1 Dynamic Games

Definition 1.1. *A **dynamic game on a graph** is six tuple*

$$\Sigma = (\mathcal{X}, \mathcal{Y}, \prec; \boldsymbol{q}; f, u) \tag{1.1}$$

with the data explained as follows. $(\mathcal{X}, \prec)$ *is a graph without loops and circles.* f *and* u *are families of intermediate and final reward functions adapted to the set of players* $\boldsymbol{I}$*. Next,*

$$\mathcal{Y} \;=\; \left(\mathcal{Y}^i\right)_{i \in \boldsymbol{I}}$$

is a system of finite sets, $\mathcal{Y}^i$ *is called player* i *'s **action set**. Finally, the mapping*

$$\boldsymbol{q} \;:\; \mathcal{X} \setminus \partial\mathcal{X} \times \mathcal{Y}^1 \times \ldots \times \mathcal{Y}^n \to \mathcal{X} \tag{1.2}$$

*(the **transition function**) specifies the choice of a successor, it has to satisfy*

$$\boldsymbol{q}(\xi; y_1, \ldots, y_n) \in \boldsymbol{N}(\xi)$$

for all $\xi \in \mathcal{X} \setminus \partial\mathcal{X}$ *and* $\boldsymbol{y} \in \mathcal{Y}^1 \times \ldots \times \mathcal{Y}^n$.
We write

$$\underline{\overline{\mathbf{Y}}} \;:=\; \mathcal{Y}^1 \times \ldots \times \mathcal{Y}^n$$

and

$$\underline{\overline{\mathbf{X}}} \;=\; \{\boldsymbol{x} = (x_1, \ldots, x_n) \,|\, \boldsymbol{x} \text{ is a path in } \mathcal{X}\}$$

as previously.

Note that for $n = 2$ and fixed $\xi \in \mathcal{X} \setminus \partial\mathcal{X}$ the mapping

$$\boldsymbol{q}(\xi; \bullet) \;=\; \boldsymbol{q}(\xi; y_1, y_2)_{y_1 \in \mathcal{Y}^1, y_2 \in \mathcal{Y}^2}$$

can be regarded as a matrix: the rows correspond to the actions of player 1 and the columns to the actions of player 2. Thus, imagine that at each node (with the exception of the boundary nodes) we find a matrix of this type. At a particular node, player 1 chooses a row, player 2 chooses a column and the process moves to the successor specified by the corresponding entry of the matrix.

examtreewww **Example 1.2.** Let $n = 2$ and let $\mathcal{Y}^1 = \{o, u\}$, $\mathcal{Y}^2 = \{\ell, r\}$. The tree $(\mathcal{X}, \prec)$ as well as the final rewards are given by the following sketch; the intermediate rewards are assumed to be 0.

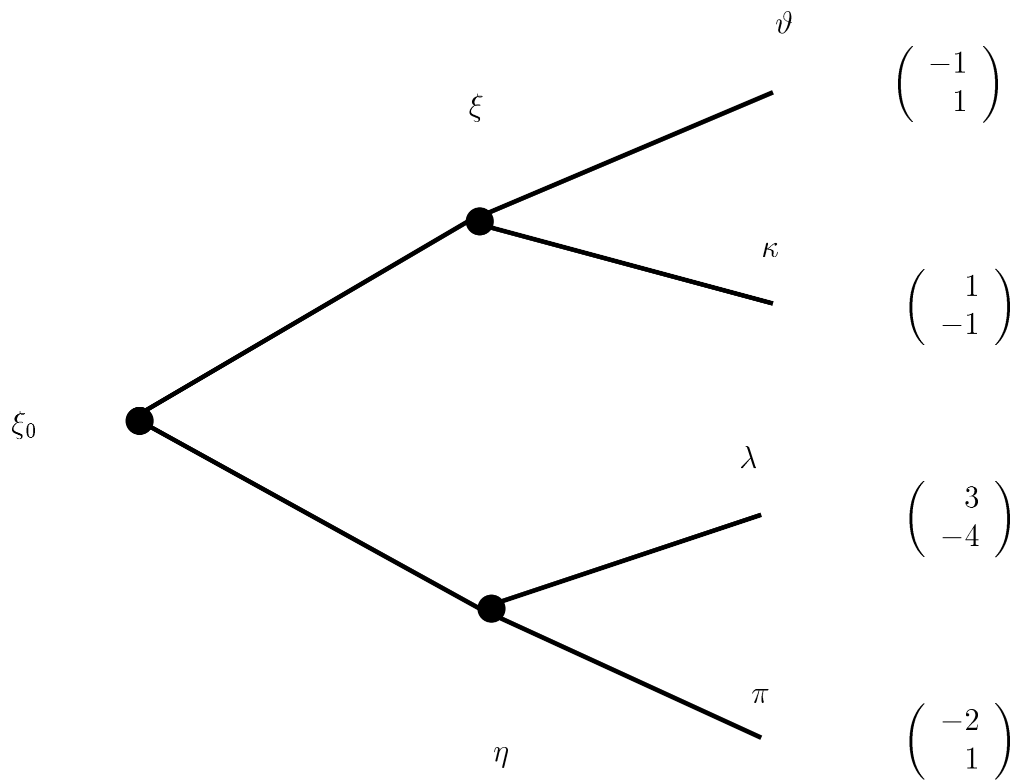

Figure 1.1: A dynamic game on a tree

□

The transition function is given as follows

$$
\begin{aligned}
\boldsymbol{q}(\xi_0, \bullet) &= \begin{matrix} & \ell & r \\ o & \xi & \xi \\ u & \eta & \eta \end{matrix} \\
\boldsymbol{q}(\xi, \bullet) &= \begin{matrix} & \ell & r \\ o & \vartheta & \kappa \\ u & \vartheta & \kappa \end{matrix} \\
\boldsymbol{q}(\eta, \bullet) &= \begin{matrix} & \ell & r \\ o & \lambda & \pi \\ u & \pi & \lambda \end{matrix}
\end{aligned}
$$

Observe that player 1 decides for ξ or η at ξ_0 regardless of the choice of actions of player 2. Similarly, player 2 is in charge at node ξ. However, at node η players have to take the opponents choice into account.

Thus, while a value function can be established recursively at node x, there are conflicting interests at node η. Now, whenever the players find themselves at node η, the choice of o by player 1 and l by player 2 obviously leadss to a payoff vector $(1, -1)$. Performing this computation for all possible pairs of action we see that the players at node η are actually facing a bimatrix game which is given by the two matrices

$$u^1(\eta, \boldsymbol{q}(\eta, \bullet)) = \begin{array}{c} \\ o \\ u \end{array} \begin{array}{c} \quad \ell \quad\ r \\ \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \end{array}$$

$$u^2(\eta, \boldsymbol{q}(\eta, \bullet)) = \begin{array}{c} \\ o \\ u \end{array} \begin{array}{c} \quad \ell \quad\ r \\ \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix} \end{array}$$

This bimatrix game has no pure equilibrium strategies. Hence, in order to define the equilibrium behavior as well as a (recursive) value function we have to randomize over the players actions. This naturally leads to the following definition.

Definition 1.3. *A* ***behavioral strategy*** *for player $i \in \boldsymbol{I}$ is a system of probability distributions on $\mathcal{Y}^i$*

$$\boldsymbol{A}^i(\xi, \bullet) \quad (\xi \in \mathcal{X} \setminus \partial\mathcal{X}) \tag{1.3}$$

or, in other words, a kernel

$$\boldsymbol{A}^i \mid \mathcal{X} \setminus \partial\mathcal{X} \Rightarrow \mathcal{Y}^i \ . \tag{1.4}$$

Of course, an n–tuple of behavioral strategies $\boldsymbol{A}^1, \ldots, \boldsymbol{A}^n$ generates a kernel

$$\boldsymbol{A} \mid \mathcal{X} \setminus \partial\mathcal{X} \Rightarrow \underline{\overline{\mathbf{Y}}} \tag{1.5}$$

via

productker (1.6)

$$\boldsymbol{A}(\xi; y_1, \ldots, y_n) = \boldsymbol{A}^1(\xi; y_1) \cdot \ldots \cdot \boldsymbol{A}^n(\xi; y^n) =: \boldsymbol{A}^1 \otimes \ldots \otimes \boldsymbol{A}^n(\xi; y^1, \ldots, y^n)$$

$\boldsymbol{A}$ reflects the stochastically independent application of a random mechanism of ech player based on his observation of the node at hand.

Clearly, an n–tuple of behavioral strategies (we shall just say "a BS") induces a transition kernel on the nodes of the graph as well. This kernel controlls the transition from one node to one of its successors – depending on the BS – and is written

$$\begin{aligned} \boldsymbol{Q} = \boldsymbol{Q}^{\boldsymbol{A}} \quad &| \quad \mathcal{X} \setminus \partial\mathcal{X} \Rightarrow \mathcal{X} \\ \boldsymbol{Q}(\xi; \eta) &= \boldsymbol{A}(\xi; \{\boldsymbol{y} | \boldsymbol{q}(\xi, \boldsymbol{y}) = \eta\}) \\ &= \boldsymbol{A}(\xi; \boldsymbol{q}^{-1}(\xi, \bullet)(\{\eta\})) \\ &= (\boldsymbol{q}(\xi, \bullet)\boldsymbol{A}(\xi, \bullet)) \{\eta\} \ . \end{aligned} \tag{1.7}$$

In other words, $\boldsymbol{Q}^{\boldsymbol{A}}(\xi, \bullet)$ is obtained by transporting $\boldsymbol{A}(\xi, \bullet)$ from $\underline{\overline{\mathbf{Y}}}$ to $\boldsymbol{N}(\xi)$ via $\boldsymbol{q}(\xi, \bullet)$. Now, clearly $\boldsymbol{Q} = \boldsymbol{Q}^{\boldsymbol{A}}$ generates a Markovian probability measure on $\underline{\overline{\mathbf{X}}}$, this is denoted by $\boldsymbol{m}^{\boldsymbol{A}}_{\xi_0}$.

In what follows we – somewhat sloppily – use the notation $\boldsymbol{A}$ not just for the product kernel defined via equation (1.6) but as well for the strategy n–tuple. No confusion will occur.

payoffwww **Definition 1.4.** *Let Σ by a dynamic game on a tree and let $\boldsymbol{A}$ be a VHS. Let τ_0 denote the length of a path and let $\tau := \tau_0 \circ X$.*

1. *A process*

$$X : \Omega \to \underline{\overline{\mathbf{X}}}$$

*is said to be **generated** or **controlled** by $\boldsymbol{A}$ if X has the distribution $\boldsymbol{m}_{\xi_0}^{\boldsymbol{A}}$, i.e., if*

$$X\mathbb{P} = \mathbb{P} \circ X^{-1} = \boldsymbol{m}_{\xi_0}^{\boldsymbol{A}} \tag{1.8}$$

holds true.

2. *The **payoff** to player $i \in \boldsymbol{I}$ resulting from $\boldsymbol{A}$ (via X) is given by*

$$\begin{aligned} \boldsymbol{C}_{\boldsymbol{A}}^{i\xi_0} &:= \mathbb{E}\left(\sum_{t=1}^{\tau} f^i(X_{t-1}, X_t) + u^i(X_\tau)\right) \\ &= \mathbb{E}(C^i \circ X) = \int_\Omega C^i \circ X dP \\ &= \int_{\underline{\overline{\mathbf{X}}}} C^i \, d\boldsymbol{m}_{\xi_0}^{\boldsymbol{A}} = \mathbb{E}_{\boldsymbol{m}_{\xi_0}^{\boldsymbol{A}}}(C^i) \\ &= \mathbb{E}_{\boldsymbol{m}_{\xi_0}^{\boldsymbol{A}}}\left(\sum_{t=1}^{\tau_0} f^i(x_{t-1}, x_t) + u^i(x_{\tau_0})\right), \end{aligned} \tag{1.9}$$

where the 4. equation uses a transformation of the variables.

By the above definition, the function $\boldsymbol{C}_{\bullet}^{i\xi_0}$ is constituted as a mapping on n–tuples of behavioral strategies. Therefore, we have an n–person non–cooperative game in normal form at hand. More precisely,

Definition 1.5. *Let $\mathcal{A}^i$ denote the set of player i^s behavioral strategies and denote by*

$$\mathcal{A} = \mathcal{A}^1 \times \ldots \times \mathcal{A}^n \tag{1.10}$$

the set of BS n–tupels (or just BS). Let $\boldsymbol{C}_{\bullet}^{i\xi_0} : \mathcal{A} \to \mathbb{R}$ be player i^s payoff as given by Definition 1.4. Then

$$\Gamma = \Gamma_{\Sigma\xi_0} = \left(\mathcal{A}^1, \ldots, \mathcal{A}^n; C_{\bullet}^{1\xi_0}, \ldots C_{\bullet}^{n\xi_0}\right) \tag{1.11}$$

*is the **normal form in BS** derived from Σ.*

Remark 1.6. *As previously we can operate the same apparatus when the process starts at some $\xi \in \mathfrak{X}$. Then there is a function τ_0^ξ indicating the length of a path starting at ξ as well as the random length $\tau^\xi = \tau_0^\xi \circ X$. Also, the Markovian measure $\boldsymbol{m}_\xi^{\boldsymbol{A}}$ is generated by means of a BS $\boldsymbol{A}$. This BS may be defined for all* *nodes of the subtree generated by ξ. Or else, one could take a BS defined for the whole tree and restrict it to the subtree. The payoff is then*

$$\begin{aligned} C_A^{i\xi} &:= \mathbb{E}\left(\sum_{s=1}^{\tau^\xi} f^i(X_{s-1}, X_s) + u^i(X_{\tau^\xi})\right) \\ &= \mathbb{E}_{m_\xi^A}\left(\sum_{s=1}^{\tau_0^\xi} f^i(x_{t-1}, x_t) + u^i(x_{\tau_0^\xi})\right). \end{aligned} \tag{1.12}$$

On the other hand, we can formulate this payoff within the framework of the full tree using a BS as well as the generated measure $m_{\xi_0}^A$ and the process X distributed accordingly. If $(\mathfrak{X}, \prec)$ is a tree (and all paths passing ξ have length t), then we have as well

$$\begin{aligned} C_A^{i\xi} &:= \mathbb{E}\left(\sum_{s=t}^{\tau} f^i(X_{s-1}, X_s) + u^i(X_\tau) \,\middle|\, X_{t-1} = \xi\right) \\ &= \mathbb{E}_{m_{\xi_0}^A}\left(\sum_{s=t}^{\tau_0} f^i(x_{t-1}, x_t) + u^i(x_\tau) \,\middle|\, x_{t-1} = \xi\right). \end{aligned} \tag{1.13}$$

The bothersome details are left to the reader.

Lemma 1.7. *Let A be a BS and let X be a process distributed according to $m_{\xi_0}^A$. Then, for any $\xi \in \mathfrak{X} \setminus \partial\mathfrak{X}$ we have*

$$\mathbb{E}\left(f^i(X_{t-1}, X_t) \,\middle|\, X_{t-1} = \xi\right) = \int_{N(\xi)} f^i(\xi, \eta) Q^A(\xi, d\eta) = \int_{\underline{\mathbf{Y}}} f^i(\xi, q(\xi, \eta)) A(\xi, dy)$$

If $(\mathfrak{X}, \prec)$ is not a tree, then we can also formulate a more general version. For, in this case a path may hit a node ξ at varying time instances. Let

$$\theta_\xi^0(x) := \begin{cases} t & \text{if } \xi \in \{x_0, \dots, x_T\} \text{ and } x_t = \xi \\ \infty & \text{otherwise} \end{cases}$$

define the hitting time of a path and let $\theta := \theta_\xi^0 \circ X$, then we have the version:

$$\begin{aligned} \mathbb{E}\left(f^i(X_\theta, X_{\theta+1}) \,\middle|\, \xi \in \{X_0, \dots, X_\tau\}\right) &= \mathbb{E}\left(f^i(X_\theta, X_{\theta+1}) \,\middle|\, Z_\xi = 1\right) \\ &= \int_{N(\xi)} f^i(\xi, \eta) Q^A(\xi, d\eta) \\ &= \int_{\underline{\mathbf{Y}}} f^i(\xi, q(\xi, \eta)) A(\xi, dy) . \end{aligned}$$

Proof:

The first equation is due to the Markovian property of the process: the conditional probabilities of the process are given by the transitional kernels.

The second equation follows from a transformation of the variable:

$$\begin{aligned}\int\limits_{\boldsymbol{N}(\xi)} f^i(\xi,\eta)\boldsymbol{Q}^{\boldsymbol{A}}(\xi,d\eta) &= \int\limits_{\boldsymbol{N}(\xi)} f^i(\xi,\eta)\left(\boldsymbol{q}(\xi,\bullet)\boldsymbol{A}(\xi,\bullet)\right)\{d\eta\} \\ &= \int\limits_{\underline{\overline{\mathbf{Y}}}} \big(\boldsymbol{q}(\xi,\bullet)f^i(\xi,\bullet)\big)\,\boldsymbol{A}(\xi,d\boldsymbol{y}) \\ &= \int\limits_{\underline{\overline{\mathbf{Y}}}} f^i(\xi,\boldsymbol{q}(\xi,\eta))\boldsymbol{A}(\xi,d\boldsymbol{y}),\end{aligned}$$

q.e.d.

2 Backwards Induction

The following lemma relates the payoff at some node $\xi \in \mathfrak{X} \setminus \partial\mathfrak{X}$ to the payoff received when the process has moved one step ahead.

conestep **Lemma 2.1.** *Let Σ be a dynamic game and let $\boldsymbol{A}$ be a BS (n–tuple). Then, for $\xi \in \mathfrak{X} \setminus \partial\mathfrak{X}$ we have*

reclemmawww (2.1)

$$\begin{aligned} \boldsymbol{C}_{\boldsymbol{A}}^{i\xi} &= \int_{\underline{\overline{\mathbf{Y}}}} \left\{ \boldsymbol{C}^{i\boldsymbol{q}(\xi,\eta)} + f^i(\xi, \boldsymbol{q}(\xi,\eta)) \right\} \boldsymbol{A}(\xi, d\eta) \\ &= \boldsymbol{A}(\xi, \bullet) \left\{ \boldsymbol{C}^{i\boldsymbol{q}(\xi,\bullet)} + f^i(\xi, \boldsymbol{q}(\xi,\bullet)) \right\} . \end{aligned}$$

Here we have written the last equation as a scalar product for vectors (with coordinates indexed by $\mathfrak{Y}$) in order to have a shorthand for the following proof. Within the formula, $\boldsymbol{A}$ of course denotes the product kernel.

Proof:

We have
(2.2)

$$\begin{aligned} C_{\boldsymbol{A}}^{i\xi} &= \mathbb{E}\left(\sum_{s=t}^{\tau} f^i(X_{s-1}, X_s) + u^i(X_\tau) \,\middle|\, X_{t-1} = \xi \right) \\ &= \mathbb{E}\left(\sum_{s=t+1}^{\tau} f^i(X_{s-1}, X_s) + u^i(X_\tau) + f^i(X_{t-1}, X_t) \,\middle|\, X_{t-1} = \xi \right) \\ &= \mathbb{E}\left(\sum_{s=t+1}^{\tau} f^i(X_{s-1}, X_s) + u^i(X_\tau) + f^i(\xi, X_t) \,\middle|\, X_{t-1} = \xi \right) \\ &= \int_{\boldsymbol{N}(\xi)} \mathbb{E}\left(\sum_{s=t+1}^{\tau} f^i(X_{s-1}, X_s) + u^i(X_\tau) + f^i(\xi, \eta) \,\middle|\, X_t = \eta \right) \boldsymbol{Q}^{\boldsymbol{A}}(\xi, d\eta) \\ &= \int_{\boldsymbol{N}(\xi)} \left(\boldsymbol{C}_{\boldsymbol{A}}^{i\eta} + f^i(\xi, \eta) \right) \boldsymbol{Q}^{\boldsymbol{A}}(\xi, d\eta) \\ &= \int_{\boldsymbol{N}(\xi)} \left(\boldsymbol{C}_{\boldsymbol{A}}^{i\boldsymbol{q}(\xi,\eta)} + f^i(\xi, \boldsymbol{q}(\xi,\eta)) \right) \boldsymbol{A}(\xi, d\boldsymbol{y}). \end{aligned}$$

q.e.d.

We denote by $\mathcal{M}_0^i$ the set of probabilities on $\mathfrak{Y}^i$, i.e.,

$$\mathcal{M}_0^i := \left\{ \boldsymbol{a}^i \in \mathbb{R}^{\mathfrak{Y}^i} \,\middle|\, \boldsymbol{a}^i \geq \boldsymbol{0}, \ \sum_{w \in \mathfrak{Y}^i} a_w^i = 1 \right\}. \tag{2.3}$$

Now the following definition specifies the idea of backwards induction in the context of stochastic processes induced by behavioral strategies.

Definition 2.2. *A behavioral strategy (n–tuple) is said to be obtained by* ***backwards induction*** *or to be* ***subgame perfect*** *if there exists a family of functions* $\boldsymbol{v}^i : \mathcal{X} \to \mathbb{R} \quad (i \in \boldsymbol{I})$ *such that the following holds true.*

backwww (2.4)
$$\begin{aligned} \boldsymbol{v}^i(\xi) &= \overline{\boldsymbol{A}}(\xi,\bullet)\left\{\boldsymbol{v}^i(\boldsymbol{q}(\xi,\bullet)) + f^i(\xi,\boldsymbol{q}(\xi,\bullet))\right\} \\ &= \max_{\boldsymbol{a}^i \in \mathcal{M}_0^i} \boldsymbol{a}^i \otimes \overline{\boldsymbol{A}}^{(-i)}(\xi,\bullet)\left\{\boldsymbol{v}^i(\boldsymbol{q}(\xi,\bullet)) + f^i(\xi,\boldsymbol{q}(\xi,\bullet))\right\} \\ &\qquad (i \in \boldsymbol{I}, \xi \in \mathcal{X} \setminus \partial\mathcal{X}) \end{aligned}$$

randwww (2.5)
$$\boldsymbol{v}^i(\xi) = u^i(\xi) \quad (i \in \boldsymbol{I}, \xi \in \partial\mathcal{X})$$

Remark 2.3. *Let us denote by* $\Gamma^0(\xi)$ *the "one stage game at* ξ*" i.e., the game*

$$\Gamma^0(\xi) \;:=\; (\mathcal{M}_0^1, \ldots, \mathcal{M}_0^n; V^{1\xi}, \ldots, V^{n,\xi}) \tag{2.6}$$

with

$$V^{i\xi} \;:=\; \boldsymbol{v}^i(\boldsymbol{q}(\xi,\bullet)) + f^i(\xi,\boldsymbol{q}(\xi,\bullet))$$

for $x \in \mathcal{X} \setminus \partial\mathcal{X}$*. Note that* $V^{i\xi}$ *is a function on* $\mathcal{Y}$ *but can be extended to be a function on* $\mathcal{M}_0 \;:=\; \mathcal{M}_0^1 \times \ldots \times \mathcal{M}_0^n$.

Then equation (2.4) *shows that* $\overline{\boldsymbol{A}}(\xi,\bullet)$ *is an equilibrium in* $\Gamma^0(\xi)$*. Equation* (2.5) *is of course the boundary condition.*

Theorem 2.4. *Let* $\Sigma = (\mathcal{X}, \mathcal{Y}, \prec; \boldsymbol{q}; f, u)$ *be a dynamic game on a graph and suppose that* $\overline{\boldsymbol{A}}$ *is obtained by backwards induction.Then the following holds true:*

1. *For every* $\xi \in \mathcal{X}$, $\overline{\boldsymbol{A}}$ *is a Nash equilibrium in* $\Gamma_{\Sigma\xi}$.
2. *For every* $\xi \in \mathcal{X}$, $v^i(\xi) = C^{i\xi}_{\overline{\boldsymbol{A}}}$ *is the equilibrium payoff.*

Proof: As previously, we proceed by induction. Thus, given $\xi \in \mathcal{X}$, we have the following equations.

1^st^STEP :

$$\begin{aligned} \boldsymbol{v}^i(\xi) &= \overline{\boldsymbol{A}}(\xi,\bullet)\left\{\boldsymbol{v}^i(\boldsymbol{q}(\xi,\bullet)) + f^i(\xi,\boldsymbol{q}(\xi,\bullet))\right\} \\ &\qquad \text{(by definition of backwards induction)} \\ &= \overline{\boldsymbol{A}}(\xi,\bullet)\left\{C^{i\boldsymbol{q}(\xi,\bullet)}_{\overline{\boldsymbol{A}}} + f^i(\xi,\boldsymbol{q}(\xi,\bullet))\right\} \\ &\qquad \text{(by induction)} \\ &= C^{i\xi}_{\overline{\boldsymbol{A}}} \\ &\qquad \text{(by Lemma 2.1)} \end{aligned} \tag{2.7}$$

2^nd^STEP : Now, assume that player i deviates from his strategy $\overline{\boldsymbol{A}}^i$, hence switches to some alternative strategy $\widehat{\boldsymbol{A}}^i$. We have to check that he does not improve his payoff. Denote by $\widehat{\boldsymbol{A}} = (\overline{\boldsymbol{A}}^1, \dots \overline{\boldsymbol{A}}^{i-1}, \widehat{\boldsymbol{A}}^i, \overline{\boldsymbol{A}}^{i+1} \dots, \overline{\boldsymbol{A}}^n)$, then we have

$$
\begin{aligned}
C^{i\xi}_{\overline{\boldsymbol{A}}} &= \overline{\boldsymbol{A}}(\xi,\bullet)\left\{C^{i\boldsymbol{q}(\xi,\bullet)}_{\overline{\boldsymbol{A}}} + f^i(\xi,\boldsymbol{q}(\xi,\bullet))\right\} \\
&\qquad \text{(as above)} \\
&\geq \widehat{\boldsymbol{A}}(\xi,\bullet)\left\{C^{i\boldsymbol{q}(\xi,\bullet)}_{\overline{\boldsymbol{A}}} + f^i(\xi,\boldsymbol{q}(\xi,\bullet))\right\} \\
&\qquad \text{(by definition of backwards induction)} \\
&\geq \widehat{\boldsymbol{A}}(\xi,\bullet)\left\{C^{i\boldsymbol{q}(\xi,\bullet)}_{\widehat{\boldsymbol{A}}} + f^i(\xi,\boldsymbol{q}(\xi,\bullet))\right\} \\
&\qquad \text{(by induction)} \\
&= C^{i\xi}_{\widehat{\boldsymbol{A}}}\ . \\
&\qquad \text{(by Lemma 2.1)}
\end{aligned}
\tag{2.8}
$$

thus, $\overline{\boldsymbol{A}}$ is indeed an equilibrium strategy in $\Gamma_{\Sigma\xi}$. **q.e.d.**

Corollary 2.5. *For every dynamic game on a tree the normal form in behavioral strategies has subgame perfect Nash equilibria.*

Proof: Clearly, equation (2.5) defines the the functions $\boldsymbol{v}^i$ on the boundary $\partial\mathcal{X}$ of $\mathcal{X}$. By means of equation (2.4) one can define the $\boldsymbol{v}^i$ recursively. This is done via a mixed equilibrium in the game $\Gamma^0(\xi)$ which exists according to Nash's Theorem.

q.e.d.

Example 2.6. We return to Example 1.2 which was given by the following tree and the data as specified previously.

The one step game at node η is given by

$$\Gamma^0(\eta) \;=\; \left(\mathcal{M}^1_0, \mathcal{M}^2_0; V^{1\eta}, V^{2,\eta}\right)$$

with

$$V^{1,\eta} \;=\; u^1(\eta, \boldsymbol{q}(\eta,\bullet)) \;=\; \begin{array}{c} \\ o \\ u \end{array}\begin{array}{c} \begin{array}{cc} \ell & r \end{array} \\ \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \end{array}$$

and

$$V^{2\eta} \;=\; u^2(\eta, \boldsymbol{q}(\eta,\bullet)) \;=\; \begin{array}{c} \\ o \\ u \end{array}\begin{array}{c} \begin{array}{cc} \ell & r \end{array} \\ \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix} \end{array}$$

as pointed out already. It is seen at once that equilibrium strategies of this bimatrix game are given by $(\frac{1}{2}, \frac{1}{2})$ for each player. The payoff at this pair of strategies is $(\frac{1}{2}, -\frac{3}{2})$. Thus we put $\overline{\boldsymbol{A}}^1(\eta,\bullet) = \overline{\boldsymbol{A}}^2(\eta,\bullet) = (\frac{1}{2}, \frac{1}{2})$ and $\boldsymbol{v}(\eta) = (\boldsymbol{v}^1(\eta), \boldsymbol{v}^2(\eta)) = (\frac{1}{2}, -\frac{3}{2})$. The tree is augmented accordingly by attaching $\boldsymbol{v}(\eta)$ to the node η.

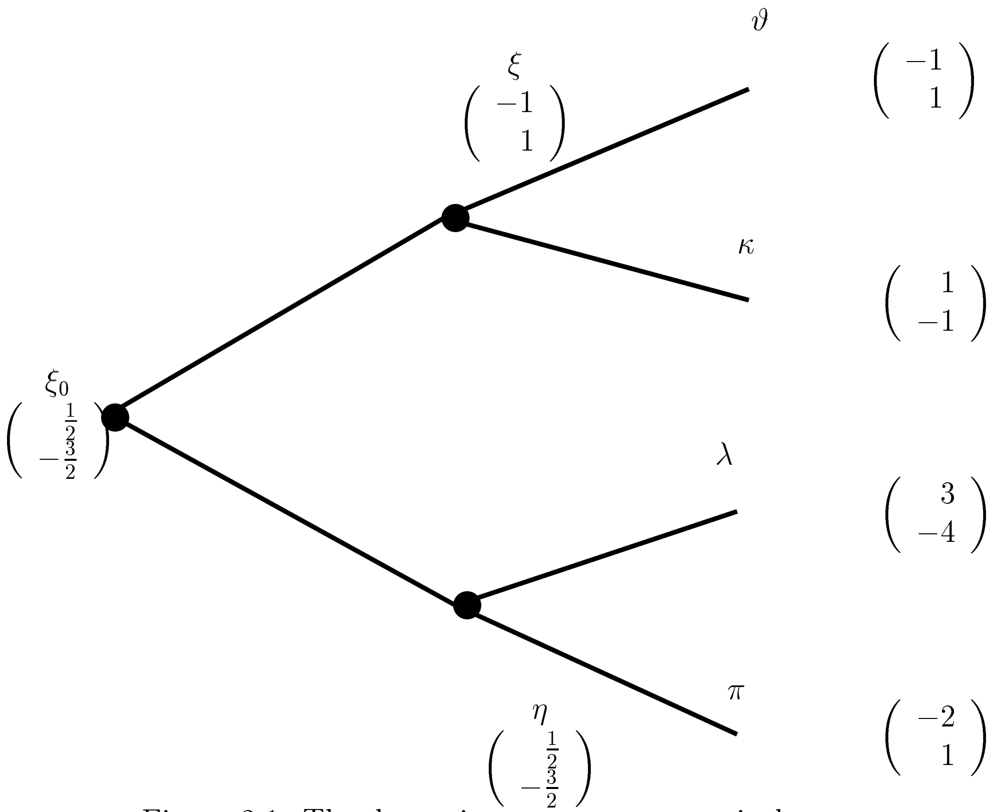

Figure 2.1: The dynamic game game recursively □

A similar reasoning leads to the valuefunction evaluated at ξ. Of course, at ξ it is obvious, that player 2 has the choice and will move to ϑ in order to get a unit. Formally we have

$$V^{1,\xi} \;=\; u^1(\xi, \boldsymbol{q}(\xi,\bullet)) \;=\; \begin{array}{c} \\ o \\ u \end{array} \begin{array}{c} \begin{array}{cc} \ell & r \end{array} \\ \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \end{array}$$

and

$$V^{2\xi} \;=\; u^2(\xi, \boldsymbol{q}(\eta,\bullet)) \;=\; \begin{array}{c} \\ o \\ u \end{array} \begin{array}{c} \begin{array}{cc} \ell & r \end{array} \\ \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \end{array}.$$

We conclude that player 2's equilibrium strategy is to choose ℓ with probability 1, written

$$\boldsymbol{A}^2(\xi,\bullet) = \delta_\ell = (1,0)$$

(the Dirac measure or unit mass concentrated at ℓ). Player 1's strategy at ξ is arbitrary. The value function at ξ is $\boldsymbol{v}(\xi) = (-1,1)$.

Finally, player 1 at node ξ_0 observes that he has complete control concerning the transition and that, by choosing u (or rather $\delta_u = (0,1)$) he will achieve $\frac{1}{2}$, hence his equilibrium strategy is δ_u and the value function at ξ_0 is $(\frac{1}{2}, -\frac{3}{2})$. Player 2's equilibrium strategy at ξ_0 is arbitrary.

Example 2.7. In passing we remark that any bimatrix game can bee senn as the normal form of a dynamic game using our present model. We restrict ourselves to an example.

Consider the following dynamic game on a tree.

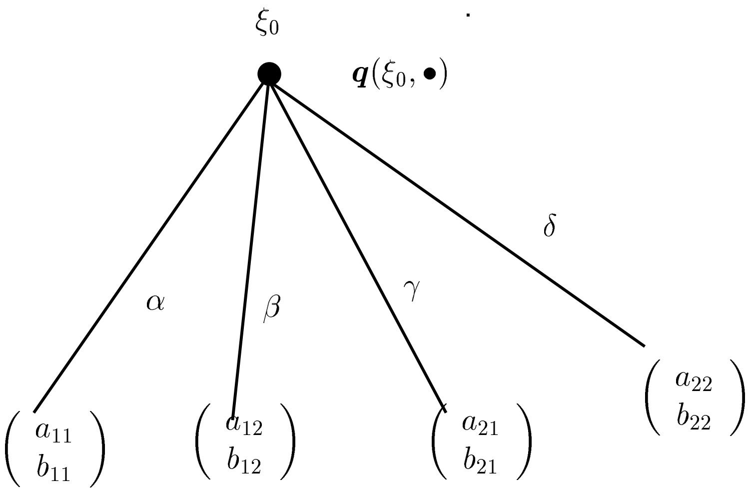

Figure 2.2: An dynamic game with a transition function

trans4baum*eps

The transition function is given via

$$\boldsymbol{q}(\xi_0, \bullet) = \begin{array}{c} \\ o \\ u \end{array} \begin{array}{c} \quad \ell \quad r \\ \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \end{array} .$$

and the normal form is

$$\boldsymbol{C}^{1\xi_0} = \begin{array}{c} \\ o \\ u \end{array} \begin{array}{c} \quad \ell \quad\quad r \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \end{array} , \quad \boldsymbol{C}^{2\xi_0} = \begin{array}{c} \\ o \\ u \end{array} \begin{array}{c} \quad \ell \quad\quad r \\ \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \end{array} .$$

3 Incomplete Information

treeipf

We consider a model with *imperfect information* of the players concerning the state of the game. Thus, players may not be informed about the current state of the process or the current node the play has been moved into. Nevertheless, the player in charge has to make a decision concerning the next node.

A consistent and precise model containing the exact notion of imperfect or incomplete information is an intricate device. To say that a player cannot distinguish between several nodes of his command is to say that he receives a signal depending on the nodes a process may acquire. The signal may be the same for everal nodes. Equivalently, we may describe a set of nodes resulting in the same signal, called an "information set".

In what follows we include a player named "chance " into the model. Hence, the player set is always meant to be $\boldsymbol{I}_0 := \boldsymbol{I} \cup \{0\}$. Our intuition is that player "chance" does not act strategically but ads to randomness: at some node it chooses a successor by a random device represented by probability attached to that node.

The further details are specified successively by a sequence of definitions. The first task is to descibe the extensive form.

treedef **Definition 3.1.** *A* ***dynamic game with i.i.*** *is a tuple*

$$\Sigma = (\mathcal{X}, \mathcal{Y}, \mathcal{K}, \prec; \boldsymbol{\iota}, \boldsymbol{\mu}^{\bullet}; \boldsymbol{q}, \boldsymbol{\kappa}; f, u) \tag{3.1}$$

the ingredients of which are defined as follows:

$(\mathcal{X}, \prec; \boldsymbol{\iota}; f, u)$ is a dynamic game in the sense of Definition **??**. In particular $\boldsymbol{\iota} : \mathcal{X} - \partial\mathcal{X} \to \boldsymbol{I}_0$ assigns every node $\xi \in \mathcal{X} - \partial\mathcal{X}$ to either some player $i \in \boldsymbol{I}$ or chance $(i = 0)$. As previously, we write

$$\mathcal{X}^i := \boldsymbol{\iota}^{-1}(\{i\}) = \{\xi \mid \boldsymbol{\iota}(\xi) = i\} \tag{3.2}$$

for the set of nodes that are controlled by player $i \in \boldsymbol{I}$. In particular, $\mathcal{X}^0$ is the set of ***chance nodes***. Recall that $(\mathcal{X}^i)_{i \in I_0}$ constitutes a partition which is referred to as the ***player partition***.

$\boldsymbol{\mu}^{\bullet} = (\boldsymbol{\mu}^{\xi})_{\xi \in \mathcal{X}^0}$ is a family of probabilities. To each chance node $\xi \in \mathcal{X}^0$ there is attached a probability $\boldsymbol{\mu}^{\xi}$ defined on the set $\mathbf{N}(\xi) = \{\eta \mid \xi \prec \eta\}$ of successors of ξ.

Intuitively, if the process of playing the game reaches ξ, then a random device represented by $\boldsymbol{\mu}^{\xi}$ chooses the successor of ξ, the probability for $\eta \in \mathbf{N}(\xi)$ is denoted by $\boldsymbol{\mu}^{\xi}_{\eta}$.

Next, the $\mathcal{K} = (\mathcal{K}^i)_{i \in \boldsymbol{I}}$ is a family of finite, nonempty sets; $\mathcal{K}^i$ is the set of ***signals*** to player $i \in \boldsymbol{I}$. Accordingly, $\boldsymbol{\kappa} = (\boldsymbol{\kappa}^i)_{i \in \boldsymbol{I}}$ is a family of mappings

$$\boldsymbol{\kappa}^i : \mathcal{X}^i \to \mathcal{K}^i. \tag{3.3}$$

Here $\mathcal{K}^i$ is a finite set, called the set of player i's ***signals***. A set

(3.4) $$\mathcal{X}^i_\kappa := (\boldsymbol{\kappa}^i)^{-1}(\kappa) \quad \{\kappa \in \mathcal{K}^i\}$$

is called an ***information set*** of player i and the system

(3.5) $$\underline{\underline{\mathcal{X}}}^i := \{\mathcal{X}^i_\kappa \mid \kappa \in \mathcal{K}^i\}$$

constitute a partition of $\mathcal{X}^i$, this system is called the ***information partition*** of player i.

Intuitively, whenever some $\xi \in \mathcal{X}^i$ occurs, player i is informed about the signal $\kappa \in \boldsymbol{\kappa}^i(\xi)$ only and not about the actual state. Thus, player i cannot distinguish between $\xi, \eta \in K^i_\kappa = (\boldsymbol{\kappa}^i)^{-1}(\kappa)$ as he receives the same signal κ. Consequently, he will have to specify his "strategic behavior" depending on his observed signals and not on the "true" states of the process.

In order to provide a well defined model, the mapping $\boldsymbol{\kappa}^0$ is assumed to be the identity, – "chance has full information".

Now, that we observe the information available to player i, we have to specify the procedure according to which this player chooses a successor, even though he does not know at which state exactly the process is located. In particular, if all states compatible with his information would have the same number of successors, then a player could just draw a number in order to specify the next node. This idea is captured by the transition function $\boldsymbol{q} = (\boldsymbol{q}(\xi, \bullet)_{\xi \in \mathcal{X} - \partial\mathcal{X}}$, which in the second variable contains actions.

Let $\mathcal{Y}^i_\kappa$ denote a finite nonempty set called the (feasible) ***actions*** of player i when he observes the signal $\kappa \in \mathcal{K}^i$ – so $\mathcal{Y}$ is the family of all the action sets.

We require that, for $\xi \in \mathcal{X}^i_\kappa$ there is the mapping

(3.6) $$\boldsymbol{q}(\xi, \bullet) : \mathcal{Y}^i_\kappa \to \mathbf{N}(\xi) \quad (\xi \in \mathcal{X}^i_\kappa)$$

which, given the "true state" ξ and an action y of player i chooses a successor of ξ. The image $\boldsymbol{q}(\xi, y)$specifies exactly one successor $\eta \in \mathbf{N}(\xi)$, which is thought of to be the next state resulting from player i's action.

Frequently it is assumed that each $\boldsymbol{q}(\xi, \bullet)$ maps the actions in $\mathcal{Y}^i_\kappa$ ***bijectively*** onto the successors of ξ – *simultaneously for all* $\xi \in \mathcal{X}^i_k$. In this case, all $\xi \in \mathcal{X}^i_\kappa = (\boldsymbol{\kappa}^i)^{-1}(\kappa)$ must necessarily have the same number of successors, this number is

$$\mid \mathbf{N}(\xi) \mid = \mid \mathcal{Y}^i_\kappa \mid \quad (\xi \in (\boldsymbol{\kappa}^i)^{-1}(\kappa)).$$

∘ ∼∼∼∼∼∼ ∘

The sketch in Figure 3.1 represents the typical situation at the information set $\mathcal{X}^i_\kappa$ of player i.

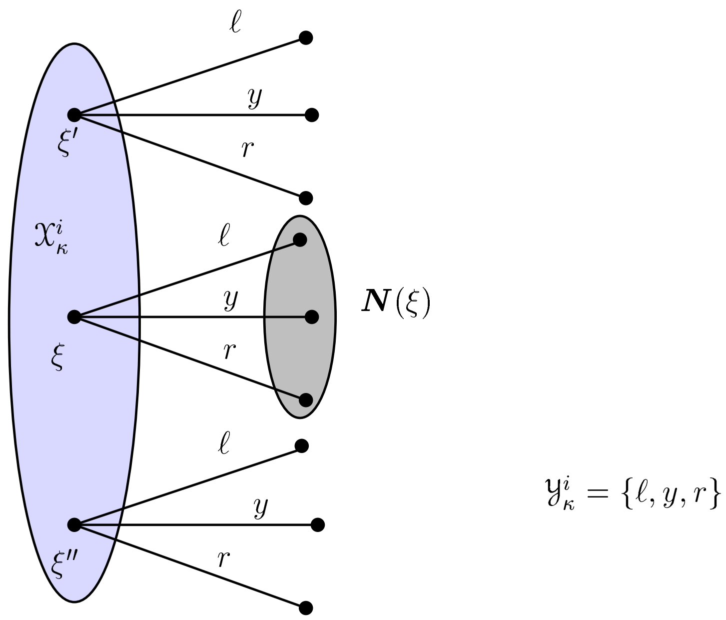

Figure 3.1: An information set of player i

localtree*eps

In this context, then the set

$$C_y \;:=\; \{\boldsymbol{q}(\xi, y) \,|\, \xi \in \mathcal{X}^i_\kappa\} \tag{3.7}$$

of all successors that are chosen by applying some $y \in \mathcal{Y}^i_\kappa$ is called a ***choice***. All successors in some C_y are "identified" in a straightforward manner. These are what player i observes to be the consequences of choosing $y \in \mathcal{Y}^i_\kappa$. Clearly, all choices have the same cardinality which is

$$|\, C_y \,| = |\; \mathcal{X}^i_\kappa \,| \;. \tag{3.8}$$

Figure 3.2 depicts some essentials. It is common to represent choices C_y by attaching the action y to those successor which are identified by their common value of the transition function function $\boldsymbol{q}$. I.e., we write r to all elements of C_r indicating that these successors will result from their predecessor if the player in charge chooses "right"...

We will now describe various types of **strategic behavior** , i.e., classes of ***strategies*** for a player. Accordingly, there will be the corresponding "normal forms" depending on which kind of strategies we focus upon.

Player i is not capable of observing the states $\xi \in \mathcal{X}$. Indeed, his observation is restricted to the elements of $\mathcal{K}^i$. We postulate that by therefore, player i is forced to restrict his behavior on strategies depending on $\mathcal{K}^i$ at most.

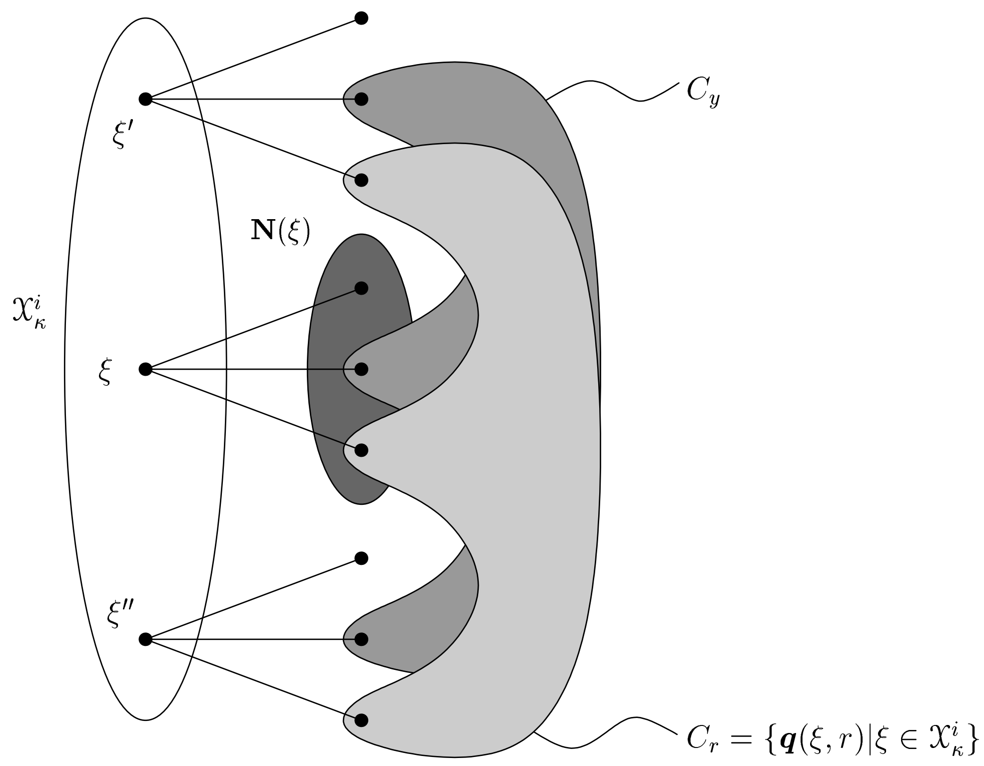

Figure 3.2: An information set and choices tree0*eps

treepure **Definition 3.2.** *Let Σ be a tree game. A **pure strategy** for player i is a mapping*

treepuref (3.9) $$\boldsymbol{\alpha}^i : \mathcal{K}^i \to \bigcup_{\kappa \in \mathcal{K}^i} \mathcal{Y}^i_\kappa$$

such that

treepureff (3.10) $$\boldsymbol{\alpha}^i(\kappa) \in \mathcal{Y}^i_\kappa \quad (\kappa \in \mathcal{K}^i)$$

holds true. We denote by

treepured (3.11) $$\mathfrak{S}^i := \{\boldsymbol{\alpha}^i \mid \boldsymbol{\alpha}^i \text{ satisfies } (3.9) \text{ and } (3.10)\}$$

the set of pure strategies for player i.

The mapping $\boldsymbol{\alpha}^i$ may also be written $\boldsymbol{\alpha}^i = (\boldsymbol{\alpha}^i(\kappa))_{\kappa \in \mathcal{K}^i}$; we prefer this version within the next definition.

treevhs **Definition 3.3.** *Let Σ be a tree game. A **behavioral strategy (b.s.)** for player i is a family*

$$\boldsymbol{A}^i = (\boldsymbol{A}^i(\kappa, \bullet))_{\kappa \in \mathcal{K}^i}$$

such that, for every $\kappa \in \mathcal{K}^i$

treevhsff (3.12) $$\boldsymbol{A}^i(\kappa, \bullet) \text{ is a probability on } \mathcal{Y}^i_\kappa.$$

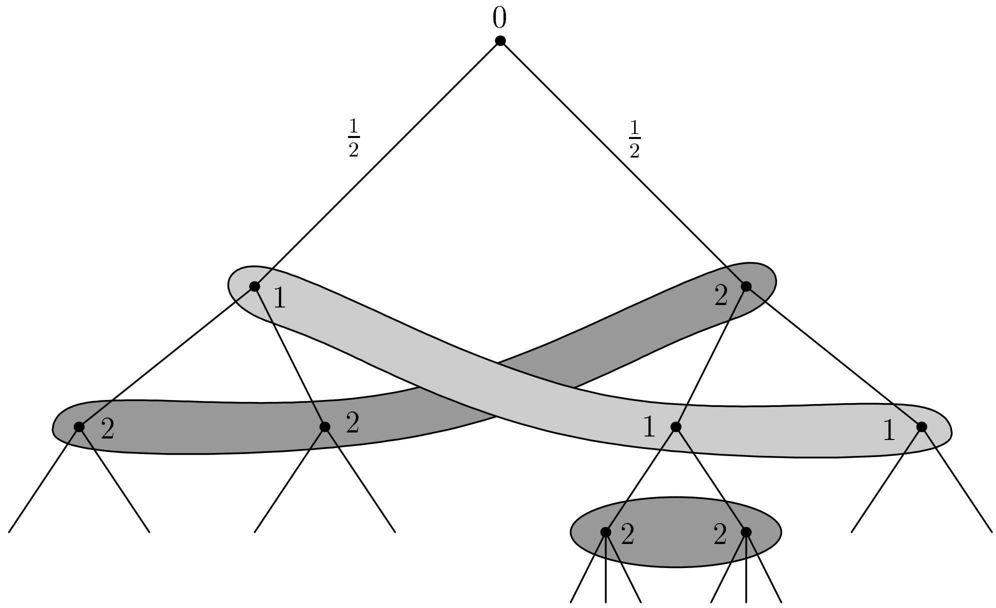

Figure 3.3: An Example with I.I. f-tree2c*eps

That is, at each information κ player i selects a random device over the actions available to him.

treevhsd

$$\mathfrak{A}^i := \{\boldsymbol{A}^i \mid \boldsymbol{A}^i \ \ \textit{satisfies } (3.12)\} \tag{3.13}$$

is the set of behavioral strategies of player i.

Finally there is the concept of a mixed strategy which reflects the idea of randomizing over the pure strategies.

treemix **Definition 3.4.** *Let Σ be a tree game and let $\mathfrak{S}^i$ denote the pure strategies of player i. A* ***mixed strategy*** *for player i is a probability $\boldsymbol{M}^i$ on $\mathfrak{S}^i$.*

treemixd

$$\mathfrak{M}^i = \{\boldsymbol{M}^i \mid \boldsymbol{M}^i \ \textit{is a mixed strategy for player } i\} \tag{3.14}$$

denotes the corresponding set.

Our standard notation assumes that the Cartesian product is taken if we omit the upper index i, thus

$$\begin{aligned} \mathfrak{S} &= \mathfrak{S}^1 \times \cdots \times \mathfrak{S}^n \\ \mathfrak{A} &= \mathfrak{A}^1 \times \cdots \times \mathfrak{A}^n \\ \mathfrak{M} &= \mathfrak{M}^1 \times \cdots \times \mathfrak{M}^n \end{aligned}$$

denotes sets of strategy n-tuples.

treelist

Remark 3.5.

treeeininfo

1. *If a player i has just one information set, then mixed strategies and behavioral strategies coincide, that is we have $\mathfrak{M}^i = \mathfrak{A}^i$.*

treeimbed

2. *We may regard pure strategies as a particular type of behavioral ones as well as of mixed ones; in other words, pure strategies can be embedded into behavioral strategies (as well as into mixed strategies).*

   For, if $\boldsymbol{\alpha}^i$ is a pure strategy then the corresponding behavioral strategy $\boldsymbol{A}^{i,\boldsymbol{\alpha}^i}$ is defined by

$$\boldsymbol{A}^i(\kappa, \bullet) = \delta_{\boldsymbol{\alpha}^i(\kappa)}(\bullet) = \boldsymbol{e}^{\boldsymbol{\alpha}^i(\kappa)}$$

   This is a behavioral strategy which puts probability 1 on the action provided by $\boldsymbol{\alpha}^i(q)$. In this sense we can now state that $\mathfrak{S}^i \subseteq \mathfrak{A}^i$ or $\mathfrak{S} \subseteq \mathfrak{A}$ holds true.

   The embedding of pure strategies into mixed strategies is also the canonical one; we do skip the discussion of the details.

3. *Therefore, if we discuss the distribution induced by behavioral strategies on the paths of a tree, then the distribution induced by pure strategies is as well defined. Note that the latter distribution is not necessarily $\delta-$shaped, as chance moves may be incorporated in the structure of the game tree.*

4. *As it turns out mixed strategies eventually can be disposed of; it is sufficient to consider behavioral strategies. The background is provided by* KUHN*'s Theorem (cf. Theorem* **??***), which however, in the present context requires the additional assumption of perfect recall. To develop the details will be the aim of this section.*

5. *Finally we would like to note that the actions chosen by chance (by player 0) can also be viewed as a behavioral strategy (of player 0). More precisely, if we consider the family of distributions provided by $\mu^\bullet$ and keep in mind that the information partition of $\mathfrak{X}^0$ is the one into singletons, i.e.*

$$\mathfrak{X}^0 = \sum_{\xi \in \mathfrak{X}^0} \{\xi\},$$

   then for $\kappa = \xi \in \mathfrak{X}^0$ the notation

$$\boldsymbol{A}^0(\kappa, \bullet) = \mu^\xi_\bullet$$

   obviously defines a behavioral strategy for the chance player.

distribA

Definition 3.6. *Let Σ be a tree game and let*

$$\boldsymbol{A} = (\boldsymbol{A}^i)_{i \in I}$$

be an $n-$tuple of behavioral strategies . Then a probability $\boldsymbol{m}^{\boldsymbol{A}}_{\boldsymbol{\mu}}$ in $\underline{\overline{\mathbf{X}}}$ (involving the chance distributions $\boldsymbol{\mu}^\bullet$) is provided as follows.

1. *Define transition probabilities via*

treeq (3.15)

$$\begin{aligned}
\boldsymbol{\mu}^\xi_\eta \;=\; \boldsymbol{\mu}^{\boldsymbol{A}\xi}_\eta \;:=&\quad \boldsymbol{A}^{\iota(\xi)}(\boldsymbol{\kappa}^{\boldsymbol{\iota}(\xi)}, \{y | \boldsymbol{q}(\xi, y) = \eta\}) \\
=&\quad \boldsymbol{A}^{\iota(\xi)}(\boldsymbol{\kappa}^{\boldsymbol{\iota}(\xi)}, \bullet) \circ \boldsymbol{q}(\xi, \bullet)^{-1}(\{\eta\}) \\
=&\quad \left\{ \boldsymbol{q}(\xi, \bullet) \boldsymbol{A}^{\iota(\xi)}(\boldsymbol{\kappa}^{\boldsymbol{\iota}(\xi)}, \bullet) \right\} (\{\eta\}) \quad (\xi \in \mathfrak{X} \setminus \partial\mathfrak{X}, \eta \in \mathbf{N}(\xi))
\end{aligned}$$

   in particular, we have $\boldsymbol{\mu}^\xi_\eta = \boldsymbol{A}^0(\{\xi\}, \{\eta\})$ if $\xi \in \mathfrak{X}^0$ is a chance move.

2. *Define the distribution of a node seizure* Z *via*

$$\mathbb{P}(Z_\eta = 1 | Z_\xi = 1) = \boldsymbol{\mu}_\eta^\xi \quad (\xi \prec \eta) \tag{3.16}$$

3. *Equivalently, for any play* $x = (x_0, \ldots, x_T)$ *let*

treedis
$$\boldsymbol{m}_{\boldsymbol{\mu}}^{\boldsymbol{A}}(\{x\}) := \boldsymbol{\mu}_{x_1}^{x_0} \quad \boldsymbol{\mu}_{x_2}^{x_1} \ldots \boldsymbol{\mu}_{x_T}^{x_{T-1}}, \tag{3.17}$$

and define a process $X : \Omega \to \underline{\overline{\mathbf{X}}}$ *to be* ***controlled*** *by* $\boldsymbol{A}$ *if* X *has distribution* $\boldsymbol{m}_{\boldsymbol{\mu}}^{\boldsymbol{A}}$.

4. *The* ***payoff*** *to player* i *given* $\boldsymbol{A}$ *is*

treepayx
$$C_{\boldsymbol{A}}^{i\boldsymbol{\mu}} := \int\limits_{\underline{\overline{\mathbf{X}}}} \left(\sum_{t=1}^{\tau_0} f^i(x_{t-1} x_t) + u^i(x_{\tau_0}) \right) d\boldsymbol{m}_{\boldsymbol{\mu}}^{\boldsymbol{A}}(x). \tag{3.18}$$

here $\tau_0 : \underline{\overline{\mathbf{X}}} \to \mathbb{N}$ *is the length of the path, i.e., defined by* $\tau_0(x) = T$ *if* $x = (x_0, \ldots, x_T)$.

The payoff to player i resulting from A can also computed in terms of a process X with distribution $\boldsymbol{m}_{\boldsymbol{\mu}}^{\boldsymbol{A}}$. We can write

treepayoff
$$\begin{aligned} C_{\boldsymbol{A}}^{i\boldsymbol{\mu}} &:= \mathbb{E} \left(\sum_{t=1}^{\boldsymbol{\tau}} f^i(X_{t-1}, X_t) + u^i(X_\tau) \right) \\ &= \int \left(\sum_{t=1}^{\tau} f^i(x_{t-1}, x_t) + u^i(x_\tau) \right) d\boldsymbol{m}_{\boldsymbol{\mu}}^{\boldsymbol{A}}(x) \end{aligned} \tag{3.19}$$

Here we have used $\boldsymbol{\tau} = \tau_0 \circ X$ to indicate the *random* length of a play. Not unusual, the argument is provided by introducing the ***evaluation***

treeeval
$$\begin{aligned} C^i &: \underline{\overline{\mathbf{X}}} \to \mathbb{R} \\ C^i(x) &= \sum_{t=1}^{\tau(x)} f^i(x_{t-1}, x_t) + u^i(x_{\tau_0(x)}) \end{aligned} \tag{3.20}$$

Since $\tau = \tau_0 \circ X$ it is seen that (3.19) defines $C_{\boldsymbol{A}}^{i\boldsymbol{\mu}}$ via $C_{\boldsymbol{A}}^{i\boldsymbol{\mu}} = E\ C^i \circ X$ while (3.18) claims that $C_{\boldsymbol{A}}^{i\boldsymbol{\mu}} = \int_{\underline{\overline{\mathbf{X}}}} C^i(x) d\boldsymbol{m}_{\boldsymbol{\mu}}^{\boldsymbol{A}}(x)$ holds true. The claim is correct by the formula for "transformation of variables" since $\boldsymbol{m}_{\boldsymbol{\mu}}^{\boldsymbol{A}}$ is the distribution of X and hence

$$\mathbb{E}\ C^i \circ X = \int\limits_{\Omega} (C^i \circ X) d\mathbb{P}$$

$$= \int\limits_{\underline{\overline{\mathbf{X}}}} C^i d(X\mathbb{P}) = \int C^i d\boldsymbol{m}_{\boldsymbol{\mu}}^{\boldsymbol{A}}$$

treepmix

Remark 3.7.

1. *As we have already remarked a pure strategy is a special behavioral strategy. Therefore the induced distribution called m^α_μ is a well defined quantity.*
2. *It should be clear that for a game tree without chance moves the distribution induced by a pure strategy α is a point measure concentrated on some path $x^\alpha \in \underline{\underline{\mathbf{X}}}$.*

We are now in the position to define the normal form corresponding to behavioral strategies.

treenfv **Definition 3.8.** *Let Σ be a tree game and let $\mathfrak{A}^1, \ldots, \mathfrak{A}^n$ denote the sets of behavioral strategies of the players. Define*

$$C^{i\boldsymbol{\mu}}_\bullet : \mathfrak{A}^1 \times \cdots \times \mathfrak{A}^n \to \mathbb{R}$$

by either (3.19) or (3.18). The **normal form** *game induced by Σ (in behavioral strategies) is the (noncooperative) n-person game*

$$\Gamma = \Gamma_{\Sigma^{\boldsymbol{\mu}}} := (\mathfrak{A}^1, \ldots, \mathfrak{A}^n;\ C^{1\boldsymbol{\mu}}_\bullet, \ldots, C^{n\boldsymbol{\mu}}_\bullet)$$

Pure strategies can be seen as a particular case of behavioral strategies via the embedding

$$\alpha \to \boldsymbol{A}^\alpha$$

that is described in (3.5). Hence, if $\mathfrak{S}^i$ is the space of pure strategies of player i and if $C^{i\boldsymbol{\mu}}_\alpha$ is given by $C^{i\boldsymbol{\mu}}_{\boldsymbol{A}^\alpha}$, then we have an immediate definition of the normal form in pure strategies induced by Σ.

Definition 3.9. *Let Σ be a tree game and let $\mathfrak{S}^1, \ldots, \mathfrak{S}^n$ denote the sets of pure strategies of the players. Define*

$$C^{i\boldsymbol{\mu}}_\bullet : \mathfrak{S}^1 \times \cdots \times \mathfrak{S}^n \to \mathbb{R}$$

by $C^{i\boldsymbol{\mu}}_\alpha := C^{i\boldsymbol{\mu}}_{\boldsymbol{A}^\alpha}$ $(\alpha \in \mathfrak{S})$. Then the **normal form** *game induced by Σ (in pure strategies) is the (noncooperative) n-person game*

treepurenf (3.21) $$\tilde{\Gamma} = \tilde{\Gamma}_{\Sigma\boldsymbol{\mu}} = (\mathfrak{S}^1, \ldots, \mathfrak{S}^n; C^{1\boldsymbol{\mu}}_\bullet, \ldots, C^{n\boldsymbol{\mu}}_\bullet).$$

Example 3.10. The first example reflects a 2–Person game. We consider both alternatives. In the original version, players have complete information, they observe the actual state the process is located in.

In the second version player 2 cannot distinguish between the two nodes he owns. He receives the same signal, thus he has to base his decision on just this observation. Observe that the normal form is reduced as a consequence of “fewer” actions available to him, which results in “fewer” pure stratgies as well.

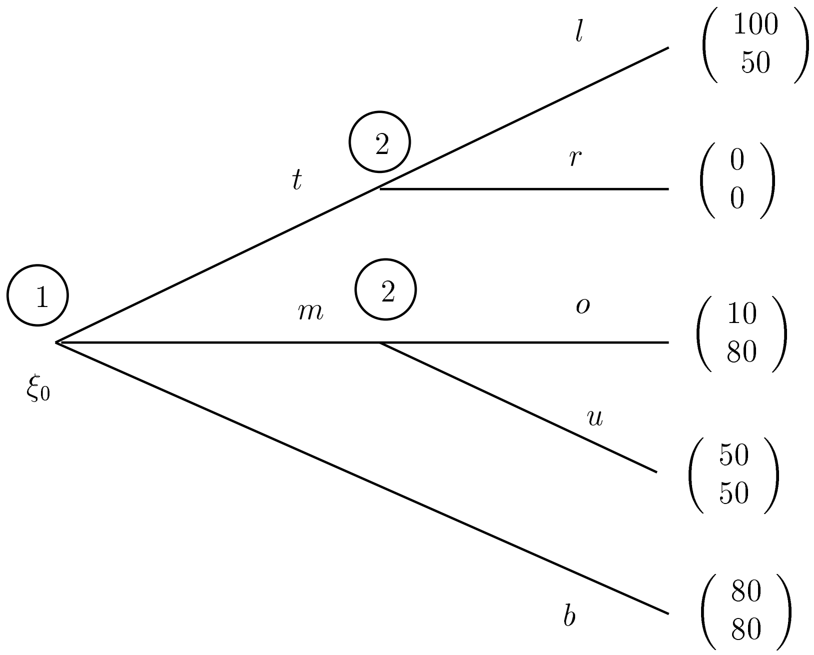

Figure 3.4: A dynamic game with complete information

neubaum*eps

The normal form is given by

$$\boldsymbol{C}^{1\xi_0} = \begin{array}{c} \\ t \\ m \\ b \end{array} \begin{array}{c} \begin{array}{cccc} lo & lu & ro & ru \end{array} \\ \begin{pmatrix} 100 & 100 & 0 & 0 \\ 10 & 50 & 10 & 50 \\ 80 & 80 & 80 & 80 \end{pmatrix} \end{array}$$

and

$$\boldsymbol{C}^{2\xi_0} = \begin{array}{c} \\ t \\ m \\ b \end{array} \begin{array}{c} \begin{array}{cccc} lo & lu & ro & ru \end{array} \\ \begin{pmatrix} 50 & 50 & 0 & 0 \\ 80 & 50 & 80 & 50 \\ 80 & 80 & 80 & 80 \end{pmatrix} \end{array}$$

Now we restrict the information for player 2:

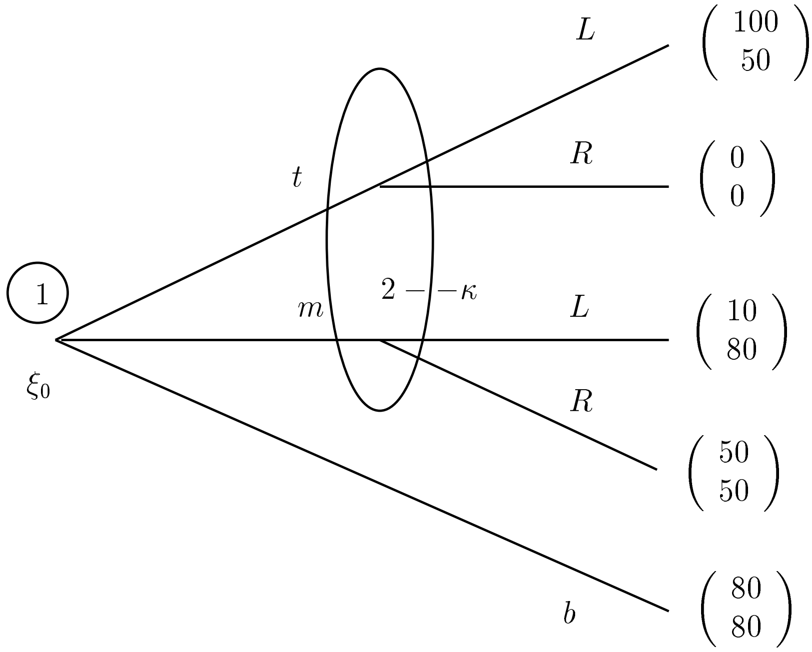

Figure 3.5: Restricting the information of player 2

neubaum2*eps

Now the normal form is

$$
\boldsymbol{C}^{1\xi_0} = \begin{array}{c} \\ t \\ m \\ b \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 100 & 0 \\ 10 & 50 \\ 80 & 80 \end{pmatrix} \end{array}
$$

and

$$
\boldsymbol{C}^{2\xi_0} = \begin{array}{c} \\ t \\ m \\ b \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 50 & 0 \\ 80 & 50 \\ 80 & 80 \end{pmatrix} \end{array}
$$

Example 3.11. The following example demonstrates that an arbitrary bimatrix game can be the normal form of a dynamic game with incomplete information. We rextrict the demonstration on a 2×2–bimatrix game, but the generalization to any nomal form is not too difficult. We choose arbitrary matrices

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

and the dynamic game represented by the tree as follows.

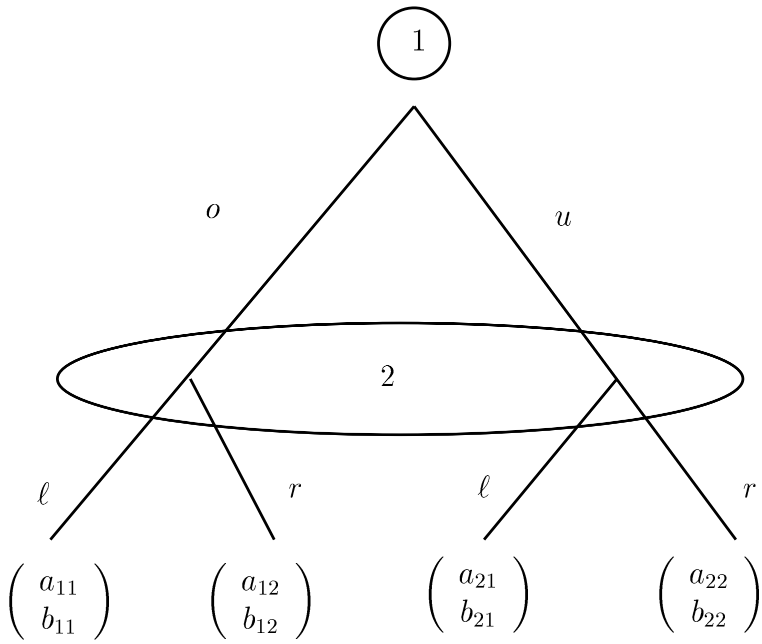

Figure 3.6: An extensive form that yields the two matrices as normal form allbims*eps

Then it is seen at once that normal form of this tree game is given by the bimatrix game induced ba the two matrices.

The representation is by no means unique. the follwing extensive form (in which players 1 and 2 have just exchanged their roles) yields the same normal form.

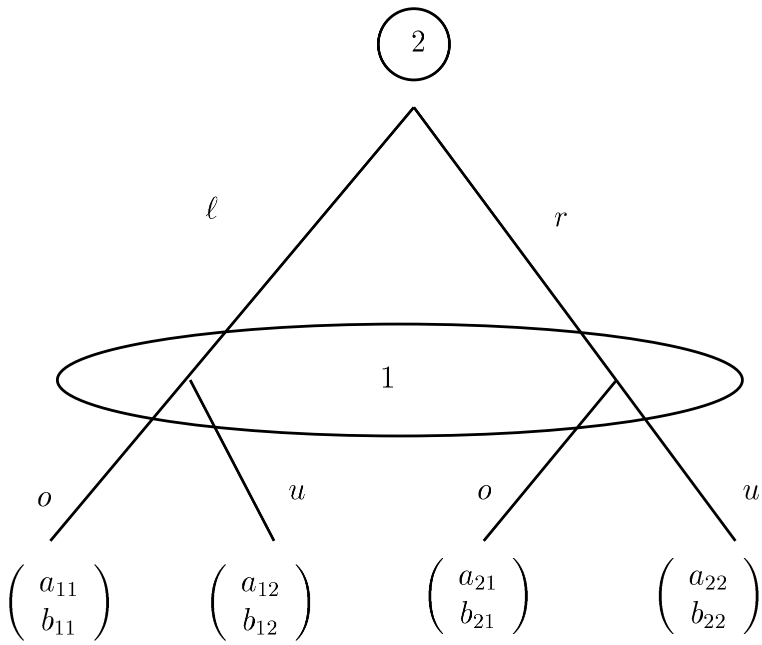

Figure 3.7: An extensive form that yields the two matrices as normal form `allbi*eps`

The model developed in the previous section involves transition functions at certain nodes that determine the next step of a process given the actions of all players – there is no player assignment. However, similarly to the above examples, we can show that the game may as well be represented in the present context by a game wih incomplete information. More precisely, we construct an I.I. game with the same normal form in pure strategies such that the actions are not performed simultaneously but successively – yet without information on moves of the player acting first.

As previously, we restrict ourselves to a simple example, which can be generalized at will.

Example 3.12. Consider the following dynamic game; the transition function at ξ_0 is represented by the matrix

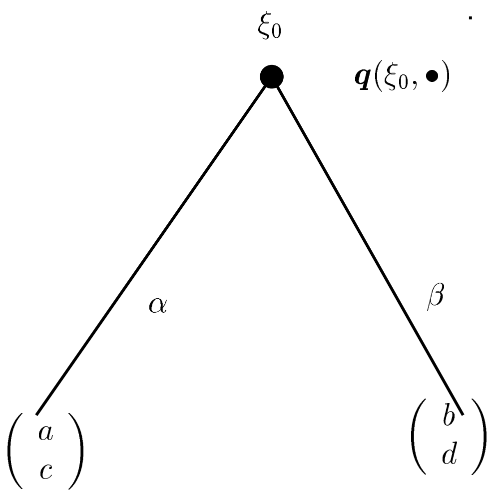

Figure 3.8: An dynamic game with a transition function `transbaum*eps`

$$\boldsymbol{q}(\xi_0, \bullet) = \begin{array}{c} \\ o \\ u \end{array} \begin{array}{c} \ell \quad r \\ \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \end{array} .$$

The normal form is

$$\boldsymbol{C}^{1\xi_0} = \begin{array}{c} \\ o \\ u \end{array} \begin{array}{c} \ell \quad r \\ \begin{pmatrix} a & b \\ b & a \end{pmatrix} \end{array} , \quad \boldsymbol{C}^{2\xi_0} = \begin{array}{c} \\ o \\ u \end{array} \begin{array}{c} \ell \quad r \\ \begin{pmatrix} c & d \\ d & c \end{pmatrix} \end{array}$$

The following I.I. game yields the same normal form:

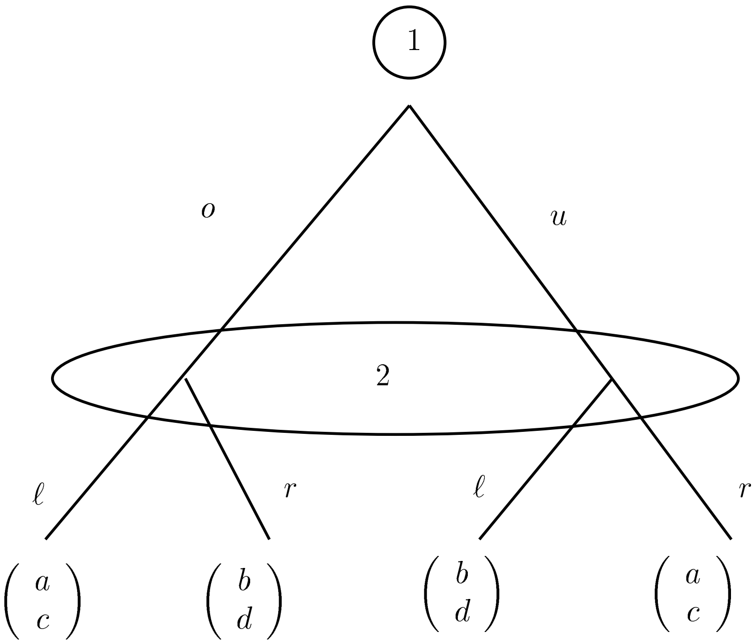

Figure 3.9: An extensive I.I. game with the same normal form transbaum2*eps

Obviously, any bimatrix game can be seen as the normal form

Now we turn to the normal form to be derived from a dynamic game (an extensive form). The first approach to the normal form game *in mixed strategies* is based on the concept of the mixed extension: a mixed strategy $\boldsymbol{M}^i$ for player i is a probability on $\mathfrak{S}^i$ $(i \in I)$. If

(3.22) $$\boldsymbol{M} = \boldsymbol{M}^1 \otimes \cdots \otimes \boldsymbol{M}^n$$

denotes the stochastically independent mixture of all players' mixed strategies, then the naive concept for the payoff results in the corresponding expectation. That is, we have

treemixex (3.23) $$C^{i\boldsymbol{\mu}}_{\boldsymbol{M}} := \int_{\mathfrak{S}} C^{i\boldsymbol{\mu}}_{\alpha} d\boldsymbol{M}\ (\alpha).$$

On the other hand, there should be the concept of a process resulting from the application of $\boldsymbol{M}$ or controlled by $\boldsymbol{M}$.

In order to formalize this idea, the first task is to define the distribution induced by a mixed strategy. To this end we naturally attempt to define the mixture of the distributions that result from pure strategies.

We have seen in Remark (3.7) that a pure strategy α results in a distribution $\boldsymbol{m}^{\alpha}_{\boldsymbol{\mu}}$. Mixing these distributions is formally described as follows.

mixkern **Definition 3.13.** *Let Σ be a tree game (with chance distributions $\boldsymbol{\mu}^{\bullet}$). Define a transition kernel*

$$\boldsymbol{K}^{\boldsymbol{\mu}} \mid \mathfrak{S} \Rightarrow \underline{\overline{\mathbf{X}}}$$

by

$$\boldsymbol{K}^{\boldsymbol{\mu}}(\alpha, \bullet) \;=\; \boldsymbol{m}^{\alpha}_{\boldsymbol{\mu}} \quad (\alpha \in \mathfrak{S}). \tag{3.24}$$

Also, let $\boldsymbol{M} \in \mathfrak{M}$ be a mixed strategy and $X : \Omega \to \underline{\overline{\mathbf{X}}}$ a process. Then X is **controlled** *by $\boldsymbol{M}$ (or results from the application of $\boldsymbol{M}$) if the distribution of X is given by*

$$X\mathbb{P} \;=\; \boldsymbol{K}^{\boldsymbol{\mu}}\boldsymbol{M} \;=\; \int_{\mathfrak{S}} \boldsymbol{m}^{\alpha}_{\boldsymbol{\mu}}(\bullet)\boldsymbol{M}(d\alpha). \tag{3.25}$$

If we adopt a process distributed according to $\boldsymbol{K}^{\boldsymbol{\mu}}\boldsymbol{M}$ as to "be controlled by $\boldsymbol{M}$", then corresponding payoff should be given by

treemixpay
$$\begin{aligned} C^{i\boldsymbol{\mu}}_{\boldsymbol{M}} =& \mathbb{E}\left(\sum_{t=1}^{\tau} f^i(X_{t-1}, X_t) + u^i(X_\tau)\right) \\ =& \int_{\underline{\overline{\mathbf{X}}}} C^i(x)(\boldsymbol{K}^{\boldsymbol{\mu}}M)(dx) \\ =& \int_{\underline{\overline{\mathbf{X}}}} C^i d(\boldsymbol{K}^{\boldsymbol{\mu}}\boldsymbol{M}), \end{aligned} \tag{3.26}$$

where C^i is the "evaluation" defined in (3.20); compare the corresponding formula for the behavioral strategies which is (3.19). We hasten to show that (3.26) does not contradict (3.23), for we prove

treemixpx **Theorem 3.14.** *If Σ is a tree game and $\boldsymbol{M}$ a mixed strategy, then*

$$\int_{\mathfrak{S}} C^{i\boldsymbol{\mu}}_{\alpha}\boldsymbol{M}(d\alpha) = \int_{\underline{\overline{\mathbf{X}}}} C^i(x)(\boldsymbol{K}^{\boldsymbol{\mu}}\boldsymbol{M})(dx)$$

Proof: The first step is to prove that

$$\begin{aligned} C^{i\boldsymbol{\mu}}_{\alpha} =& \int_{\underline{\overline{\mathbf{X}}}} C^i(x)\boldsymbol{m}^{\alpha}_{\boldsymbol{\mu}}(dx) \\ =& \int_{\underline{\overline{\mathbf{X}}}} C^i(x)\boldsymbol{K}^{\boldsymbol{\mu}}(\alpha, dx) \\ =& (\boldsymbol{K}^{\boldsymbol{\mu}}C^i)\;(\alpha)\;; \end{aligned}$$

holds true. The claim follows immediately in a second step which shows that

$$\begin{aligned} C^{i\boldsymbol{\mu}}_{\boldsymbol{M}} &= \int\limits_{\mathfrak{S}} C^{i\boldsymbol{\mu}}_{\alpha} \boldsymbol{M}(d\alpha) \\ &= \int\limits_{\mathfrak{S}} (\boldsymbol{K}^{\boldsymbol{\mu}} C^i)(\alpha) \boldsymbol{M}(d\alpha) \\ &= \int\limits_{\underline{\overline{\mathbf{X}}}} C^i(x)(\boldsymbol{K}^{\boldsymbol{\mu}}\boldsymbol{M})(dx) \quad . \end{aligned}$$

The last equation is due to the familiar formula for the transformation of variables. **q.e.d.**

We have now obtained two consistent intuitive ways of regarding a mixed strategy; we may either regard the mixture over the pure strategies and the corresponding expectations or we may consider the mixture over the distributions resulting from pure strategies and take the payoff from a process distributed accordingly. Both versions lead to the same payoffs. Hence we have a non-contradictory definition of the normal form in mixed strategies.

Definition 3.15. *Let Σ be a tree game and let $\mathfrak{M}^1, \ldots, \mathfrak{M}^n$ denote the sets of mixed strategies of the players. Define*

$$C^{i\boldsymbol{\mu}}_{\bullet} : \mathfrak{M}^1 \times \cdots \times \mathfrak{M}^n \to \mathbb{R}$$

by (3.23) or (3.26) Then the **normal form** *game induced by Σ (in mixed strategies) is the (noncooperative) n-person game*

treemixexten (3.27) $$\bar{\Gamma} = \bar{\Gamma}_{\Sigma\boldsymbol{\mu}} = (\mathfrak{M}^1, \ldots, \mathfrak{M}^n; C^{1\boldsymbol{\mu}}_{\bullet}, \ldots, C^{n\boldsymbol{\mu}}_{\bullet}),$$

which is also the mixed extension of $\tilde{\Gamma}_{\Sigma\boldsymbol{\mu}}$.

Before we compare behavioral and mixed strategies we have to consider the relation between a fixed path or play and a pure strategy; clearly not all pure strategies are compatible with a given path.

treesup **Definition 3.16.** *Let $\bar{\alpha}^i \in \mathfrak{S}^i$ be a pure strategy for player i and let $\bar{x} = (x_0, \ldots, x_T)$ be a path. We shall say that $\bar{\alpha}^i$* ***supports*** *$\bar{x}$ if, for all $t = 1, \ldots, T$ and $\bar{x}_{t-1} \in \mathfrak{X}^i_\kappa$, it follows that*

(3.28) $$\boldsymbol{q}(\bar{x}_{t-1}, \bar{\alpha}^i(\kappa)) = \bar{x}_t \ .$$

That is, if player i at $\bar{x}_{t-1}$ observes $\kappa = \boldsymbol{\kappa}^i(\bar{x}_{t-1})$ and decides for some $y = \bar{\alpha}^i(\kappa)$, then $\bar{x}_t$ is the uniquely defined next node resulting from this decision.

treebem3 **Remark 3.17.** *Generally the distribution induced by a pure strategy on the paths of the tree can now be represented as follows:*

(3.29) $$m^{\alpha}_{\mu}(\{x\}) := \left\{ \begin{array}{ll} \prod_{\{t | x_{t-1} \in \mathfrak{X}^0\}} \mu^{x_{t-1}}_{x_t} & \textit{if } \alpha \textit{ supports } x \\ 0 & \textit{otherwise} \end{array} \right\}.$$

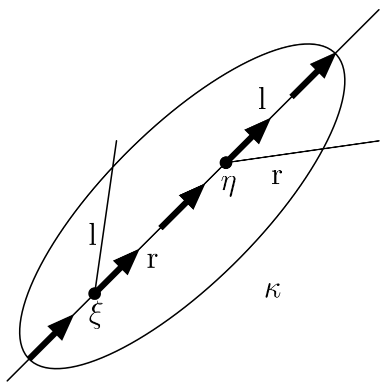

Figure 3.10: Inconsistent histories

f-tree-3*eps

Now it turns out that there occurs a serious problem with our model if a path visits an information set more than once. In such a context we can easily construct a path that is not supported by any pure strategy. Also the specification of a behavioral strategy at κ, i.e., $\boldsymbol{A}^i(\kappa, \bullet)$ has no obvious interpretation if the information set $\mathfrak{X}^i_\kappa$ is visited twice. Is the random device applied twice independently? Or should the result of one random choice be applied both times?

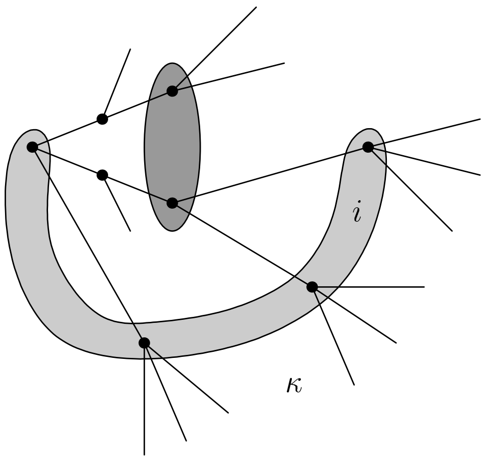

Figure 3.11: Inconsistent Signals

f-tree-4*eps

We would like to avoid some problems resulting from this possibility. So far, however, we have not made an assumption to prevent that a player observes a signal twice because the process reenters an information set or stays therein for several steps.

Definition 3.18. *To every path* $\boldsymbol{x} = (x_0, \ldots, x_t)$ *we associate player* i*'s* ***history*** $\mathcal{H}^i(\boldsymbol{x}) = (\kappa_0, \ldots, \kappa_s)$ *with* $s \leq t$ *and* $\kappa_r \in \mathcal{K}^i \cup \{\emptyset\}$ $(r = 0, \ldots, s)$

defined recursively as follows.

history (3.30)

$$\mathcal{H}^i(x_0) = \begin{cases} (\emptyset) & x_0 \notin \mathcal{X}^i \\ (\boldsymbol{\kappa}^i(x_0)) & x_0 \in \mathcal{X}^i \end{cases}$$

$$\mathcal{H}^i(x_0,\ldots,x_t) = \begin{cases} (\boldsymbol{\kappa}^i(x_t)) & \mathcal{H}^i(x_0,\ldots,x_{t-1}) = (\emptyset),\ x_t \in \mathcal{X}^i \\ \mathcal{H}^i(x_0,\ldots,x_{t-1}),\boldsymbol{\kappa}^i(x_t) & \mathcal{H}^i(x_0,\ldots,x_{t-1}) \neq (\emptyset),\ x_t \in \mathcal{X}^i \\ \mathcal{H}^i(x_0,\ldots,x_{t-1}) & x_t \notin \mathcal{X}^i \end{cases}$$

We shall say that Σ admits of ***consistent signals*** *if, for any two paths*

$$\boldsymbol{x} = (x_0,\ldots,x_t),\quad \widehat{\boldsymbol{x}} = (\hat{x}_0,\ldots,\hat{x}_{t'})$$

satisfying

$$\boldsymbol{\kappa}^i(x_t) = \boldsymbol{\kappa}^i(\hat{x}_{t'})$$

it follows that

$$\mathcal{H}^i(\boldsymbol{x}) = \mathcal{H}^i(\widehat{\boldsymbol{x}})$$

holds true.

Thus, to any path we associate the corresponding sequence of signals player i observes. Feasibly, he observes nothing. However, if he keeps track of his observations, then this will not change his state of information when he observes a new signal. For, if he observes some $\kappa = \boldsymbol{\kappa}^i(\xi)$, then he can compute all possible path that could lead to this signal in the sense, that the endpoint x_t satisfies $\boldsymbol{\kappa}^i(x_t)) = \kappa = \boldsymbol{\kappa}^i(\xi)$. All these paths yield the same sequence of signals. Hence, player i can at once determine the past history he observed based on his last signal – both informations are consistent.

Our first observation is as follows.

treebsmd **Theorem 3.19.** *Let Σ be a tree game, $\boldsymbol{A}$ a behavioral strategy and $\boldsymbol{M}$ a mixed strategy. The processes controlled by $\boldsymbol{A}$ and $\boldsymbol{M}$ have the same distribution if*

treemdisa (3.31)

$$\boldsymbol{M}^i(\{\boldsymbol{\alpha}^i \mid \boldsymbol{\alpha}^i \text{ supports } \bar{x}\}) = \prod_{\{t \mid \bar{x}_{t-1}\in\mathcal{X}^i\}} \left(\boldsymbol{q}(\bar{x}_{t-1},\bullet)\boldsymbol{A}^i(\boldsymbol{\kappa}^i(\bar{x}_{t-1}),\bullet)\right)(\bar{x}_t)$$

holds true for any play $\bar{x} = (\bar{x}_0,\ldots,\bar{x}_T)$ and for any $i \in I$.

Proof: The distribution resulting from the application of $\boldsymbol{M}$ puts on any $x \in \overline{\mathbf{X}}$ the weight

treef1 (3.32)

$$\begin{aligned}\boldsymbol{K}^{\boldsymbol{\mu}}\boldsymbol{M}(\{x\}) &= \sum_{\alpha\in\mathfrak{S}} \boldsymbol{m}^{\alpha}_{\boldsymbol{\mu}}(\{x\})\boldsymbol{M}(\{\alpha\}) \\ &= \sum_{\{\alpha\in\mathfrak{S}\mid\alpha \text{ supports } x\}} \left(\prod_{\{t\mid x_{t-1}\in\mathcal{X}^0\}} \boldsymbol{\mu}^{x_{t-1}}_{x_t}\right) \boldsymbol{M}(\{\alpha\})\end{aligned}$$

in view of Remark (3.17). We continue (3.32) by

treef2 (3.33)

$$\begin{aligned}\cdots &= \prod_{\{t\mid x_{t-1}\in\mathcal{X}^0\}} \boldsymbol{\mu}^{x_{t-1}}_{x_t}\ \boldsymbol{M}(\{\alpha \mid \alpha \text{ supports } x\}) \\ &= \prod_{\{t\mid x_{t-1}\in\mathcal{X}^0\}} \boldsymbol{\mu}^{x_{t-1}}_{x_t}\ \prod_{i\neq 0}\boldsymbol{M}^i(\{\boldsymbol{\alpha}^i \mid \boldsymbol{\alpha}^i \text{ supports } x\})\end{aligned}$$

this is so since the players' mixed strategies are combined to the product measure and the event involved has a product character, i.e.,

$$\{\alpha \mid \alpha \ \text{ supports } \ x\} = \bigcap_{\substack{i \in I \\ i \neq 0}} \{\alpha \mid \boldsymbol{\alpha}^i \ \text{ supports } \ x\}$$

According to the assumption (3.31) we may continue formula (3.33) by

$$\cdots = \prod_{\{t \mid x_{t-1} \in \mathcal{X}^0\}} \boldsymbol{\mu}_{x_t}^{x_{t-1}} \prod_{i \neq 0} \prod_{\{t \mid x_{t-1} \in \mathcal{X}^i\}} \left(\boldsymbol{q}(\bar{x}_{t-1}, \bullet) \boldsymbol{A}^i(\boldsymbol{\kappa}^i(\bar{x}_{t-1}), \bullet) \right)(\bar{x}_t)$$

which, by Definition (3.6) formula (3.15) and formula (3.17) is indeed $\boldsymbol{m}_{\boldsymbol{\mu}}^{\boldsymbol{A}}(\{x\})$, **q.e.d.**

Next let us focus on the imitation of mixed strategies by behavioral strategies.

Definition 3.20. *Let $\boldsymbol{A}$ be a b.s. Then a mixed strategy $\boldsymbol{M}^{\boldsymbol{A}} \in \mathfrak{M}$ is given by*

$$\boldsymbol{M}^{i\boldsymbol{A}^i}(\{\alpha\}) = \prod_{\kappa \in \mathcal{K}^i} \boldsymbol{A}^i(\kappa, \boldsymbol{\alpha}^i(\kappa)) \quad (i \in I) \quad .$$

That is, player i considers his random devices $\boldsymbol{A}^i(\kappa, \bullet)$ as stochastically independent and computes the probability that $\boldsymbol{A}^i$ decides as $\boldsymbol{\alpha}^i$ would do.

Lemma 3.21. *Assume that Σ admits of consistent signals. Let $\boldsymbol{A}$ be a b.s. and $\bar{x} = (x_1, \ldots, x_T)$ a path. Then, for the corresponding mixed strategy $\boldsymbol{M}^{\boldsymbol{A}}$*

$$\boldsymbol{M}^{i\boldsymbol{A}^i}(\{\boldsymbol{\alpha}^i \mid \boldsymbol{\alpha}^i \ \textit{ supports } \ \bar{x}\}) = \prod_{\{t \mid \bar{x}_{t-1} \in \mathcal{X}^i\}} \left(\boldsymbol{q}(\bar{x}_{t-1}, \bullet) \boldsymbol{A}^i(\boldsymbol{\kappa}^i(\bar{x}_{t-1}), \bullet) \right)(\bar{x}_t)$$

holds true for $i \in I$.

Proof: By definition of $\boldsymbol{M}^{\boldsymbol{A}}$ we have

$$\begin{aligned} &\boldsymbol{M}^{i\boldsymbol{A}^i}(\{\boldsymbol{\alpha}^i \mid \ \text{ supports } \ \bar{x}\}) \\ &\qquad = \sum_{\{\boldsymbol{\alpha}^i \mid \boldsymbol{\alpha}^i \ \text{ supports } \ \bar{x}\}} \prod_{\kappa \in \mathcal{K}^i} \boldsymbol{A}^i \ (\kappa, \boldsymbol{\alpha}^i(\kappa)) \\ &\qquad = \cdots \end{aligned}$$

With respect to the last product, consider some $\bar{\kappa} \in \mathcal{K}^i$ such that *no* $\bar{x}_t$ yields the signal $\bar{\kappa}$. Separate this factor and split the sum according to whether some $\boldsymbol{\alpha}^i$ decides for some $y \in \mathcal{Y}^i_{\bar{\kappa}}$ (every $\boldsymbol{\alpha}^i$ decides for some unique y since we have consistent signals!). This yields

$$\begin{aligned} \cdots \ &= \sum_{\{\boldsymbol{\alpha}^i \mid \boldsymbol{\alpha}^i \ \text{ supports } \ \bar{x}\}} \boldsymbol{A}^i(\bar{\kappa}, \boldsymbol{\alpha}^i(\bar{\kappa})) \prod_{\{\kappa \in \mathcal{K}^i \mid \kappa \neq \bar{\kappa}\}} \boldsymbol{A}^i(\kappa, \boldsymbol{\alpha}^i(\kappa)) \\ &= \sum_{y \in \mathcal{Y}^i_{\bar{\kappa}}} \sum_{\{\boldsymbol{\alpha}^i \mid \boldsymbol{\alpha}^i \ \text{ supports } \ \bar{x}, \boldsymbol{\alpha}^i(\bar{\kappa}) = y\}} \boldsymbol{A}^i(\bar{\kappa}, \boldsymbol{\alpha}^i(\bar{\kappa})) \prod_{\{\kappa \in \mathcal{K}^i \mid \kappa \neq \bar{\kappa}\}} \boldsymbol{A}^i(\kappa, \boldsymbol{\alpha}^i(\kappa)) \\ &= \sum_{y \in \mathcal{Y}^i_{\bar{\kappa}}} \boldsymbol{A}^i(\bar{\kappa}, y) \left(\sum_{\{\boldsymbol{\alpha}^i \mid \boldsymbol{\alpha}^i \ \text{ supports } \ \bar{x}, \alpha^i(\bar{\kappa}) = y\}} \prod_{\{\kappa \in \mathcal{K}^i \mid \kappa \neq \bar{\kappa}\}} \boldsymbol{A}^i(\kappa, \boldsymbol{\alpha}^i(\kappa)) \right) \end{aligned}$$

Consider the last sum, in which the product does not depend on y. To every $\boldsymbol{\alpha}^i$ supporting $\bar{x}$ which decides for y, i.e., yields $\boldsymbol{\alpha}^i(\kappa) = y$, there corresponds an $\boldsymbol{\alpha}'^i$ which decides $\boldsymbol{\alpha}'^i(\kappa) = y'$ and otherwise looks exactly like $\boldsymbol{\alpha}^i$. This is so far all $y' \neq y \in \mathfrak{Y}^i_{\bar{\kappa}}$ and hence the sum yields the same value for each $y \in \mathfrak{Y}^i_{\bar{\kappa}}$. Actually, the common value can be written

$$\sum_{\{\hat{\boldsymbol{\alpha}}^i \in \mathfrak{S}^{i0}\}} \quad \prod_{\{\kappa \in \mathcal{K}^i \mid \kappa \neq \bar{\kappa}\}} \boldsymbol{A}^i(\kappa, \boldsymbol{\alpha}^i(\kappa))$$

where we have introduced a set

$$\mathfrak{S}^{i0} := \{\hat{\boldsymbol{\alpha}}^i \in \mathfrak{S}^i \mid \ \hat{\boldsymbol{\alpha}}^i \ \text{ is a mapping on } \ \mathcal{K}^i - \{\bar{\kappa}\}, \ \hat{\boldsymbol{\alpha}}^i \ \text{ supports } \ \bar{x}\}$$

for convenience.

As this common value does not depend on κ, we can put it as a common factor in front of the sum

$$\sum_{y \in \mathfrak{Y}^i_{\bar{\kappa}}} \boldsymbol{A}^i(\bar{\kappa}, y) \quad ,$$

which, however, equals 1.

Now, consider the case that $\boldsymbol{\kappa}^i(\bar{x}_{t-1}) = \bar{\kappa}$ for $\bar{\kappa} \in \mathcal{K}^i$, i.e., the path yields (once!) the signal $\bar{\kappa}$. Since $\bar{x}_{t-1}$ yields $\bar{\kappa}$ we know that $\bar{x}_t$ does not yield $\bar{\kappa}$ and there is exactly one $\bar{y}$ such that $\boldsymbol{q}(\bar{x}_{t-1}, \bar{y}) = \bar{x}_t$ (we have assumed consistent signals). Hence, any pure strategy supporting $\bar{x}$ (and hence satisfying $\boldsymbol{\alpha}^i(\bar{\kappa}) = \bar{y}$) can be split into $(\boldsymbol{\alpha}^i(\bar{\kappa}), \hat{\boldsymbol{\alpha}}^i)$ such that $\hat{\boldsymbol{\alpha}}^i$ is a mapping on $\mathcal{K}^i - \{\bar{\kappa}\}$ (a "restricted strategy"). Vice versa, any mapping $\hat{\boldsymbol{\alpha}}^i$ on $\mathcal{K}^i - \{\bar{\kappa}\}$ induces a strategy indicated by $(\bar{\kappa}, \hat{\boldsymbol{\alpha}}^i)$.

This way we have a partition

$$\begin{aligned}
\{\boldsymbol{\alpha}^i \mid \boldsymbol{\alpha}^i \ \text{ supports } \ \bar{x}\} & \\
&= \{(\bar{y}, \hat{\boldsymbol{\alpha}}^i) \mid \hat{\boldsymbol{\alpha}}^i \in \mathfrak{S}^{i0}\} \\
&= \sum \{\hat{\boldsymbol{\alpha}}^i \mid \hat{\boldsymbol{\alpha}}^i \in \mathfrak{S}^{i0}\} \times \{\bar{y}\}
\end{aligned}$$

The remaining arguments are the same, so the factor that appears this time is not 1 but

$$\boldsymbol{A}^i(\bar{\kappa}, \bar{y}) \;=\; \boldsymbol{A}^i\left(\bar{\kappa}, \boldsymbol{q}(\bar{x}_{t-1}, \bullet)^{-1}(\{\bar{x}_t\})\right) \;=\; \left(\boldsymbol{q}(\bar{x}_{t-1}, \bullet)\boldsymbol{A}^i\boldsymbol{\kappa}^i(\bar{x}_{t-1}), \bullet)\right)(\bar{x}_t)$$

The further reduction of the quantity follows along the same path and yields the desired product, **q.e.d.**

The reverse direction requires more structure: usually we cannot expect that behavioral strategies are sufficient to imitate mixed ones unless we impose additional requirements on Σ.

conschoice **Definition 3.22.** *Let Σ be a game tree admitting consistent signals. Σ allows for* ***consistent actions*** *if the following holds true.*

Let

$$\overline{\boldsymbol{x}} = (\bar{x}_0, \dots, \bar{x}_t), \hat{\boldsymbol{x}} = (\hat{x}_0, \dots, \hat{x}_{t'})$$

be two paths satisfying

$$\bar{x}_t, \hat{x}_{t'} \in \mathcal{X}^i, \quad \boldsymbol{\kappa}^i(\bar{x}_t) = \boldsymbol{\kappa}^i(\hat{x}_{t'}).$$

Also let

$$\mathcal{H}^i(\overline{\boldsymbol{x}}) = \mathcal{H}^i(\hat{\boldsymbol{x}}) = (\kappa_1, \dots, \kappa_l, \kappa)$$

be the sequence of signals observed. (see (3.30) for the definition of $\mathcal{K}(\bullet)$). Suppose for some h, t, t', u, r that

$$\boldsymbol{\kappa}^i(\bar{x}_{u-1}) = \kappa_h = \boldsymbol{\kappa}^i(\hat{x}_{r-1})$$

holds true. Then it follows that

samesignal

$$\boldsymbol{q}(\bar{x}_{u-1}, \bullet)^{-1}(\bar{x}_u) = \boldsymbol{q}(\hat{x}_{r-1}, \bullet)^{-1}(\hat{x}_r) \tag{3.34}$$

holds true.

We offer two interpretations. The first one is provided by a direct observation: player i finds himself in the position of observing κ. Intermediately, he observed some κ_h.If he could choose different actions and still arrive at observing κ, then he might be able to distinguish between having passed $\bar{x}_u$ and $\hat{x}_r$ – which would contradict our general tendency to provide consistent observation in the model.

Another interpretation is provided by the following Lemma, according to which a strategy can be employed to precisely point out those paths that it is caompatible with.

conssupport

Lemma 3.23. *Let Σ be a tree game admitting of consistent actions. Let $\overline{\boldsymbol{x}} = (\bar{x}_0, \dots, \bar{x}_T)$ be a path with $\boldsymbol{\kappa}^i(\bar{x}_T) = \kappa$. Then, if some $\boldsymbol{\alpha}^i$ supports $\overline{\boldsymbol{x}}$, it supports any $\hat{\boldsymbol{x}} = (\hat{x}_0, \dots, \hat{x}_s)$ with $\boldsymbol{\kappa}^i(\hat{x}_S) = \kappa$.*

Proof: Since Σ admits of consistent signals, the paths $\overline{\boldsymbol{x}}$ and $\hat{\boldsymbol{x}}$ generate the same sequence of signals, say

$$\mathcal{H}^i(\overline{\boldsymbol{x}}) = \mathcal{H}^i(\hat{\boldsymbol{x}}) = (\kappa_1, \dots, \kappa_l, \kappa)$$

Let $\hat{x}_{r-1} \in \mathcal{X}^i$ and assume that $\boldsymbol{\kappa}^i(\hat{x}_{r-1}) = \kappa_h$. Then there is $\bar{x}_{u-1}$ such that $\boldsymbol{\kappa}^i(\bar{x}_{u-1}) = \kappa_h$ holds true as well. Suppose the strategy $\boldsymbol{\alpha}^i$ decides for $\boldsymbol{\alpha}^i(\kappa_h) = y$. As $\boldsymbol{\alpha}^i$ supports $\overline{\boldsymbol{x}}$ we know that $\boldsymbol{q}(\bar{x}_{t-1}, y) = \bar{x}_t$ and because of (3.34) we have $\boldsymbol{q}(\hat{x}_{t-1}, y) = \hat{x}_t$. Thus, $\boldsymbol{\alpha}^i$ supports $\hat{\boldsymbol{x}}$ as well.

q.e.d.

Definition 3.24. *A tree game Σ allows for* ***perfect recall*** *if the following holds true:*

1. *Σ admits of consistent signals.*

2. *Σ admits of consistent actions.*

That is, the history player i observes is consistent with the information set this history leads to as well as with any strategy player i has chosen to generate this history.

Accordingly, we may consider the set of strategies that are consistent with a certain signal. This is the set of all strategies supporting any path that leads to that signal, a set which is now unambiguously specified by the signal.

defmaba **Definition 3.25.** *Let Σ be a tree game with perfect recall.*

1. *Define for $i \in \boldsymbol{I}$*

stratsig
$$\mathfrak{S}^i(\kappa) := \{\boldsymbol{\alpha}^i \in \mathfrak{S}^i \mid \boldsymbol{\alpha}^i \text{ supports } x = (x_0, \ldots, x_t), x_t \in \mathcal{Y}^i_\kappa\}. \tag{3.35}$$

   to be the set of strategies that ***support a signal*** *$\kappa \in \mathcal{K}^i$.*

2. *Let $\boldsymbol{M}^i$ be a mixed strategy for player $i \in \boldsymbol{I}$. The corresponding behavioral strategy for player i is denoted by $\boldsymbol{A}^i = \boldsymbol{A}^{i\boldsymbol{M}^i}$ and defined by*

behamix
$$\boldsymbol{A}^i(\kappa, y) = \frac{\boldsymbol{M}^i(\{\boldsymbol{\alpha}^i \mid \boldsymbol{\alpha}^i \in \mathfrak{S}^i(\kappa),\ \boldsymbol{\alpha}^i(\kappa) = y\})}{\boldsymbol{M}^i(\mathfrak{S}^i(\kappa))}, \tag{3.36}$$

   provided the denominator is positive. Otherwise the definition is arbitrary.

Theorem 3.26. *Let Σ be a tree game with perfect recall, and let $\boldsymbol{M}^i$ a mixed strategy for player $i \in \boldsymbol{I}$. Let $\boldsymbol{A}^i = \boldsymbol{A}^{i\boldsymbol{M}^i}$ correspond to $\boldsymbol{M}^i$ via Definition 3.25. Then, for any path $\bar{x}$*

$$\boldsymbol{M}^i(\{\boldsymbol{\alpha}^i \mid \boldsymbol{\alpha}^i \text{supports } \bar{x}\}) = \prod_{\{t \in \{1,\ldots,T\} \mid \bar{x}_{t-1} \in \mathfrak{X}^i\}} \boldsymbol{A}^i(\boldsymbol{\kappa}^i(\bar{x}_{t-1}), \boldsymbol{q}(\bar{x}_{t-1}, \bullet)^{-1}(\bar{x}_t))$$

holds true.

Proof: Consider the path $(\bar{x}_0, \ldots, \bar{x}_t)$ and let

$$\{\bar{x}_0, \ldots, \bar{x}_t\} \cap \mathfrak{X}^i = \{\bar{x}_{r_1}, \ldots, \bar{x}_{r_s}\}$$

with $\bar{x}_{r_1}$ preceding $\bar{x}_{r_2,\ldots}$ etc. Correspondingly, there is a unique sequence of information sets or signals

$$\mathcal{H}^i(\bar{x}_{r_1}, \ldots, \bar{x}_{r_{s-1}}) = (\kappa_1, \ldots, \kappa_{s-1}), \quad \boldsymbol{\kappa}^i(\bar{x}_{r_l}) = \kappa_l \ (l = 1, \ldots, s-1)$$

as well as actions $y_2, \ldots, y_s$ generated such that

$$\boldsymbol{q}(\bar{x}_{r_1}, y_2) = \bar{x}_{r_2}, \ldots, \boldsymbol{q}(x_{r_{s-1}}, y_s) = x_{r_s}$$

holds true.

Therefore, we have

treequot (3.37)

$$
\begin{aligned}
&= \prod_{\{r \mid \bar{x}_{r-1} \in \mathfrak{X}^i\}} \boldsymbol{A}^i(\boldsymbol{\kappa}^i(\bar{x}_{r-1}), \boldsymbol{q}(\bar{x}_{t-1}, \bullet)^{-1}(\bar{x}_r) \\
&= \boldsymbol{A}^i(\kappa_1, y_2) \cdot \boldsymbol{A}^i(\kappa_2, y_3) \cdot \dots \cdot \boldsymbol{A}^i(\kappa_{s-1}, y_s) \\
&= \frac{\boldsymbol{M}^i(\{\boldsymbol{\alpha}^i \mid \boldsymbol{\alpha}^i \in \mathfrak{S}^i(\kappa_1),\ \boldsymbol{\alpha}^i(\kappa_1) = s_y\})}{\boldsymbol{M}^i(\mathfrak{S}^i(\kappa_1))} \cdot \\
&\quad \frac{\boldsymbol{M}^i(\{\boldsymbol{\alpha}^i \mid \boldsymbol{\alpha}^i \in \mathfrak{S}^i(\kappa_2),\ \boldsymbol{\alpha}^i(\kappa_2) = y_3\})}{\boldsymbol{M}^i(\mathfrak{S}^i(\kappa_2))} \cdot \\
&\quad \dots \cdot \frac{\boldsymbol{M}^i(\{\boldsymbol{\alpha}^i \mid \boldsymbol{\alpha}^i \in \mathfrak{S}^i(\kappa_{s-1}),\ \boldsymbol{\alpha}^i(\kappa_{s-1}) = y_s\})}{\boldsymbol{M}^i(\mathfrak{S}^i(\kappa_{s-1}))}
\end{aligned}
$$

Consider the second denominator. We obtain

$$
\begin{aligned}
\boldsymbol{M}^i(\mathfrak{S}^i(\kappa_2)) &= \boldsymbol{M}^i(\{\boldsymbol{\alpha}^i \mid \boldsymbol{\alpha}^i(\kappa_1) = y_1\}) \\
&= \boldsymbol{M}^i(\{\boldsymbol{\alpha}^i \mid \boldsymbol{\alpha}^i \in \mathfrak{S}^i(\kappa_1), \alpha^i(\kappa_1) = s_1\})
\end{aligned}
$$

as every $\boldsymbol{\alpha}^i$ is compatible with observing κ_1: it is the first time player 1 observes something and he has not made any decision so far; hence actually $\mathfrak{S}^i(\kappa_1) = \mathfrak{S}^i$ and the first denominator in (3.37) equals 1. It follows that the second denominator equals the first enumerator etc., so after cancelling we obtain the last enumerator, i.e.,

$$
\begin{aligned}
\dots &= \boldsymbol{M}^i(\{\boldsymbol{\alpha}^i \mid \boldsymbol{\alpha}^i \in \mathfrak{S}^i(\kappa_{s-1}), \boldsymbol{\alpha}^i(\kappa_{s-1}) = y_s\}) \\
&= \boldsymbol{M}^i(\{\boldsymbol{\alpha}^i \mid \boldsymbol{\alpha}^i(\kappa_1) = y_1, \dots, \alpha^i(\kappa_{s-1}) = y_s\}) \\
&= \boldsymbol{M}^i(\{\boldsymbol{\alpha}^i \mid \boldsymbol{\alpha}^i \text{ supports } \bar{x}\}),
\end{aligned}
$$

q.e.d.

It is well known that the theorem is wrong without the assumption of perfect recall. The standard counterexample can be constructed using a game with one player only:

Example 3.27. Consider the game represented by the following sketch

There is just one player involved and the set of his pure strategies can be written to be

$$\mathfrak{S}^1 = \{lL, lR, rL, rR\}. \tag{3.38}$$

We consider the mixed strategy $\boldsymbol{M} = (\frac{1}{2}, 0, 0, \frac{1}{2})$ and compute the distribution $\boldsymbol{KM}$ induced by $\boldsymbol{M}$ on the paths of our tree. To this end, observe that the distributions $\boldsymbol{m}^\alpha$ induced by pure strategies $\alpha \in \mathfrak{S}^1$ are

$$
\begin{aligned}
\boldsymbol{m}^{lL} &= \boldsymbol{\delta}_{\{x_0, x_1, x_2\}} \quad \text{for} \quad \alpha = lL \\
\boldsymbol{m}^{rR} &= \boldsymbol{\delta}_{\{x_0, \xi_1, \zeta_2\}} \quad \text{for} \quad \alpha = rR.
\end{aligned} \tag{3.39}
$$

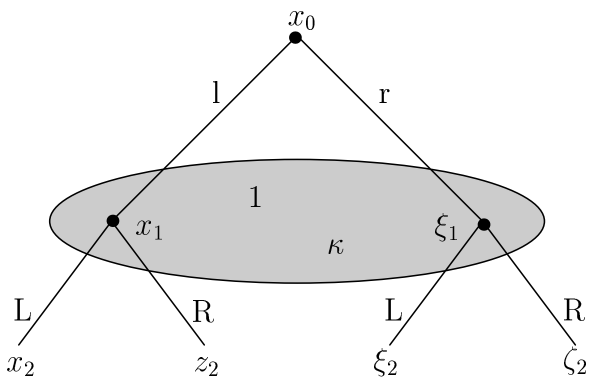

Figure 3.12: A Counterexample f-tree-5*eps

From this the distribution induced by $\boldsymbol{M}$ is computed to be

$$\boldsymbol{KM} \;=\; \int_{\mathfrak{S}^1} \boldsymbol{m}^{\alpha}(\bullet)\boldsymbol{M}(d\alpha) \;=\; \frac{1}{2}\left(\boldsymbol{\delta}_{\{x_0,x_1,x_2\}} + \boldsymbol{\delta}_{\{x_0,\xi_1,\zeta_2\}}\right). \tag{3.40}$$

We now claim, that this distribution cannot be generated by a behavioral strategy. Indeed, take an arbitrary behavioral strategy $\boldsymbol{A}$ which can be written

$$\begin{aligned} \boldsymbol{A} \;&=\; (A(x_0,\bullet), A(\kappa,\bullet)) \\ &=\; ((\underset{l}{p}, \underset{r}{1-p}), (\underset{L}{q}, \underset{R}{1-q})) \end{aligned} \tag{3.41}$$

The distribution generated by $\boldsymbol{A}$ may then be described as follows:

$$\begin{aligned} \boldsymbol{m}^{\boldsymbol{A}}(\{x_0,x_1,x_2\}) \;&=\; pq \\ \boldsymbol{m}^{\boldsymbol{A}}(\{x_0,x_1,z_2\}) \;&=\; p(1-q) \\ \boldsymbol{m}^{\boldsymbol{A}}(\{x_0,\xi_1,\xi_2\}) \;&=\; (1-p)q \\ \boldsymbol{m}^{\boldsymbol{A}}(\{x_0,\xi_1,\zeta_2\}) \;&=\; (1-p)(1-q) \end{aligned} \tag{3.42}$$

Now, $\boldsymbol{KM}$ puts mass 0 on both the "inner" paths. Therefore, in order for $\boldsymbol{A}$ to imitate $\boldsymbol{M}$ with respect to the distribution, we must necessarily have

$$0 \;=\; p(1-q) \;=\; (1-p)q, \tag{3.43}$$

that is, we have either

$$p \;=\; 0 \;\text{ and }\; q \;=\; 0$$

or

$$p \;=\; 1 \;\text{ and }\; q \;=\; 1.$$

Accordingly it follows that either

$$\boldsymbol{m}^{\boldsymbol{A}} \;=\; \boldsymbol{\delta}_{\{x_0,\xi_1,\zeta_2\}}$$

or

$$\boldsymbol{m}^{\boldsymbol{A}} \;=\; \boldsymbol{\delta}_{\{x_0,x_1,x_2\}},$$

which is never $\boldsymbol{KM}$. Thus there is no behavioral strategy which can imitate $\boldsymbol{M}$ by generating the same distribution on the paths of our tree.

samedistrib **Theorem 3.28.** *Let Σ be a tree game with perfect recall. Let $\boldsymbol{A}$ a behavioral strategy and $\boldsymbol{M}$ a mixed strategy. Suppose that both $\boldsymbol{A}$ and $\boldsymbol{M}$ generate the same distribution, i.e., we have*

$$\boldsymbol{K}^{\boldsymbol{\mu}}\boldsymbol{M} = \boldsymbol{m}_{\boldsymbol{\mu}}^{\boldsymbol{A}} \quad .$$

Then it follows that the payoffs to all players are the same, i.e.,

$$C_{\boldsymbol{M}}^{i\boldsymbol{\mu}} = C_{\boldsymbol{A}}^{i\boldsymbol{\mu}} \quad (i \in \boldsymbol{I})$$

holds true.

Proof: once.

Recall the "evaluation", i.e., the mapping

$$\begin{aligned} &C^i : \underline{\overline{\mathbf{X}}} \longrightarrow \mathbb{R} \\ &C^i(x) = u^i(x_{\boldsymbol{\tau}}) + \sum_{t=1}^{\boldsymbol{\tau}} f^i(x_t), \end{aligned} \tag{3.44}$$

where $\boldsymbol{\tau}$ is the length of a play. This mapping evaluates the payoffs for player i along a path. Then we have

eq:5.18
$$C_{\boldsymbol{A}}^{i\boldsymbol{\mu}} = E_{\boldsymbol{m}_{\boldsymbol{\mu}}^{\boldsymbol{A}}} C^i = \int C^i \, d\boldsymbol{m}_{\boldsymbol{\mu}}^{\boldsymbol{A}}. \tag{3.45}$$

Similarly, for every $\alpha \in \mathfrak{S}$

eq:5.19
$$\begin{aligned} C_{\alpha}^{i\boldsymbol{\mu}} = C_{\boldsymbol{A}^{\alpha}}^{i\boldsymbol{\mu}} &= \int C^i(x) \, d\boldsymbol{m}_{\boldsymbol{\mu}}^{\boldsymbol{A}^{\alpha}}(x) \\ &= \int C^i(x) \boldsymbol{K}^{\boldsymbol{\mu}}(\alpha, dx) = \boldsymbol{K}^{\boldsymbol{\mu}} C^i(\alpha). \end{aligned} \tag{3.46}$$

By the general formula for the transformation of variables we find

eq:5.20
$$\begin{aligned} C_{\boldsymbol{M}}^{i\boldsymbol{\mu}} &= \int C_{\alpha}^{i\boldsymbol{\mu}} \, d\boldsymbol{M}(\alpha) \\ &= \int (\boldsymbol{K}^{\boldsymbol{\mu}} C^i) \, d\boldsymbol{M} \\ &= \int C^i \, d(\boldsymbol{K}^{\boldsymbol{\mu}}\boldsymbol{M}) = \int C^i \, d\boldsymbol{m}_{\boldsymbol{\mu}}^{\boldsymbol{A}} \\ &= C_{\boldsymbol{A}}^{i\boldsymbol{\mu}}, \end{aligned} \tag{3.47}$$

q.e.d.

Concluding we observe the following Theorem which shows that it suffices to consider behavioral strategies.

th:treekuhns **Theorem 3.29 (KUHN'S Theorem).** *Let Σ be a tree game with perfect recall and let $\boldsymbol{\mu}^{\bullet}$ be the corresponding family of distributions of the random moves.*

1. *Let* $\bar{\boldsymbol{A}} \in \mathfrak{A}$ *be a Nash equilibrium in behavioral strategies (i.e., for* $\Gamma_{\Sigma,\boldsymbol{\mu}}$*). Then* $\boldsymbol{M}^{\bar{\boldsymbol{A}}} \in \mathfrak{M}$ *is a Nash equilibrium in mixed strategies (i.e. for* $\bar{\Gamma}_{\Sigma,\boldsymbol{\mu}}$*). In particular, if* Σ *is a zero-sum two person game then (so are* $\bar{\Gamma}_{\Sigma,\boldsymbol{\mu}}$ *and* $\Gamma_{\Sigma,\boldsymbol{\mu}}$ *and)*
$$v_{\Gamma_{\Sigma,\boldsymbol{\mu}}} = v_{\bar{\Gamma}_{\Sigma,\boldsymbol{\mu}}}$$
2. *If* $\bar{\boldsymbol{M}}$ *is a Nash equilibrium (for* $\bar{\Gamma}_{\Sigma,\boldsymbol{\mu}}$*) then* $\boldsymbol{A}^{\bar{\boldsymbol{M}}}$ *is a Nash equilibrium (for* $\Gamma_{\Sigma,\boldsymbol{\mu}}$*).*
3. *In any equilibrium a player cannot improve upon his payoff by deviating to the other type of strategy.*

Proof: Consider the first statement. Observe that we have defined the mappings
$$\alpha \to \boldsymbol{A}^{\alpha} \quad , \quad \boldsymbol{A} \to \boldsymbol{M}^{\boldsymbol{A}} \quad , \quad \alpha \to \boldsymbol{M}^{\alpha}$$
in such a way that
$$\boldsymbol{M}^{\boldsymbol{A}^{\alpha}} = \boldsymbol{M}^{\alpha} = \boldsymbol{\delta}_{\alpha}$$
is verified at once. We know by Theorem 3.19 and Theorem 3.28 that $\boldsymbol{A}$ and $\boldsymbol{M}^{\boldsymbol{A}}$ always yield the same payoff.

Next, let $\bar{\boldsymbol{A}}$ be an equilibrium in $\Gamma_{\Sigma,\boldsymbol{\mu}}$. We are to show that $\boldsymbol{M}^{\bar{\boldsymbol{A}}}$ is an equilibrium in $\bar{\Gamma}_{\Sigma,\boldsymbol{\mu}}$ (the statement concerning the values follows then immediately). For simplicity, let us assume $n = 2$, the general n is just as simple. Assume now that "player 1 deviates" (in $\bar{\Gamma}_{\Sigma,\boldsymbol{\mu}}$) i.e., he plays a mixed strategy $\hat{\boldsymbol{M}}^{1}$. We have to show that

eq:5.21 (3.48)
$$C^{1\boldsymbol{\mu}}_{\boldsymbol{M}^{\bar{\boldsymbol{A}}}} \geq C^{1\boldsymbol{\mu}}_{\hat{\boldsymbol{M}}^{1},\boldsymbol{M}^{2,\bar{\boldsymbol{A}}^{2}}}$$

(and a similar relation for player 2, however, the argument is just symmetric).

Now, as $\bar{\boldsymbol{A}}$ is an equilibrium,

eq:5.22 (3.49)
$$C^{1\boldsymbol{\mu}}_{\boldsymbol{M}^{\bar{\boldsymbol{A}}}} = C^{1\boldsymbol{\mu}}_{\bar{\boldsymbol{A}}} \geq C^{1\boldsymbol{\mu}}_{(\boldsymbol{A}^{1},\bar{\boldsymbol{A}}^{2})} \qquad \left(\boldsymbol{A}^{1} \in \mathfrak{A}^{1}\right)$$

for any b.s. $\boldsymbol{A}^{1}$ of player 1. In particular, player 1 may "play pure", i.e., we may choose $\boldsymbol{A}^{1,\alpha^{1}}$ for any $\alpha^{1} \in \mathfrak{S}^{1}$. Let us identify α^{1}, $\boldsymbol{A}^{1,\alpha^{1}}$, $\boldsymbol{M}^{1,\alpha^{1}}$ etc. and insert this in (3.49), thus obtaining

eq:5.23 (3.50)
$$C^{1\boldsymbol{\mu}}_{\boldsymbol{M}^{\bar{\boldsymbol{A}}}} \geq C^{1\boldsymbol{\mu}}_{(\alpha^{1},\bar{\boldsymbol{A}}^{2})}$$

Define for the moment $\hat{\boldsymbol{A}} := (\alpha^{1}, \bar{\boldsymbol{A}}^{2})$ such that (3.50) may be extended to

eq:5.24 (3.51)
$$\begin{aligned}
C^{1\boldsymbol{\mu}}_{\boldsymbol{M}^{\bar{\boldsymbol{A}}}} \geq C^{1\boldsymbol{\mu}}_{\hat{\boldsymbol{A}}} & \qquad = C^{1\boldsymbol{\mu}}_{\boldsymbol{M}^{\hat{\boldsymbol{A}}}} \\
= \int C^{1\boldsymbol{\mu}}_{\alpha} \boldsymbol{M}^{\hat{\boldsymbol{A}}}(d\alpha) & \qquad = E_{\boldsymbol{M}^{\hat{\boldsymbol{A}}}} C^{1\boldsymbol{\mu}}_{\bullet} \\
= E_{\boldsymbol{\delta}_{\alpha^{1}} \otimes \boldsymbol{M}^{2,\bar{\boldsymbol{A}}^{2}}} C^{1\boldsymbol{\mu}}_{\bullet} & \qquad = E_{\boldsymbol{M}^{2,\bar{\boldsymbol{A}}^{2}}} \left(E_{\boldsymbol{\delta}_{\alpha^{1}}} C^{1\boldsymbol{\mu}}_{\bullet}\right) \\
& \qquad (\text{"Fubini's theorem"}) \\
= E_{\boldsymbol{M}^{2,\bar{\boldsymbol{A}}^{2}}} C^{1\boldsymbol{\mu}}_{(\alpha^{1},\bullet)}. &
\end{aligned}$$

As (3.51) holds true for all $\alpha^1 \in \mathfrak{A}_1$, we "integrate" with the mixed strategy $\hat{\boldsymbol{M}}^1$ player 1 is using to "deviate" (the left side is a constant w.r.t. α_1 and thus taking the integral does not change its value). We obtain

$$\begin{aligned} C^{1\boldsymbol{\mu}}_{\boldsymbol{M}\bar{\boldsymbol{A}}} \geq & \int E_{\boldsymbol{M}^2,\bar{\boldsymbol{A}}^2} C^{1\boldsymbol{\mu}}_{(\alpha^1,\bullet)} \hat{\boldsymbol{M}}^1(d\alpha^1) \\ =& E_{\hat{\boldsymbol{M}}^1 \times \boldsymbol{M}^2,\bar{\boldsymbol{A}}^2} C^{1\boldsymbol{\mu}}_{\bullet} \\ =& \, C^{1\boldsymbol{\mu}}_{\hat{\boldsymbol{M}}^1,\boldsymbol{M}^2,\bar{\boldsymbol{A}}^2} \end{aligned}$$

Thus, we have proved (3.48) and we are finished with the first statement of the theorem.

Statements 2 and 3 however, follow just as easily by applying Theorem 3.28 accordingly. **q.e.d.**

Corollary 3.30. *Let Σ be a tree game with perfect recall. Then $\Gamma_{\Sigma\boldsymbol{\mu}}$ has Nash equilibria.*

Proof: Since the game in pure strategies $\tilde{\Gamma}_{\Sigma\boldsymbol{\mu}}$ has finitely many strategies only, its mixed extension $\bar{\Gamma}_{\Sigma\boldsymbol{\mu}}$ is essentially a generalized matrix game, its strategy sets are simplices in Euclidean spaces. This game has Nash equilibria according to NASH's Theorem. According to KUHN's Theorem these Nash equilibria in mixed strategies induce Nash equilibria in behavioral strategies, **q.e.d.**

Proof: A second proof can be enginiered by means of the one deviation priciple (see CHAPTER I, SECTION 3). The version for an imperfect information setup puts an agent at each information set. Accordingly, one has to prove that a Nash equilibrium of the agent normal form yields a Nash equilibrium of the original normal form. This is exactly the statement of ODV. Now, if a player sits at each information set, then mixed strategies and behavioral ones are equivalent and the existence for behavioral strategies follows again from Nash's theorem. **q.e.d.**

References