}}(T) \geq \alpha$. This shows (10) and hence the elements described by (9) are contained in $\mathcal{X}$ for small ε. Now in view of

$$\bar{\boldsymbol{x}} = \frac{1}{2}\left(\boldsymbol{x}^{+\varepsilon} + \boldsymbol{x}^{-\varepsilon}\right) \quad (\varepsilon \in \mathbb{R})$$

we established a convex combination of $\bar{\boldsymbol{x}}$ by elements of $\mathcal{X}$ once ε is sufficiently small and not zero. Therefore the basic assumption of this step has been led *ad absurdum*. We know that $\bar{\boldsymbol{x}}_k = 0$ holds true for some $k \in \boldsymbol{I} - \Delta$.

3rd STEP :

As $k \notin \Delta$ holds true there exists $S \in \underline{\underline{\mathbf{W}}}^{min}$ such that $k \in S$ and $S - k \in \underline{\underline{\mathbf{L}}}$ is satisfied. In view of the constant sum property we have $(S - k)^c \in \underline{\underline{\mathbf{W}}}$ and

hence

$$\begin{aligned} \bar{\boldsymbol{x}}(S) &\geq \alpha \\ \bar{\boldsymbol{x}}(S^c) &= \bar{\boldsymbol{x}}(S^c + k) \\ &= \bar{\boldsymbol{x}}((S-k)^c) \quad \geq \quad \alpha \ . \end{aligned}$$

From this we obtain at once

$$1 = \bar{\boldsymbol{x}}(S) + \bar{\boldsymbol{x}}(S^c) \geq 2\alpha$$

or

$$\alpha \leq \frac{1}{2}.$$

Now, S is minimal winning, hence satisfies $\boldsymbol{m}(S) = \alpha$. For S^c we obtain

$$\boldsymbol{m}(S^c) = 1 - \boldsymbol{m}(S) = 1 - \alpha \geq \alpha,$$

thus $\boldsymbol{v}(S) = \boldsymbol{v}(S^c) = 1$.

But $\boldsymbol{v}$ is a constant sum game and hence both, S and S^c cannot be winning coalitions. Consequently we have

$$\mathcal{X} \ = \ \{\boldsymbol{m}\}, \quad \beta = \boldsymbol{\mu}(S) = \boldsymbol{m}(S) = \alpha \quad (S \in \underline{\underline{\mathbf{Q}}}_\alpha) \ ,$$

q.e.d.

For completeness we add a further definition concerning measures or distributions of votes that can also be seen as a "finite version of nonatomicity". This is the notion of non–degeneracy.

Remark 1.5. Given the conditions of Theorem 1.4, we find that

$$\boldsymbol{m} \text{ nd. } \alpha \tag{11}$$

holds true. Indeed, if $\boldsymbol{x}$ is a solution of the equation

$$\sum_{i \in S} x_i \ = \ \alpha \quad (S \in \underline{\underline{\mathbf{Q}}}_\alpha), \tag{12}$$

then it is seen at once that

$$\boldsymbol{m}_i = 0 \Longrightarrow \boldsymbol{x}_i = 0 \quad (i \in I)$$

holds true as well.

Now define, for small $\varepsilon > 0$,

$$\boldsymbol{\mu}^\varepsilon \ := \ \frac{\boldsymbol{m} + \varepsilon \boldsymbol{x}}{1 + \varepsilon \boldsymbol{x}(\boldsymbol{I})} \ , \quad \alpha^\varepsilon \ := \ \frac{\alpha + \varepsilon}{1 + \varepsilon \boldsymbol{x}(\boldsymbol{I})} \ ,$$

such that

$$\boldsymbol{\mu}^{\varepsilon}(S) \begin{array}{c} > \\ < \\ = \end{array} \alpha^{\varepsilon} \Longleftrightarrow \boldsymbol{m}(S) \begin{array}{c} > \\ < \\ = \end{array} \varepsilon$$

is satisfied. For small positive ε the quantities $\boldsymbol{\mu}^{\varepsilon}$ and $\boldsymbol{\alpha}^{\varepsilon}$ are nonnegative and yield

$$\boldsymbol{v} = \boldsymbol{v}_{\alpha}^{\boldsymbol{m}} = \boldsymbol{v}_{\beta^{\varepsilon}}^{\boldsymbol{m}^{\varepsilon}}.$$

Hence, in view of the previous uniqueness theorem, we obtain $\boldsymbol{\mu}^{\varepsilon} = \boldsymbol{m}$ and $\beta^{\varepsilon} = \alpha$ from which $\boldsymbol{x} = \boldsymbol{m}$ follows easily.

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Remark 1.6. 1. Historically the above theorem establishes the importance of the theory of homogeneous games, the proof is due to von Neumann–Morgenstern [4]. Obviously the tools applied are those of convex analysis, we employ a standard procedure based on the existence of extreme points uf a compact convex polyhedron. The development to be presented in the following sections, however, is of a purely combinatorial nature.

2. As a consequence of the previous theorem we observe that, for any simple constant sum, superadditive, and homogeneous game without dummies, the system of minimal winning coalitions and the defining system coincide.

3. Similarly the ***incidence matrix***, i.e., the matrix listing the indicator functions (vectors) of the minimal winning coalitions,

$$I(\boldsymbol{v}) \;=\; (\mathbf{1}_S)_{S \in \underline{\underline{\mathbf{W}}}^{min}}$$

equals the coefficient matrix of the defining system. Now, if we consider a normalized representation $(\boldsymbol{m}, \alpha)$ then clearly, in view of our previous results we know that $\boldsymbol{m}$ and α are rationals (Lemma 1.3). Now the solutions of the defining system form a one dimensional manifold;the positive ones of those are representation of $\boldsymbol{v}$ and multiples of the normalized representation. As $\boldsymbol{m}$ and α are rationals, we can therefore, by an appropriate multiplication, find an integer valued representation.

4. Among all integer valued representations $(\boldsymbol{M}, \lambda)$ clearly the one with minimal total mass $\boldsymbol{M}(\boldsymbol{I})$ is uniquely defined, this representation constitutes the "minimal committee" representing the homogeneous game $\boldsymbol{v}$.

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The following definition compares the strength of two players. It is generally valid for all cooperative TU–games.

Definition 1.7. *Let $(\boldsymbol{I}, \boldsymbol{v})$ be a TU –game. Player $i \in \boldsymbol{I}$ is said to be* ***stronger*** *than player $j \in \boldsymbol{I}$ if*

$$\boldsymbol{v}(S+i) \geq \boldsymbol{v}(S+j) \quad (S \in \underline{\underline{\mathbf{P}}}, \{i,j\} \cap S = \emptyset) \tag{13}$$

holds true. We write $i \succ j$ or $i \succ_{\boldsymbol{v}} j$, this way establishing a (not necessarily complete) binary relation on $\boldsymbol{I}$.

For simple games this means that It is not hard to verify that, whenever player j changes a losing coalition S to a winning coalition $S+j$ by joining it, then so does player j. Now, whenever a simple game $\boldsymbol{v}$ admits of a representation $(\boldsymbol{m}, \boldsymbol{v})$, then the ordering $\succ$ is complete and is given by

$$1 \succ 2 \succ \ldots \succ n \quad \Leftrightarrow \quad m_1 \geq, \ldots, \geq m_n. \tag{14}$$

Remark 1.8. 1. In what follows we shall deal with the general non constant sum case. In view of the above definition we shall assume that our representation is integer valued and that the players are ordered according to strength,that is, we assume

$$m_1 \geq, \ldots, \geq m_n, \tag{15}$$

or equivalently

$$1 \succ 2 \succ \ldots \succ n. \tag{16}$$

For some of the following definitions we will assume that $\delta \boldsymbol{I} = \{1, \ldots, n\}$ is the set of players. However, most definitions have to be valid also for a more general set of players with a specified total ordering. It will be our policy to mention (and explain) the general definition, but not always to write it down. The reason is obvious: once the reader has understood the "natural" version, it is rather clear how to extend it – on the other hand it is not actually clarifying to provide the definition in the general case.

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Lemma 1.9 (The principle of *cutting the tail*)**.** *Let $\boldsymbol{m}$ be homogeneous with respect to α. If $S \in \underline{\underline{\mathbf{P}}}$ satisfies $\boldsymbol{m}(S) > \alpha$, then there is $k \in \boldsymbol{I}$ such that $\boldsymbol{m}(S \cap [1,k]) = \alpha$ is satisfied.*

The meaning is as follows: we can reduce the measure of S down to α by throwing out successively the smallest players (i.e., the "tail") until the level α is hit.

Proof: Accordingly, the proof is quite obvious: throw out the smallest player of S and continue successively until the level α is hit. Necessarily, this level *has* to be hit. For otherwise, should we proceed with the currently smallest player and, by eliminating him, fall *below* α, then there is no way of hitting α at all, as every other player has larger mass – a contradiction to homogeneity.

q.e.d.

Definition 1.10. *Let $i \notin \Delta$ be a non–dummy player.*

1. *The quantity*

   (17) $$l^{(i)} \;:=\; min\left(l(S) \,\middle|\, i \in S, S \in \underline{\underline{\mathbf{W}}}^{min}\right)$$

   is player i's ***delimiter****. Moreover, the coalition*

   (18) $$C^{(i)} \;:=\; [l^{(i)}+1, n]$$

   constitute player i's ***satellites****.*

2. *player i has the* ***character*** *of a* ***sum*** *if*

   (19) $$\boldsymbol{m}_i \leq \boldsymbol{m}(C^{(i)})$$

   holds true. Otherwise player i has the character of a ***step****. We write $\Sigma = \Sigma(\boldsymbol{v})$ for the coalition of sums in (with respect to) the game $\boldsymbol{v}$ and $\mathrm{T} = \mathrm{T}(\boldsymbol{v})$ for the coalition of steps. This way we obtain a decomposition of the set of players*

   (20) $$\boldsymbol{I} \;=\; \Delta + \Sigma + \mathrm{T}$$

   into dummies, sums, and steps.

3. *For any player $i \in \Sigma$ the restriction*

   $$\boldsymbol{m}^{(i)} \;:=\; \boldsymbol{m}|_{C^{(i)}}$$

   is called player i's ***satellite measure*** *and the game*

   (21) $$\boldsymbol{v}^{(i)} \;:=\; \boldsymbol{v}_{\boldsymbol{m}_i}^{\boldsymbol{m}^{(i)}}$$

   is player i's ***satellite game****.*

4. *the* ***character vector*** *of* $\boldsymbol{v}$ *is the vector* $\chi = \chi(\boldsymbol{v})$ *given by*

$$\chi(\boldsymbol{v})_i := \left\{ \begin{array}{lll} 0 & \textit{if} & i \textit{ is a dummy} \\ 1 & \textit{if} & i \textit{ is a sum} \\ 2 & \textit{if} & i \textit{ is a step} \end{array} \right\} \tag{22}$$

Remark 1.11. 1. A player $i \notin \Delta$ is a sum if and only if there exists a coalition $S \in \underline{\underline{\mathbf{W}}}^{min}$ and a player $k \geq l(S) + 1$ such that

$$S - i + [l(S) + 1, k] \in \underline{\underline{\mathbf{W}}}^{min} \tag{23}$$

holds true. Indeed, by definition we know that $C^{(i)} = [l(S) + 1, k]$ has mass exceeding $\boldsymbol{m}_i$ and by cutting the tail we find player k as desired.

This means that a player i is a sum if and only if there exists a coalition of small players (beyond his delimiter and within his satellites) which can replace player i in a minimal winning coalition. From this it follows that the property of being a sum (and hence the definition of characters) is independent of the representation: *the assignment of characters to the players is a property of the game.*

2. Similarly, a step is player who cannot be replaced by his satellites in any minimal winning coalition.

3. In particular, every veto player (i.e., a player who belongs to every winning coalition) is a step.

4. The smallest nondummy player is always a step, for his satellites consist of dummies only.

5. The above definition of characters rests on the assumption that $\boldsymbol{I} = \{1, \ldots, n\}$ has been chosen. In this case $l^{(i)}$ is the length of the shortest min–win coalition containing player i. If $\boldsymbol{I}$ is a finite set endowed with some total ordering $\succ$, then the name of the player located at position $l^{(i)}$ (which is computed by means of $\succ$) may well differ from this position. It should be clear that the definition of characters nevertheless can be extended in a canonical manner.

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Example 1.12. Let $\boldsymbol{m} = (5, 4, 3, 2, 1, 1)$ and $\alpha = 12$, the $\boldsymbol{v} = \boldsymbol{v}_\alpha^{\boldsymbol{m}}$ is homogeneous. Instead of the coalitions of $\underline{\underline{\mathbf{W}}}^{min}$ we list the *incidence matrix*

$$\boldsymbol{I} = \boldsymbol{I}(\boldsymbol{v}) = (1_S)_{S \in \underline{\underline{\mathbf{W}}}^{min}} = (1_S)_{S \in \underline{\underline{\mathbf{Q}}}_\alpha} = \boldsymbol{Q}_\alpha$$

(the equations hold true as there are no dummies in the game):

$$\boldsymbol{I} \;=\; \begin{pmatrix} 1\,1\,1\,0\,0\,0 \\ 1\,1\,0\,1\,1\,0 \\ 1\,0\,1\,1\,1\,1 \end{pmatrix} \tag{24}$$

The rows of $\boldsymbol{I}$ appear in lexicographic order, this way we can see rather easy who can possibly be replaced by his satellites in a min–win coalition and who not.

Now, player 1 (with weight $\boldsymbol{m}_1 = 5$) has a delimiter $l^{(1)} = 3$ and as $\boldsymbol{m}(\{4,5,6\}) = 4 < 5$ is the case, player 1 is a *step*. Next we observe that $l^{(2)} = l^{(3)} = 3$ holds true and as we have $\boldsymbol{m}(\{4,5,6\}) = 4 \geq \boldsymbol{m}_2, \boldsymbol{m}_3$, it follows at once that players 2 and 3 are both *sums*.

players 4 and 5 have delimiters $l^{(i)} = l^{(i)} = 5$. Player 4 (weight 2) cannot be replaced by player 6 but player 5 can. Thus, the former is a *step* and the latter a *sum*. Finally, player 6 as the smallest (and non–dummy) is clearly a *step*.

We indicate this, writing t for steps and s for sums (d for dummies does not appear in this example) as follows:

$$\begin{array}{rcl} \boldsymbol{m} & = & (5,4,3,2,1,1) \\ & & t\;\;s\;\;s\;\;t\;\;s\;\;t\;\;, \end{array} \qquad \alpha \;=\; 12$$

or else by presenting the character vector which is $\chi(\boldsymbol{v}) \;=\; (2,1,1,2,1,2)$.

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The lexicographic ordering of min–win coalitions as represented in the incidence matrix $\boldsymbol{I}$ starts out with a coalition which has the shape of an interval $[1, i_0]$. Clearly, whenever a homogeneous game $\boldsymbol{v} = \boldsymbol{v}_\alpha^{\boldsymbol{m}}$ is represented homogenously by $\boldsymbol{m}$ and α, then player i_0 is obtained from the grand coalition I by cutting the tail.I.e., we can write

$$\begin{aligned} i_0 \;&:=\; \min\{i \in \boldsymbol{I} \mid \boldsymbol{m}([1,i]) \geq \alpha\} \\ S^{(0)} \;&:=\; [1, i_0] \end{aligned} \tag{25}$$

in order to obtain the lexicographic first min–win coalition. Of course, this coalition depends on the game and not on the particular representation.

Definition 1.13. *Let $\boldsymbol{v}$ be a homogeneous game. The* ***lex–max min–win*** *coalition is the lexicographically first minimal winning coalition and is written*

$$S^{(0)} \;:=\; [1, i_0] \;. \tag{26}$$

The player (or position) i_0 *is the* ***zero delimiter*** *and the coalition*

$$C^{(0)} \ := \ [i_0 + 1, n] \tag{27}$$

is the coalition of ***zero satellites****. We write*

$$\boldsymbol{m}^{(0)} \ := \ \boldsymbol{m}|_{C^{(0)}} \tag{28}$$

for the restriction of $\boldsymbol{m}$ *onto the zero satellites and call* $\boldsymbol{m}^{(0)}$ *the* ***zero satellite measure*** *–which is quite in accordance with Definition 1.10.*

Remark 1.14. For any player $j \in S^{(0)}$ we find that

$$l^{(j)} \ = \ i_0, \ C^{(j)} \ = \ C^{(0)}, \ \boldsymbol{m}^{(j)} \ = \ \boldsymbol{m}^{(0)}$$

holds true. Thus, $j \in S^{(0)}$ is a sum if and only if $\boldsymbol{m}_j \leq \boldsymbol{m}(C^{(0)}) \ = \ \boldsymbol{m}^{(0)}(C^{(0)})$ holds true. Otherwise, player j is a step in which case he is necessarily a veto player. The veto players are *exactly* the steps in the lex–max min–win coalition $S^{(0)}$.

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Now we have the following

Lemma 1.15 (The Basic Lemma)**.** *Let* $(\boldsymbol{m}, \alpha) \in \mathbb{A} \times \mathbb{R}_+$ *and assume* $\boldsymbol{m}(\boldsymbol{I}) \geq \alpha$*. Then* $\boldsymbol{m}$ *hom* α *holds true if and only if the following is satisfied:*

1. $\boldsymbol{m}(S^{(0)}) \ = \ \alpha$,
2. *for all* $j \in S^{(0)}$ *the relation* $\boldsymbol{m}^{(0)}$ *hom* $\boldsymbol{m}_j$ *holds true.*

We can imagine that, for each sum in $S^{(0)}$, i.e., each player $j \in S^{(0)}$ with $\boldsymbol{m}_j \leq \boldsymbol{m}(C^{(0)})$, the players in $C^{(0)}$ play a game $\boldsymbol{v}^j \ := \ \boldsymbol{v}_{\boldsymbol{m}_j}^{\boldsymbol{m}^{(0)}}$ in order to replace this player in the lexicographic first coalition $S^{(0)}$. The Basic Lemma requires that these games are homogeneous. In particular, if some minimal winning coalition, say S^j, of the game $\boldsymbol{v}^j$ replaces player j in $S^{(0)}$, then the result is a *min–win* coalition of the original game,i.e., $S^{(0)} - j + S^j \in \underline{\underline{\mathbf{W}}}^{min}$.

Proof of Lemma 1.15

1st STEP : Assume first of all that $\boldsymbol{m}$ *hom* α holds true. Then, condition *1.* is satisfied by the very definition of $S^{(0)}$ and i_0 in view of the principle of cutting the tail. Hence, it remains to verify condition *2.*

To this end, pick $j \in S^{(0)}$ and $T \subseteq C^{(0)}$ with $\boldsymbol{m}(T) > \boldsymbol{m}_j$, we have to show that the mass of T can be cut down to $\boldsymbol{m}_j$. Now define

$$T' \ := \ S^{(0)} - j + T \ ,$$

then clearly $\boldsymbol{m}(T') > \alpha$ and $\boldsymbol{m}(T' - T) < \alpha$ is the case. By cutting the tail we remove the players of smallest weight from T', i.e., we define

$$r \ := \ \min\{\rho | \boldsymbol{m}(T' \cap [1, \rho])\} \geq \alpha$$

and obtain

$$S' \ := \ T' \cap [1, r]$$

with $\boldsymbol{m}(S') = \alpha$. Clearly, $S' = S^{(0)} - j + S$ with a suitable nonempty $S \subseteq T$ as only small players outside of $S^{(0)} - j$ have been omitted.

Because of

$$\boldsymbol{m}(S^{(0)}) = \alpha = \boldsymbol{m}(S^{(0)} - j + S) = \boldsymbol{m}(S^{(0)}) - \boldsymbol{m}_j + \boldsymbol{m}(S),$$

we obtain

$$\boldsymbol{m}(S) = \boldsymbol{m}^{(0)}(S) = \boldsymbol{m}_j.$$

That is, we have indeed cut down the mass of T down to the one of S, which proves the desired relation $\boldsymbol{m}^{(0)}$ *hom* α. This finishes the first step.

2ndSTEP :

Assume now that conditions *1.* and *2.* are satisfied. We are going to show that $\boldsymbol{m}$ *hom* α holds true. To this end we pick some $T \in \underline{\underline{\mathbf{P}}}$ such that $\boldsymbol{m}(T) > \alpha$ holds true. Now, if it so happens that $S^{(0)} \subseteq T$ is the case, then nothing has to be shown as $S^{(0)}$ is the desired *min–win* coalition in T with measure α by assumption *1.*

So we assume that we have $S^{(0)} \cap T^c \neq \emptyset$. Consider the decomposition suggested by

$$\alpha < \boldsymbol{m}(T) \ = \ \boldsymbol{m}(S^{(0)} \cap T) + \boldsymbol{m}(C^{(0)} \cap T)$$

and observe that we have

$$\alpha \ = \ \boldsymbol{m}(S^{(0)} \cap T) + \boldsymbol{m}(S^{(0)} \cap T^c) \ .$$

By comparing we obtain

$$\boldsymbol{m}^{(0)}(C^{(0)} \cap T) = \boldsymbol{m}(C^{(0)} \cap T) > \boldsymbol{m}(S^{(0)} \cap T^c) = \sum_{j \in S^{(0)}, j \notin T} \boldsymbol{m}_j \ .$$

Now, in view of *2.* we can successively pick coalitions $S^j \subseteq C^{(0)} \cap T$ which are disjoint and have *exactly* measure $\boldsymbol{m}_j$. Joining these coalitions with the players of T in $S^{(0)}$ we obtain

$$S := S^{(0)} \cap T + \sum_{j \in S^{(0)}, j \notin T} S^j \ .$$

This coalition satisfies $S \subseteq T$ and

$$\begin{aligned} \boldsymbol{m}(S) &= \boldsymbol{m}(S^{(0)} \cap T) + \sum_{j \in S^{(0)}, j \notin T} \boldsymbol{m}(S^j) \\ &= \boldsymbol{m}(S^{(0)} \cap T) + \sum_{j \in S^{(0)}, j \notin T} \boldsymbol{m}_j \\ &= \boldsymbol{m}(S^{(0)} \cap T) + \boldsymbol{m}(S^{(0)} \cap T^c) \\ &= \boldsymbol{m}(S^{(0)}) = \alpha, \end{aligned} \tag{29}$$

and hence we have reduced the measure of T to α by means of a suitable subset,

q.e.d.

Corollary 1.16. *1. Let $\boldsymbol{v} = \boldsymbol{v}_\alpha^{\boldsymbol{m}}$ be homogeneously represented by $\boldsymbol{m}$ and α. Then the satellite games $\boldsymbol{v}^{(j)}$ ($j \in S^{(0)}$) are homogeneous games. On the other hand, if, for some representation condition* 2. *of Lemma 1.15 is satisfied and the satellite games are homogeneous, then so is the game $\boldsymbol{v}$.*

2. Let $\boldsymbol{v}$ be a homogeneous game. Then, given $j \in S^{(0)}$, a coalition $S \subseteq C^{(0)} = C^{(j)}$ is min–win with respect to $\boldsymbol{v}^{(j)}$ if and only if $S^{(0)} - i + S$ is min–win *with respect to $\boldsymbol{v}$.*

Example 1.17. Consider the pair $\boldsymbol{m}, \alpha$ given by

$$\boldsymbol{m} = (19, 15, 15, 4, 4, 4, 3, 1, 1, 1,) \ , \quad \alpha = 34 \tag{30}$$

Indeed $\boldsymbol{m}$ is homogeneous with respect to α and the incidence matrix of the

resulting game $\boldsymbol{v}_{\alpha}^{\boldsymbol{m}}$ is given by

$$\boldsymbol{I}(\boldsymbol{v}) \;=\; \begin{pmatrix} 1&1&0&0&0&0&0&0&0&0 \\ 0&1&1&1&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0&0 \\ 1&0&0&1&1&1&1&0&0&0 \\ 0&1&1&0&1&0&0&0&0&0 \\ 0&1&1&0&0&1&0&0&0&0 \\ 0&1&1&0&0&0&1&1&0&0 \\ 1&0&0&1&1&1&0&1&1&1 \\ 0&1&1&0&0&0&1&0&1&0 \\ 0&1&1&0&0&0&1&0&0&1 \end{pmatrix}$$

The *lex–max min–win* coalition is indicated by the first line of this matrix and, as the total weight of the remaining players easily exceeds their weight, both players are sums. The incidence matrix actually shows that each player (apart from the smallest one) is a sum. First, take the lexicographic first coalition in which a player appears (the first row exhibiting a digit 1 in this players position). Next observe that there is lexicographically later coalition in which he does *not* appear but the shape of the incidence vector is the same *regarding all preceding players* – meaning that this player has been replaced by smaller ones.

The first players satellite game is constructed by

$$\boldsymbol{m}^{(1)} \;=\; (15, 4, 4, 4, 3, 1, 1, 1,) \text{ and } \boldsymbol{m}_1 = 19,$$

this game is $\boldsymbol{v}^{(1)} \;=\; \boldsymbol{v}_{\boldsymbol{m}_1}^{\boldsymbol{m}^{(1)}}$ and the incidence matrix is

$$\boldsymbol{I}(\boldsymbol{v}^{(1)}) = \begin{pmatrix} 1&1&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0 \\ 1&0&0&1&0&0&0&0 \\ 1&0&0&0&1&1&0&0 \\ 1&0&0&0&1&0&1&0 \\ 1&0&0&0&1&0&0&1 \end{pmatrix}$$

Naturally, this is a homogeneous game, again we may be concerned about the characters. E.g, the first player in this game is a step (actually a veto player as he appears in all coalitions). His name originally was player 3, the second player with weight 15. Here we encounter the enumeration problem: as the player set of the satellite game is $\{3, 4, 5, 6, 7, 8, 9, 10\}$, the length of a coalition and the name of the player do not coincide.

The next player (originally player 4) is obviously a sum: his weight is 4 and the remaining player have weights $(4, 4, 3, 1, 1, 1)$, hence the resulting satellite game – suitably called $\boldsymbol{v}^{(14)}$ is described by the incidence matrix

$$\boldsymbol{I}(\boldsymbol{v}^{(14)}) \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

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Example 1.18. Another nice example is given by

$$\boldsymbol{m} \;=\; (170, 70, 70, 31, 29, 10, 4, 4, 2, 2, 2, 1) \text{ and } \alpha = 240$$

The incidence matrix is given by

$$\boldsymbol{I}(\boldsymbol{v}) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and shows the following characters listed via the character vector as

$$\chi \;=\; \chi(\boldsymbol{v}) \;=\; (2, 1, 1, 2, 2, 1, 1, 1, 1, 1, 2, 0)$$

(recall that 0 is interpreted as "dummy", 1 is interpreted as "sum" and 2 as "step"). Now, if we consider

$$\boldsymbol{m}' \;=\; (50, 21, 21, 8, 8, 5, 2, 2, 1, 1, 1, 0) \text{ and } \alpha' = 71,$$

then the incidence matrix is the same; both pairs represent the same homogeneous game homogeneously. The second representation, however, shows much more lucidly that the smallest player is a dummy and that players 4 and 5 are of the same type.

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Let us now study a particular procedure according to which sums are replaced within minimal winning coalitions.

Definition 1.19.

Let $T \in \underline{\underline{\mathbf{W}}}^{min}$ and let $i \notin T$ be such that

$$\emptyset \neq [i+1, l(T)] \subseteq T \tag{31}$$

holds true. Then player i is called the ***last dropout*** *of T.*

Let $S \in \underline{\underline{\mathbf{W}}}^{min}$ and let $i \in S$ be such that

$$[i, l(S)] \subseteq S, \quad \boldsymbol{m}([l(S)+1, n]) \geq \boldsymbol{m}_i \tag{32}$$

holds true. Then player i is called ***dispensable*** *in S or with respect to S.*

Theorem 1.20. *Let i be the last dropout of some $T \in \underline{\underline{\mathbf{W}}}^{min}$. Then i is necessarily a sum. In particular, there exists $S \in \underline{\underline{\mathbf{W}}}^{min}$ containing i with the following properties:*

1. *Player i is dispensable in S.*
2. $S \cap [1, i-1] \;=\; T \cap [1, i-1]$.

In other words, coalition S is constructed from coalition T by inserting the last dropout i and cutting the tail as necessary; by this procedure the preceding players are not altered.

Proof: The formal proof corroborates the obvious intuitive construction: Define $S' \;:=\; T + i$, then $\boldsymbol{m}(S') > \alpha$ is a necessary consequence. By cutting the tail, we obtain a minimal winning coalition $S \subseteq S'$, i.e., $\boldsymbol{m}(S) \;=\; \alpha$. As there are players behind the gap denoted by i in T, we know that $\boldsymbol{m}(T \cap [1, i-1]) < \alpha$ is the case, this means that the cutting procedure stops earlier and we have $i \in S$. Clearly, there are no changes in the presence of players *before* i. If we write tentatively $s \;:=\; l(S)$, $t \;:=\; l(T)$, then, as the players behind i in T form an interval, we observe that $T - S \;=\; [s+1, t]$ holds true and of course $\boldsymbol{m}(T - S) \;=\; \boldsymbol{m}_i$ follows at once. Obviously, player i has the character of a sum.

q.e.d.

Definition 1.21. *Let T be a* min-win *coalition having a last dropout. Then $\boldsymbol{\iota}(T) \in \underline{\underline{\mathbf{W}}}^{min}$ is the coalition obtained by inserting the dropout and cutting the tail, i.e., the coalition specified by Theorem 1.20.*

Theorem 1.22. *Let i be dispensable in some $S \in \underline{\underline{\mathbf{W}}}^{min}$. Then i is necessarily a sum and there exists $T \in \underline{\underline{\mathbf{W}}}^{min}$ with the following properties:*

1. *Player i is the last dropout in T.*
2. $S \cap [1, i-1] \;=\; T \cap [1, i-1]$.

Proof: The procedure is obvious: we cancel i from S and replace this player by smaller players from outside of S. Formally, define

$$T' \;=:\; S - i + [l(S)+1, n],$$

which yields $\boldsymbol{m}(T') \geq \alpha$. By cutting the tail we obtain $T \subseteq T'$ with $\boldsymbol{m}(T) = \alpha$. The claims are obviously satisfied.

q.e.d.

Definition 1.23. *Let $S \in \underline{\underline{\mathbf{W}}}^{min}$ and let r be a player dispensable in S. Then $\boldsymbol{\delta}_r(S) \in \underline{\underline{\mathbf{W}}}^{min}$ denotes the coalition obtained by **dropping** r and replacing him by smaller players following S, i.e., the coalition described by Theorem 1.22.*

Corollary 1.24. *If r is the last dropout in some $T \in \underline{\underline{\mathbf{W}}}^{min}$, then r is dispensable in $\boldsymbol{\iota}(T)$ and*

$$\boldsymbol{\delta}_r(\boldsymbol{\iota}(T)) = T \tag{33}$$

holds true. Also, whenever j is dispensable in some $S \in \underline{\underline{\mathbf{W}}}^{min}$, then j is the last dropout in $\boldsymbol{\delta}_j(S)$ and

$$\boldsymbol{\iota}(\boldsymbol{\delta}_j(S)) = S \tag{34}$$

holds true. In this sense, whenever one operation is defined, then so is the other one and both operations are inverse to each other .

Theorem 1.25. *The following statements are equivalent:*

1. *player $i \in I$ is a sum.*
2. *player $i \in I$ is a dropout in some coalition $T \in \underline{\underline{\mathbf{W}}}^{min}$, i.e., $i \notin S$ and $l(S) > i$ holds true.*
3. *player i is the last dropout in some $T \in \underline{\underline{\mathbf{W}}}^{min}$.*

4. *There exists a pair of coalitions $S, T \in \underline{\underline{\mathbf{W}}}^{min}$ with the following properties.*

   (a) $i \in S,\ i \notin T$,

   (b) $[1, i-1] \cap S = [1, i-1] \cap T$,

   (c) $[i, l(S)] \subseteq S,\ [i+1, l(T)] \subseteq T$.

It may be useful to visualize the last statement with respect to the structure of the incidence matrix $\boldsymbol{I}(\boldsymbol{v})$: this matrix has to contain two rows of the shape

$$\begin{array}{lcl} * * * \ o * * \ o \ o * * * * & 1 & * * * \ldots\ldots\ldots\ldots\ldots\ldots \\ * * * \ o * * \ o \ o * * * * & 0 & * * * * * * * * \ldots\ldots\ldots \end{array}$$

Proof: *1 ⇒ 2*:
If i is a sum, then there exists a minimal winning S such that $i \in S$ and $m([l(S)+1, n] \geq m_i$ holds true. As previously, we throw i out of S and, by cutting the tail, we obtain a coalition $T \subseteq T' := S - i + [l(S)+1, n]$ which is minimal winning and in which i constitutes a dropout.

2 ⇒ 3:
If i constitutes a dropout whatsoever and i is not the last dropout, then some smaller player is the last dropout. Hence $\boldsymbol{\iota}(T)$ (in which the last dropout of T has been inserted) has a smaller number of dropouts. By successively filling up this way we obtain eventually a coalition in which i is the last dropout.

3 ⇒ 4:
Choose T such that i is the last dropout of T. Then obviously $S := \boldsymbol{\iota}(T)$ supplies the desired partner so that the pair S, T is as stated.

4 ⇒ 1:
If a pair S, T is provided with the properties indicated, then it is seen that $T - S \subseteq [l(S)+1, n]$ satisfies $m(T-S) = m_i$. Hence, we have $m[l(S)+1, n] \geq m_i$, which shows that i is a sum.

q.e.d.

Corollary 1.26. 1. *Steps rule their followers: that is, if j is a step and $i > j$ is a player following j (hence has no larger weight, i.e., $m_j \geq m_i$ or $j \succ i$), then, for any $S \in \underline{\underline{\mathbf{W}}}^{min}$ containing i, it follows that S contains j. In other words, with any player i all the preceding steps are contained in a* min–win *coalition.*

2. *Let i be a sum and let (according to the definition of sums) $S \in \underline{\underline{\mathbf{W}}}^{min}$ be such that $i \in S, l^{(i)} = l(S)$ is true. Then i is the last dropout in $\boldsymbol{\delta}_i(S)$.*

Proof: The second statement follows obviously as a special case of Theorem 1.22 (and Definition 1.23). The first statement is true as, in view of Theorem 1.25, a step can never constitute a dropout in any *min–win* coalition.

q.e.d.

Theorem 1.27. *Let $\boldsymbol{v}$ be a simple game which is homogeneous and constant sum. Then $\boldsymbol{v}$ has exactly one step, this is the smallest non–dummy player.*

Proof: Without loss of generality we may assume that $\boldsymbol{v}$ has no dummies and hence player n is a step. Now, assume *per absurdum* that player $j < n$ is a step as well. Pick a *min–win* coalition S which n is a member of, then, as *steps rule their followers*, j is a member of S as well. Now consider the coalition

$$T \;:=\; S^c + n \;=\; (S-n)^c.$$

As $S - n$ is losing, certainly T is winning (we have a constant sum game).

Indeed, T is even minimal winning. For, $T - n \;=\; S^c$ is not winning as S is winning and, for any other $k \in T$, the coalition $T - k$ is *a fortiori* not winning as n is the weakest player in the game.

But looking at T we have found a coalition which is *min–win*, contains n and does not contain j – which is impossible as *steps rule their followers.*

q.e.d.

Corollary 1.28. *Let $\boldsymbol{v}$ be a simple game which is homogeneous and constant sum. Also, let τ be the smallest nondummy (which is the only step of the game). Then there exists a homogeneous integer–valued representation $(\boldsymbol{m}, \alpha)$ of $\boldsymbol{v}$ with the following properties:*

$$\begin{array}{lcll} \boldsymbol{m}_i & = & 0 & \qquad (i \in \Delta) \\ \boldsymbol{m}_\tau & = & 1 & \\ \boldsymbol{m}_{\tau-1} & = & 1 & \\ \boldsymbol{m}(\boldsymbol{I}) & = & 2\alpha - 1 & . \end{array} \tag{35}$$

Any further homogeneous representation that assigns zero weight to the dummies is a multiple of the above one.

The reader will realize that this corollary is a slight extension of our first theorem within this section, i.e., of Theorem 1.4. The present version, however, is of a purely combinatoric nature while the proof of Theorem 1.4 rests

on convex analysis procedures. The present version is hence supported by a completely independent proof due to A. Ostmann ([1]).

Proof: Again, without loss of generality, we assume that there are no dummies in the game. Also, we may start out with a representation $(\boldsymbol{m}, \alpha)$ satisfying $\boldsymbol{m}_n = 1$.

In view of Theorem 1.27 each player apart from the last one is a sum. Therefore, for each player $i \neq n$, there is a pair of *min–win* coalitions S, T such that

1. $i \in S,\ i \notin T$,
2. $[1, i-1] \cap S = [1, i-1] \cap T$,

holds true. Obviously it follows that

$$\boldsymbol{m}_i \ = \ \boldsymbol{m}(T - S) \tag{36}$$

holds true, the weight of each player (apart from n) is uniquely defined by the weights of his smaller successors. More precisely, we must have

$$\begin{array}{rcl} \boldsymbol{m}_{n-1} & = & \boldsymbol{m}_n \\ . & = & . \\ . & = & . \\ . & = & . \\ \boldsymbol{m}_i & = & \boldsymbol{m}(T-S) \quad (S, T\, suitable) \\ . & = & . \\ . & = & . \\ . & = & . \\ \alpha & = & \boldsymbol{m}(S^{(0)}) \end{array} \tag{37}$$

Now we see that all weights $\boldsymbol{m}_i$ as well as α are multiples of $\boldsymbol{m}_n$. Thus, all these numbers are integers if we choose $\boldsymbol{m}_n = 1$. The essential uniqueness of this representation is also clear at once.

It remains to establish the shape of α. However, for a *min–win* coalition S and its losing complement S^c we find

$$\boldsymbol{m}(\boldsymbol{I}) = \boldsymbol{m}(S) + \boldsymbol{m}(S^c) \leq \alpha + \alpha - 1 = 2\alpha + 1\ . \tag{38}$$

On the other hand, we may pick a *min–win* coalition T containing player n. Then we know that $\boldsymbol{m}(T - n) = \boldsymbol{m}(T) - \boldsymbol{m}_n = \alpha - 1$ holds true; hence $(T - n)^c$ is winning. This implies

$$\boldsymbol{m}(\boldsymbol{I}) = \boldsymbol{m}(T - n) + \boldsymbol{m}((T - n)^c) \geq \alpha - 1 + \alpha = 2\alpha - 1 \tag{39}$$

holds true. Combining (38) and (39) yields $\boldsymbol{m}(\boldsymbol{I}) = 2\alpha - 1$ as desired.

q.e.d.

2 The Minimal Representation

Within this section we study homogeneous representations of a general homogeneous game, not necessarily zero sum. We demonstrate that a homogeneous game has a unique integer representation with minimal total weight $\boldsymbol{m}(I)$. This representation is also minimal in the sense that $\boldsymbol{m}$ – when seen as a vector – is coordinate–wise smaller than any vector provided by any other representation. For short, a homogeneous game has a unique minimal integer representation.

The presentation offerd in this section essentially follows the proofs given by A. Ostmann [1], who exhibited the existence of the minimal representation.

Some preparations are necessary. We shall first continue to discuss the mechanisms of replacement that were already sucessfully employed in Section 1. These procedures will then be suitably extended so as to be applicable for maximal losing coalitions as well.

Recall that we consider $\underline{\underline{\mathbf{W}}}^{min}$ to be endowed with the lexicographical ordering, i.e., we write $S \prec_L T$ if 1_S precedes 1_T lexicographically.

Remark 2.1. The operations $\boldsymbol{\delta}$ and $\boldsymbol{\iota}$ can be extended to describe a natural relation between *min–win* coalitions and *max–los* coalitions as follows.

Let $S \in \underline{\underline{\mathbf{W}}}^{min}$ and let $r \in S$. We call player r ***indispensable*** if

1. $[r+1, l(S)] \subseteq S$
2. $\boldsymbol{m}([l(S)+1, n]) < \boldsymbol{m}_r$

holds true. The second property is of course independent of the representation: it means that

$$S - r + [l(S)+1, n] \in \underline{\underline{\mathbf{L}}}^{max}$$

is the case. In other words, there is no gap behind r and the small players behind S do not have enough weight to replace r. Note that player n is indispensable in the sense of the definition for any *min–win* coalition he is a member of.

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Definition 2.2. *If r is indispensable in S, the we use the notation*

$$\boldsymbol{\delta}_r(S) \;:=\; S - r + [l(S)+1, n] \in \underline{\underline{\mathbf{L}}}^{max}. \tag{1}$$

On the other hand, the following holds true.

Lemma 2.3. *Let $T \in \underline{\underline{\mathbf{L}}}^{max}$. Then there exists a player $j \notin T$ with the following properties:*

1. $[j+1, n] \subseteq T$.
2. *There exists $S \in \underline{\underline{\mathbf{W}}}^{min}$ such that $\boldsymbol{\delta}_j(S) = T$ holds true.*

The lemma tells us that we may look for some last "blank" or "gap" in coaliton T and then find a *min–win* coalition S which generates T via the dropping procedure of the indispensable player j.

In particular, it follows from the lemma that any *max–los* coalition contains player n or player $n-1$ (or both). If player n is not a member, then he is the "last blank". The precise definition of this term follows after the proof of the Lemma.

Proof: Define

$$j \;:=\; \max\left\{i \in I \,\middle|\, T \cup [i, n] \in \underline{\underline{\mathbf{W}}}\right\}, \tag{2}$$

then, as $T \cup [j, n] \in \underline{\underline{\mathbf{W}}}$ and $T \cup [j+1, n] \notin \underline{\underline{\mathbf{W}}}$, clearly j cannot be a member of T. We know that $T \cup [j+1, n]$ is losing. But T is supposed to be *maximal* losing, consequently we have

$$[j+1, n] \subseteq T. \tag{3}$$

On the other hand, we have $T + [j, n] \in \underline{\underline{\mathbf{W}}}$ and by cutting the tail of this coalition we obtain some coalition $S \in \underline{\underline{\mathbf{W}}}^{min}$. The cutting procedure cannot affect player j, for otherwise it would result in a subset of T. Hence

$$j \in S \subseteq T + j \tag{4}$$

holds true.

q.e.d.

Definition 2.4. *Let $T \in \underline{\underline{\mathbf{L}}}^{max}$. We shall call player $j \notin T$ as defined by Lemma 2.3 the* ***last blank*** *of T. Note that the last blank may be player n. Also, the* min–win *coalition S specified by Lemma 2.3 is denoted by $\boldsymbol{\iota}(T)$*

Let us now define the class of candidates for representation of a given homogeneous game.

Definition 2.5. *Let $\boldsymbol{v}$ be a homogeneous game and let $\boldsymbol{m}^\circ$ be a measure and $\alpha_\circ$ a positive real number. We shall say that the pair $(\boldsymbol{m}^\circ, \alpha_\circ)$ is* ***compatible*** *with $\boldsymbol{v}$ if the following two conditions hold true:*

1. *The* lex–max min–win *coalition satisfies $\boldsymbol{m}^\circ(S^{(0)}) = \alpha_\circ$.*
2. *For every sum $i \in I$ the weight is*

$$\boldsymbol{m}_i^\circ \;=\; \boldsymbol{m}^\circ(S^{(i)}). \tag{5}$$

3. *For every step $j \in I$ the weight satisfies*

$$\boldsymbol{m}_j^\circ > \boldsymbol{m}^\circ(C^{(j)})\ . \tag{6}$$

Thus, the weight of a sum is always determined by the total weight of his satellites while the weight of a step exceeds the weight of the satellites. This suggest that there is a certain degree of freedom in choosing compatible representations which corresponds to the number of steps available.

Note that the smallest nondummy player is always a step, hence we require in particular that the weight of the smallest nondummy player exceeds the total weight off all the dummies.

Now the first kind of relationship between homogeneous representations and compatible ones is easy to establish.

Theorem 2.6. *Let v be a homogeneous game which is homogeneously represented by $(\boldsymbol{m}, \alpha)$. Then $(\boldsymbol{m}, \alpha)$ is compatible.*

Proof: Condition *1.* is obvious from homogeneity. Condition *2.* follows as any player $i \in \Sigma$ in the shortes *min–win* coalition he is a member of (i.e., in $S_{(i)}$ is replaced by $S^{(i)}$), formaly

$$\boldsymbol{\delta}_i(S_{(i)}) = S_{(i)} - i + S^{(i)}.$$

Both, $S^{(i)}$ and $S_{(i)} - i + S^{(i)}$ are minimal wining, hence have the same measure with respect to $\boldsymbol{m}$, from which obtain Condition *2.*

Similarly, if we take the shortest coalition $S_{(j)}$ containing a step j, then $C^{(j)} = [l^{(j)} + 1, n]$ is disjoint; by throwing j out and bringin in all players of $C^{(j)}$ we must obtain a losing coalition (steps rule their followers), hence the weight of j is larger than the one of the satellites. This proves Condition *3.*

q.e.d.

Lemma 2.7. *Let $\boldsymbol{v}$ be a homogeneous game and let the pair $(\boldsymbol{m}^\circ, \alpha_\circ)$ satisfy conditions 1. and 2. of Defininition 2.5. Then, for every $S \in \underline{\underline{\mathbf{W}}}^{min}$ the equation $\boldsymbol{m}^\circ(S) = \alpha_\circ$ holds true.*

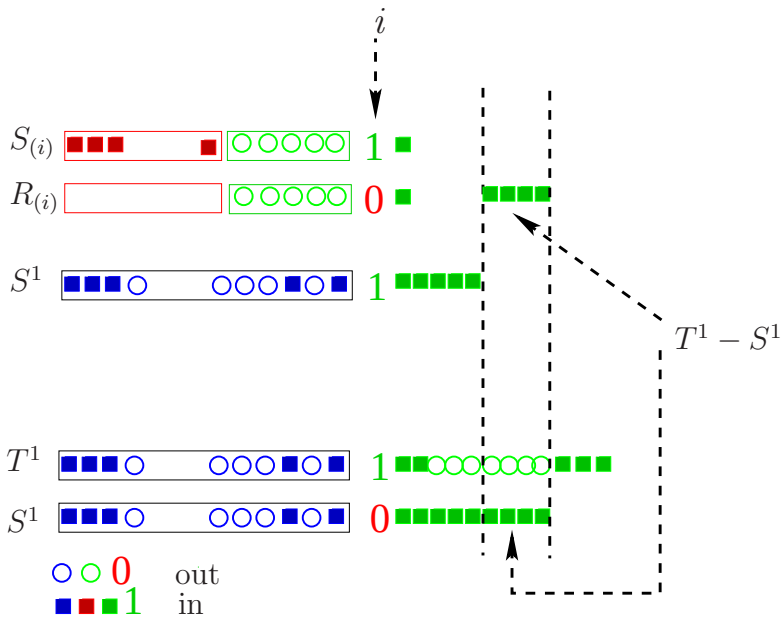

Figure 2.1: Determining $\boldsymbol{m}^\circ(T^1)$

Proof: As frequently we assume that there are no dummies in the game, thus player n constitute the last step. There is no danger in this assumption as dummies never appear in minimal winning coalitions. By definition we have immediately $\boldsymbol{m}^\circ(S^{(0)}) = \alpha^\circ$. Therefore, we proceed by induction and we are going to prove:

> Let $T^1 \in \underline{\underline{\mathbf{W}}}^{min}$ and suppose that, for all coalitions $S \in \underline{\underline{\mathbf{W}}}^{min}, S \prec_L T^1$ we know that $\boldsymbol{m}^\circ(S) = \alpha_\circ$ holds true. Then $\boldsymbol{m}^\circ(T^1) = \alpha_\circ$ holds true as well.

To this end, let S^1 be the immediate lexicographic predecessor of T^1. Also, let i be the first player not common to S^1 and T^1, then clearly $i \in S^1$, $i \notin T^1$ and

$$[1, i-1] \cap S^1 \;=\; [1, i-1] \cap T^1$$

is true. Moreover, as T^1 follows immediately on S^1 lexicographically, i is the last dropout of T^1. Hence, i is a sum according to Theorem 1.25.

Now insert player i and cut the tail, i.e., define $S^0 := \boldsymbol{\iota}(T^1)$. We know that player i is dispensable in coalition S^0 (i.e., $\boldsymbol{\delta}_i(S^0) = T^1$) and that

$$[1, i-1] \cap S^0 \;=\; [1, i-1] \cap T^1$$

holds true. Clearly we have

$$S^0 \preceq_L S^1 \prec_L T^1, \;\; \boldsymbol{m}^\circ(S^0) \;=\; \boldsymbol{m}^\circ(S^1) \;=\; \alpha_\circ \tag{7}$$

by the induction hypothesis. Also we have

$$T^1 - S^0 \subseteq [l(S^0) + 1, n] \tag{8}$$

and

$$S^0 - T^1 = i, \tag{9}$$

in other words, coalition $T^1 - S^0$ substitutes player i.

Now consider the *shortest* coalition which is *min–win* and contains i, this is $S_{(i)}$. Clearly the length is not exceeding the one of S^0, hence this coalition is disjoint to $T^1 - S^0$ by (8). Therefore, we can substitute player i in $S_{(i)}$ by $T^1 - S^0$, i.e.,

$$R_{(i)} \;:=\; S_{(i)} - i + (T^1 - S^0) \in \underline{\underline{\mathbf{W}}}^{min} \tag{10}$$

holds true. Consequently we obtain

$$\begin{aligned} \boldsymbol{m}^\circ(T^1) \;&=\; \boldsymbol{m}^\circ(S^0) - \boldsymbol{m}^\circ_i + \boldsymbol{m}^\circ(T^1 - S^0) \\ &=\; \alpha_\circ - \boldsymbol{m}^\circ_i + \boldsymbol{m}^\circ(R_{(i)} - S_{(i)}) \\ &=\; \alpha_\circ \;\; ; \end{aligned}$$

this completes our induction.

q.e.d.

Theorem 2.8. *Let $\boldsymbol{v}$ be a homogeneous game and let the pair $(\boldsymbol{m}^\circ, \alpha_\circ)$ be compatible with $\boldsymbol{v}$. Then, for every $T \in \underline{\underline{\mathbf{L}}}^{max}$ the inequality $\boldsymbol{m}^\circ(T) < \alpha_\circ$ holds true. Hence, $(\boldsymbol{m}^\circ, \alpha_\circ)$ constitutes a representation of $\boldsymbol{v}$.*

Proof: 1$^{\mathbf{st}}$STEP : Let $j_T \notin T$ denote the last blank of coalition T and let $\boldsymbol{\iota}(T) \;=:\; S_T \in \underline{\underline{\mathbf{W}}}^{min}$ be the *min–win* coalition obtained by inserting the last blank and cutting the tail (see Lemma 2.3 and Definition 2.4). Then we have

$$T \subseteq S_T - i_T + (T - S_T), \tag{11}$$

and hence

$$\boldsymbol{m}^\circ(T) \leq \boldsymbol{m}^\circ(S_T) - \boldsymbol{m}^\circ_{i_T} + \boldsymbol{m}^\circ(T - S_T) \tag{12}$$

2ndSTEP : Suppose that j_T happens to be a step. Then, as $j_T \in S_T$ and S_T is minimal winning, we know that necessarily $[l(S_T), n] \subseteq C^{(j_T)}$ holds true, therefore we have

$$\boldsymbol{m}^\circ_{j_T} > \boldsymbol{m}^\circ(C^{(j_T)}) \geq \boldsymbol{m}^\circ([l(S_T), n]) = \boldsymbol{m}^\circ(T - S_T). \tag{13}$$

Now we see from (12) that $\boldsymbol{m}^\circ(T) < \alpha_\circ$ is true indeed.

3rdSTEP : We still have to deal with the case that j_T is a sum. Instead and more generally we shall show by induction that $\boldsymbol{m}^\circ(S) < \alpha_\circ$ is true (we perform this task using the second step). The induction proceeds backwards according to j_T.

However, for all *max–los* coalitions T with $j_T = n$, our claim follows from the $2^{nd}STEP$ as player n is a step.

Suppose now that, for all *max–los* coalitions T with $j_T > k$ it has been verified that $\boldsymbol{m}^\circ(S) < \alpha_\circ$ is true, we have to show it for any T with $j_T = k$. Again, if j_T is a step, then nothing has to be shown in view of the $2^{nd}STEP$. So we can now proceed assuming that k is a sum.

Recall that according to Lemma 2.3 we have $[k+1, n] \subseteq T$. Hence, T contains all players following $k = j_T$, in particular those of $C^{(k)}$. By Lemma 2.7 we know that

$$\boldsymbol{m}^\circ_k = \boldsymbol{m}^\circ(C^{(k)}) \tag{14}$$

is true. Therefore, the coalition

$$T' \ := \ T - k + C^{(k)}$$

has the same weight as coalition T, hence it is losing. Now T' is not necessary maximal losing but we may add successively (from behind) small players until we obtain a coalition

$$T'' \supseteq T, \quad T'' \in \underline{\underline{\mathbf{L}}}^{max}.$$

As T'' contains k it is clear that $T'' \cup [k+1, n]$ is winning. Therefore, the last blank of T'' has to be larger than k (compare the construction of the last blank in Lemma 2.3), i.e., we have

$$j_{T''} \geq k + 1 \quad .$$

So the induction hypothesis applies to T'', that is we have, $\boldsymbol{m}^\circ(T'') < \alpha_\circ$. Collecting the pieces we come up with

$$\boldsymbol{m}^\circ(T) \;=\; \boldsymbol{m}^\circ(T') \leq \boldsymbol{m}^\circ(T'') < \alpha_\circ.$$

This completes the induction and proves the theorem,

q.e.d.

Corollary 2.9. *A homogeneous game has a unique homogeneous integer representation* $(\boldsymbol{m}^\star, \alpha^\star)$ *with minimal weight* $\boldsymbol{m}^\star(I)$*. This representation is given as follows:*

1. *The* lex–max min–win *coalition satisfies* $\boldsymbol{m}^\star(S^{(0)}) = \alpha_\star$.
2. *For every sum* $i \in I$ *the weight is*

$$\boldsymbol{m}_i^\star \;=\; \boldsymbol{m}^\star(S^{(i)}). \tag{15}$$

3. *For every step* $j \in I$ *the weight satisfies*

$$\boldsymbol{m}_j^\star \;=\; \boldsymbol{m}^\star(C^{(j)}) + 1\ . \tag{16}$$

4. *For every dummy* $k \in I$ *the weight satisfies*

$$\boldsymbol{m}_k^\star \;=\; 0\ . \tag{17}$$

The representation $(\boldsymbol{m}^\star, \alpha^\star)$ *is also minimal when the vectors (measures) are compared coordinate–wise.*

Proof: Obviously we proceed by induction: assigning 0 to the dummies reduces a representataion in total measure. As we restrict ourselves to integer representations, the smallest nondummy (the last step) has to obtain 1 and the inductive procedure proceeds backwards: for every sum the weight is defined recursively and obviously minimal when defined according to 2, provided the weight was minimal for all smaller players. Similarly for the steps. Note that Theorem 2.6 as well as Theorem 2.8 are employed.

q.e.d.

Theorem 2.10. *A homogeneous game has a unique integer representation* $(\boldsymbol{m}^\star, \alpha^\star)$ *with minimal weight* $\boldsymbol{m}^\star(I)$*. This is the unique minimal integer homogeneous representation as defined by Corollary 2.9.*

Proof: Let $(\boldsymbol{m}, \alpha)$ be an integer representation of $\boldsymbol{v}$, not necessarily homogeneous. Clearly, all the dummies get zero weight. Also, the smallest nondummy (a step) obtains at least weight 1.

Let us proceed by induction: pick player $i \in I$ and assume that, for all smaller players $j \in I$, it has been verified that $\boldsymbol{m}_j \geq \boldsymbol{m}_j^\star$ holds true. We have to show that $\boldsymbol{m}_i \geq \boldsymbol{m}_i^\star$ holds true as a consequence.

Suppose player i is a step. Let $S_{(i)}$ be the shortest *min–win* coalition he is a member of and let $C^{(i)}$ be his satellites. We know that $T := S^{(i)} - i + C^{(i)}$ loses, hence

$$\begin{aligned} \alpha &> \boldsymbol{m}(T) = \boldsymbol{m}(S^{(i)} - i + C^{(i)}) \\ &= \boldsymbol{m}(S^{(i)}) - \boldsymbol{m}_i + \boldsymbol{m}(C^{(i)}) \geq \alpha - \boldsymbol{m}_i + \boldsymbol{m}(C^{(i)}) \end{aligned}$$

implies

$$\boldsymbol{m}_i \geq \boldsymbol{m}(C^{(i)}) + 1 \geq \boldsymbol{m}^\star(C^{(i)}) + 1 = \boldsymbol{m}_i^\star \quad .$$

So we have to consider the case that i is a sum. Again, let $S_{(i)}$ be the shortest *min–win* coalition he is a member of and let $k := l^{(i)}$ be the delimiter (the last player in $S^{(i)}$). Cons ider the system of minimal winning coalitions given by

$$\underline{\underline{\mathbf{T}}} := \left\{ T \in \underline{\underline{\mathbf{W}}}^{min} \,\middle|\, T \cap [1,k] = S^{(i)} - i \right\} . \tag{18}$$

This system is nonempty, for certainly $\delta_i(S_{(i)})$ is a member. Let $\widehat{T}$ be the longest coalition in the system and let $l = l(\widehat{T})$ be the last player in $\widehat{T}$. Then l is a step. For, if l was a sum, then his satellites $C^{(l)}$ would be disjoint to $\widehat{T}$, hence we could replace him in $\widehat{T}$, thereby increasing the length of $\widehat{T}$ without leaving $\underline{\underline{\mathbf{T}}}$ – a contradiction. So l is a step with satellites $C^{(l)} = [l+1, n]$.

Now replacing l by his satellites, we obtain

$$R := \widehat{T} - l + C^{(l)} \in \underline{\underline{\mathbf{L}}} \quad . \tag{19}$$

Certainly R is losing but it is actually *max–los*: the smallest player that could be inserted is l and that would render the result winning – so any other addition of an outside player would result in a winning coalition. And indeed, we find

$$\boldsymbol{m}^\star(R) = \boldsymbol{m}^\star(\widehat{T}) - \boldsymbol{m}_l^\star + \boldsymbol{m}^\star(C^{(l)}) = \alpha^\star - 1 \tag{20}$$

as $\boldsymbol{m}_l^\star = \boldsymbol{m}^\star(C^{(l)}) + 1$ (l is a step). Consequently, we come up with

$$\boldsymbol{m}^\star(R - S_{(i)}) = \boldsymbol{m}^\star(R) - (\alpha^\star - \boldsymbol{m}_i^\star)\boldsymbol{m}^\star(R) - \alpha^\star + \boldsymbol{m}_i^\star = \boldsymbol{m}_i^\star - 1 \quad . \tag{21}$$

This we keep in mind for moment while we turn to the competing representation $(\boldsymbol{m}, \alpha)$.

Here we know that

$$\boldsymbol{m}(S_{(i)}) \geq \alpha > \boldsymbol{m}(R)$$

holds true. However, both coalitions involved coincide up to $i-1$ and again from $i+1$ to k in view of our construction. It is only player i that is left in $S_{(i)}$ and we obtain

$$\boldsymbol{m}_i > \boldsymbol{m}(R \cap [k+1, n]) = \boldsymbol{m}(R - S_{(i)}) \geq \boldsymbol{m}^\star(R - S_{(i)}) = \boldsymbol{m}_i^\star - 1. \tag{22}$$

The last inequality stems from induction and the last equation is given by (21),

q.e.d.

3 The Recursive Structure

Within this section we discuss an alternative approach towards the assignement of characters and the construction of the minimal representation of a homogeneous game. This approach is based on the *Basic Lemma* (Lemma 1.15)which can be used for an alternative definition of satellite games as compared to Definition 1.10. If this definition is employed, there is also a way to recursively define the characters which, eventually, leads to the minimal representation.

We will represent this approach in a somewhat dichotomic way: In order to clarify the connections we start out with some properties of the satellite structure which look alike in either approach. Then we shall tentatively ask the reader to forget about the original aproach presented in the previous sections in order to indicate that both approaches are independent.

Theorem 3.1. *Let $\boldsymbol{v}$ be a homogeneous game and let $i \notin S^{(0)}$. Then $i \in \Sigma$ if and only if $i \in \Sigma^{(j)}$ holds true for* some *$j \in S^{(0)}$.*

Proof:

1st STEP : Let $i \in \Sigma^{(j)}$ for some $j \in S^{(0)}$. In particular this implies that $v^{(j)}$ is not trivial, hence $j \in \Sigma$ holds true. Now, there exists a coalition $\widetilde{S}^{(j)} \subseteq C^{(j)}$ ($= S^{(0)c}$) satisfying

1. $i \in \widetilde{S}^{(j)}$,
2. $\boldsymbol{m}_j = \boldsymbol{m}^{(j)}(\widetilde{S}^{(j)})$,
3. $\boldsymbol{m}^{(j)}([l(\widetilde{S}^{(j)}) + 1, n]) \geq \boldsymbol{m}_i$.

Actually, the superscript $^{(j)}$ at $\boldsymbol{m}$ can be omitted as $\widetilde{S}^{(j)}$ is a subset of $C^{(j)}$. Now consider the coalition

$$\widetilde{S} \;:=\; S^{(0)} - j + \widetilde{S}^{(j)}$$

which is obtained when we replace player $j \in S^{(0)}$ by $\widetilde{S}^{(j)}$ (both have the same mass). Obviously we have

$$l(\widetilde{S}) \;=\; l(\widetilde{S}^{(j)}). \tag{1}$$

In addition we may conclude the following:

4. $i \in \widetilde{S}$ (in view of 1.),

5. $\alpha = \boldsymbol{m}(\widetilde{S})$ (in view of 2.)

6. $\boldsymbol{m}^{(j)}([l(\widetilde{S})+1,n] \geq \boldsymbol{m}_i$ in view of 3. and (1).

From this we conclude immediately that $i \in \Sigma$ holds true indeed.

2ndSTEP : On the other hand, consider now the situation that $i \in \Sigma$ is true. Let $\widehat{S} \in \underline{\underline{\mathbf{W}}}^{min}$ be a coalition satisfying $i \in \widehat{S}$ and $\boldsymbol{m}([l(\widehat{S})+1,n]) \geq \boldsymbol{m}_i$. Surely, $\widehat{S} \supseteq S^{(0)}$ cannot be true, for the total mass would exceed α in view of $\boldsymbol{m}(\widehat{S}) \geq \boldsymbol{m}(S^{(0)}) + \boldsymbol{m}_i > \alpha$. Similarly, $S^{(0)} \supseteq \widehat{S}$ cannot be true as we have $i \in \widehat{S}$. Hence

$$S^{(0)} - \widehat{S} \neq \emptyset \neq \widehat{S} - S^{(0)}$$

is satisfied. As both, $\widehat{S}$ and $S^{(0)}$ have the same measure α, we infer the equation

$$\boldsymbol{m}(S^{(0)} - \widehat{S}) \;=\; \boldsymbol{m}(\widehat{S} - S^{(0)}) \tag{2}$$

for the corresponding domains outside the intersection. This we can as well write

$$\sum_{j \in S^{(0)}, j \notin \widehat{S}} \boldsymbol{m}_j \;=\; \boldsymbol{m}(\widehat{S} - S^{(0)}). \tag{3}$$

Now we use the fact that

$$\widehat{S} - S^{(0)} \subseteq C^{(j)} = C^{(0)} = S^{(0)c}$$

holds true for all $j \in S^{(0)}$. According to the *Basic Lemma* (Lemma 1.15) it follows that $\boldsymbol{m}^{(j)} = \boldsymbol{m}|_{S^{(0)},c}$ is homogeneous with respect to all the $\boldsymbol{m}^{(j)}$, $(j \in S^{(0)})$. Therefore, equation (3) tells us that we can extract successively (beginning with the largest $\boldsymbol{m}^{(j)}$ – use the principle of cutting the tail) coalitions $\widehat{S}^{(j)}$ of mass $\boldsymbol{m}(\widehat{S}^{(j)}) = \boldsymbol{m}_j$ $(j \in S^{(0)} - \widehat{S})$ out of $\widehat{S} - S^{(0)}$. One of these coalitions must contain player i.

Now, all of these coalitions have a last index not exceeding the length (last index) of $\widehat{S}$. In particular the one containing i satisfies therefore

$$\begin{aligned} i &\in \widetilde{S}^{(j)}, \\ \boldsymbol{m}(\widetilde{S}^{(j)}) &= \boldsymbol{m}_j, \\ \boldsymbol{m}^{(j)}([l(\widetilde{S}^{(j)})+1,n]) &\geq \boldsymbol{m}_i. \end{aligned} \tag{4}$$

The last inequality results directly from the definition of $\widehat{S}$. Now we may attach a superscript $^{(j)}$ at all terms involving $\boldsymbol{m}$ in (4), as as the set in the arguments are contained in $C^{(j)} = S^{(0)c}$. Then (4) shows exactly that $i \in \Sigma^{(j)}$ holds true.

q.e.d.

Theorem 3.2. *Let $\boldsymbol{v}$ be a homogeneous game and let $i \notin S^{(0)}$. Then $i \in \Delta$ if and only if $i \in \Delta^{(j)}$ holds true for* all *$j \in S^{(0)}$.*

Proof: This proof is almost immediate along the beaten path that has been cleared with the proof of Theorem 3.1. If i is *not* a dummy with respect to some $\boldsymbol{v}^{(j)}$, then pick a *min–win* coalition $\widetilde{S}^{(j)}$ of $\boldsymbol{v}^{(j)}$ containing i. The coalition

$$S^{(0)} - j + \widetilde{S}^{(j)}$$

contains i and is *min–win* with respect to $\boldsymbol{v}$. Thus i is no dummy "in $\boldsymbol{v}$".

On the other hand, if i is not a dummy with respect to $\boldsymbol{v}$, then let S be a *min–win* coalition containing i. Not all of the players $j \in S^{(0)}$ can be contained in S. Again, the coalition $S - S^{(0)}$ decomposes in coalitions $\widetilde{S}^{(j)}$ $(j \in S^{(0)} - S)$ with mass $\boldsymbol{m}(\widetilde{S}^{(j)}) = \boldsymbol{m}_j$ $(j \in S^{(0)} - S)$. One of these has to contain i and constitutes a *min–win* coalition of $\boldsymbol{v}^{(j)}$,

q.e.d.

The above results suggest a second approach to the structure of homogeneous games and in particular towards the unique minimal representation of these games. This approach is based on the recursive structure indicated by the satellite games.

Within the framework of this approach, the characters are defined by induction via the satellite games and the minimal representation is obtained from the minimal representation of the satellite games.

This approach is described in Rosenmüller [2]. However, the paper mentioned establishes the formation of characters for *types*, while in this presentation we have adopted the original version (due to Ostmann) which assigns characters to players.

We describe the alternative approach by introducing (recursively) a new version of characters which, in view of the previous theorems, later turns out to be the old one. To this end, we assume that we are given homogeneous games by means of a homogeneous representation $\boldsymbol{v} = \boldsymbol{v}_\alpha^{\boldsymbol{m}}$. Again the definition of characters superficially refers to the representation at hand but it

is seen rather early that characters, satellites etc. depend on the game (the set function $\boldsymbol{v}$) only.

To have a notation, the alternative versions of characters are tentatively referred to as ***h-characters***.

Definition 3.3. *Let $\boldsymbol{v} = \boldsymbol{v}_\alpha^{\boldsymbol{m}}$ be a homogeneous game represented by $\boldsymbol{m}$ and α homogenously.*

1. *Let $n = |I| = 1$. Let $\boldsymbol{v}$ be non-zero, i.e., $\boldsymbol{v} = \boldsymbol{\delta}_1$ (the dirac measure or unit measure concentrated on the single element 1 of I). Then the character of player 1 is **h–step**. The set of steps is $\widehat{T} := \{1\}$.*

   *On the other hand, if $\boldsymbol{v} = 0$ is the trivial game for 1 player, then the character of player 1 is **h–dummy**.*

2. *Let $n > 1$ and assume that for all homogeneous games with less than n players, characters for every player are well defined. Let $S^{(0)}$ denote the* lex–max min–win *coalition. Then the character of player $i \in I$ is defined as follows.*

   (a) *Let $i \in S^{(0)}$. If $\boldsymbol{m}(I - S^{(0)}) \geq \boldsymbol{m}_i$ holds true, then player i's character is **h–sum**. Otherwise, player i's character is **h–step**.*

   (b) *Let $i \notin S^{(0)}$. If i is an h–sum in some $\boldsymbol{v}^{(j)}$ $(j \in S^{(0)})$, the i is an **h–sum**. If i is an h–dummy in all $\boldsymbol{v}(j)$, then i is an **h–dummy**. In the remaining case (i.e., if i is never an h–sum in some $\boldsymbol{v}^{(j)}$ and an h–step at least once), then i is an **h–step**.*

Now it is immediately clear that we actually did *not* introduce a different version of characters:

Theorem 3.4. *Let $\boldsymbol{v}$ be a homogeneous game. The the decompositions induced by h–characters is the same as the one induced by characters. In other words, $i \in I$ is an h–sum if and only if it is a sum, and the same holds true for steps and dummies respectively.*

We continue by pointing out the alternative approach towards the unique minimal representation. This approach uses the concept of the satellite games and is based on an inductive procedure similarely as in Definition 3.3. The reader should, therefore think in terms of h–characters during the following exposition. The ideal attitude would be to forget about the results of Sections 1 and 2 (with the exception of the *Basic Lemma*) and adopt

just Definition 3.3 as a starting point. The fact that we just proved both definitions of characters to be equivalent presently serves only to simplify the notation: we may omit the prefix h– for convenience.

Theorem 3.5. *Let $\boldsymbol{v}$ be a homogeneous game. Let $j \in S^{(0)}$ be a player in the* min–win *coalition and let $i \in C^{(0)}$ be a player outside of it. Denote by*

$$l^{(j,i)} \;:=\; \min\left\{ l(S) \,\middle|\, S \subseteq C^{(j)}, i \in S, \boldsymbol{m}([l(S)+1,n]) \geq \boldsymbol{m}_i \right\} \tag{5}$$

the minimal lenght of a min–win *coalition in which i is dispensable with respect to $\boldsymbol{v}^{(j)}$. Then*

$$l^{(i)} \;=\; \min\left\{ l^{(j,i)} \,\middle|\, j \in S^{(0)},\; i \in \Sigma^{(j)} \right\} \tag{6}$$

holds true.

Theorem 3.6. *Let $\boldsymbol{v}$ be a homogeneous game and let the pair $(\boldsymbol{m}^\circ, \alpha_\circ)$ be compatible with $\boldsymbol{v}$. Then, $(\boldsymbol{m}^\circ, \alpha_\circ)$ constitutes a representation of $\boldsymbol{v}$.*

Proof:

1stSTEP : As usual, let us assume that there are no dummies in the game. Canonically the notation $\boldsymbol{m}^{\circ(i)} \;:=\; \boldsymbol{m}^\circ\big|_{C^{(i)}}$ $(i \in I)$ is used for the restricted versions of $\boldsymbol{m}^\circ$. We shall prove by induction ("backward") that

$$(\boldsymbol{m}^{\circ(i)}, \boldsymbol{m}_i^\circ) \text{ represents } \boldsymbol{v}^{(i)} \;\; (i \in I) \tag{7}$$

is satisfied. This is a trivial statement for $i = n$, which constitutes the induction beginning.

2ndSTEP :

assume now that $1 \leq j \leq n-1$ holds true and that (7) has been verified for all i with $j > i$. That is, we know that

$$v^{(i)} \;=\; v^{\boldsymbol{m}^{\circ(i)}} \;\; (j > i) \tag{8}$$

is satisfied. Consider in particular a player i, $j < i$, who is a sum with respect to $\boldsymbol{v}^{(j)}$, formally $i \in \Sigma^{(j)}$. Now, if we compute the minimal length of a *min-win* coalition containing i with respect to $\boldsymbol{v}^{(j)}$, i.e. $l^{(j,i)}$, then we find $l^{(j,i)} \geq l^{(i)}$ in view of Theorem 3.5. Also, the carier $C^{(j,i)}$ is certainly an interval located more to the right than $C^{(i)}$, i.e.,

$$C^{(j,i)} \subseteq C^{(i)} \tag{9}$$

is true. From this we would like to conclude that

$$(\boldsymbol{m}^{\circ(j,i)}, \boldsymbol{m}_i^\circ) \text{ represents } \boldsymbol{v}^{(j,i)} \tag{10}$$

as well. And indeed, for $S \subseteq C^{(j,i)}$ it follows that $S \subseteq C^{(i)}$ is true. Hence $\boldsymbol{m}^{\circ(j,i)} = \boldsymbol{m}_i^\circ$ and $\boldsymbol{m}^{\circ(i)} = \boldsymbol{m}_i^\circ$ are equivalent statements. The same holds true for the $<$-sign, so indeed (10) is true.

3rdSTEP : Therefore it is sufficient to show the following:

> If, for all $i > j$, $(\boldsymbol{m}^{\circ(j,i)}, \boldsymbol{m}_i^\circ)$ represents $\boldsymbol{v}^{(j,i)}$, then $(\boldsymbol{m}^{\circ(j)}, \boldsymbol{m}_j^\circ)$ represents $\boldsymbol{v}^{(j)}$.

4thSTEP : It constitutes just a simplification to omit the index ^{j} and to prove :

$$\begin{array}{lll} \star & \text{If, for all } i \in I,\ (\boldsymbol{m}^{\circ(i)}, \boldsymbol{m}_i^\circ) \text{ represents } \boldsymbol{v}^{(i)}, & \star \\ \star & \text{then } (\boldsymbol{m}^\circ, \alpha_\circ) \text{ represents } \boldsymbol{v}. & \star \end{array} \tag{11}$$

For indeed, $\alpha_\circ = \boldsymbol{m}^\circ(S^{(0)})$ quite similarly as $\boldsymbol{m}_i^\circ = \boldsymbol{m}^\circ(S^{(i)})$. (Or, for short, it suffices so to speak to verify the last step of our induction procedure). This statement we are now going to prove within the next steps.

5thSTEP :

To this end, for any $S \in \underline{\underline{\mathbf{P}}}, S \neq S^{(0)}$, we consider the decomposition indicated by

$$S \;=\; S \cap S^{(0)} + S \cap C^{(0)}. \tag{12}$$

According to whether S is winning or losing we shall try (a not unfamiliar procedure) to cut out multiples of $\boldsymbol{m}_j^\circ$ $(j \in S^{(0)},\ j \notin S)$ out of $S \cap C^{(0)}$. If, for comparison, we refer to any homogeneous representation $(\boldsymbol{m}, \alpha)$ of $\boldsymbol{v}$, then the procedure works with respect to $\boldsymbol{m}$ as follows:

If $\boldsymbol{m}(S) = \alpha$ (i.e., if S is minimal winning), then we have

$$\boldsymbol{m}(S \cap C^{(0)}) \;=\; \boldsymbol{m}(S - S^{(0)}) \;=\; \boldsymbol{m}(S^{(0)} - S) \;=\; \sum_{j \in S^{(0)} - S} \boldsymbol{m}_j.$$

As we have homogeneous games $\boldsymbol{v}^{(j)}$ $(j \in S^{(0)})$ (represented by $\boldsymbol{m}^{(j)}, \boldsymbol{m}_j$), we can decompose $S \cap C^{(0)}$ via

$$S \cap C^{(0)} \;=\; \sum_{j \in S^{(0)} - S} \widetilde{S}^j$$

satisfying

$$\boldsymbol{m}(\widetilde{S}^j) = \boldsymbol{m}^{(j)}(\widetilde{S}^j) = \boldsymbol{m}_j.$$

Now turn to $\boldsymbol{m}^\circ$. In view of our assumptions in (11) we have also

$$\boldsymbol{m}^\circ(\widetilde{S}^j) = \boldsymbol{m}^{\circ(j)}(\widetilde{S}^j) = \boldsymbol{m}^\circ_j.$$

and hence, moving backwards

$$\begin{aligned}
\boldsymbol{m}^\circ(S) &= \boldsymbol{m}^\circ(S \cap S^{(0)}) + \boldsymbol{m}^\circ(S \cap C^{(0)}) \\
&= \boldsymbol{m}^\circ(S \cap S^{(0)}) + \sum_{j \in S^{(0)} - S} \boldsymbol{m}^\circ(\widetilde{S}^j) \\
&= \boldsymbol{m}^\circ(S \cap S^{(0)}) + \sum_{j \in S^{(0)} - S} \boldsymbol{m}^\circ_j \\
&= \boldsymbol{m}^\circ(S \cap S^{(0)}) + \boldsymbol{m}^\circ(S^{(0)} - S) \\
&= \boldsymbol{m}^\circ(S^{(0)}) \\
&= \alpha_\circ,
\end{aligned} \tag{13}$$

where the last equation uses the compatibility of $(\boldsymbol{m}^\circ, \alpha_\circ)$. This way we have proved that minimal winning coalitions are represented correctly by $(\boldsymbol{m}^\circ, \alpha_\circ)$.

6thSTEP : Next, if $\boldsymbol{m}(S) < \alpha$ holds true, then we can extract certain coalitions $\widetilde{S}^{(j)}$ for certain $j \in S^{(0)} - S$ satisfying $\boldsymbol{m}(\widetilde{S}^{(j)}) = \boldsymbol{m}_j$. However, there remains a $j \in S^{(0)}$ and a remainder coalition, say $\widetilde{R}^{(j)}$ such that $\boldsymbol{m}(\widetilde{R}^{(j)}) < \boldsymbol{m}_j$ holds true. This again implies $\boldsymbol{m}^\circ(\widetilde{R}^{(j)}) < \boldsymbol{m}^\circ_j$. Consequently we obtain quite similar as in the previous step

$$\boldsymbol{m}^\circ(S \cap C^{(0)}) < \sum_{j \in S^{(0)} - S} \boldsymbol{m}^\circ_j = \boldsymbol{m}^\circ(S^{(0)} - S)$$

and hence $\boldsymbol{m}^\circ(S) < \alpha_\circ$.

7thSTEP : Finally we verify that $\boldsymbol{m}^\circ$ is indeed homogenous w.r. to $\alpha_\circ$. this is quite easy to see: if $\boldsymbol{m}^\circ(S) > \alpha_\circ$ for some $S \in \underline{\underline{\mathbf{P}}}$, then we know by the previous steps that S is winning, hence there is $T \subseteq S$ with $\boldsymbol{m}(T) = \alpha$. This implies $\boldsymbol{m}^\circ(T) = \alpha_\circ$ and we have indeed $\boldsymbol{m}^\circ$ hom $\alpha_\circ$.

8thSTEP : This way we have verified (11). On one hand this completes the induction step announced in the $1^{st} STEP$. On the other hand we have to show that (7) implies that $(\boldsymbol{m}^\circ, \alpha_\circ)$ represents $\boldsymbol{v}$ homogenously – a task we have just performed.

q.e.d.

4 The Incidence Vector

This section follows the exposition of Sudhölter [3]. We consider a sequence of k *min–win* coalitions $S^{(0)}, S^{(1)}, \ldots, S^{(k-1)}$ which we suppose to be generated successively via the procedure of dropping one player at a time. Let us make this more precise as follows.

Definition 4.1. *A sequence of* min–win *coalitions* $S^{(0)}, S^{(1)}, \ldots, S^{(k-1)}$ *of length* k *is said to be obtained by* ***successive dropping*** *of players if the following holds true:*

1. $S^{(0)}$ *is the* lex–max min–win *coalition*
2. *For every* $j \in \{1, \ldots, k-1\}$ *there exists* $i < j$ *such that*
   (a) $j \in S^{(i)}$,
   (b) j *is dispensable in* $S^{(i)}$,
   (c) $S^{(j)} \;=\; \boldsymbol{\delta}_j(S^{(i)})$

   holds true.

Thus the sequence has a shape indicated as follows:

$$
\begin{array}{rcl}
S^{(0)} & & \\
S^{(1)} & = & \boldsymbol{\delta}_1(S^{(0)}) \\
\ldots & = & \ldots \\
S^{(j)} & = & \boldsymbol{\delta}_j(S^{(i)}) \\
\ldots & = & \ldots
\end{array}
\tag{1}
$$

such that $S^{(i)}$ is among the coalitions constructed previously. The question arises whether this procedure can be continued for the "next" player k. Indeed, we have

Theorem 4.2. *Let* $k < n$. *For any sequence of* min–win *coalitions* $S^{(0)}, S^{(1)}, \ldots, S^{(k-1)}$ *of length* k *obtained by successive dropping, there is* $l \in \{1, \ldots, k-1\}$ *such that* k *is dispensable in* $S^{(l)}$.

Thus, the sequence can be continued by putting

$$S^{(k)} \;:=\; \boldsymbol{\delta}_k(S^{(l)}),$$

such that the extended sequence again is obtained by successive dropping in the sense of Definition 4.1. Of course, this means that there exists a sequence of n coalitions obtained by successive dropping as indicated by the scheme (1). In other words, it is posssible to construct a sequence of *min–win* coalitions, each one obtained from the previous ones by successively dropping players in their natural order.

Proof: 1stSTEP :

Suppose the statement is wrong and we have found a player k who is *not* dispensable from any of the previously constructed coalitions $S^{(0)}, S^{(1)}, \ldots, S^{(k-1)}$.

Now, as player k is a sum, there exists a coalition S such that player k *is* dispensable in S (cf Theorem 1.25). This coalition cannot be the *lex–max min–win* coalition as it is the first in the sequence. Any such S will therefore have a last dropout $\boldsymbol{\delta}(S)$.

Now, among all such coalitions S choose the one, say $\overline{S}$, which has the *largest* last dropout, say $\overline{r} = \boldsymbol{\delta}(\overline{S})$. Note that $\overline{r} < k$ is necessarily true (cf. Definition 1.19). Hence $S^{(\overline{r}+1)}$ is well defined (and obtained by dropping $\overline{r}$ from some $S^{(j)} = \boldsymbol{\iota}(S^{(\overline{r}+1)})$).

Next we continue by discussing two possible alternatives.

2ndSTEP : Assume that

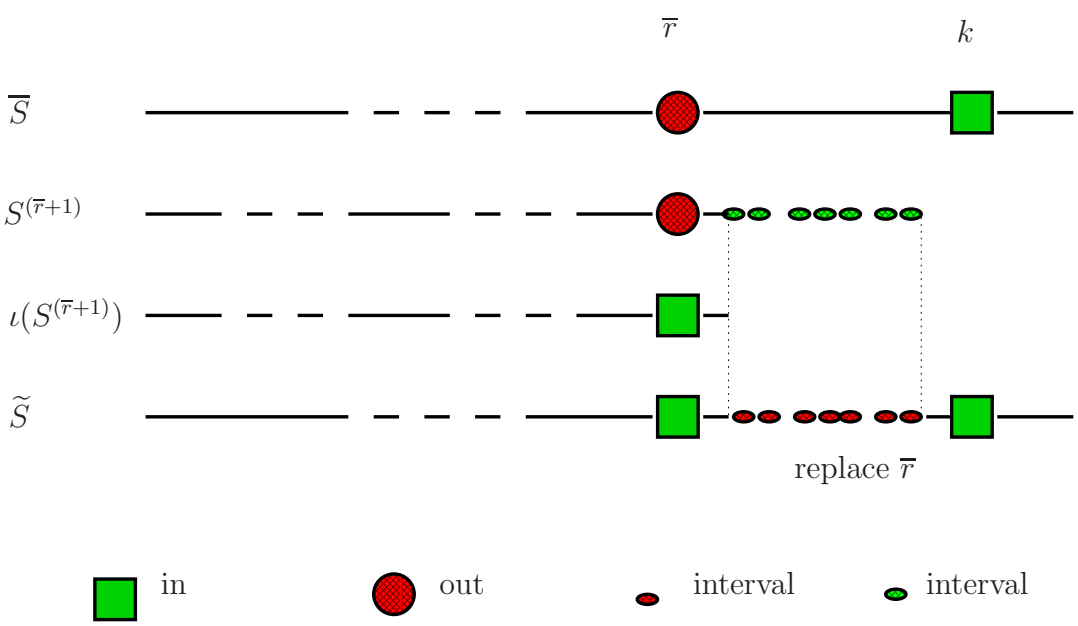

Figure 4.1: A long $\overline{S}$

$$l(S^{(\overline{r}+1)}) < l(\overline{S}) \tag{2}$$

holds true (cf. Figure 4.1).

As k is not dispensable in $S^{(\overline{r}+1)}$, we must have

$$l(S^{(\overline{r}+1)}) < k \tag{3}$$

(recall that we can drop k even from $\overline{S}$ which is "longer"). Now consider the coalition

$$\widetilde{S} \;:=\; \overline{S} \cup \{\overline{r}\} - [l(\boldsymbol{\iota}(S^{(\overline{r}+1)})) + 1, l(S^{(\overline{r}+1)})]. \tag{4}$$

Coalition $\widetilde{S}$ is *min–win* and contains k because of (3). Moreover, it contains a last dropout exceeding $\overline{r}$. This contradicts the maximality required in choosing $\overline{r}$.

3rdSTEP : Alternatively, assume that

$$l(S^{(\overline{r}+1)}) \geq l(\overline{S}) \tag{5}$$

holds true. (cf. Figure 4.2) Recall that k has never been dropped from all our $S^{(j)}$. Hence the $=$ –sign cannot prevail, for otherwise k would be dispensable in $S^{(\overline{r}+1)}$.

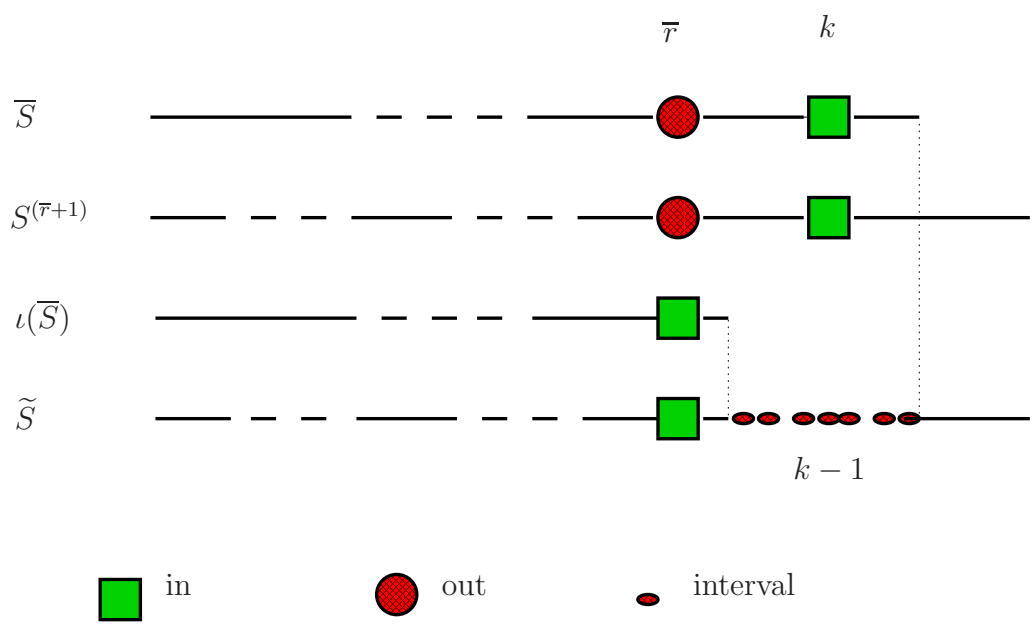

Figure 4.2: A short $\overline{S}$

So we have indeed

$$l(S^{(\overline{r}+1)}) > l(\overline{S}). \tag{6}$$

By a similar argument we see that

$$l(\boldsymbol{\iota}(S^{(\overline{r}+1)})) < k \tag{7}$$

is the case. Now we consider the coalition

$$\widetilde{S} \;:=\; S^{(\overline{r}+1)} \cup \{\overline{r}\} - [l(\boldsymbol{\iota}(\overline{S})) + 1, l(\overline{S})] \tag{8}$$

which again is *min–win* and yields

$$\widetilde{S} \cap [k-1, l(\overline{S})] \;=\; \emptyset. \tag{9}$$

Thus, for coalition $\widetilde{S}$ it is seen that the full intervall $[k-1, l(\overline{S})]$ is missing and $l(\overline{S})$ is the last dropout in view of (6). So we can first insert $l(\overline{S})$ and then successively the preceding players until we reach k. I. e., we apply the operation $\boldsymbol{\iota}$ exactly $l(\overline{S}) - k$ times on $\widetilde{S}$, thus obtaining the coalition

$$\boldsymbol{\iota}^{(l(\overline{S})-k)}(\widetilde{S}). \tag{10}$$

Clearly, in this coalition player k is dispensable and $k-1$ is the last dropout. But $k-1 > \overline{r}$ follows from equation (7) . Hence, we have again a contradiction to the asserted maximality of $\overline{r}$. This finishes our proof.

q.e.d.

The above theorem suggests a procedure which generates a sequence of "characteristic" coalitions by dropping players in the natural order. This procedure is described verbally as follows:

First of all, drop player 1 from $S^{(0)}$.

Next, if players $1, \ldots, k-1$ have ben dropped and $S^{(0)}, \ldots, S^{(k-1)}$ have been defined, then we know by Theorem 4.2 that player k is dispensable in at least one of these coalitions. Let $l^{(j)} \;:=\; l(S^{(j)})$ denote the length of the coalitions constructed as yet. Necessarily we have $k \leq l^{(j)}$ for at least some $j < k$. Now, the (any) shortest coalition among those constructed so far having a length of at least k is *a fortiori* of the nature that k is dispensable. The shortest length of a coalition of this nature is

$$\pi_k \;:=\; \min\left\{ l^{(j)} \,\middle|\, j < k,\; k \leq l^{(j)} \right\}.$$

Still, there could be more than one coalition having this shortest length and containing k; if so, then we choose the first in the order established by the procedure, i.e., we choose $S^{(i_0)}$ such that i_0 satisfies

$$i_0 \;:=\; \min\left\{j \,\middle|\, l^{(j)} = \pi_k\right\}.$$

The formal definition of this procedure is now obvious. We assume as usual that the min of an empty set of integers is 0, and proceed as follows.

Definition 4.3. *1. For every $K \in \mathbb{N}$ we define the mapping $\boldsymbol{\pi} \;:\; \mathbb{N}^K \to \mathbb{N}^K$ (on vectors written $l = (l_0, \ldots, l_{K-1})$) by*

$$\boldsymbol{\pi}^l_k \;:=\; \min\{l_j \,|\, j < k \le l_j\} \quad (l \in \mathbb{N}^K, k = 1, \ldots K). \tag{11}$$

2. Based on this, we define for every $K \in \mathbb{N}$ a mapping $\boldsymbol{\omega} \;:\; \mathbb{N}^K \to \mathbb{N}^K$ by

$$\boldsymbol{\omega}^l_k \;:=\; \min\left\{j \,\middle|\, j < k,\; l_j = \boldsymbol{\pi}^l_k\right\} \quad (l \in \mathbb{N}^K, k = 1, \ldots, K). \tag{12}$$

*3. For any homogeneous game $\boldsymbol{v}$ the **lex–max min–win system** of coalitions*

$$\underline{\underline{\mathbf{S}}}^{(0)} = \underline{\underline{\mathbf{S}}}^{(0)}(\boldsymbol{v}) = \left\{S^{(0)}, S^{(1)}, \ldots, S^{(n-1)}\right\}$$

is recursively defined as follows:

(a) $S^{(0)}$ is the lex–max min–win *coalition.*

(b) Given $k < n$ and $S^{(0)}, \ldots, S^{(k-1)}$, write $l \;:=\; (l(S^{(0)}), \ldots, l(S^{(k-1)}))$ and define $i_0 \;:=\; \boldsymbol{\pi}^l_k$ and

$$S^{(k)} \;:=\; \boldsymbol{\delta}_k(S^{(i_0)}).$$

4. The incidence matrix corresponding to $\underline{\underline{\mathbf{S}}}^{(0)}$, i.e., the matrix

$$\boldsymbol{I}^{(0)}(\boldsymbol{v}) \;:=\; (1_S)_{S \in \underline{\underline{\mathbf{S}}}^{(0)}}$$

*is called the **characterizing incidence matrix** of $\boldsymbol{v}$.*

We mention again that quantities discussed above are well defined in view of Theorem 4.2. Note that a superscript K at the mappings $\boldsymbol{\pi}$ and $\boldsymbol{\omega}$ can be omitted because for varying length of the argument l the various versions are defined "consistently". The coordinate mappings $\boldsymbol{\pi}_k$ yield the same result for the restriction of some $l \in \mathbb{N}^K$ to $\mathbb{N}^M$ with $k \le M \le K$.

Example 4.4. The following show the *lex–max min–win* system of a homogeneous game. Suitably this is represented by the corresponding incidence matrix $\boldsymbol{I}^{(0)}(\boldsymbol{v})$

First of all, if we take $\boldsymbol{m} = (1,1,1,1,1,1,1,1)$ and $\lambda = 4$, the the system $\underline{\underline{\mathbf{W}}}^{min}$ of all *min–win* coalitions consists of all coalitions of 4 players. The matrix $\boldsymbol{I}^{(0)}(\boldsymbol{v})$ is the following one:

```
                1 1 1 1 0 0 0 0
                0 1 1 1 1 0 0 0
 (0)            1 0 1 1 1 0 0 0
I   (v) =       1 1 0 1 1 0 0 0
                1 1 1 0 1 0 0 0
                0 1 1 1 0 1 0 0
                0 1 1 1 0 0 1 0
                0 1 1 1 0 0 0 1
```

Another example is obtained my taking $\boldsymbol{m} = (6,5,4,2,2,1,1,1)$ and $\lambda = 15$. We obtain for $\boldsymbol{I}^{(0)}(\boldsymbol{v})$ the matrix

```
                1 1 1 0 0 0 0 0
                0 1 1 1 1 1 1 0
                1 0 1 1 1 1 0 0
 (0)            1 1 0 1 1 0 0 0
I   (v) =       1 1 0 0 1 1 1 0
                1 1 0 1 0 1 1 0
                1 0 1 1 1 0 1 0
                0 1 1 1 1 1 0 1
```

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In $\mathbb{APL}$ – code this reads as follows:

```
     The following program generates the
     lex--max min--win system of
     a homogeneous game

     1 1 1 1 1 1 1 1  MINFIX 4


            1 1 1 1 1 0 0 0 0
            0 1 1 1 1 0 0 0
            1 0 1 1 1 0 0 0
 (0)        1 1 0 1 1 0 0 0
I   (v) =   1 1 1 0 1 0 0 0
            0 1 1 1 0 1 0 0
            0 1 1 1 0 0 1 0
            0 1 1 1 0 0 0 1
     GPS ← 6 5 4 2 2 1 1 1
     LPS ← 15
       GPS MINFIX LPS
1 1 1 0 0 0 0 0
0 1 1 1 1 1 1 0
1 0 1 1 1 1 0 0
1 1 0 1 1 0 0 0
1 1 0 0 1 1 1 0
1 1 0 1 0 1 1 0
1 0 1 1 1 0 1 0
0 1 1 1 1 1 0 1
```

Recall that $\mathbb{H}_n$ denotes the set of all homogeneous $n-$person games without dummies, steps, and one–person winning coalitions. Now it turns out that the mapping which carries such a game into its *min–win lex–max* system is injective.

Theorem 4.5. *Let* $\boldsymbol{v}, \boldsymbol{v}' \in \mathbb{H}_n$ *and suppose that the* min–win lex–max *systems* $\underline{\underline{\mathbf{S}}}^{(0)}$, $\underline{\underline{\mathbf{S}}}^{(0)\prime}$ *concide. Then* $\boldsymbol{v} = \boldsymbol{v}'$ *holds true.*

Proof:

1stSTEP : We know that the minimal representation uniquely determines the game and *vice versa*. Hence it suffices to show that the *min–win lex–max*

system uniquely determines the minimal representation $\boldsymbol{m}^\star$ of the game.

Now, given $\underline{\underline{\mathbf{S}}}^{(0)}$, we define recursively a measure $\boldsymbol{m}$ as follows:

1. Define $\boldsymbol{m}_n \ := \ 1$.

2. Given $\boldsymbol{m}_n, \ldots, \boldsymbol{m}_{k-1}$ there is a unique $i_0 < k$ such that $S^{(i_0)} = \boldsymbol{\iota}(S^{(k)})$ is satisfied. That is, $S^{(k)}$ has been obtained by dropping k from $S^{(i_0)}$ or formally $\boldsymbol{\delta}_k(S^{(i_0)} \ = \ S^{(k)}$. Clearly, $S^{(k)} - S^{(i_0)}$ contains just the players replacing k in $S^{(k)}$, so all of them succeed k, we may set

$$\boldsymbol{m}_k \ := \ \boldsymbol{m}(S^{(k)} - S^{(i_0)}).$$

3. Finally define $\lambda \ := \ \boldsymbol{m}(S^{(1)}$.

As k is the first player at which both coalitions under considerations differ, we have

$$\boldsymbol{m}\left(S^{(k)} - S^{(i_0)}\right) = \boldsymbol{m}_k = \boldsymbol{m}\left(S^{(i_0)} - S^{(k)}\right)$$

and hence $S^{(k)} = S^{(i_0)}$. We conclude immediately that all values $(\boldsymbol{m}(S))_{S \in \underline{\underline{\mathbf{S}}}^{(0)}}$ are equal to λ.

2ndSTEP : Consider the linear system of equations in variables $x = (x_1, \ldots, x_n)$ and x_{n+1} given by

$$\begin{aligned} \sum_{i \in S} x_i = & \quad x_{n+1} \\ x_1 = & \quad 1. \end{aligned} \tag{13}$$

Obviously, $(\boldsymbol{m}, \lambda)$ is the unique solution of this system in view of the above successive construction which yields this unique solution.

This system can also be written with the aid of the characterizing incidence matrix

$$\begin{aligned} \boldsymbol{I}^{(0)}(\boldsymbol{v})x \ = & \quad x_{n+1} \\ x_1 \ = & \quad 1. \end{aligned}$$

Observe that we have obtained a subsystem of the linear system of equations set up by the full incidence matrix, i.e., of

$$\begin{aligned} \boldsymbol{I}(\boldsymbol{v})x \ = & \quad x_{n+1} \\ x_1 \ = & \quad 1. \end{aligned} \tag{14}$$

However, we know that the unique minimal representation $(\boldsymbol{m}^\star, \alpha^\star)$ is the unique solution of (14). Hence we have necessarily

$$(\boldsymbol{m}, \lambda) = (\boldsymbol{m}^\star, \alpha^\star)$$

which shows that the *min–win lex–max* system $\underline{\underline{\mathbf{S}}}^{(0)}$ uniquely determines the minimal representation of the game,

q.e.d.

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