# Operations Research A:

## Linear Programming

and

## Bimatrix Games

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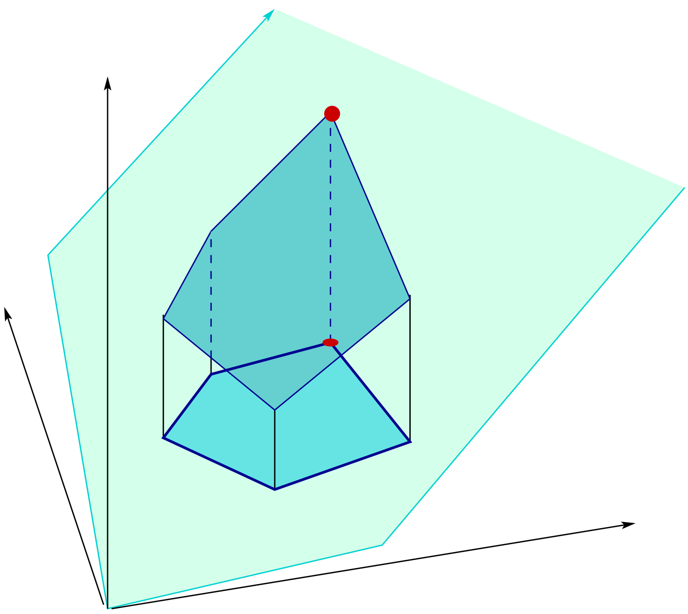
Lecture Notes
Winter 2003 – 04





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# Introduction

# 0 Introduction

The basic problem of linear programming is described as follows:

There is a **production mechanism** (a plant, firm, factory) available to a decision maker. This mechanism is capable of converting

"raw materials " or **"factors"** – labeled $i = 1, \ldots, m$

into

"goods" or **"products "** – labeled $j = 1, \ldots, n$.

The production mechanism has the following characteristics.

1. For one unit of product $j$ the amount of $a_{ij} \quad (i = 1, \ldots, m)$ units of the various factors is necessarily used.

2. The production process is "linear", or shows "constant returns to scale". That is, in order to generate $r > 0$ units of product $j$ it is necessary to use $ra_{ij} \quad (i = 1, \ldots, m)$ units of the factors.

Two further data serve to describe the situation faced by the decision maker.

1. The decision maker has $b_i \quad (i = 1, \ldots, m)$ units of factor $i$ at his disposal.

2. There is an external market at which the decision maker can sell his products; for a unit of product $j$ he will receive the amount of $c_j \quad (j = 1, \ldots, n)$ units in money.

We assume that it is the task of the decision maker to maximize his total income or expenditure, given his restriction of factors and the production possibilities available to him. This is, however a rather vague objective, a precise and mathematically tractable version has to be formulated. Within the following definition we provide the exact mathematical formulation of certain objects as well as interpretations – the latter are not really part of the definition.

The notation is much smoother if we use matrix–vector notation. We write

$$(1) \qquad I := \{1, \ldots, m\}, \quad J = \{1, \ldots, n\},$$

and introduce the matrix

$$\boldsymbol{A} \;:=\; (a_{ij})_{i\in I, j\in J}\,. \tag{2}$$

The ***rows*** of this matrix will generally be denoted by

$$\boldsymbol{A}_{i\bullet} \;:=\; (a_{i1},\ldots,a_{in}) \quad (i \in I) \tag{3}$$

and the ***columns*** by

$$\boldsymbol{A}_{\bullet j} \;:=\; (a_{1j},\ldots,a_{mj}) \quad (j \in J). \tag{4}$$

Then we combine the above vague descriptions by means of the following definition (and interpretation).

**Definition 0.1.** *A **Linear Program (LP)** is a triple*

$$(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c}) \tag{5}$$

*consisting of an $m \times n$–matrix $\boldsymbol{A}$, and vectors $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{c} \in \mathbb{R}^n$.*

*The interpretation as described above justifies the terminology: $\boldsymbol{A}$ is the **input–output matrix**, $\boldsymbol{b}$ is the **constraint vector**, and $\boldsymbol{c}$ represents the **objective function** (i.e., the function $\boldsymbol{x} \to \boldsymbol{cx}$ $(\boldsymbol{x} \in \mathbb{R}^n)$).*

Let us now formulate the task of the decision maker.

**Definition 0.2.**
1. *A **production plan** is given by a vector $\boldsymbol{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n_+$.*

   *That is, the decision maker ("we") plan to produce the quantity $x_j$ of product $j$.*

2. *A production plan $\boldsymbol{x}$ is **feasible** whenever*

$$\boldsymbol{Ax} \le \boldsymbol{b} \tag{6}$$

*holds true.*

We note that the total consumption of factor $i$, when the production plan $\boldsymbol{x}$ is implemented is

$$\sum_{j=1}^{n} a_{ij}x_j \;.$$

Therefore, the inequalities

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i = 1, \ldots, m)$$

state that our production plan does not consume more than what is available of each factor $i$.

Hence, using matrix–vector notation, we read equation (6)

$$\boldsymbol{A}_{i\bullet}\boldsymbol{x} \leq b_i \quad (i = 1, \ldots, m)$$

and find that a production plan is feasible if it obeys the restriction imposed by the scarceness of the factors, i.e., by the constraints.

The set of feasible production plans or just the ***feasible set*** is

$$C := \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geq \boldsymbol{0}, \quad \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}. \tag{7}$$

The expenditure (= profits as we see no costs) obtained from a production plan $\boldsymbol{x}$ is

$$\sum_{j=1}^{n} c_j x_j = \boldsymbol{c}\boldsymbol{x}. \tag{8}$$

We may proceed with the description of the decision makers task by providing a concise definition as follows:

**Definition 0.3.** *Given the above data, i.e., a linear program* $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ *a production plan* $\bar{\boldsymbol{x}} \in \mathbb{R}^n_+$ *is* ***optimal*** *if it is feasible (i.e.,* $\bar{\boldsymbol{x}} \in C$*), and satisfies*

$$\begin{aligned} \boldsymbol{c}\bar{\boldsymbol{x}} &= \max\{\boldsymbol{c}\boldsymbol{x} \mid \boldsymbol{x} \in C\} \\ &= \max\{\boldsymbol{c}\boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{R}^n, \quad \boldsymbol{x} \geq \boldsymbol{0}, \quad \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}. \end{aligned} \tag{9}$$

*This way we maximize the profits given the restriction on production plans that are dictated by scarce resources (and nonnegativity...). It is by no means clear that the maximum exists. Indeed, it may happen that there are no feasible vectors at all. Or else, the objective function may be unbounded on the feasible set. This will be part of a discussion to be initiated later.*

*If the maximum indicated in* (9) *exists we may refer to it as to the* ***value*** *of the linear program* $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$.

Now we try to develop a *geometrical* view of this problem. This is most easy for $n = 2$. For $0 \neq \boldsymbol{a} \in \mathbb{R}^2$ and $\beta \in \mathbb{R}$, the set

$$\{\boldsymbol{x} \in \mathbb{R}^2, \mid \boldsymbol{ax} \leq \beta\}$$

is a half–plane; $\boldsymbol{a}$ is the normal of the bounding straight line (cf. Figure 0.1).

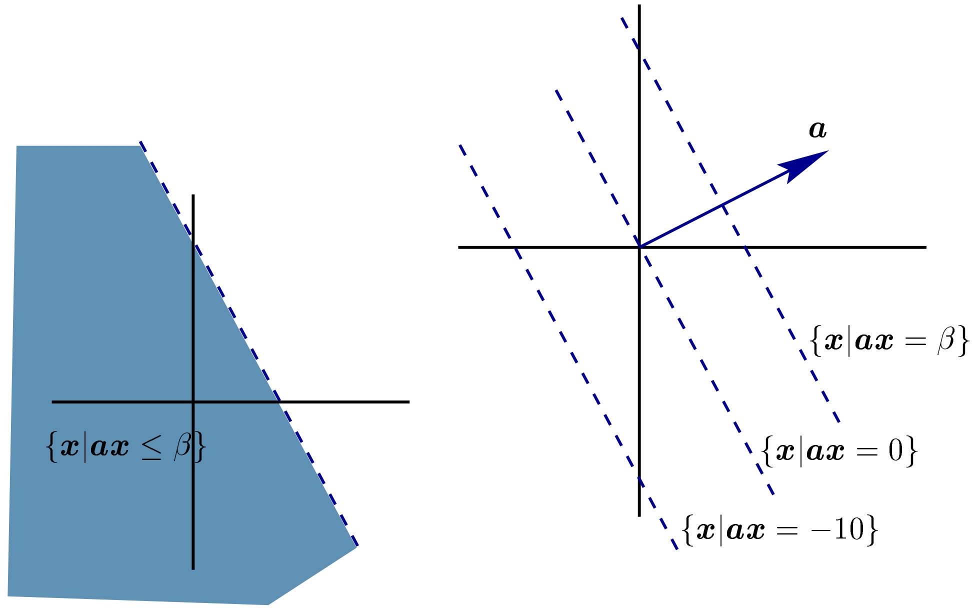



Figure 0.1: Straight lines and half spaces

Therefore, the general form of a set

$$
\begin{aligned}
C &= \left\{ \boldsymbol{x} \in \mathbb{R}^2 \,\middle|\, \boldsymbol{x} \geq \boldsymbol{0}, \quad \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \right\} \\
&= \left\{ \boldsymbol{x} \in \mathbb{R}^2 \,\middle|\, x_j \geq 0 \ \ (j = 1, 2), \quad \boldsymbol{A}_{i\bullet}\boldsymbol{x} \leq b_i \ \ (i = 1, \ldots, m) \right\} \\
&= \bigcap_{j=1,2} \left\{ \boldsymbol{x} \in \mathbb{R}^2 \,\middle|\, x_j \geq 0 \right\} \cap \bigcap_{i=1,\ldots,m} \left\{ \boldsymbol{x} \in \mathbb{R}^2 \,\middle|\, \boldsymbol{A}_{i\bullet}\boldsymbol{x} \leq b_i \ \right\}
\end{aligned}
$$

is described by Figure 0.2.

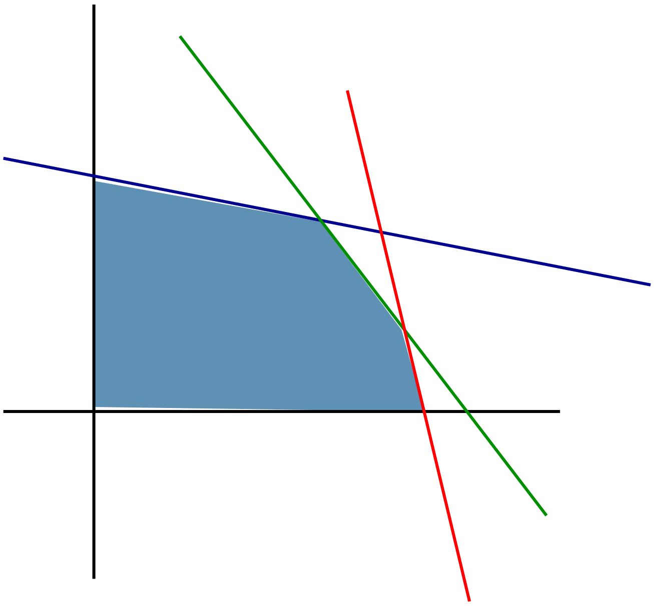


Figure 0.2: A feasible set in $\mathbb{R}^2$

The linear function given by

$$\boldsymbol{x} \to \boldsymbol{c}\boldsymbol{x}$$

on $\mathbb{R}^2$ may be represented by its graph which is a plane in $\mathbb{R}^3$. Therefore we obtain the picture offered in Figure 0.3 as the geometrical paradigm of the maximization problem suggested by Definitions 0.1 and 0.3.

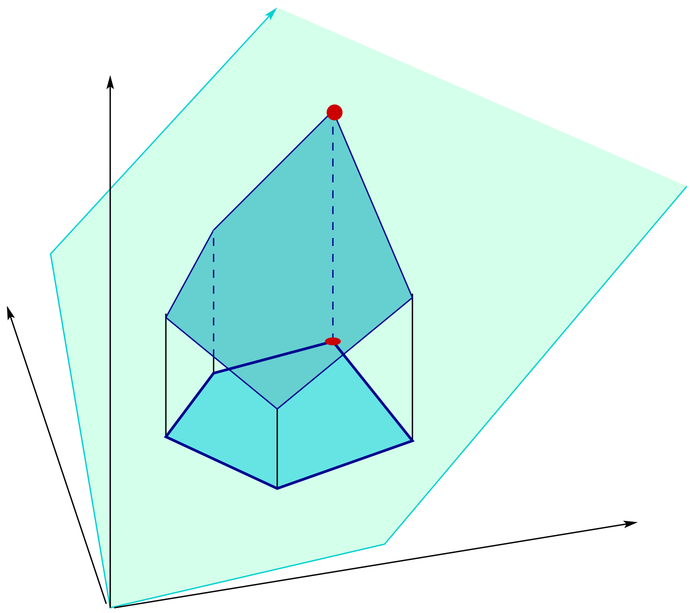


Figure 0.3: Maximizing a linear function in $\mathbb{R}^2$

It is rather easy to find maximizers as well as the maximum in 2 dimensions. However, we will have to present methods or algorithms that provide the value of the program and optimal solutions (if they exist) for any dimension without the appeal to imagination. Moreover, the question of existence and conditions to ensure that the algorithm reacts properly to linear programs without optimal solutions etc. will have to be treated.

# Chapter 1

# Prerequisites: Convex Analysis

Within this chapter we shall present some basic topics of Convex Analysis. These topics provede the tools which are permanently used in various contexts of Linear Programming and Game Theory. We discuss convex sets and their extremal points, convex and linear functions on such sets, and separation theorems.

# 1 Convex Sets and Convex Polyhedra

Intuitively, a convex set is a subset of $\mathbb{R}^n$ which shows no 'holes' or 'cavities' in its geometric structure. More precisely, a convex is a subset of $\mathbb{R}^n$ with the property that any two of its elements can be connected by a straight line segment. The term "line segment" or "interval" between two points of $\mathbb{R}^n$ can be given a precise meaning. From this the resulting definitions follow at once.

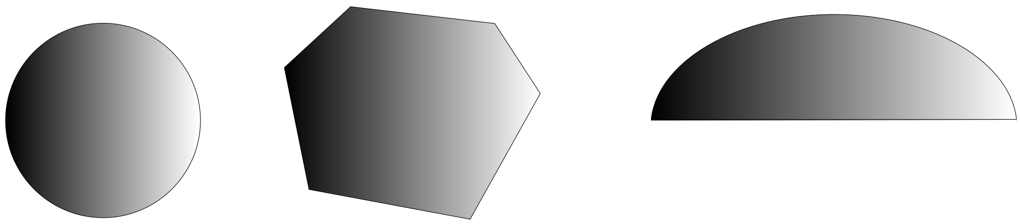


Figure 1.1: The Shape of Convex Sets – intuitive pictures

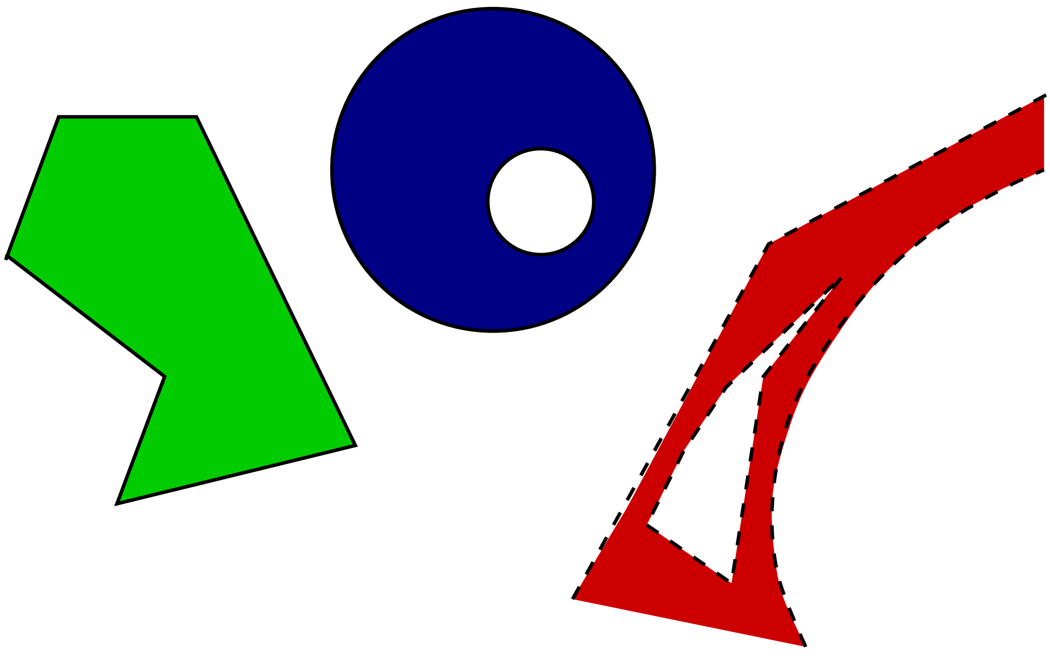


Figure 1.2: Non–Convex Sets

**Definition 1.1.** *1. Let $\boldsymbol{x}^0, \boldsymbol{x}^1 \in \mathbb{R}^n$ be two points or vectors and let $\lambda \in \mathbb{R}$ be a real number. We write*

$$\begin{aligned} \boldsymbol{x}^\lambda &:= (1-\lambda)\boldsymbol{x}^0 + \lambda\boldsymbol{x}^1 \\ [\boldsymbol{x}^0, \boldsymbol{x}^1] &:= \{\boldsymbol{x}^\lambda | \lambda \in [0,1]\} \\ (\boldsymbol{x}^0, \boldsymbol{x}^1) &:= \{\boldsymbol{x}^\lambda | \lambda \in (0,1)\} \end{aligned} \quad .$$

*If $\lambda \in [0,1]$ is true, then we call $\boldsymbol{x}^\lambda$ a **convex combination** of $\boldsymbol{x}^0$ and $\boldsymbol{x}^1$. The set $[\boldsymbol{x}^0, \boldsymbol{x}^1]$ is called the **closed interval** generated by $\boldsymbol{x}^0$ and $\boldsymbol{x}^1$. Similarly, $(\boldsymbol{x}^0, \boldsymbol{x}^1)$ is the **open interval** generated by $\boldsymbol{x}^0$ and $\boldsymbol{x}^1$.*

2. *A set $C \subseteq \mathbb{R}^n$ is called **convex** if, for any two vectors, $\boldsymbol{x}^0, \boldsymbol{x}^1 \in C$ it follows that*

   $$[\boldsymbol{x}^0, \boldsymbol{x}^1] \subseteq C \tag{1}$$

   *holds true.*

3. *A set $C \subseteq \mathbb{R}^n$ is called a **convex polyhedron**, if there exists a nonempty set of vectors $\boldsymbol{a}^1, ..., \boldsymbol{a}^m \in \mathbb{R}^n$ (not all of them $\mathbf{0}$) and real numbers $b_1, ..., b_m \in \mathbb{R}$ such that*

   $$\{C = \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{a}^i\boldsymbol{x} \leq b_i \quad (i = 1...m)\} \tag{2}$$

   *holds true. Equivalently, we can say that there exists an $m \times n$ matrix $\boldsymbol{A} \neq \mathbf{0}$ and a vector $\boldsymbol{b} \in \mathbb{R}^m$ such that*

   $$C = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\} \tag{3}$$

   *holds true.*

4. *A set $C \subseteq \mathbb{R}^n$ is a **half-space** if there is a vector $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{R}^n$ and a real number $b \in \mathbb{R}$ such that*

   $$C = \{\boldsymbol{x} \mid \boldsymbol{a}\boldsymbol{x} \leq b\} \tag{4}$$

   *holds true.*

5. *A set $C \subseteq \mathbb{R}^n$ is a **hyperplane** if there is a vector $\mathbf{0} \neq \boldsymbol{a} \in \mathbb{R}^n$ and a real number $b \in \mathbb{R}$ such that*

   $$C = \{\boldsymbol{x} \mid \boldsymbol{a}\boldsymbol{x} = b\} \tag{5}$$

   *holds true.*

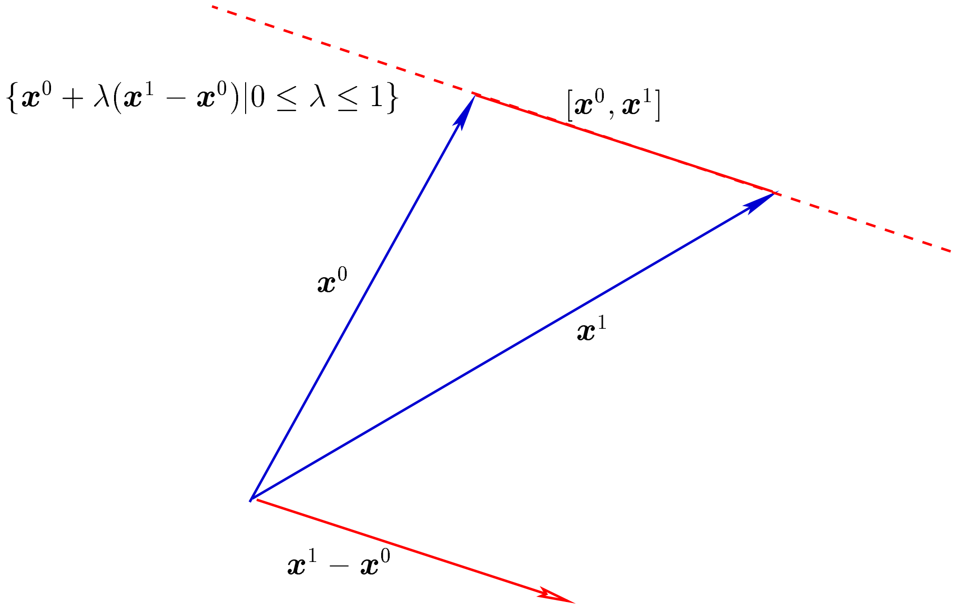



Figure 1.3: The construction of an interval

We offer a detailed interpretation of the above concepts (Figure 1.3):

Consider two *different* points $x^0, x^1 \in \mathbb{R}^n$. Note that $\boldsymbol{x}^\lambda = \boldsymbol{x}^0 + \lambda(\boldsymbol{x}^1 - \boldsymbol{x}^0)$ holds true, hence up to translation by $\boldsymbol{x}^0$ we find $\boldsymbol{x}^\lambda$ to be a multiple of $\boldsymbol{x}^1 - \boldsymbol{x}^0$. Thus, while $\lambda$ is running through the reals, $\boldsymbol{x}^\lambda$ is describing a line that passes through $\boldsymbol{x}^0$ and $\boldsymbol{x}^1$ ( the injective mapping

$$\boldsymbol{x}^\bullet \ : \ \mathbb{R} \to \mathbb{R}^n, \tag{6}$$

defined by $\lambda \to \boldsymbol{x}^\lambda$ $(\lambda \in \mathbb{R})$, provides a ***parametrization*** of this line).

Clearly, as $\lambda$ ranges through $[0, 1]$, $\boldsymbol{x}^\lambda$ is running from $\boldsymbol{x}^0$ to $\boldsymbol{x}^1$ thus describing the intervall $[\boldsymbol{x}^0, \boldsymbol{x}^1]$.

Next, as $|\boldsymbol{x}^\lambda - \boldsymbol{x}^0| = \lambda |\boldsymbol{x}^1 - \boldsymbol{x}^0|$ is true, we note that

$$\lambda = \frac{|\boldsymbol{x}^\lambda - \boldsymbol{x}^0|}{|\boldsymbol{x}^1 - \boldsymbol{x}^0|}$$

follows and analogously

$$1 - \lambda = \frac{|\boldsymbol{x}^\lambda - \boldsymbol{x}^1|}{|\boldsymbol{x}^1 - \boldsymbol{x}^0|}$$

holds true. This means that $\boldsymbol{x}^\lambda$ bisects the intervall $[\boldsymbol{x}^0, \boldsymbol{x}^1]$ proportionally to $\lambda/(1 - \lambda)$. This way we can identify the vector $\boldsymbol{x}^\lambda$ on the intervall $[\boldsymbol{x}^0, \boldsymbol{x}^1]$

within a geometrical context. In particular, $\boldsymbol{x}^{\frac{1}{2}}$ is the midpoint of the interval $[\boldsymbol{x}^0, \boldsymbol{x}^1]$.

The idea of a convex set has now a precise mathematical formulation. A convex set $C$ contains any interval which can be formed by any two of its elements; this is the correct elaboration of the fact that $C$ should not contain 'holes' or 'cavities'. See Figure 1.4.

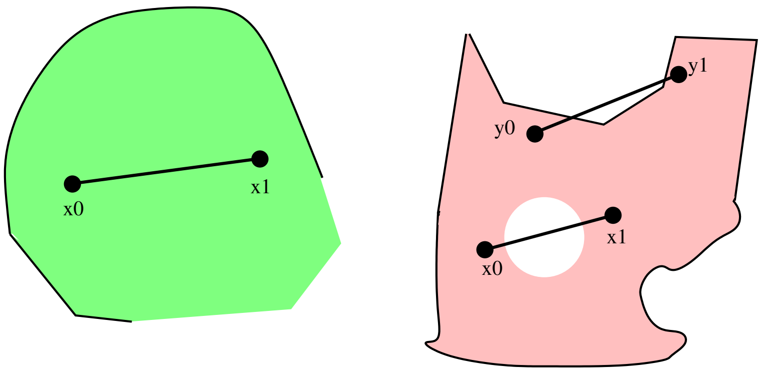



Figure 1.4: A convex set and a non convex set

As to the further concepts mentioned in the above definition, we assume that the reader is familiar with a hyperplane. The (geometrical) nature of a half space should be derived from the one of a hyperplane: it containes all vectors "on one side" of a hyperplane. A convex polyhedron can be seen as a finite intersection of half spaces; thus in $\mathbb{R}^2$ its typical shape is depicted in Figure 1.5.

Here are some families of convex sets.

**Theorem 1.2.** *1. A convex polyhedron is convex.*

*2. Hyperplanes and half-spaces are convex polyhedra.*

*3. Linear subspaces and linear manifolds of $\mathbb{R}^n$ are convex.*

*4. An interval is a convex polyhedron.*

**Proof:**

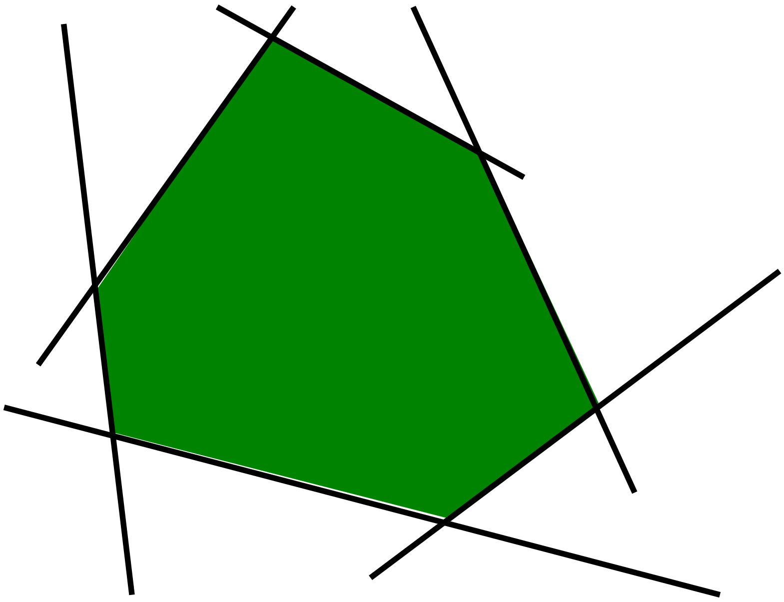


Figure 1.5: A convex polyhedron

1. If two vectors $\boldsymbol{x}^0$ and $\boldsymbol{x}^1$ satisfy an inequality, then so does any convex combination.

2. Obvious. Note that an equation can be replaced by two inequalities. Also, any $\geq$–inequality can be replaced by a $\leq$–inequality by reversing the sign.

3. Obvious.

4. Assume that $\boldsymbol{x}^0 \neq \boldsymbol{x}^1$ is true. Then the straight line $\{\boldsymbol{x}^\lambda \mid \lambda \in \mathbb{R}\}$ is a linear manifold that may be represented as the solution space of a system of linear equations. That is, there is a matrix $\boldsymbol{A}$ of rank $n-1$ and a vector $\boldsymbol{b}$ such that

$$\{\boldsymbol{x}^\lambda \mid \lambda \in \mathbb{R}\} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$$

holds true. Let $\boldsymbol{a} \in \mathbb{R}^n$ be a vector which is independent on the rows of $\boldsymbol{A}$. Assume (without loss of generality) $\boldsymbol{a}\boldsymbol{x}^0 < \boldsymbol{a}\boldsymbol{x}^1$ and consider the convex polyhedron

$$\left\{x \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \quad \boldsymbol{a}\boldsymbol{x}^0 \leq \boldsymbol{a}\boldsymbol{x} \leq \boldsymbol{a}\boldsymbol{x}^1\right\}. \tag{7}$$

This set is a subset of the above mentioned straight line. It containes the two points $\boldsymbol{x}^0$ and $\boldsymbol{x}^1$, hence the interval $[\boldsymbol{x}^0, \boldsymbol{x}^1]$. It is easy to see that any point on the straight line which is not located within the interval violates exactly one of the two inequalities involving $\boldsymbol{a}$.

**q.e.d.**

Convex sets can be manipulated in a way that the class of convex sets is not left, that is, convexity is preserved.

**Theorem 1.3.** *1. If $C \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$ are convex sets then $C \cap B$ is convex. Arbitrary intersections of convex sets are convex.*

*2. If $C$ and $B$ are convex polyhedra, then so is $C \cap B$. Finite intersections of convex polyhedra are convex polyhedra.*

*3. A convex polyhedron is the intersection of finitely many half-spaces.*

*4. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping and $C \subseteq \mathbb{R}^n$ is convex, then*

$$f(C) := \{f(\boldsymbol{x}) \mid \boldsymbol{x} \in C\}$$

*is convex.*

*5. If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping and $C \subseteq \mathbb{R}^n$ is a convex polyhedron, then*

$$f(C) := \{f(\boldsymbol{x}) \mid \boldsymbol{x} \in C\}$$

*is a convex polyhedron.*

**Proof:** All statements are obvious except, perhaps, the last one. For this we sketch the following proof.

We restrict our argument to the case of a a half space $\{\boldsymbol{x}|\boldsymbol{ax} \leq b\}$, obviously this is sufficient.

**1$^{st}$STEP** : If $f$ is surjective and represent by some (invertible) matrix $\boldsymbol{A}$ via $f(\boldsymbol{x}) = \boldsymbol{Ax} \quad (\boldsymbol{x} \in IR^n)$, then clearly

$$\{\boldsymbol{Ax} \mid \boldsymbol{ax} \leq b\} = \left\{\boldsymbol{y} \,\middle|\, \boldsymbol{aA}^{-1}\boldsymbol{y} \leq b\}\right\},$$

which proves everything.

Suppose, therefore that $f$ is not invertible. Consider the linear subspace

$$U^0 \;:=\; \{\boldsymbol{x} \mid \boldsymbol{Ax} = 0\}$$

and let $U^{\perp}$ be the orthogonal subspace of $\mathbb{R}^n$.

**2ndSTEP** : Assume that there is $\boldsymbol{x}^0 \in U^0$ with $\boldsymbol{a}\boldsymbol{x}^0 \neq 0$. Then, for arbitrary $\boldsymbol{x}^\perp \in U^\perp$ and real $t$ we have

$$\boldsymbol{A}\boldsymbol{x}^\perp = \boldsymbol{A}(\boldsymbol{x}^\perp + t\boldsymbol{x}^0)$$

while

$$\boldsymbol{a}(\boldsymbol{x}^\perp + t\boldsymbol{x}^0) = \boldsymbol{a}\boldsymbol{x}^\perp + t\boldsymbol{a}\boldsymbol{x}^0 \leq b,$$

provided $t$ is chosen sufficiently large and with the right sign. Therefore

(8) $$\boldsymbol{A}U^\perp \subseteq \{\boldsymbol{A}\boldsymbol{x} \mid \boldsymbol{a}\boldsymbol{x} \leq b\} \subseteq \{\boldsymbol{A}\boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{R}^n\} \subseteq \boldsymbol{A}U^\perp$$

holds true. Necessarily we have the =-sign prevails, thus our set is a linear subspace. Hence it is a convex polyhedron in view of the previous theorem.

**3rdSTEP** : Suppose that, on the other hand $\boldsymbol{a}\boldsymbol{x}^0 = 0$ holds true for all $\boldsymbol{x}^0 \in U^0$. Now any $\boldsymbol{x} \in \mathbb{R}^n$ can uniquely be decomposed into $\boldsymbol{x} = \boldsymbol{x}^0 + \boldsymbol{x}^\perp$ each component being in one of the orthogonal subspaces.

Now $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{a}(\boldsymbol{x}^0 + \boldsymbol{x}^\perp) = \boldsymbol{a}\boldsymbol{x}^\perp$, hence

$$\{\boldsymbol{A}\boldsymbol{x} \mid \boldsymbol{a}\boldsymbol{x} \leq b\} = \left\{\boldsymbol{A}\boldsymbol{x}^\perp \,\middle|\, \boldsymbol{x}^\perp \in U^\perp,\ \boldsymbol{a}\boldsymbol{x}^\perp \leq b\right\}.$$

But on $U^\perp$ the mapping $f$ represented by $\boldsymbol{A}$ is invertible, hence, up to a change of basis, we may apply the result of the $1^{st}STEP$,

**q.e.d.**

**Example 1.4.** Now we test some sets for convexity formally. E.g., the unit sphere

(9) $$S \;=\; \{\boldsymbol{x} \in \mathbb{R}^n \mid |\boldsymbol{x}| \leq 1\}$$

can be seen to be a convex set. For, if $\boldsymbol{x}^0, \boldsymbol{x}^1 \in S$ holds true, i.e., if we have

$$|\boldsymbol{x}^0| \leq 1,\ |\boldsymbol{x}^1| \leq 1,$$

then it follows that

$$\begin{aligned} |\boldsymbol{x}^\lambda| = &\quad |\lambda\boldsymbol{x}^1 + (1-\lambda)\boldsymbol{x}^0| \leq |\lambda\boldsymbol{x}^1| + |(1-\lambda)\boldsymbol{x}^0| \\ = &\quad \lambda|\boldsymbol{x}^1| + (1-\lambda)|\boldsymbol{x}^0| \leq \lambda + (1-\lambda) = 1, \end{aligned}$$

for $\lambda \in [0,1]$. This shows that $\boldsymbol{x}^\lambda \in S$ holds true as well, hence $S$ is convex.

The following is a particular convex set: a simplex. Simplices are of multiple use for convex analysis.

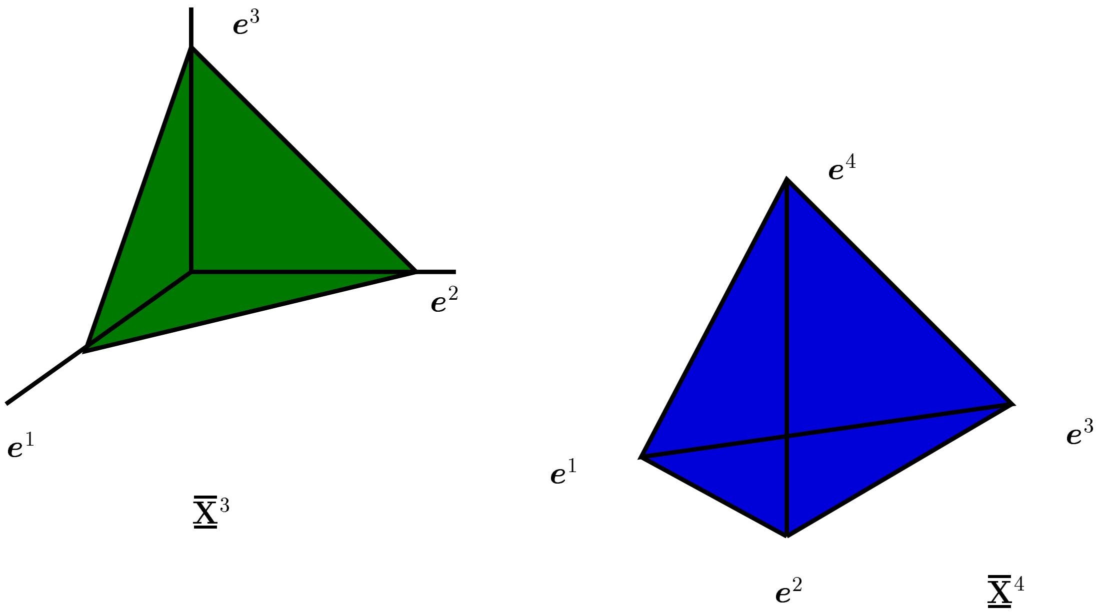



Figure 1.6: The Simplices in $\mathbb{R}^3, \mathbb{R}^4$

**Definition 1.5.** *For any natural number $k$ the set*

$$\underline{\overline{\mathbf{X}}} = \underline{\overline{\mathbf{X}}}^k = \left\{ \boldsymbol{x} \in \mathbb{R}^k \mid \boldsymbol{x} \geq 0, \sum_{l=1}^{k} x_i = 1 \right\}$$

*is called the* ***unit simplex*** *of* $\mathbb{R}^k$.

A simplex is a convex set which can be proved easily. We now use the notion of the simplex in order to extend the idea of convexity.

**Theorem 1.6.** *A set $C \subseteq \mathbb{R}^n$ is convex if and only if the following holds true: For any $\boldsymbol{x}^1, ..., \boldsymbol{x}^k \in C$ and any $\boldsymbol{\alpha} \in \underline{\overline{\mathbf{X}}}^k$ it follows that*

$$\sum_{l=1}^{k} \alpha_l \boldsymbol{x}^l \in C \tag{10}$$

*holds true.*

**Proof:** The 'if'-part is obvious, since condition (1) of Definition 1.2 is implied by condition (10) of Theorem 1.6.

On the other hand, suppose that $C$ is convex. We proceed by induction. For $k = 2$ condition (10) is satisfied since $\boldsymbol{\alpha} \in \underline{\overline{\mathbf{X}}}^2$ means $\alpha_2 = 1 - \alpha_1$.

If $k > 2$ and condition (10) holds true for 1,2,...,k-1, then observe that

$$\sum_{l=1}^{k} \alpha_l \boldsymbol{x}^l = \left( \sum_{\rho=1}^{k-1} \alpha_\rho \right) \left( \sum_{l=1}^{k-1} \frac{\alpha_l}{\sum_{\rho=1}^{k-1} \alpha_\rho} \boldsymbol{x}^l \right) + \alpha_k \boldsymbol{x}^k$$

holds true.

Now, the last sum consists of $k-1$ summands with coefficients that establish a vector in $\underline{\overline{\mathbf{X}}}^{k-1}$. This sum therefore yields an element of $C$ by induction. Thus, the term on the right side is a sum of *two* elements of $C$ with nonnegative coefficients summing up to 1 (i.e., constituting a vector in $\underline{\overline{\mathbf{X}}}^2$). This, again by induction, is an element of $C$, **q.e.d.**

**Definition 1.7.** *1. If $\boldsymbol{x}^1, ...\boldsymbol{x}^k \in \mathbb{R}^n$ and $\boldsymbol{\alpha} \in \underline{\overline{\mathbf{X}}}^k$ then*

$$\boldsymbol{x}^{\boldsymbol{\alpha}} := \sum_{l=1}^{k} \alpha_l \boldsymbol{x}^l \tag{11}$$

*is called a **convex combination** of $\boldsymbol{x}^1, ..., \boldsymbol{x}^k$. In this context the coordinates of $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_k) \in \underline{\overline{\mathbf{X}}}^k$ may be referred to as **convexifying coefficients***

*2. If $E \subseteq \mathbb{R}^n$, then*

$$\boldsymbol{CnvH}\, E := \{\boldsymbol{x}^{\boldsymbol{\alpha}} \mid \boldsymbol{x}^1, ..., \boldsymbol{x}^k \in E, \boldsymbol{\alpha} \in \underline{\overline{\mathbf{X}}}^k,\ k \in \mathbb{N}\} \tag{12}$$

*is the **convex hull** of $E$.*

In passing we note that by (11) we have implicitely defined a mapping

$$\boldsymbol{x}^{\bullet} : \underline{\overline{\mathbf{X}}}^k \to \mathbb{R}^n \tag{13}$$

which generalizes the one given by formula (6), thus could be considered as to provide a *parametrization* of $\boldsymbol{CnvH}\{\boldsymbol{x}^0, \ldots, \boldsymbol{x}^k\}$. However, the mapping does not have to be injective unless the vectors involved are linearly independent.

**Remark 1.8.** *1. Let $\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3 \in \mathbb{R}^2$. Following the proof of Theorem 1.6 we observe that*

$$\begin{aligned}\boldsymbol{x}^{\boldsymbol{\alpha}} &= \alpha_1\boldsymbol{x}^1 + \alpha_2\boldsymbol{x}^2 + \alpha_3\boldsymbol{x}^3 \\ &= (\alpha^1 + \alpha_2)\underbrace{\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\boldsymbol{x}^1 + \frac{\alpha_2}{\alpha_1+\alpha_2}\boldsymbol{x}^2\right)}_{=:\boldsymbol{x}^{12} \text{ in the interval } [\boldsymbol{x}^1,\boldsymbol{x}^2]} + \alpha_3\boldsymbol{x}^3 \\ &= (1-\alpha_3)\boldsymbol{x}^{12} + \alpha_3\boldsymbol{x}^3\end{aligned}$$

*is located within the interval $[\boldsymbol{x}^{12}, \boldsymbol{x}^3]$ where $\boldsymbol{x}^{12} \in [\boldsymbol{x}^1, \boldsymbol{x}^2]$. See Figure 1.7 which allows to visualize the role of the coefficients $\boldsymbol{\alpha}$.*

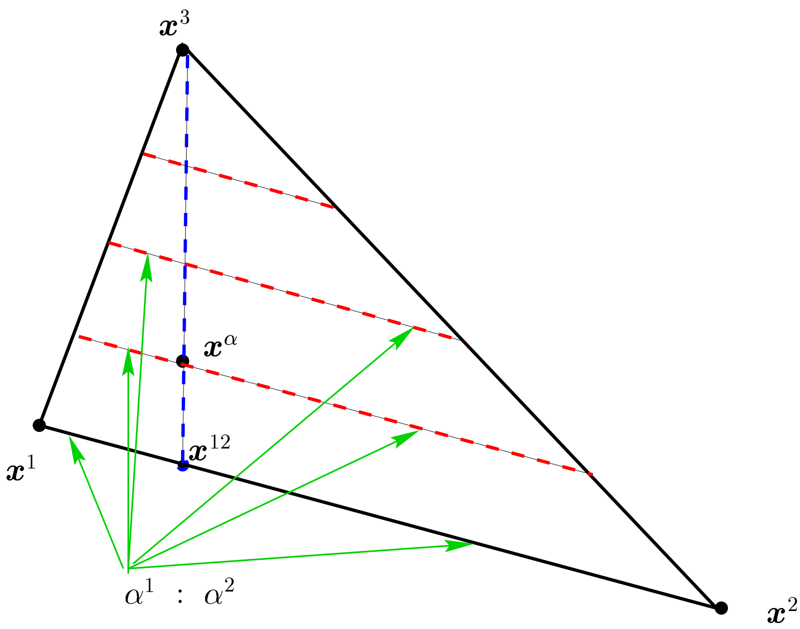



Figure 1.7: Interpreting $\boldsymbol{x}^{\boldsymbol{\alpha}}$

*Thus, intuitively, while $\alpha$ runs through $\overline{\mathbf{X}}^3$, $\boldsymbol{x}^{\boldsymbol{\alpha}}$ describes the elements of the triangle 'spanned' by $\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3$ - the convex hull $\boldsymbol{CnvH}\{\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3\}$.*

2. *In a general setup the convex hull of finitely many points may look as described in Figure 1.8*

3. *Note that for the **unit vectors** $e^i = (0, \cdots, 1, \cdots, 0) \in \mathbb{R}^k$ $(i = 1, \ldots, k)$,*

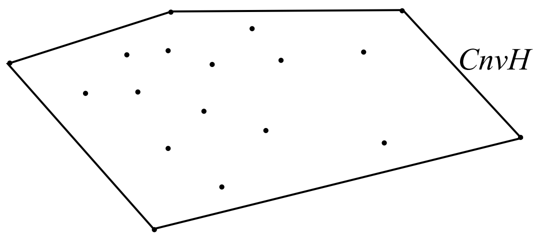



Figure 1.8: The Convex Hull of finitely many points

*we have*

$$\textbf{\textit{CnvH}}\{e^1,...,e^k\} = \left\{\sum_{l=1}^{k} x_l e^l \mid \boldsymbol{x} \in \underline{\overline{\mathbf{X}}}^k\right\} = \left\{\boldsymbol{x} \mid \boldsymbol{x} \in \underline{\overline{\mathbf{X}}}^k\right\} = \underline{\overline{\mathbf{X}}}^k \tag{14}$$

**Theorem 1.9.** *1. For any $E \subseteq \mathbb{R}^n$, the set **CnvH** $E$ is convex.*

*2. If $E \subseteq \mathbb{R}^n$ and $D \subseteq \mathbb{R}^n$ is convex such that $E \subseteq D$ holds true, then **CnvH** $E \subseteq D$ holds true as well.*

*3. **CnvH** $E = \bigcap\{C \subseteq \mathbb{R}^n \mid C \text{ convex}, E \subseteq C\}$*
*That is, **CnvH** $E$ is the **smallest convex set containing** $E$.*

*4. If $C$ is a convex set, then **CnvH** $C = C$ holds true.*

**Proof:**

1. Let $\boldsymbol{x}^{\boldsymbol{\alpha}} \in \textbf{\textit{CnvH}}\, E$ and $\boldsymbol{y}^{\boldsymbol{\beta}} \in \textbf{\textit{CnvH}}\, E$, i.e.

$$\boldsymbol{x}^{\boldsymbol{\alpha}} = \alpha_1 \boldsymbol{x}^1 + \cdots + \alpha_k \boldsymbol{x}^k, \quad \boldsymbol{y}^{\boldsymbol{\beta}} = \beta_1 \boldsymbol{y}^1 + \cdots + \beta_r \boldsymbol{y}^r$$

where $\boldsymbol{x}^l, \boldsymbol{y}^s \in E(l = 1 \cdots k, s = 1 \cdots r)$, and $\boldsymbol{\alpha} \in \underline{\overline{\mathbf{X}}}^k, \boldsymbol{\beta} \in \underline{\overline{\mathbf{X}}}^r$ holds true.

Then we obtain

$$\lambda \boldsymbol{x}^{\boldsymbol{\alpha}} + (1-\lambda)\boldsymbol{y}^{\boldsymbol{\beta}} = \sum_{l=1}^{k} \lambda\alpha_l\ \boldsymbol{x}^l + \sum_{s=1}^{r} (1-\lambda)\beta_s \boldsymbol{y}^s.$$

Now, the convexifying coefficients may be combined to form a vector

$$\boldsymbol{\gamma} = (\lambda\alpha_1, \cdots, \lambda\alpha_k, (1-\lambda)\beta_1, \cdots, (1-\lambda)\beta_r) \in \underline{\overline{\mathbf{X}}}^{k+r},$$

since they are all nonnegative and add up to

$$\sum_{\sigma=1}^{k+r} \gamma_\sigma = 1.$$

This, however shows that

$$\lambda \boldsymbol{x}^{\boldsymbol{\alpha}} + (1-\lambda)\boldsymbol{y}^{\boldsymbol{\beta}} = \boldsymbol{z}^{\gamma}$$

is an element of $\boldsymbol{CnvH}\, E$ with obvious choice of $z^1, \cdots, z^{k+r}$.

2. This statement follows from Theorem 1.6

3. This follows from the second statement and the fact that $E \subseteq \boldsymbol{CnvH}\, E$ holds always true,

**q.e.d.**

The following lemma involves a reference to a *topological* property as well.

**Lemma 1.10.** *Let $C$ be a nonempty, convex, and closed subset of $\mathbb{R}^n$. Then there exists a unique vector $\bar{\boldsymbol{x}} \in C$ satisfying*

$$| \,\bar{\boldsymbol{x}}\, | = \; min \;\; \{|\boldsymbol{x}| \mid \boldsymbol{x} \in C\} \,. \tag{15}$$

**Proof:**

**1stSTEP** :

Let $\alpha := inf\, \{| \,\boldsymbol{x}\, || \,\boldsymbol{x} \in C\}$ and let $(\boldsymbol{x}^k)_{k\in\mathbb{N}}$ be a sequence of elements of $C$ such that $|\boldsymbol{x}^k| \to \alpha \;\; (k \to \infty)$ holds true. Let $(\boldsymbol{x}^k)_{k\in\bar{\mathbb{N}}\subseteq\mathbb{N}}$ be a convergent subsequence, say

$$\boldsymbol{x}^k \to \bar{\boldsymbol{x}} \;\; (k \in \bar{\mathbb{N}})$$

with suitable $\bar{\boldsymbol{x}} \in \mathbb{R}^n$. Then $\bar{\boldsymbol{x}} \in C$ as $C$ is closed and $| \,\bar{\boldsymbol{x}}\, | = \alpha$ as $| \bullet |$ is continuous as a function on $\mathbb{R}^n$. This proves the existence of the required vector.

**2ndSTEP** :

In order to prove uniqueness, suppose $| \,\bar{\boldsymbol{x}}\, | = \alpha$ and $| \,\bar{\boldsymbol{y}}\, | = \alpha$ holds true for $\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}} \in C$. Then $\frac{\bar{\boldsymbol{x}}+\bar{\boldsymbol{y}}}{2} \in C$ and

$$\alpha^2 \leq \left|\frac{\bar{\boldsymbol{x}}+\bar{\boldsymbol{y}}}{2}\right|^2 = \frac{1}{2}| \,\bar{\boldsymbol{x}}\, |^2 + \frac{1}{2}| \,\bar{\boldsymbol{y}}\, |^2 - \left|\frac{\bar{\boldsymbol{x}}-\bar{\boldsymbol{y}}}{2}\right|^2 = \alpha^2 \underbrace{- \left|\frac{\bar{\boldsymbol{x}}-\bar{\boldsymbol{y}}}{2}\right|^2}_{\leq 0} \tag{16}$$

Hence we have necessarily $|\bar{\boldsymbol{x}} - \bar{\boldsymbol{y}}| = 0, \bar{\boldsymbol{x}} = \bar{\boldsymbol{y}}$, **q.e.d.**

The following theorem establishes an important property of hyperplanes in the context of convex analysis: the ability to ***separate*** points from convex sets or to separate convex set from each other. We start out with a formal description.

**Theorem 1.11 (The Separation Theorem).** *Let $\emptyset \neq C \subseteq \mathbb{R}^n$ be convex and closed and let $\boldsymbol{x}^0 \notin C$. Then there is $0 \neq \boldsymbol{p} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that*

$$\boldsymbol{p}\boldsymbol{x}^0 < \alpha < \min\{\boldsymbol{p}\boldsymbol{x} | \boldsymbol{x} \in C\} \tag{17}$$

*holds true.*

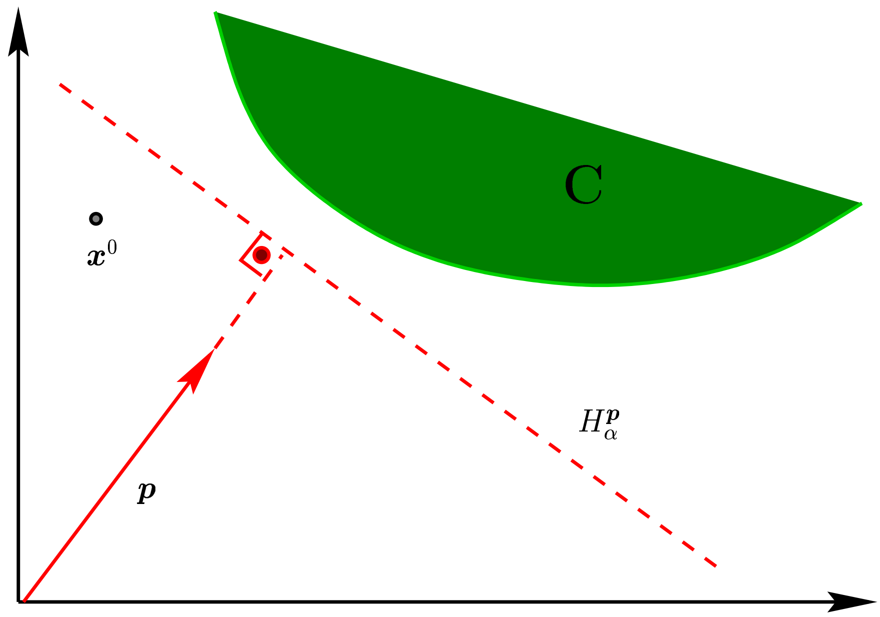



Figure 1.9: The Separation Theorem

An interpretation of the term “separation theorem” runs as follows: (cf. Figure 1.9)

Let $H^{\boldsymbol{p}}_\alpha = \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{p}\boldsymbol{x} = \alpha\}$ denote the hyperplane defined by $\boldsymbol{p}$ and $\alpha$. Then (17) shows that $\boldsymbol{x}^0 \notin H^{\boldsymbol{p}}_\alpha$ and $x \notin H^{\boldsymbol{p}}_\alpha$ $(x \in C)$ is the case. Moreover, $\boldsymbol{x}^0$ and $\boldsymbol{x} \in C$ are 'on different sides' of $H^{\boldsymbol{p}}_\alpha$. Thus, the hyperplane $H^{\boldsymbol{p}}_\alpha$ decomposes $\mathbb{R}^n$ into two half spaces, on of them containing $\boldsymbol{x}^0$ and the other one $C$ 'strictly'. That is $H^{\boldsymbol{p}}_\alpha$ ***separates*** $C$ and $\boldsymbol{x}^0$.

**Proof of Theorem 1.11**

**1$^{\text{st}}$STEP** :

First of all let us consider the case that $\boldsymbol{x}^0 = \mathbf{0}$ is true. Let $\bar{\boldsymbol{x}} \in C$ be such that

$$| \bar{\boldsymbol{x}} | = \min\{| \boldsymbol{x} | \mid \boldsymbol{x} \in C\},$$

this can be achieved in view of Lemma 1.10. Note that $\bar{\boldsymbol{x}} \neq 0$ and $|\bar{\boldsymbol{x}}| > 0$ follow from $\mathbf{0} = \boldsymbol{x}^0 \notin C$.

Now, for any $\boldsymbol{x} \in C$ and $\lambda \in [0,1]$, we have $\lambda\boldsymbol{x} + (1-\lambda)\bar{\boldsymbol{x}} \in C$. Therefore we obtain

$$\begin{aligned} |\bar{\boldsymbol{x}}|^2 &\leq |\lambda\boldsymbol{x} + (1-\lambda)\bar{\boldsymbol{x}}|^2 \\ &= |\bar{\boldsymbol{x}} + \lambda(\boldsymbol{x} - \bar{\boldsymbol{x}})|^2 \\ &= |\bar{\boldsymbol{x}}|^2 + 2\lambda\bar{\boldsymbol{x}}(\boldsymbol{x} - \bar{\boldsymbol{x}}) + \lambda^2|\boldsymbol{x} - \bar{\boldsymbol{x}}|^2 \ . \end{aligned}$$

Hence, for $\lambda \in (0,1]$ we obtain the inequality

$$0 \leq \bar{\boldsymbol{x}}(\boldsymbol{x} - \bar{\boldsymbol{x}}) + \frac{\lambda}{2}|\boldsymbol{x} - \bar{\boldsymbol{x}}|^2.$$

But as $\lambda$ can be arbitrarily small we obtain immediately the inequality

$$0 < \bar{\boldsymbol{x}}^2 \leq \bar{\boldsymbol{x}}\boldsymbol{x} \quad (\boldsymbol{x} \in C). \tag{18}$$

Now we put $\alpha := \frac{\bar{\boldsymbol{x}}^2}{2}$ and $\boldsymbol{p} := \bar{\boldsymbol{x}}$. Rewriting (18) we find an inequality

$$0 = \boldsymbol{p}\boldsymbol{x}^0 < \alpha < 2\alpha \leq \boldsymbol{p}\boldsymbol{x} \quad (\boldsymbol{x} \in C), \tag{19}$$

as asserted.

**2$^{\text{nd}}$STEP** :

Now, let $\boldsymbol{x}^0 \in \mathbb{R}^n$ be arbitrary. Define

$$C^0 := C - \boldsymbol{x}^0 = \{\boldsymbol{x} - \boldsymbol{x}^0 | \boldsymbol{x} \in C\}.$$

Then $C^0$ is nonempty, closed, and convex as it is the image of $C$ under the linear function (translation) $f : \mathbb{R}^n \to \mathbb{R}^n, \ f(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{x}^0$.

Clearly, $\mathbf{0} \notin C^0$ holds true. Hence, in view of our first step and in particular by (19), there is $\alpha_0$ and $\boldsymbol{p}$ such that

$$\boldsymbol{p}\ \mathbf{0} = 0 < \alpha_0 < 2\alpha_0 \leq \boldsymbol{p}\boldsymbol{y}$$

for all $\boldsymbol{y} \in C^0$. But for all $\boldsymbol{x} \in C$ we have $\boldsymbol{x} - \boldsymbol{x}^0 \in C^0$, hence we obtain

$$0 < \alpha_0 < 2\alpha_0 \leq \boldsymbol{p}(\boldsymbol{x} - \boldsymbol{x}^0) \quad (\boldsymbol{x} \in C)$$

or

$$\boldsymbol{p}\boldsymbol{x}^0 < \boldsymbol{p}\boldsymbol{x}^0 + \alpha_0 < \boldsymbol{p}\boldsymbol{x}^0 + 2\alpha_0 \leq \boldsymbol{p}\boldsymbol{x} \quad (\boldsymbol{x} \in C). \tag{20}$$

That is, using $\alpha := \boldsymbol{p}\boldsymbol{x}^0 + \alpha_0$ we obtain finally a series of inequalities

$$\boldsymbol{p}\boldsymbol{x}^0 < \alpha < \alpha + \alpha_0 \leq \boldsymbol{p}\boldsymbol{x} \quad (\boldsymbol{x} \in C), \tag{21}$$

which verifies our theorem for the general case. **q.e.d.**

We present some further versions of separation theorems.

**Theorem 1.12.** *1. Let $C \subseteq \mathbb{R}^n$ be a nonempty, open and convex set and let $\boldsymbol{x}^0 \notin C$. Then there is $0 \neq \boldsymbol{p} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that*

$$\boldsymbol{p}\boldsymbol{x}^0 = \alpha < \boldsymbol{p}\boldsymbol{x} \quad (\boldsymbol{x} \in C)$$

*holds true.*

*2. Let $C \subseteq \mathbb{R}^n$ be nonempty, closed and convex set and let $\boldsymbol{x}$ be a point in the boundary of $C$. Then there is $0 \neq \boldsymbol{p} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that*

$$\boldsymbol{p}\boldsymbol{x}^0 = \alpha \leq \boldsymbol{p}\boldsymbol{x} \quad (\boldsymbol{x} \in C)$$

*holds true.*

**Example 1.13.** Let

$$C = \left\{\boldsymbol{x} | x_2 \geq (x_1 - 1)^2 - 1\right\} \subseteq \mathbb{R}^2$$

$C$ is easily seen to be convex. Geometrically, this set is located above the graph of $f : \mathbb{R}^1 \to \mathbb{R}^1,\ f(x_1) = (x_1 - 1)^2 - 1$, see Figure 1.10

Furthermore, let us put $F : \mathbb{R}^2 \to \mathbb{R}$

$$F(\boldsymbol{x}) = x_2 + 1 - (x_1 - 1)^2,$$

then we obtain $C = \{\boldsymbol{x} | F(\boldsymbol{x}) \geq 0\}$. Consider the boundary of $C$ which is $\partial C = \{\boldsymbol{x} | F(\boldsymbol{x}) = 0\}$. Then, for some $\boldsymbol{x} \in \partial C$, we consider the ***gradient***

$$\frac{\partial F}{\partial \boldsymbol{x}}(\boldsymbol{x}) = \left(\frac{\partial F}{\partial x_1}(\boldsymbol{x}), \frac{\partial F}{\partial x_2}(\boldsymbol{x})\right) = (-2(x_1 - 1), 1)$$

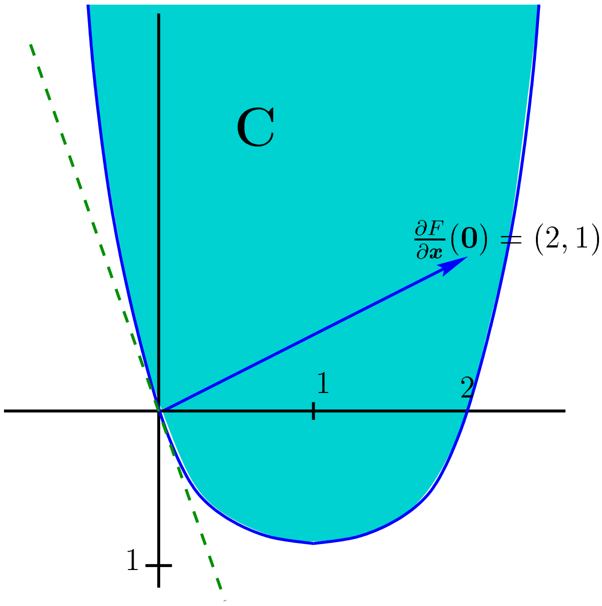



Figure 1.10: The Gradient Constitutes a Supporting Hyperplane

is perperdicular on $\partial C$, since $0 = F(\boldsymbol{x}) - F(\boldsymbol{x}') = (\boldsymbol{x} - \boldsymbol{x}')\frac{\partial F}{\partial \boldsymbol{x}}(\boldsymbol{x}) + \mathcal{O}(|\boldsymbol{x} - \boldsymbol{x}'|^2)$ for $\boldsymbol{x}, \boldsymbol{x}' \in \partial C$. For example, the gradient

$$\frac{\partial F}{\partial \boldsymbol{x}}(\mathbf{0}) = (2,1) =: \boldsymbol{p}$$

constitutes a hyperplane separating $\mathbf{0}$ from $C$. Indeed, we have $\boldsymbol{p}\mathbf{0} = 0$ on one hand and $\boldsymbol{p}\boldsymbol{x} = 2x_1 + x_2 \geq 2x_1 + x_2 - x_1^2 \geq 0$ for $\boldsymbol{x} \in C$ on the other hand.

Thus, if the boundary of $C$ is given via a differentiable function, for any point on the boundary the gradient of this function constitutes a separating hyperplane.

However, the function $F$ defined on $\mathbb{R}^2$ by

$$F(\boldsymbol{x}) = x_2 - |x_1| \quad (\boldsymbol{x} \in \mathbb{R}^2)$$

yields a set

$$D := \{\boldsymbol{x} | F(\boldsymbol{x}) \geq 0\} = \{\boldsymbol{x} | x_2 \geq |x_1|\}$$

(see Figure 1.11).

With respect to this set, $\mathbf{0}$ is a boundary point and any $\boldsymbol{p} \in \mathbb{R}^2$ with $p_2 \geq |p_1|$ constitutes a separating hyperplane. As we will see, separating or supporting

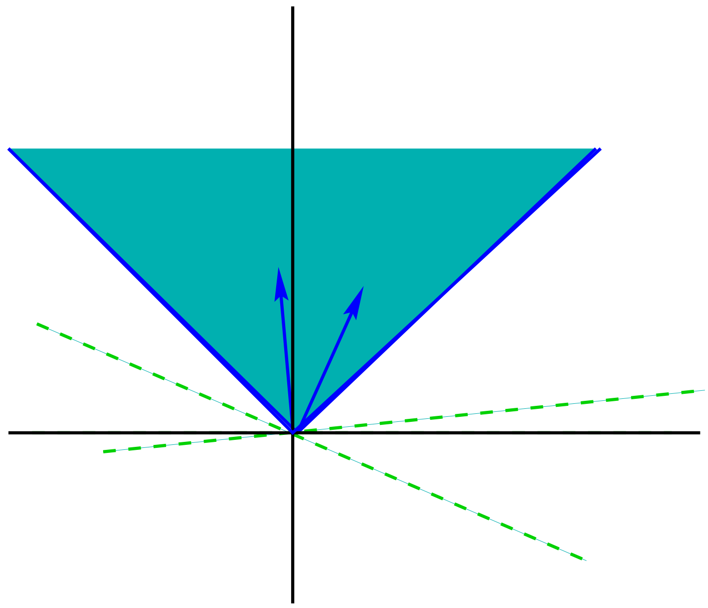


Figure 1.11: The Gradient Constitutes a Supporting Hyperplane

hyperplanes may serve as generalized derivatives in the context of Convex Analysis.

**Definition 1.14.** *1. Let $C$ and $D$ be subsets of $\mathbb{R}^n$, $\mathbf{0} \neq \boldsymbol{p} \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$. Put*

$$H_\alpha^{\boldsymbol{p}} = \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{px} = \alpha\}$$

*We shall say that $H_\alpha^{\boldsymbol{p}}$ separates $C$ and $D$*

*(a)* ***weakly*** *if*

$$\boldsymbol{pc} \leq \alpha \leq \boldsymbol{pd} \quad \boldsymbol{c} \in C, \boldsymbol{d} \in D$$

*(b)* ***strictly*** *if*

$$\boldsymbol{pc} \leq \alpha < \boldsymbol{pd} \quad (\boldsymbol{c} \in C, \boldsymbol{d} \in D)$$

*or*

$$\boldsymbol{pc} < \alpha \leq \boldsymbol{pd} \quad (\boldsymbol{c} \in C, \boldsymbol{d} \in D)$$

*(c)* ***strongly*** *if*

$$\boldsymbol{pc} < \alpha < \boldsymbol{pd} \quad (\boldsymbol{c} \in C, \boldsymbol{b} \in D)$$

*(d)* ***strongly***$^+$ *if*

$$\sup\{\boldsymbol{pc} | \boldsymbol{c} \in C\} < \alpha < \inf\{\boldsymbol{pd} | \boldsymbol{d} \in D\}$$

*holds true.*

2. *If $C \subseteq \mathbb{R}^n$ is convex and $\boldsymbol{x}^0$ is a boundary point of $C$, then a hyperplane separating $\boldsymbol{x}^0$ and $C$ (weakly, strictly,...) is also called (weakly, strictly,...)* ***supporting*** *$C$ at $\boldsymbol{x}^0$. (Of course, separating $\boldsymbol{x}^0$ and $C$ and separating $\{\boldsymbol{x}^0\}$ and $C$ is the same notion.)*

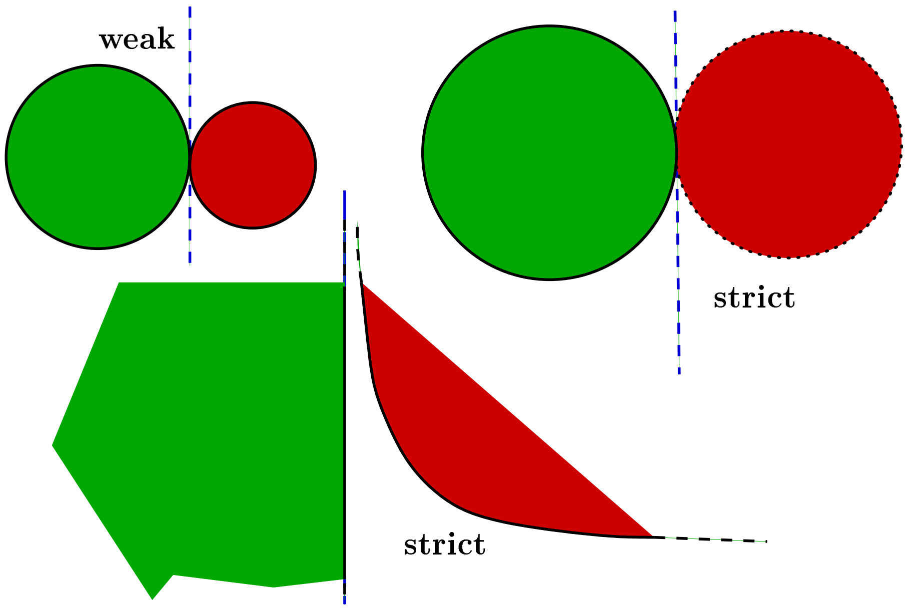



Figure 1.12: Versions of the Separation Theorem

Thus, Theorem 1.11 means that a closed convex set $C$ can be strongly$^+$ separated from any $\boldsymbol{x}^0 \notin C$. In Theorem 1.12 the second statement is that any point on the boundary of a closed convex set admits of a (weakly) supporting hyperplane. And Example 1.13 suggests that the gradient constitutes the unique supporting hyperplane at some point if the boundary of $C$ is smoothely represented by some differentiable function.

**Theorem 1.15.** *Let $C$ and $D \subseteq \mathbb{R}^n$ be nonempty and convex sets satisfying $C \cap D = \emptyset$. Assume that $C$ is open. Then $C$ and $D$ can be strictly separated.*

**Proof:** (Sketch) Define $F := C - D = \{\boldsymbol{c} - \boldsymbol{d} | \boldsymbol{c} \in C, \boldsymbol{d} \in D\}$.

It is not hard to see that $F$ is open and convex. Now, as $\boldsymbol{0} \notin F$, $\boldsymbol{0}$ and $F$ may be strictly separated (Theorem 1.12), i.e., we find $\boldsymbol{0} \neq \boldsymbol{p} \in \mathbb{R}^n$ and $0 = \alpha' \in \mathbb{R}$ such that

$$\boldsymbol{p}\boldsymbol{x} > 0 \quad (\boldsymbol{x} \in F)$$

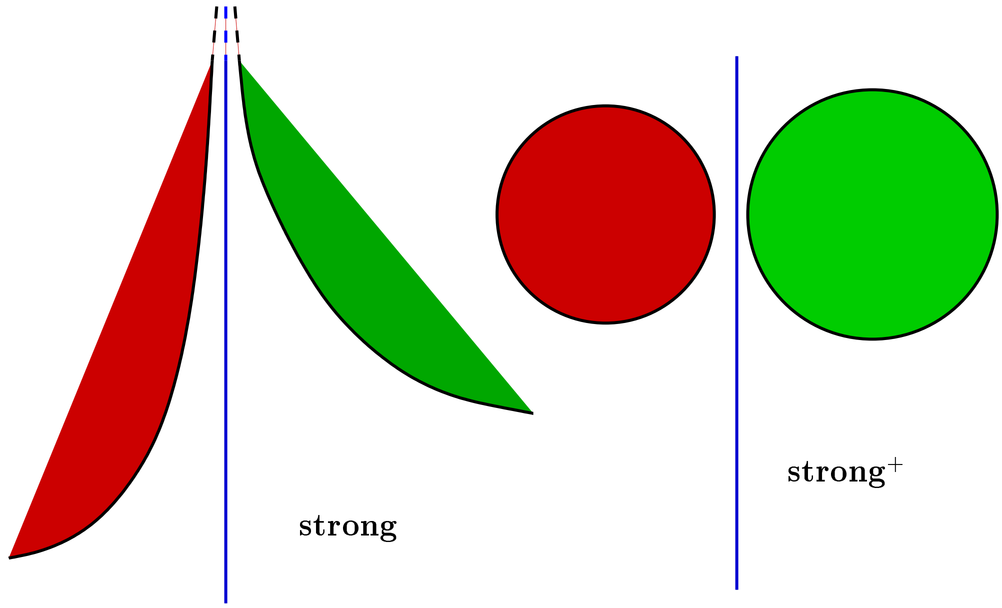



Figure 1.13: More Versions of the Separation Theorem

holds true, i.e. we have

$$\boldsymbol{p}(\boldsymbol{c}-\boldsymbol{d}) > 0 \quad \boldsymbol{c} \in C, \boldsymbol{d} \in D,$$

or

$$\boldsymbol{pc} > \boldsymbol{pd} \quad \boldsymbol{c} \in C, \boldsymbol{d} \in D$$

Let $\alpha := \sup\{\boldsymbol{pd} | \boldsymbol{d} \in D\}$. Then clearly

$$\boldsymbol{pc} \geq \alpha \geq \boldsymbol{pd} \quad (\boldsymbol{c} \in C, \boldsymbol{d} \in D).$$

folows at once. However, we claim that $\boldsymbol{pc} > \alpha \ \ (\boldsymbol{c} \in C)$ holds true. Indeed, if $\boldsymbol{px} = \alpha$ for some $\boldsymbol{x} \in C$, then, assuming w.l.o.g. $p_1 \neq 0$, the vector $\boldsymbol{x}^\varepsilon := \boldsymbol{x} - \frac{p_1}{|p_1|^2}(\varepsilon, 0, \cdots, 0)$ yields

$$\boldsymbol{px}^\varepsilon = \boldsymbol{px} - \varepsilon < \boldsymbol{px} = \alpha$$

for all $\varepsilon > 0$, thus $\boldsymbol{x}^\varepsilon \notin C$ for all $\varepsilon > 0$. But as $\boldsymbol{x}^\varepsilon \to \boldsymbol{x} \ \ (\varepsilon \to 0)$, $\boldsymbol{x}$ is a boundary point of $C$, thus $\boldsymbol{x} \notin C$ as $C$ is open. **q.e.d.**

We provide some further versions of separation theorems.

**Theorem 1.16.** *Let $C$ and $D$ be convex sets in $\mathbb{R}^n$ satisfying $C \cap D = \emptyset$.*

1. *If $C$ is closed and $D$ is compact then both sets may be strongly$^+$ separated.*

2. *If both sets are closed then they may be strictly separated.*

3. *If $C$ is closed and $\boldsymbol{x}^0$ is a boundary point, then $\boldsymbol{x}^0$ admits of a (weakly) supporting hyperplane, i.e., $C$ and $\boldsymbol{x}^0$ may be weakly separated.*

4. *If $C$ is a convex polyhedron and $D$ is an open half-line of the form*

$$D = \{t\boldsymbol{b} | t > 0\}$$

*for some $\boldsymbol{b} \in \mathbb{R}^n$, then strict separation is achieved, i.e., there is $\boldsymbol{0} \neq \boldsymbol{p} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ such that*

$$\boldsymbol{pa} \leq \alpha < \boldsymbol{p}t\boldsymbol{b} \quad (\boldsymbol{a} \in C, t > 0).$$

We shall only prove the last statement.

**Proof:**

**1$^{\text{st}}$STEP** :

First of all assume $\boldsymbol{0} \in C$. As $C$ is a convex polyhedron, there is a matrix $\boldsymbol{A}$ and a vector $\boldsymbol{d}$ such that

$$C = \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{Ax} \leq \boldsymbol{d}\}$$

holds true.

Since $t\boldsymbol{b} \notin C$ $(t > 0)$ we can find, for any $t > 0$, some $k$ such that $\boldsymbol{A}_{k\bullet}t\boldsymbol{b} > d_k$ holds true. Since $k$ attains only finitely many values (i.e., the rows of $\boldsymbol{A}$), we can choose a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to 0$ $(n \in \mathbb{N})$ and

$$\boldsymbol{A}_{k\bullet}t_n\boldsymbol{b} > d_k$$

for some fixed $k$. Thus, $d_k \leq 0$ as $t_n \to 0$ $(n \in \mathbb{N})$. And as $\boldsymbol{A}_{k\bullet}t_n\boldsymbol{b} > 0$ holds true, we have $\boldsymbol{A}_{k\bullet}\boldsymbol{b} > 0$ or $\boldsymbol{A}_{k\bullet}t\boldsymbol{b} > 0$ $(t > 0)$. I.e., choosing $\boldsymbol{p} := \boldsymbol{A}_{k\bullet}$ and $\alpha := 0$ we achieve our goal.

**2$^{\text{nd}}$STEP** :

Next assume $\boldsymbol{0} \notin C$. Define

$$\alpha \ := \ \min\left\{|t\boldsymbol{b} - \boldsymbol{x}|^2 | \ t \geq 0, \ \boldsymbol{x} \in C\right\} \ := \ \bar{t}\boldsymbol{b} - \bar{\boldsymbol{x}},$$

with suitable $\bar{\boldsymbol{t}} \geq 0,\ \bar{\boldsymbol{x}} \in C$.

We have necessarily $\alpha > 0$. For, if $\alpha = 0$ prevails, then $\bar{\boldsymbol{t}}\boldsymbol{b} = \bar{\boldsymbol{x}} \in C$ and as $\mathbf{0} \notin C$ we must have $\bar{\boldsymbol{t}} > 0$. This implies $\bar{\boldsymbol{x}} \in D \cap C$, which we have excluded.

Now $\bar{\boldsymbol{x}}$ minimizes the distance of $\bar{\boldsymbol{t}}\boldsymbol{b}$ towards $C$ and, vice versa, $\bar{\boldsymbol{t}}\boldsymbol{b}$ minimizes the distance of $\bar{\boldsymbol{x}}$ towards $D$. As in Theorem 1.11 we see by a simple computation that $\bar{\boldsymbol{p}} := \bar{\boldsymbol{t}}\boldsymbol{b} - \bar{\boldsymbol{x}}$ yields a hyperplane

$$H^{\bar{\boldsymbol{p}}}_{\frac{\alpha}{2}} := \left\{ \boldsymbol{x} \in \mathbb{R}^n \,\middle|\, \bar{\boldsymbol{p}}\boldsymbol{x} = \frac{\alpha}{2} \right\}$$

that separates $B$ and $C$ strongly$^+$. (In fact, the hyperplane separates $C$ and $\overline{D} = D \cup \{\mathbf{0}\}$ strongly$^+$). **q.e.d.**

The next series of statements can be seen as a set of applications of separation procedures.

As preparation we prove a simple Lemma (1.17), the technique of which is pure linear algebra. The following (1.18) is then the analogue concerning inequalities or convexity theory.

**Lemma 1.17.** *Let $\boldsymbol{A}$ be an $m \times n$ matrix and let $\boldsymbol{b} \in \mathbb{R}^m$. Then one and only one of the following two statements is true.*

*(a) There exists $\bar{\boldsymbol{x}} \in \mathbb{R}^n$ such that*

$$\boldsymbol{A}\bar{\boldsymbol{x}} = \boldsymbol{b}$$

*holds true.*
*(b) There is $\bar{\boldsymbol{u}} \in \mathbb{R}^m$ such that*

$$\bar{\boldsymbol{u}}\boldsymbol{A} = \mathbf{0}, \bar{\boldsymbol{u}}\boldsymbol{b} = 1$$

*holds true.*

**Proof:**

**1$^{\text{st}}$STEP** :

Clearly, not both statements can be true. For if $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{u}}$, as described in (a) and (b) exist simultaneously, then we have:

$$\mathbf{0} = \bar{\boldsymbol{u}}\boldsymbol{A}$$

hence

$$0 = \bar{\boldsymbol{u}}\boldsymbol{A}\ \bar{\boldsymbol{x}} = \bar{\boldsymbol{u}}\ \boldsymbol{A}\bar{\boldsymbol{x}} = \bar{\boldsymbol{u}}\boldsymbol{b} = 1,$$

a contradiction.

**$2^{\text{nd}}$STEP** : Suppose, statement (a) is wrong, that is, there is no solution of the linear system of equations $\boldsymbol{Ax} = \boldsymbol{b}$. Then we have

$$\begin{aligned}
\boldsymbol{rank} \begin{pmatrix} \boldsymbol{A} & \begin{matrix} b_1 \\ \vdots \\ b_m \end{matrix} \end{pmatrix} \quad &= \boldsymbol{rank}\ \boldsymbol{A} + 1 \\
= \boldsymbol{rank}\ \boldsymbol{A}^T + 1 \quad &= \boldsymbol{rank} \begin{pmatrix} \boldsymbol{A}^T & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{pmatrix} + 1 \\
= \boldsymbol{rank} \begin{pmatrix} \boldsymbol{A}^T & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ b_1 \dots b_m & 1 \end{pmatrix} \quad &\geq \boldsymbol{rank} \begin{pmatrix} \boldsymbol{A}^T \\ b_1 \dots b_m \end{pmatrix} \\
&= \boldsymbol{rank} \begin{pmatrix} \boldsymbol{A} & \begin{matrix} b_1 \\ \vdots \\ b_m \end{matrix} \end{pmatrix}
\end{aligned}$$

Hence, all inequalities involved are necessarily equations; in particular we obtain

$$\boldsymbol{rank} \begin{pmatrix} \boldsymbol{A}^T \\ b_1 \dots b_m \end{pmatrix} = \boldsymbol{rank} \begin{pmatrix} \boldsymbol{A}^T & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ b_1 \dots b_m & 1 \end{pmatrix}$$

meaning that the linear system of equations

$$\boldsymbol{uA} = 0, \quad \boldsymbol{ub} = 1$$

does have a solution, i.e., statement (b) is true. **q.e.d.**

Note that (b) can be replaced by a formulation that calls for $\boldsymbol{ub} < 0$.

The above lemma is a topic of Linear Algebra. We now discuss the analogous version within the territory of Convex Analysis.

**Theorem 1.18 (Farkas' Lemma, Theorem of the Alternative).** *Let $\boldsymbol{A}$ be an $m \times n$ matrix and let $\boldsymbol{b} \in \mathbb{R}^m$. Then one and only one of the following*

*statements is true.*
*(a) There exists* $\bar{\boldsymbol{x}} \in \mathbb{R}^n$ *such that*

$$\boldsymbol{A}\bar{\boldsymbol{x}} = \boldsymbol{b} \quad , \quad \bar{\boldsymbol{x}} \geq \boldsymbol{0}$$

*holds true.*
*(b) There exists* $\bar{\boldsymbol{u}} \in \mathbb{R}^m$ *such that*

$$\bar{\boldsymbol{u}}\boldsymbol{A} \geq \boldsymbol{0} \quad , \quad \bar{\boldsymbol{u}}\boldsymbol{b} < 0$$

*holds true.*

**Proof:** We may assume $\boldsymbol{b} \neq \boldsymbol{0}$ - otherwise the theorem is obviously true.

**1$^{\text{st}}$STEP** : Not both statements can be true simultaneously. For, if $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{u}}$ as described in (a) and (b) exist simultaneously, then

$$\bar{\boldsymbol{u}}\boldsymbol{A} \geq \boldsymbol{0}$$

hence

$$0 \leq \bar{\boldsymbol{u}}\boldsymbol{A}\ \bar{\boldsymbol{x}} = \bar{\boldsymbol{u}}\ \boldsymbol{A}\bar{\boldsymbol{x}} = \bar{\boldsymbol{u}}\boldsymbol{b} < 0$$

- a contradiction.

**2$^{\text{nd}}$STEP** : Assume now that (a) is wrong. Consider the set

$$C = \{\boldsymbol{A}\boldsymbol{x} | \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{x} \geq \boldsymbol{0}\}$$

then clearly, $\boldsymbol{b} \notin C$. Equivalently we can say

$$t\boldsymbol{b} \notin C \qquad (t > 0),$$

or, in other words

$$C \cap B = \emptyset$$

with $B = \{t\boldsymbol{b} | t > 0\}$. The sets C and B are convex (and C is a convex polyhedron, see Remark 1.3, *No. 5* ). Using Theorem 1.16 (*No. 4*), we find a strictly separating hyperplane, i.e., $\boldsymbol{p} \neq \boldsymbol{0}$ and $\alpha \in \mathbb{R}$ such that

$$\boldsymbol{p}\boldsymbol{z} \leq \alpha < \boldsymbol{p}t\boldsymbol{b} \qquad (\boldsymbol{z} \in C,\ t > 0).$$

Clearly $\alpha = 0$ as $t$ may be arbitrarily small and $\boldsymbol{0} \in C$.

Next, for the basis vectors $\boldsymbol{e}^j \in \mathbb{R}^n_+$ we have $\boldsymbol{A}\boldsymbol{e}^j = \boldsymbol{A}_{\bullet j} \in C$, thus

$$\boldsymbol{p}\boldsymbol{A}_{\bullet j} \leq 0 \quad (j = 1 \dots n)$$

or

$$\boldsymbol{p}\boldsymbol{A} \leq 0 \quad , \quad \boldsymbol{p}\boldsymbol{b} > 0 \; .$$

Now $\bar{\boldsymbol{u}} := -\boldsymbol{p}$ satisfies the desired condition, hence statement (b) is true. **q.e.d.**

There are several versions of *Farkas' Lemma*, which are more or less equivalent. The proofs are, therefore, derived from the one version we have already checked. Let us mention the following

**Theorem 1.19.** *Let $\boldsymbol{A}$ be an $m \times n$ matrix and let $\boldsymbol{b} \in \mathbb{R}^m$. One and only one of the following statements is true.*
*(a) There exists $\boldsymbol{x} \in \mathbb{R}^n$ such that*

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$

*holds true.*
*(b) There exists $\boldsymbol{u} \in \mathbb{R}^m$ such that*

$$\boldsymbol{u}\boldsymbol{A} = \boldsymbol{0} \quad , \quad \boldsymbol{u}\boldsymbol{b} < 0 \quad , \quad \boldsymbol{u} \geq 0$$

*holds true.*

**Proof:** Consider the matrix

$$\bar{\boldsymbol{A}} := (\boldsymbol{A}, -\boldsymbol{A}, \boldsymbol{I})$$

(with $m \times m$ identity matrix $\boldsymbol{I}$) and apply Theorem 1.18. Then one and only one of the following two statements is true.

($\bar{a}$) The system

$$\bar{\boldsymbol{A}}(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}) = \boldsymbol{b}, \quad (\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}) \geq \boldsymbol{0}$$

has a solution or

($\bar{b}$) the system

$$\boldsymbol{u}\bar{\boldsymbol{A}} \geq 0 \quad \boldsymbol{w}\boldsymbol{b} < 0$$

has a solution. Rewriting the details, we see that

($\bar{a}$) either the system

$$\boldsymbol{A}\boldsymbol{y} - \boldsymbol{A}\boldsymbol{z} + \boldsymbol{w} = \boldsymbol{b} \quad \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w} \geq \boldsymbol{0}$$

has a solution or else

$(\bar{b})$ the system

$$\boldsymbol{uA} \geq \mathbf{0}, \quad -\boldsymbol{uA} \geq \mathbf{0}, \quad \boldsymbol{u} \geq \mathbf{0}, \quad \boldsymbol{ub} < 0$$

has a solution. Again this can be rewritten suitably. We obtain

$(\bar{a})$

$$\boldsymbol{A}(\boldsymbol{y} - \boldsymbol{z}) + \boldsymbol{w} = \boldsymbol{b} \quad \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w} \geq \mathbf{0}$$

has a solution or

$(\bar{b})$

$$\boldsymbol{uA} = \mathbf{0}, \quad \boldsymbol{u} \geq \mathbf{0}, \quad \boldsymbol{ub} < 0$$

has a solution. However, $(\bar{a})$ is equivalent to $(a)$ as $\boldsymbol{w} \geq \mathbf{0}$ holds true. **q.e.d.**

# 2 Extremal Points

Convex analysis provides its own notion of a "boundary" which in general is different from the topogical notion.

**Definition 2.1.** *1. Let $C \subseteq \mathbb{R}^n$ be a convex set. $\bar{\boldsymbol{x}} \in C$ is called an* ***extremal point*** *of $C$ if, for all $\boldsymbol{x}^0, \boldsymbol{x}^1 \in C$ and all $\lambda \in (0,1)$ satisfying $\bar{\boldsymbol{x}} = \boldsymbol{x}^\lambda$ it follows that*

$$\boldsymbol{x}^0 = \boldsymbol{x}^1 = \bar{\boldsymbol{x}}$$

*holds true.*

2. *Extremal points of a convex polyhedron are called* ***vertices****.*

3. ***Ext*** *$C$ denotes the set of extremal points of a convex set $C \subseteq \mathbb{R}^n$.*

An extremal point of a convex set is a point that cannot be located within the interior of an interval generated by two other points of that set. Or, to put it closer to the definition, whenever an extremal point is located within an interval generated by two other points, then this interval has to be trivial.

**Example 2.2.** 1. Consider the convex polyhedron

$$C = \left\{ \boldsymbol{x} \in \mathbb{R}^3 | 0 \le x_i \le 1 \quad (i = 1,2,3), \quad \sum_{i=1}^{3} x_i \le 2 \right\}$$

which is depicted in Figure 2.1

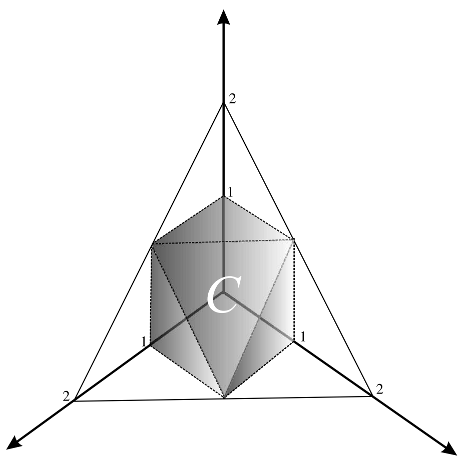



Figure 2.1: A convex polyhedron $C$

We claim that $\bar{\boldsymbol{x}} = (1, 1, 0)$ (and likewise $(0, 1, 1), (1, 0, 1)$) is extremal, i.e., a vertex. Indeed, assume that

$$\boldsymbol{x}^0, \boldsymbol{x}^1 \in C, \quad \lambda \in (0, 1)$$

is such that $\bar{\boldsymbol{x}} = \boldsymbol{x}^\lambda = (1 - \lambda)\boldsymbol{x}^0 + \lambda \boldsymbol{x}^1$.

Since $x_3^0 \geq 0$ and $x_3^1 \geq 0$ as well $\bar{x}_3 = 0$ is the case, we obtain the equations $x_3^0 = x_3^1 = 0$.

Next, if $x_1^0 < 1$ should be the case, then it would follow that

$$\begin{aligned} 1 = \bar{x}_1 \quad &= (1 - \lambda)x_1^0 + \lambda x_1^1 \\ &< (1 - \lambda)1 + \lambda x_1^1 \\ &\leq (1 - \lambda) + \lambda = 1, \end{aligned}$$

a contradiction. H ence $x_1^0 = 1$. Similarly $x_2^0 = 1$, i.e., $\boldsymbol{x}^0 = \bar{\boldsymbol{x}}$. Analogously $\boldsymbol{x}^1 = \bar{\boldsymbol{x}}$; that is $\bar{\boldsymbol{x}}$ is extremal.

2. We call the set

$$\Pi^n = \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \boldsymbol{x} \geq 0, \sum_{i=1}^{n} x_i = 1 \right\}$$

the ***unit prism*** of $\mathbb{R}^n$. Figure 2.2 shows a picture in $\mathbb{R}^3$.

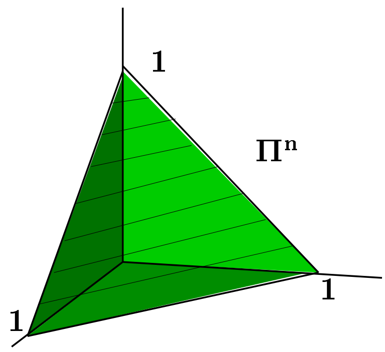



Figure 2.2: The Unit Prism of $\mathbb{R}^3$

We claim that the unit vector $\boldsymbol{e}^1 = (1, 0, 0)$ is a vertex.

Indeed, if

$$\boldsymbol{e}^1 = \boldsymbol{x}^\lambda = (1 - \lambda)\boldsymbol{x}^0 + \lambda \boldsymbol{x}^1 \quad (x^i \in C, \quad i = 1, 2)$$

for $\lambda \in (0,1)$ holds true, then, for $i \neq 1$ we have

$$0 = e_i^1 = (1-\lambda)\underbrace{x_i^0}_{\geq 0} + \lambda \underbrace{x_i^1}_{\geq 0} \geq 0,$$

which implies that $x_i^0 = x_i^1 = 0$ is the case. For $i = 1$ we have in turn

$$1 = e_1^1 = (1-\lambda)\underbrace{x_1^0}_{\leq 1} + \lambda \underbrace{x_1^1}_{\leq 1} \leq 1.$$

This implies $x_1^0 = x_1^1 = 1$; hence $\boldsymbol{x}^0 = \boldsymbol{x}^1 = \boldsymbol{e}^1$ which verifies the desired result. Similarly, it is seen that $\mathbf{0}$ is a vertex of $\Pi^n$.

3. Recall that the unit simplex in $\mathbb{R}^n$ is given by

$$\underline{\overline{\mathbf{X}}}^n = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \,\middle|\, \boldsymbol{x} \geq 0, \sum_{i=1}^{n} x_i = 1 \right\}.$$

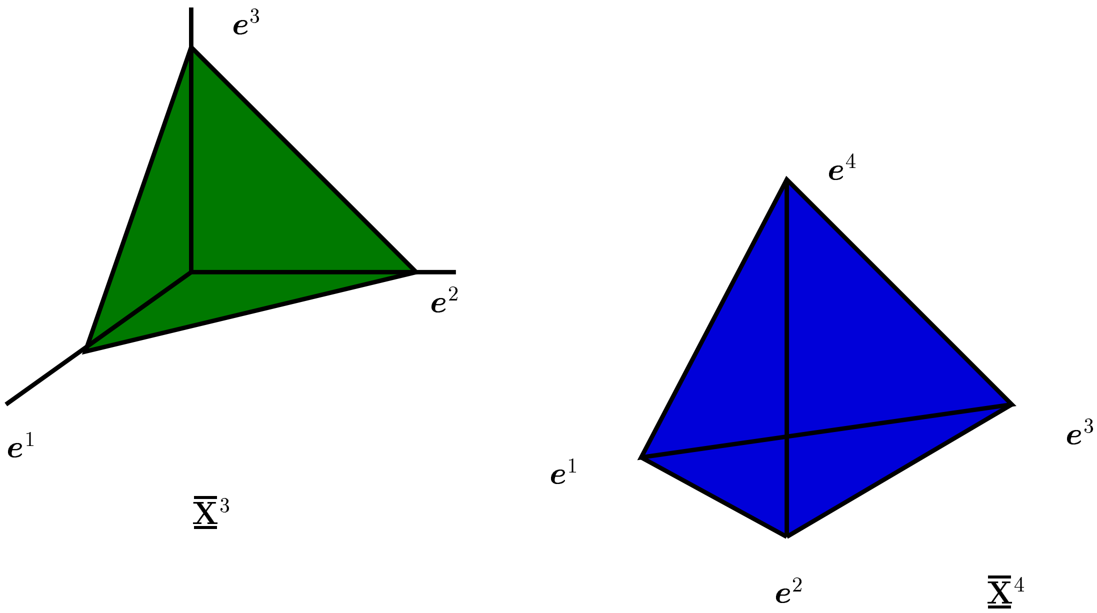



Figure 2.3: The Simplices in $\mathbb{R}^3, \mathbb{R}^4$

It is rather obvious that the unit vectors $\underbrace{\boldsymbol{e}^i = (0,\ldots,0,1,0,\ldots 0,,0)}_{i}$ are exactly the vertices.

4. The *closed* unit sphere

$$S = \{\boldsymbol{x} \in \mathbb{R}^n \mid |\boldsymbol{x}| \ \leq 1\}$$

satisfies $\boldsymbol{Ext}\, S = \{\boldsymbol{x} \in \mathbb{R}^n \mid |\boldsymbol{x}| \ = 1\}$. However, the *open* unit sphere has *no* extremal points!

**Remark 2.3.** *The relation between the boundary concept of topology and the one of convex analysis is rather involved. We denote the (topological) boundary of a set $C \subseteq \mathbb{R}^n$ by $\partial C$.*

*In the above cases we observe examples of convex sets $C$ such that $C$ as well as $\boldsymbol{Ext}\, C$ are compact sets and coincide, like the closed unit sphere. For the simplex and the prism we see that $\boldsymbol{Ext}\, C$ and $\partial C$ are compact but different.*

*In Figure 2.4 we observe a noncompact bounded convex set $C$ whith compact $\boldsymbol{Ext}\, C$. We may include the straight line*

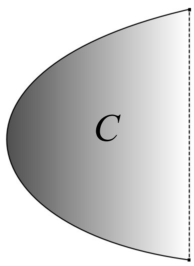



Figure 2.4: A Noncompact Bounded Convex Set $C$

*Note that the topological boundary $\partial C$ differs from $\boldsymbol{Ext}\, C$.*

*Finally, we present a famous example of a compact convex set $C$ which has a noncompact set of extremal points $\boldsymbol{Ext}\, C$. This is shown in Figure 2.5. Note that the point $x$ is not extremal, i.e., we have $x \notin \boldsymbol{Ext}\, C$, $\boldsymbol{Ext}\, C \neq \partial C$.*

In what follows, topological properties as well as convexity of sets are assumed and the relation between the two concepts is discusseed.

The basic theorem concerning the existence of extremal points is due to MINKOWSKI:

**Theorem 2.4 (Minkowski).** *For every convex, compact set $\emptyset \neq C \subseteq \mathbb{R}^n$ the set of extremal points is nonempty, i.e.,$\boldsymbol{Ext}\, C \neq \emptyset$.*

**Proof:** We proceed by induction in $n$, the dimension of the Euclidean space we are working in.

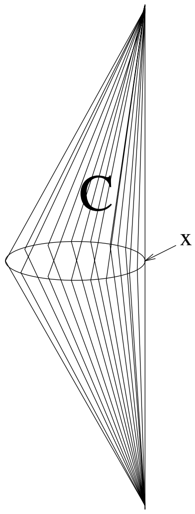



Figure 2.5: The Double–Cone

**1<sup>st</sup>STEP** : $n = 1$ In this case $C$ is a compact interval, say $C = [a, b]$ with $a \leq b$ and there is at least one extremal point, i.e., a boundary point of the interval.

**2<sup>nd</sup>STEP** : $n > 1$ Since $t$ is compact, the quantity

$$\bar{t} := max \; \{x_1 \mid \boldsymbol{x} \in C \;\; \}$$

is a well defined real number. Thus we have

$$\begin{aligned} \emptyset \neq \bar{C} &:= \{\boldsymbol{x} \in C \mid x_1 = \bar{t}\,\} \\ &= C \cap \{\boldsymbol{x} \in \mathbb{R}^n \mid x_1 = \bar{t}\,\} =: C \cap H^1_{\bar{t}} \end{aligned}$$

The set $H^1_{\bar{t}} := \{\boldsymbol{x} \in \mathbb{R}^n \mid x_1 = \bar{t}\}$ is a hyperplane, and indeed an affine subspace of $\mathbb{R}^n$ of dimension $n-1$. This space is linearly isomorphic to $\mathbb{R}^{n-1}$ - we may actually translate it so that it contains the **0**-vector or, equivalently, just assume that $\bar{t} = 0$.

Therefore, $\bar{C}$ is a convex, compact subset of an $n-1$ dimensional Euclidean space. By induction hypothesis, $\bar{C}$ has at least one extremal point, say $\bar{\boldsymbol{x}} \in \bar{C}$.

**3<sup>rd</sup>STEP** : We claim that $\bar{\boldsymbol{x}}$ is an extremal point of $C$ as well.

To this end, assume that there is $\boldsymbol{x}^0, \boldsymbol{x}^1 \in C$ and $\lambda \in (0, 1)$ such that

$$\bar{\boldsymbol{x}} = \boldsymbol{x}^{\lambda} = (1 - \lambda)\boldsymbol{x}^0 + \lambda \boldsymbol{x}^1$$

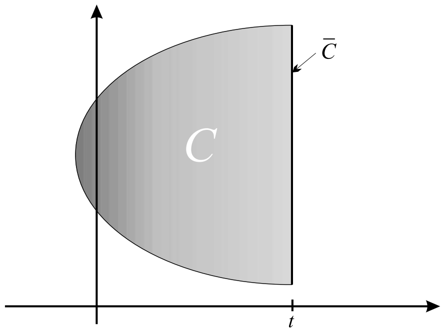



Figure 2.6: induction in $n$ (MINKOWSKI)

holds true. Then clearly $x_1^0, x_1^1 \leq \bar{t}$. If $x_1^0 < \bar{t}$ should hold true, then

$$\begin{aligned}\bar{x}_1 &= (1-\lambda)x_1^0 + \lambda x_1^1 < (1-\lambda)\bar{t} + \lambda x_1^1 \\ &\leq (1-\lambda)\bar{t} + \lambda\bar{t} = \bar{t},\end{aligned}$$

would follow, contradicting $\bar{\boldsymbol{x}} \in \bar{C}$. Thus $x_1^0 = \bar{t}$ and likewise $x_1^1 = \bar{t}$. This implies that

$$\boldsymbol{x}^0 \in \bar{C}\ ,\ \boldsymbol{x}^1 \in \bar{C}$$

is the case. However, $\bar{\boldsymbol{x}}$ is an extremal point of $\bar{C}$ which implies that $\boldsymbol{x}^0 = \boldsymbol{x}^1 = \bar{\boldsymbol{x}}$ has indeed been verified, **q.e.d.**

Let us now turn to the specific situation of convex polyhedra and analyze the properties of extremal points or ***vertices*** within this context.

We introduce the following notation: For any $m \times n$ matrix $\boldsymbol{A}$ and $\boldsymbol{b} \in \mathbb{R}^m$ let us use the notion

$$C_{\boldsymbol{A},\boldsymbol{b}} := \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}. \tag{1}$$

Recall our standard notation referring concerning the index sets referring to the rows and columns of such a matrix; i.e., we use

$$I := \{1, \ldots, m\} \quad J := \{1, \ldots, n\} \tag{2}$$

**Theorem 2.5.** *Let $\boldsymbol{A}$ be an $m \times n$ matrix and let $\boldsymbol{b} \in \mathbb{R}^m$. Assume that $\bar{\boldsymbol{x}} \in C_{\boldsymbol{A},\boldsymbol{b}}$. Then $\bar{\boldsymbol{x}}$ is a vertex of $C_{\boldsymbol{A},\boldsymbol{b}}$ if and only if the following holds true.*

*There is a subset of indices $I_0 \subseteq I$ satisfying $|I_0| = n$ such that the following holds true:*

1. *The vectors $(\boldsymbol{A}_{i\bullet})_{i \in I_0}$ are linearly independent*

2. *The equations*

$$\boldsymbol{A}_{i\bullet}\bar{\boldsymbol{x}} = b_i \quad (i \in I_0) \tag{3}$$

*hold true.*

Clearly, if the two conditions are satisfied, then $\bar{\boldsymbol{x}}$ is the *unique solution* of the system of linear equations (3). That is, an extremal points corresponds to a set of linear equations chosen among the inequalities defining the convex polyhedron, having a nonsingular coefficient matrix and defining this extremepoint uniquly.

**Proof:**

**1$^{\text{st}}$STEP** : First of all we show that an extremal point generates the corresponding system of linearly independent equations. To this end, let $\bar{\boldsymbol{x}} \in C_{\boldsymbol{A},\boldsymbol{b}}$ be a vertex of $C_{\boldsymbol{A},\boldsymbol{b}}$. Define

$$I' := \{i \in I | \boldsymbol{A}_{i\bullet}\bar{\boldsymbol{x}} = b_i\}$$

If there are $n$ linearly independent vectors among the vectors $\boldsymbol{A}_{i\bullet} \quad (i \in I')$, then we are done with the *'only if'* part of our proof. If this is *not* the case, then consider the affine subspace of $\mathbb{R}^n$ given by

$$U' = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}_{i\bullet}\boldsymbol{x} = b_i \quad (i \in I')\}\,;$$

we will use this subspace to produce a contradiction to the assumption that $\bar{\boldsymbol{x}}$ is a vertex.

Indeed, $\bar{\boldsymbol{x}} \in U'$ holds obviously true and $U'$ has dimension $\boldsymbol{dim}\ U' \geq 1$ as

$$\boldsymbol{rank}\ (\boldsymbol{A}_i)_{i\in I'} \leq n - 1$$

is the case. Choose $\hat{\boldsymbol{x}} \in U'$ such that $\hat{\boldsymbol{x}} \neq \bar{\boldsymbol{x}}$ and define for $t \in \mathbb{R}$

$$\boldsymbol{x}^{\pm t} := (1 \pm t)\bar{\boldsymbol{x}} \mp t\hat{\boldsymbol{x}} \tag{4}$$

Then clearly

$$\boldsymbol{A}_{i\bullet}\boldsymbol{x}^{\pm t} = b_i \quad (i \in I') \tag{5}$$

holds true, i.e., we have $\boldsymbol{x}^{\pm t} \in U' \quad (t \in \mathbb{R})$.

Now, for $i \notin I'$ we have $\boldsymbol{A}_{i\bullet}\bar{\boldsymbol{x}} < b_i$. Hence, for sufficiently small $t \in \mathbb{R}$

$$\boldsymbol{A}_{i\bullet}\boldsymbol{x}^{\pm t} < b_i \quad (i \notin I') \tag{6}$$

is true. But (5) and (6) imply that

$$x^{\pm t} \in C_{\boldsymbol{A},\boldsymbol{b}} \tag{7}$$

holds true for sufficiently small $t \in \mathbb{R}$. Observing

$$\bar{\boldsymbol{x}} = \frac{1}{2}(x^{+t} + x^{-t}) \tag{8}$$

we come up with a contradiction: $\bar{\boldsymbol{x}}$ is located within the interval $[x^{+t}, x^{-t}]$, both points are located in $C$ and differ from $\bar{\boldsymbol{x}}$ as $\hat{\boldsymbol{x}}$ does. This is not compatible with $\bar{\boldsymbol{x}}$ being an extremal point of $C$.

**2ndSTEP** :

On the other hand, suppose now that $\bar{\boldsymbol{x}} \in C_{\boldsymbol{A},\boldsymbol{b}}$ allows for an index set $I_0$ satisfying *items 1.* and *2.* of our theorem. We have to show that $\bar{\boldsymbol{x}}$ is extremal in $C$. Now, $\bar{\boldsymbol{x}}$ is the unique solution of the linear system of equations

$$\boldsymbol{A}_{i\bullet}\boldsymbol{x} = b_i \quad (i \in I_0). \tag{9}$$

Let $\boldsymbol{x}^0, \boldsymbol{x}^1 \in C_{\boldsymbol{A},\boldsymbol{b}}$ and $\lambda \in (0,1)$ be such that

$$\boldsymbol{x}^\lambda = \bar{\boldsymbol{x}}.$$

Then, for $i \in I_0$:

$$\begin{aligned} b_i = \boldsymbol{A}_{i.}\bar{\boldsymbol{x}} &= \boldsymbol{A}_{i.}\left((1-\lambda)\boldsymbol{x}^0 + \lambda\boldsymbol{x}^1\right) \\ &= (1-\lambda)\boldsymbol{A}_{i.}\boldsymbol{x}^0 + \lambda\boldsymbol{A}_{i.}\boldsymbol{x}^1 \\ &\leq (1-\lambda)b_i + \lambda b_i \\ &= b_i \quad (i \in I_0); \end{aligned}$$

the inequalities are satisfied since $\boldsymbol{x}^0, \boldsymbol{x}^1 \in C_{\boldsymbol{A},\boldsymbol{b}}$. Obviously, none of these inequalities can be a strict one. Hence

$$\boldsymbol{A}_{i.}\boldsymbol{x}^0 = b_i \quad , \quad \boldsymbol{A}_{i.}\boldsymbol{x}^1 = b_i \quad (i \in I_0).$$

meaning that $\boldsymbol{x}^0$ and $\boldsymbol{x}^1$ are solutions of (9). But (9) has only one solution, thus

$$\boldsymbol{x}^0 = \boldsymbol{x}^1 = \bar{\boldsymbol{x}},$$

i.e., $\bar{\boldsymbol{x}}$ is extremal in $C$. **q.e.d.**

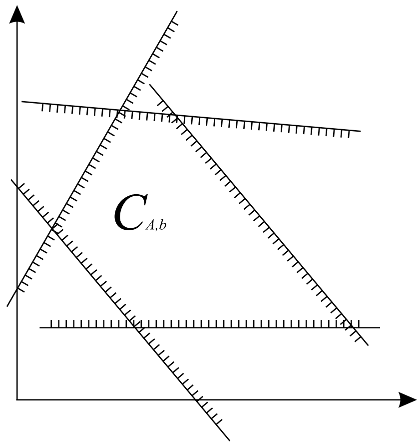



Figure 2.7: $C_{\boldsymbol{A},\boldsymbol{b}} \subseteq \mathbb{R}^2$

**Remark 2.6.** *1. A convex polyhedron has at most finitely many vertices.*

*2. Our intuition is that we reach boundary points of $C_{\boldsymbol{A},\boldsymbol{b}}$ if some of the defining inequalities become equations.*

*In $\mathbb{R}^2$ is is rather obvious that a vertex is obtained if two of the defining hyperplanes (lines) intersect. Note that two of those lines may, in addition, intersect at certain points of $\mathbb{R}^2$ that are not elements of $C_{\boldsymbol{A},\boldsymbol{b}}$. Among the conditions of Theorem 2.5 it is, therefore, explicitely stated that $\bar{\boldsymbol{x}} \in C_{\boldsymbol{A},\boldsymbol{b}}$ should be satisfied.*

*We now observe that Theorem 2.5 indicates a constructive procedure to obtain all vertices of a convex polyhedron as follows:*

**1st STEP** :

*Pick $n$ arbitrary rows of the matrix $\boldsymbol{A}$. That is, fix an index set $I_0 \subseteq I$, satisfying $|I_0| = n$.*

**2nd STEP** :

*Check whether the linear system of equations*

$$\boldsymbol{A}_{i.}\boldsymbol{x} = b_i \quad (i \in I_0) \tag{10}$$

*has a **unique** solution $\bar{\boldsymbol{x}}$ (i.e., whether **rank** $(\boldsymbol{A}_i)_{i \in I_0} = n$ holds true).*

**3rd STEP** : *If such a solution $\bar{\boldsymbol{x}}$ exists, then check whether $\bar{\boldsymbol{x}} \in C_{\boldsymbol{A},\boldsymbol{b}}$, i.e., whether*

$$\boldsymbol{A}_{i.}\bar{\boldsymbol{x}} \leq b_i \quad (i \in I - I_0)$$

*holds true. If this is the case, then $\bar{\boldsymbol{x}}$ is a vertex of $C_{\boldsymbol{A},\boldsymbol{b}}$*

**Example 2.7.** 1. Consider the convex polyhedron as defined by

$$C: \; = \; \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i \geq 0 (i = 1 \dots n), \sum_{k=1}^{n} x_i = 1\}.$$

Clearly, $C$ is the unit simplex of $\mathbb{R}^n$, in the notation introduced earlier (Definition 1.5 of SECTION 1), we have $C = \overline{\underline{\mathbf{X}}}^n$.

If we take

$$\boldsymbol{A} = \begin{pmatrix} -1 & & \\ & \ddots & \\ & & -1 \\ 1 & \cdots & 1 \\ -1 & \cdots & -1 \end{pmatrix}$$

and

$$b = (0, \dots, 0, 1, -1),$$

then we obtain $\overline{\underline{\mathbf{X}}}^n = C = C_{\boldsymbol{A},\boldsymbol{b}}$. Now we follow the procedure outlined above: we take $n$ rows of $\boldsymbol{A}$, say given by $I_0 = \{1, \dots, n-1, n+1\}$. The corresponding equations

$$\begin{aligned} x_1 &= 0 \\ &\vdots \\ x_{n-1} &= 0 \\ \sum_{k=1}^{n} x_k &= 1 \end{aligned}$$

have a unique solution $\bar{\boldsymbol{x}} = \boldsymbol{e}^n = (0, 0, \dots, 0, 1) \in \mathbb{R}^n$. This way we obtain all unit vectors as vertices of $\overline{\underline{\mathbf{X}}}^n$. Of course we can also take $I_0 = \{1, \dots, n\}$, the corresponding system of equations yields $\bar{\boldsymbol{x}} = 0$, which is not in $\overline{\underline{\mathbf{X}}}^n$.

2. Now take $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \\ 3 & 0 \\ 0 & 3 \end{pmatrix}$ and $\boldsymbol{b} = (0, 0, 1, 2, 2)$, then we have

$$C_{\boldsymbol{A},\boldsymbol{b}} = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_i \geq 0 \;\; (i = 1, 2) \; , \;\; 3x_i \leq 2 \;\; (i = 1, 2) \; , \;\; x_1 + x_2 \leq 1\}$$

If we solve $3x_1 = 2, \; x_1 + x_2 = 1$, then we find $\bar{\boldsymbol{x}} = \left(\frac{2}{3}\;,\;\frac{1}{3}\right) \in C_{\boldsymbol{A},\boldsymbol{b}}$. On the other hand, if we solve $x_1 = \frac{2}{3}\;,\; x_2 = \frac{2}{3}$ then we find $\hat{\boldsymbol{x}} = \left(\frac{2}{3}\;,\;\frac{2}{3}\right) \notin C_{\boldsymbol{A},\boldsymbol{b}}$

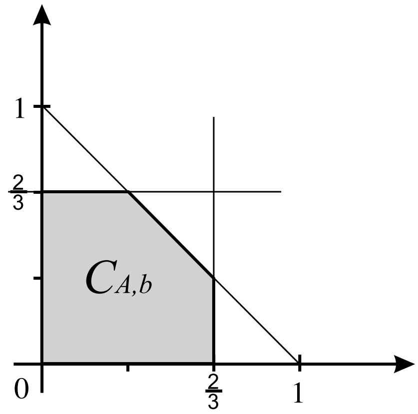



Figure 2.8: $C_{\boldsymbol{A},\boldsymbol{b}} \subseteq \mathbb{R}^2$ (Example 2.6)

Let us now turn to a slightly different version of representing a convex polyhedron. This version is particularly interesting in connection with linear programming problems.

Thus, if $\boldsymbol{A}$ is an $m \times n$ matrix and $\boldsymbol{b} \in \mathbb{R}^m$ let us write

$$D^0_{\boldsymbol{A},\boldsymbol{b}} := \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\ ,\ \boldsymbol{x} \geq 0\}. \tag{11}$$

In the framework of Theorem 2.5 we have been speaking about **row** vectors generating extremal points. After haveing changed our viewpoint concerning the representation of a partincular type of convex polyhedron, we deal with a further version. The next theorem establishes a connection between **column** vectors of $\boldsymbol{A}$ and extremal points of $D^0_{\boldsymbol{A},\boldsymbol{b}}$.

**Theorem 2.8.** *Let $\boldsymbol{A}$ be an $m \times n$ matrix and let $\boldsymbol{b} \in \mathbb{R}^m$. Let $\bar{\boldsymbol{x}} \in D^0_{\boldsymbol{A},\boldsymbol{b}}$ and define*

$$J^+ := J^+(\bar{\boldsymbol{x}}) := \{j \in J \mid \bar{x}_j > 0\}.$$

*Then $\bar{\boldsymbol{x}} \in \boldsymbol{Ext}\, D^0_{\boldsymbol{A},\boldsymbol{b}}$ if and only if the vectors $(\boldsymbol{A}_{\bullet j})_{j \in J^+}$ are linearly independent.*

There are two proofs (rather close to each other) which the reader should ponder about. The first one works by appealing to Theorem 2.5. This may not be surprising, as Theorem 2.5 is the first one establishing the connection between extremals and linear independence of the corresponding rows (!) of the defining matrix. The second version is the direct approach. Nevertheless it looks rather similar in technique if compared to the proof of 2.5.

**Proof: 1<sup>st</sup>Version** : In order to apply Theorem 2.5, write

$$D^0_{\boldsymbol{A},\boldsymbol{b}} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}_{i.}\boldsymbol{x} \leq b_i \ (i \in I), -\boldsymbol{A}_{i.}\boldsymbol{x} \leq -b_i \ (i \in I), -x_j \leq 0 \ (j \in J)\}.$$

Then $D^0_{\boldsymbol{A},\boldsymbol{b}}$ is represented as a polyhedron $C_{\boldsymbol{B},\boldsymbol{d}}$ the way we discussed in the context of Theorem 2.5 and defined in (1); more precisely, we have

$$D^0_{\boldsymbol{A},\boldsymbol{b}} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{B}\boldsymbol{x} \leq \boldsymbol{d}\} = C_{\boldsymbol{B}\boldsymbol{d}}$$

with

$$B = \begin{pmatrix} \boldsymbol{A} \\ -\boldsymbol{A} \\ -\boldsymbol{e}^j \end{pmatrix}_{j \in J}$$

and $\boldsymbol{d} = (\boldsymbol{b}, -\boldsymbol{b}, 0)$. By 2.5 we know that $\bar{\boldsymbol{x}}$ is extremal if and only if there is a system of $n$ rows $\boldsymbol{B}_{i\bullet}$ of $\boldsymbol{B}$ which are linearly independent and satisfy $\boldsymbol{B}_{i\bullet}\bar{\boldsymbol{x}} = d_i$.

However, there are the rows $(-\boldsymbol{e}^j)_{j \in J-J^+}$ satisfying $\boldsymbol{e}^j\bar{\boldsymbol{x}} = 0$. Hence in order to identify $\bar{\boldsymbol{x}} \in D^0_{\boldsymbol{A},\boldsymbol{b}}$ as extremal, it is equivalent to find $\mid J^+ \mid$ rows - say

$$(\boldsymbol{A}_{i\bullet})_{i \in I^*}$$

such that $\binom{\boldsymbol{A}_{i\bullet}}{\boldsymbol{e}^j}_{i \in I^*, j \in J-J^+}$ has rank $n$ and satisfies

$$\boldsymbol{A}_{i\bullet}\bar{\boldsymbol{x}} = b_i \quad (i \in I^*)$$

However, $\boldsymbol{A}_{i\bullet}\bar{\boldsymbol{x}} = b_i$ is satisfied by all $i \in I$ - hence it is sufficient and necessary to show that $\binom{\boldsymbol{A}_{i\bullet}}{\boldsymbol{e}^j}_{j \in J-J^+}$ has rank $n$. This is equivalent to saying that

$$(\boldsymbol{A}_{\bullet j})_{j \in J^+}$$

has rank $\mid J^+ \mid$, i.e., that the vectors $(\boldsymbol{A}_{\bullet j})_{j \in J^+}$ are linearly independent, **q.e.d.**

**Proof: 2<sup>nd</sup>Version**, the direct way:

**1<sup>st</sup>STEP** :
Let $\bar{\boldsymbol{x}}$ be a vertex of $D^0_{\boldsymbol{A},\boldsymbol{b}}$. If the vectors $(\boldsymbol{A}_{.j})_{j \in J^+}$ are linearly *dependent*, then there exists $(\bar{y}_j)_{j \in J^+} \neq 0$ such that

$$\sum_{j \in J^+} \bar{y}_j \boldsymbol{A}_{.j} = \boldsymbol{0} \tag{12}$$

*Define* $\bar{y}_j = 0, \quad (j \in J - J^+)$, then $\bar{\boldsymbol{y}} \in \mathbb{R}^n$ and $\bar{\boldsymbol{y}}$ satisfies

$$\mathbf{0} = \sum_{j \in J} \bar{y}_j \boldsymbol{A}_{\bullet j} = \boldsymbol{A}\bar{\boldsymbol{y}}. \tag{13}$$

Now, as $\bar{\boldsymbol{x}} \in D^0_{\boldsymbol{A},\boldsymbol{b}}$ and $\boldsymbol{A}\bar{\boldsymbol{x}} = b$, it follows from 13 that for all $t > 0$ the vector

$$\boldsymbol{x}^{\pm t} := \bar{\boldsymbol{x}} \pm t\bar{\boldsymbol{y}}$$

satisfies

$$\boldsymbol{A}\boldsymbol{x}^{\pm t} = \boldsymbol{A}\bar{\boldsymbol{x}} \pm t\boldsymbol{A}\bar{\boldsymbol{y}} = \boldsymbol{A}\bar{\boldsymbol{x}} = \boldsymbol{b}.$$

In addition, if $t > 0$ is sufficiently small, then $\boldsymbol{x}^{\pm t} \geq \mathbf{0}$, as $\bar{\boldsymbol{y}}$ has non-vanishing coordinates at most whenever $\bar{\boldsymbol{x}}$ has positive coordinates.

Thus, $\boldsymbol{x}^{\pm t} \in D^0_{\boldsymbol{A},\boldsymbol{b}}$ for small $t > 0$. Now, obviously

$$\bar{\boldsymbol{x}} = \frac{1}{2}(\boldsymbol{x}^{+t} + \boldsymbol{x}^{-t})$$

contradicts $\bar{\boldsymbol{x}} \in \boldsymbol{Ext}\, D^0_{\boldsymbol{A},\boldsymbol{b}}$, hence we have shown that $(\boldsymbol{A}_{\bullet j})_{j \in J^+}$ are linearly independent.

**2ndSTEP** :
Suppose now that $(\boldsymbol{A}_{.j})_{j \in J^+}$ are linearly independent. We want to show that $\bar{\boldsymbol{x}}$ is extremal in $D^0_{\boldsymbol{A},\boldsymbol{b}}$.

Assume that we can find $0 < \lambda < 1$ and $\bar{\boldsymbol{x}}^0, \bar{\boldsymbol{x}}^1 \in D^0_{\boldsymbol{A},\boldsymbol{b}}$ satisfying

$$\bar{\boldsymbol{x}} = \lambda \boldsymbol{x}^1 + (1 - \lambda)\boldsymbol{x}^0. \tag{14}$$

Now, for $j \in J - J^+$

$$0 = \bar{x}_j = \lambda x^1_j + (1 - \lambda)x^0_j \geq \lambda \cdot 0 + (1 - \lambda) \cdot 0 = 0 \ ;$$

hence none of the inequalities employed can be strict. Therefore,

$$x^0_j = x^1_j = 0 \quad (j \in J - J^+)$$

holds true. Because of $\boldsymbol{x}^k \in D^0_{\boldsymbol{A},\boldsymbol{b}}$ $(k = 1, 2)$ we know that

$$\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}^k = \sum_{j \in J} x^k_j \boldsymbol{A}_{\bullet j} = \sum_{j \in J^+} x^k_j \boldsymbol{A}_{\bullet j} \quad (k = 1, 2) \ .$$

But the matrix $(\boldsymbol{A}_{\bullet j})_{j\in J^+}$ has rank $\mid J^+ \mid$, thus $(\bar{x}_j)_{j\in J^+}$ is the only solution of the linear system of equations

$$\sum_{j\in J^+} x_j \boldsymbol{A}_{\bullet j} = \boldsymbol{b} \ ,$$

meaning that $\boldsymbol{x}^0 = \boldsymbol{x}^1 = \bar{\boldsymbol{x}}$ holds true. This shows that $\bar{\boldsymbol{x}} \in \boldsymbol{Ext}\, D^0_{\boldsymbol{A},\boldsymbol{b}}$, **q.e.d.**

**Example 2.9.** Let $m = 2,\ n = 3$ and $\boldsymbol{A} = \left(\begin{smallmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right)$ , $\boldsymbol{b} = (1, 2)$ such that

$$D^0_{\boldsymbol{A},\boldsymbol{b}} = \{\boldsymbol{x} \in \mathbb{R}^3 \mid x_3 = 1 \ ,\ x_1 + x_2 + x_3 = 2,\ \boldsymbol{x} \geq 0\}$$

is obtained.

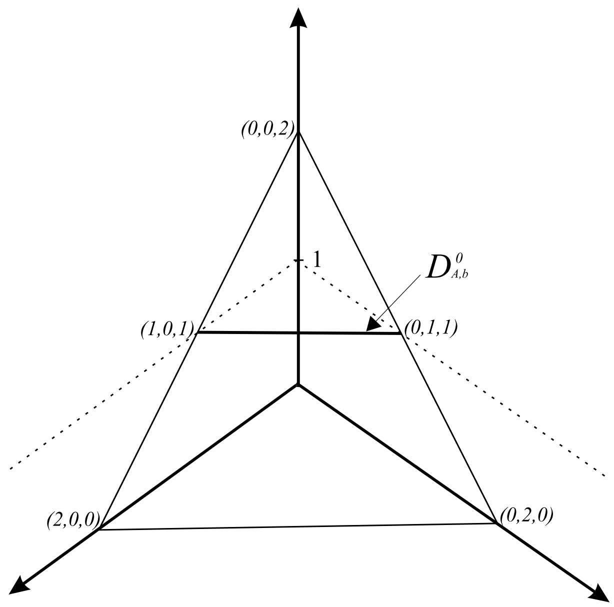



Figure 2.9: $D^0_{\boldsymbol{A},\boldsymbol{b}} = \{\boldsymbol{x} \in \mathbb{R}^3 \mid x_3 = 1 \ ,\ x_1 + x_2 + x_3 = 2,\ \boldsymbol{x} \geq 0\}$

The vertices are $\boldsymbol{x}^1 = (1, 0, 1)$ and $\boldsymbol{x}^2 = (0, 1, 1)$. The columns of $\boldsymbol{A}$ corresponding to the positive coordinates are

$$\begin{array}{lcl} \boldsymbol{x}^1 & \dashrightarrow & \boldsymbol{A}_{\bullet 1} \ ,\quad \boldsymbol{A}_{\bullet 3} \\ \boldsymbol{x}^2 & \dashrightarrow & \boldsymbol{A}_{\bullet 2} \ ,\quad \boldsymbol{A}_{\bullet 3} \end{array}$$

Observe that $\boldsymbol{A}_{\bullet 3}$ is common to both vertices, so we move from $\boldsymbol{x}^1$ to $\boldsymbol{x}^2$ by 'exchanging $\boldsymbol{A}_{\bullet 1}$ and $\boldsymbol{A}_{\bullet 2}$ .'

**Corollary 2.10.** *If* $\bar{\boldsymbol{x}}$ *is a vertex of* $D^0_{\boldsymbol{A},\boldsymbol{b}}$*, then* $\bar{\boldsymbol{x}}$ *has* ***at most*** $m$ *positive coordinates.*

We finish this section with a representation theorem. This concernes again general convex sets and is not restricted to convex polyhedra. Nevertheless it has a particular significance concerning convex polyhedra.

**Theorem 2.11 (Krein-Milman).** *Every compact, convex subset* $C$ *of* $\mathbb{R}^n$ *is the convex hull of its extremal points, more precisely*

$$C = \boldsymbol{CnvH}\,\boldsymbol{Ext}\,C.$$

**Proof: 1<sup></sup>**

**Proof:** $\mathbf{1^{st}}$**STEP** :
Since $\boldsymbol{Ext}\,C \subseteq C$ and $C$ is convex it follows at once from Theorem 1.10 that we have

$$\boldsymbol{CnvH}\,\boldsymbol{Ext}\,C \subseteq C.$$

$\mathbf{2^{nd}}$**STEP** : Hence we have to show:
If $\bar{\boldsymbol{x}} \in C$ then there is $\boldsymbol{x}^1, \cdots, \boldsymbol{x}^k \in \boldsymbol{Ext}\,C$ and $\boldsymbol{\alpha} \in \underline{\overline{\mathbf{X}}}^k$ such that

(15) $$\bar{\boldsymbol{x}} = \boldsymbol{x}^{\boldsymbol{\alpha}}$$

holds true.

The statement shall be verified by induction over $n$, the dimension of $\mathbb{R}^n$ we are working in. For $n = 1$, as $C$ is a compact interval, (15) is obviously true.

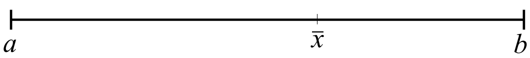



Figure 2.10: A compact interval $C$

If $\boldsymbol{a} \neq \boldsymbol{b}$ is the case, then we have for $\bar{\boldsymbol{x}} \in [\boldsymbol{a}, \boldsymbol{b}]$

$$\bar{\boldsymbol{x}} = \frac{b - \bar{\boldsymbol{x}}}{b - a}\, a \; + \; \frac{\bar{\boldsymbol{x}} - a}{b - a}\, b.$$

$\mathbf{3^{rd}}$**STEP** : Assume now $n > 1$ and consider $\bar{\boldsymbol{x}} \in C$. If $\bar{\boldsymbol{x}} \in \boldsymbol{Ext}\,C$, nothing has to be shown. Hence we may in addition assume that there is $\boldsymbol{x}^0, \boldsymbol{x}^1 \in C$ , $\boldsymbol{x}^0 \neq \boldsymbol{x}^1$, and $\bar{\lambda} \in (0,1)$ such that

(16) $$\bar{\boldsymbol{x}} = \boldsymbol{x}^{\bar{\lambda}}$$

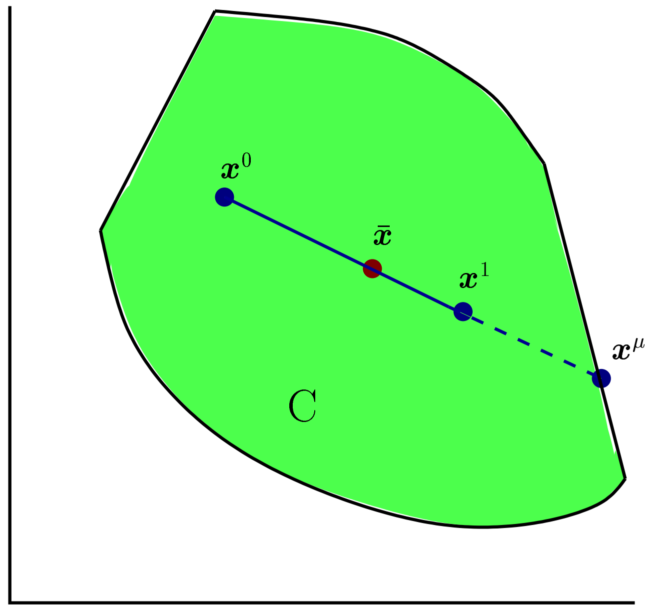



Figure 2.11: Moving $\boldsymbol{x}^1$ to the boundary

holds true. Intuitively, it is our aim to move the points $\boldsymbol{x}^1, \boldsymbol{x}^0$ to the (topological) boundary of $C$. We start with $\boldsymbol{x}^1$ (cf Figure 2.11).

As

$$\boldsymbol{x}^{\bar{\lambda}} = (1-\lambda)\boldsymbol{x}^0 + \lambda \boldsymbol{x}^1 = \boldsymbol{x}^0 + \bar{\lambda}(\boldsymbol{x}^1 - \boldsymbol{x}^0)$$

and $\boldsymbol{x}^0 \neq \boldsymbol{x}^1$ is true, we observe that the set

$$\{|\ \boldsymbol{x}^\lambda\ |\quad|\ \lambda \in \mathbb{R}_+\}$$

is unbounded. Since $C$ is compact, we may choose

$$1 \leq \mu \ := \ max\{\lambda \mid \lambda \in \mathbb{R}_+ \ , \ \boldsymbol{x}^\lambda \in C\}.$$

It is rather clear that $\boldsymbol{x}^\mu \in \partial C$ (the topological boundary of $C$).

Now we compute

$$\begin{aligned} \boldsymbol{x}^\mu &= \mu \boldsymbol{x}^1 \quad + (1-\mu)\boldsymbol{x}^0 \\ \boldsymbol{x}^1 &= \frac{\boldsymbol{x}^\mu}{\mu} \quad - \left(\frac{1-\mu}{\mu}\right)\boldsymbol{x}^0 \end{aligned}$$

and

$$\begin{aligned} \bar{\boldsymbol{x}} &= (1-\bar{\lambda})\boldsymbol{x}^0 + \bar{\lambda}\boldsymbol{x}^1 \\ &= (1-\bar{\lambda})\boldsymbol{x}^0 + \bar{\lambda}\left[\frac{\boldsymbol{x}^\mu}{\mu} - \frac{1-\mu}{\mu}\boldsymbol{x}^0\right] \\ &= \frac{(1-\bar{\lambda})\mu - \bar{\lambda}(1-\mu)}{\mu}\boldsymbol{x}^0 + \frac{\bar{\lambda}}{\mu}\boldsymbol{x}^\mu \\ &= \frac{\mu-\bar{\lambda}}{\mu}\boldsymbol{x}^0 + \frac{\bar{\lambda}}{\mu}\boldsymbol{x}^\mu \end{aligned} \tag{17}$$

As $0 < \bar{\lambda} < 1 \leq \mu$ holds true, it is seen at once that the coefficients sum up to 1 and are nonnegative. Hence we have represented $\bar{\boldsymbol{x}}$ as a convex combination of two points $\boldsymbol{x}^0$ and $\boldsymbol{x}^\mu$, one of them being in $\partial D$. We may analogously replace $\boldsymbol{x}^0$ by a point in $\partial C$, say $\boldsymbol{x}^\nu$. This completes our $3^{rd} STEP$.

**4thSTEP** : The basic idea of the proof becomes now obvious if one looks once again into the proof of MINKOWSKI's Theorem. Within that context, the induction step involves a hyperplane $H^{\frac{1}{t}}$ which was supporting $C$ appropriately. Similarly, we employ supporting hyperplanes in order to proceed with the present proof, see Figure 2.12.

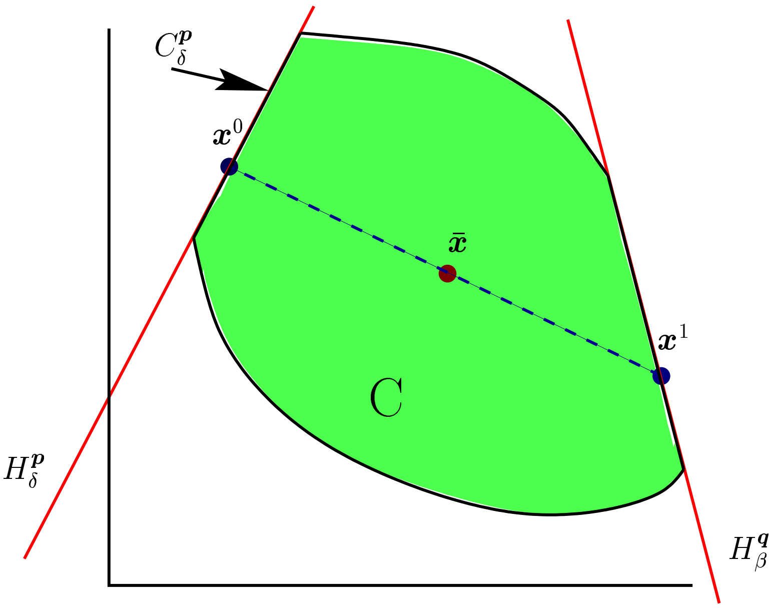



Figure 2.12: The Krein–Milman Theorem

We assume now that $\bar{\boldsymbol{x}} = \boldsymbol{x}^\lambda$ holds true with $\boldsymbol{x}^0, \boldsymbol{x}^1 \in \partial C$. As a consequence, there is a hyperplane separating $\boldsymbol{x}^0$ from $C$ weakly (a supporting hyperplane, cf. Definition 1.14 and Theorem 1.16). More precisely, there is $0 \neq \boldsymbol{p} \in \mathbb{R}^n$ and $\boldsymbol{a} \in \mathbb{R}$ such that

$$\boldsymbol{p}\boldsymbol{x} \leq \boldsymbol{p}\boldsymbol{x}^0 = \boldsymbol{a} \quad (x \in C),$$

meaning that $C$ is located completely on one side of the hyperplane

$$H^{\boldsymbol{p}}_{\boldsymbol{a}} = \{\boldsymbol{x} \mid \boldsymbol{p}\boldsymbol{x} = \boldsymbol{a}\}$$

which in turn contains $\boldsymbol{x}^0$.

**5thSTEP** : Now, $C^{\boldsymbol{p}}_{\boldsymbol{a}} := C \cap H^{\boldsymbol{p}}_{\boldsymbol{a}}$ is a convex and (relatively) compact subset of an $n-1$ dimensional affine space. Hence we may apply our induction hypothesis. Consequently, there is $\boldsymbol{y}^1, \cdots, \boldsymbol{y}^k \in \boldsymbol{Ext}\, C^{p}_{\boldsymbol{a}}$ and $\boldsymbol{\alpha} \in \underline{\overline{\mathbf{X}}}^k$ such that

$$\boldsymbol{x}^0 = \boldsymbol{y}^{\boldsymbol{\alpha}}$$

is true.

Suppose we can show that $\boldsymbol{y}^1, \cdots, \boldsymbol{y}^k \in \boldsymbol{Ext}\, C$ holds true (which is to be expected, look at the proof of Theorem 2.3), then we would perform the same procedure for $\boldsymbol{x}^1$, i.e., find $\boldsymbol{z}^1, \cdots, \boldsymbol{z}^r \in \boldsymbol{Ext}\, C^{\boldsymbol{p}}_{\alpha}$ and $\beta \in \underline{\overline{\mathbf{X}}}^r$ such that $\boldsymbol{x}^1 = \boldsymbol{z}^{\boldsymbol{\beta}}$. This way we obtain

$$\begin{aligned} \bar{\boldsymbol{x}} &= (1-\bar{\lambda})\boldsymbol{x}^0 && + \ \bar{\lambda}\boldsymbol{x}^1 \\ &= (1-\bar{\lambda})\sum_{l=1}^{k} \alpha_e \boldsymbol{y}^l && + \ \bar{\lambda}\sum_{\rho=1}^{k} \beta_\rho \boldsymbol{z}^\rho \end{aligned}$$

such that $\boldsymbol{y}^1 \cdots \boldsymbol{y}^k, \boldsymbol{z}^1 \cdots \boldsymbol{z}^l \in \boldsymbol{Ext}\, C$ and the coefficients are nonnegative and sum up to 1. Hence we will have completed the proof.

**6thSTEP** : Thus it suffices to prove that any $\tilde{\boldsymbol{x}} \in \boldsymbol{Ext}\, C^{\boldsymbol{p}}_{\alpha}$ is an element of $\boldsymbol{Ext}\, C$. The technique is by now standard:

Assume that

$$\tilde{\boldsymbol{x}} = t\boldsymbol{w}^0 + (1-t)\boldsymbol{w}^1$$

with $\boldsymbol{w}^0, \boldsymbol{w}^1 \in C$ , $t \in (0,1)$: As

$$\boldsymbol{p}\boldsymbol{w}^0 \le \alpha \ , \ \boldsymbol{p}\boldsymbol{w}^1 \le \alpha \tag{18}$$

we have

$$\begin{aligned} \boldsymbol{p}\tilde{\boldsymbol{x}} &= \ t\boldsymbol{p}\boldsymbol{w}^0 + (1-t)\boldsymbol{p}\boldsymbol{w}^1 \\ &\le \ t\alpha + (1-t)\alpha = \alpha. \end{aligned}$$

Consequently, none of the inequalities in 18 can be strict ones. That is,

$$\boldsymbol{w}^0 \in H^{\boldsymbol{p}}_{\alpha} \ , \ \boldsymbol{w}^1 \in H^{\boldsymbol{p}}_{\alpha}$$

holds true. More than that, we have $\boldsymbol{w}^0, \boldsymbol{w}^1 \in C^{\boldsymbol{p}}_{\alpha}$ and as $\tilde{\boldsymbol{x}} \in \boldsymbol{Ext}(C^{\boldsymbol{p}}_{\alpha})$ we conclude $\boldsymbol{w}^0 = \boldsymbol{w}^1 = \tilde{x}$, thus $\tilde{\boldsymbol{x}} \in \boldsymbol{Ext}\, C$, **q.e.d.**

# 3 Convex functions

**Definition 3.1.** *Let $C \subseteq \mathbb{R}^n$ be a convex set and let $f : C \longrightarrow \mathbb{R}$ be a real-valued function. $f$ is called **convex** if, for any $\boldsymbol{x}^0, \boldsymbol{x}^1 \in C$ and $\lambda \in [0,1]$ it follows that*

$$f(\boldsymbol{x}^\lambda) \leq (1-\lambda) f(\boldsymbol{x}^0) + \lambda f(\boldsymbol{x}^1) \tag{1}$$

*holds true. $f$ is **concave** if $-f$ is convex, i.e., the $\leq$-sign in (1) is replaced by the $\geq$-sign.*

As for an interpretation, consider the graph of $f$ which is a subset of $\mathbb{R}^{n+1}$. Given $\boldsymbol{x}^0$ and $\boldsymbol{x}^1$ (both in $\mathbb{R}^n$) consider the interval

$$\left\{(1-\lambda)\left[\boldsymbol{x}^0, f(\boldsymbol{x}^0)\right] + \lambda\left[\boldsymbol{x}^1, f(\boldsymbol{x}^1)\right]\right\} = \left[(\boldsymbol{x}^0, f(\boldsymbol{x}^0)), (\boldsymbol{x}^1, f(\boldsymbol{x}^1))\right]$$

Then (1) says that any point on this line is located above the corresponding point $(\boldsymbol{x}^\lambda, f(\boldsymbol{x}^\lambda))$ of the graph of $f$. Thus, the function $f$ is somehow hanging

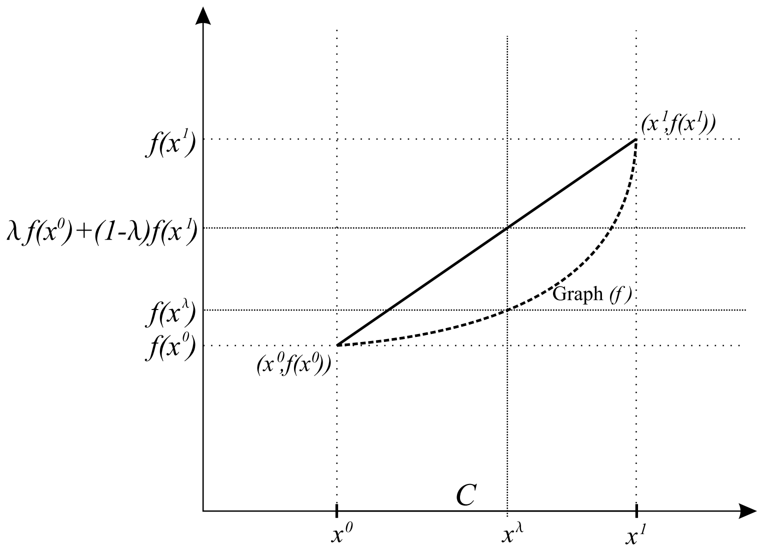



Figure 3.1: Definition of a convex function

down between any two points of the graph.

**Theorem 3.2.** *Let $C \subseteq \mathbb{R}^n$ be a convex set and let $f : C \longrightarrow \mathbb{R}$ be a function. The following statements are equivalent:*

1. *$f$ is convex*

2. *For any* $\boldsymbol{x}^1, \cdots, \boldsymbol{x}^k \in C$ *and* $\boldsymbol{\alpha} \in \underline{\overline{\mathbf{X}}}^k$ *it follows that*

$$f(\boldsymbol{x}^\alpha) \leq \sum_{=1}^{k} \alpha_l f(\boldsymbol{x}^l)$$

*holds true.*

3. *The* ***epigraph*** *of* $f$ *with respect to* $C$*, i.e., the set*

$$E_C^f f := \{(\boldsymbol{x}, t) \mid \boldsymbol{x} \in C, t \in \mathbb{R}, f(\boldsymbol{x}) \leq t\}$$

*(cf Figure 3.2) is a convex subset of* $\mathbb{R}^{n+1}$.

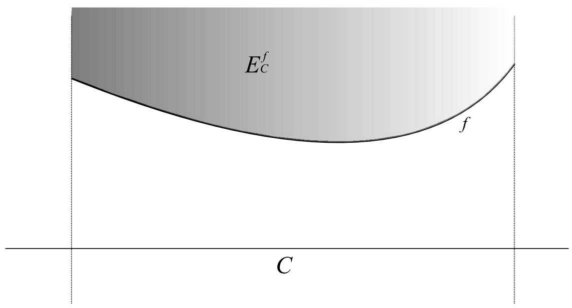



Figure 3.2: The epigraph $E_C^f$

**Theorem 3.3.** 1. *Let* $C \subseteq \mathbb{R}^1$ *be open and convex and let* $f : C \longrightarrow \mathbb{R}$ *be a differentiable function. Then* $f$ *is convex if and only if* $f'$ *is monotone increasing. If* $f$ *is twice differentiable, then* $f$ *is convex if and only if* $f''$ *is positive.*

2. $f$ *is convex and concave if and only if* $f$ *is linear.*

**Proof:**

We will only check the monotonicity of the first derivative. Let $f$ be convex and differentiable. Pick $\boldsymbol{x}^0, \boldsymbol{x}^1 \in C$ such that

$$\boldsymbol{x}^0 < \boldsymbol{x}^1 \tag{2}$$

holds true. We want to show that, as a consequence, $f'(\boldsymbol{x}^0) \leq f'(\boldsymbol{x}^1)$ holds true.

Now, for $\lambda \in [0, 1]$ we have

$$(1 - \lambda) f(\boldsymbol{x}^0) + \lambda f(\boldsymbol{x}^1) \geq f(\boldsymbol{x}^\lambda)$$

or

$$\lambda(f(\boldsymbol{x}^1) - f(\boldsymbol{x}^0)) \geq f(\boldsymbol{x}^\lambda) - f(\boldsymbol{x}^0) .$$

Thus, for $\lambda \neq 0$

$$\begin{aligned} f(\boldsymbol{x}^1) - f(\boldsymbol{x}^0) &\geq \frac{f(\boldsymbol{x}^\lambda) - f(\boldsymbol{x}^0)}{\lambda} \\ &= \frac{f(\boldsymbol{x}^0 + \lambda(\boldsymbol{x}^1 - \boldsymbol{x}^0)) - f(\boldsymbol{x}^0)}{\lambda} \end{aligned}$$

i.e.

$$\frac{f(\boldsymbol{x}^1) - f(\boldsymbol{x}^0)}{\boldsymbol{x}^1 - \boldsymbol{x}^0} \geq \frac{f(\boldsymbol{x}^0 + \lambda(\boldsymbol{x}^1 - \boldsymbol{x}^0)) - f(\boldsymbol{x}^0)}{\lambda(\boldsymbol{x}^1 - \boldsymbol{x}^0)} \tag{3}$$

and as $\lambda \longrightarrow 0$ the right side of (3) approximates $f'(\boldsymbol{x}^0)$. Thus

$$\frac{f(\boldsymbol{x}^1) - f(\boldsymbol{x}^0)}{\boldsymbol{x}^1 - \boldsymbol{x}^0} \geq f'(\boldsymbol{x}^0) \tag{4}$$

Now, exchange the role of $\boldsymbol{x}^0$ and $\boldsymbol{x}^1$. Then

$$\frac{f(\boldsymbol{x}^1) - f(\boldsymbol{x}^0)}{\boldsymbol{x}^1 - \boldsymbol{x}^0} \leq f'(\boldsymbol{x}^1) \tag{5}$$

But (4) and (5) indeed show that $f'$ is monotone. The reverse direction is omitted. **q.e.d.**

Note that the term $\frac{f(\boldsymbol{x}^1)-f(\boldsymbol{x}^0)}{\boldsymbol{x}^1-\boldsymbol{x}^0}$ that appears in equations (4) and (5) is the slope of the line segment indicated in Figure 3.1. In other words, the slope of the secant is between the derivatives at $\boldsymbol{x}^0$ and $\boldsymbol{x}^1$.

**Theorem 3.4.** *Let $C \subseteq \mathbb{R}^n$ be convex and compact, and let $f : C \to \mathbb{R}$ be a convex function. If $\max_{\boldsymbol{x}\in C} f(\boldsymbol{x})$ exists, then*

$$\max_{\boldsymbol{x}\in C} f(\boldsymbol{x}) = \max_{\boldsymbol{x}\in \boldsymbol{Ext}\, C} f(\boldsymbol{x}) \tag{6}$$

*holds true. That is, a maximum, if it exists at all, is attained on the extremal points. Hence it suffices to search for maximizers on the set $\boldsymbol{Ext}\, C$.*

**Proof:** Let $\bar{\boldsymbol{x}} \in C$ be such that $f(\bar{\boldsymbol{x}}) \geq f(\boldsymbol{x})$ $(\boldsymbol{x} \in C)$ holds true. Then there exist $k \in \mathbb{N}, \boldsymbol{\alpha} \in \underline{\overline{\mathbf{X}}}^k$, and vectors $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k \in \boldsymbol{Ext}\, C$ such that $\bar{\boldsymbol{x}} = \boldsymbol{x}^{\boldsymbol{\alpha}}$

holds true. This follows from the KREIN-MILMAN Theorem (see Theorem 2.11). Clearly we have

$$f(\bar{\boldsymbol{x}}) = f(\boldsymbol{x}^{\boldsymbol{\alpha}}) \leq \sum_{l=1}^{k} \alpha_l f(\boldsymbol{x}^l). \tag{7}$$

Now, if all values $f(\boldsymbol{x}^l)$ satisfy $f(\boldsymbol{x}^l) < f(\bar{\boldsymbol{x}})$, then it follows immediately that $\sum_{l=1}^{k} \alpha_l f(\boldsymbol{x}^l) < f(\bar{\boldsymbol{x}})$ holds true as well, a contradiction. Hence, we obtain $f(\boldsymbol{x}^l) = f(\bar{\boldsymbol{x}})$ for some $\boldsymbol{x}^l \in \boldsymbol{Ext}\, C$, **q.e.d.**

**Remark 3.5.** 1. *As a consequence of the previous theorem we observe that continuous convex functions defined on a compact convex set $C$ attain their maximum on* $\boldsymbol{Ext}\, C$. *Therefore, in order to actually compute the maximum of such a function as well as a maximizer it is sufficient to search on the extreme points.*

2. *If $n = 1$ is the case then we do actually not need the requirement of continuity. It is easily seen that a convex function on a compact interval attains its maximum on one of the boundary points.*

3. *In general this is not necessarily true for dimensions $n > 1$. The following sketch represents an example for a convex function defined on a compact set (the unit disc) which is not necessarily bounded.*

**Remark 3.6.** *We would like to point out the close relationship of derivatives and the normal of supporting hyperplanes once more. We restrict our interest to convex functions, this remark generalizes a consideration already presented in Example 1.13.*

*For convenience, let $C$ be a closed convex set and let $\bar{\boldsymbol{x}}$ be a point within the interior of $C$.*

*Consider the **epigraph** of $f$ with respect to $C$ which is a subset of $\mathbb{R}^{n+1}$ defined by*

$$E_C^f := \{(\boldsymbol{x}, t) \,|\, x \in C,\ t \in \mathbb{R},\ f(\boldsymbol{x}) \leq t\}. \tag{8}$$

*This is a closed set and the point $(\bar{\boldsymbol{x}}, f(\bar{\boldsymbol{x}}))$ is a boundary point. According to a separation theorem, $E_C^f$ and $(\bar{\boldsymbol{x}}, f(\bar{\boldsymbol{x}}))$ can be weakly separated (cf. Theorem 1.16, item 3), that is, there is a vector $\boldsymbol{0} \neq \boldsymbol{p} = (\boldsymbol{q}, r) \in \mathbb{R}^{n+1}$ such that*

$$\boldsymbol{p}(\bar{\boldsymbol{x}}, f(\bar{\boldsymbol{x}})) \leq \boldsymbol{p}(\boldsymbol{x}, t) \tag{9}$$

*holds true for all $(\boldsymbol{x}, t) \in E_C^f$.*

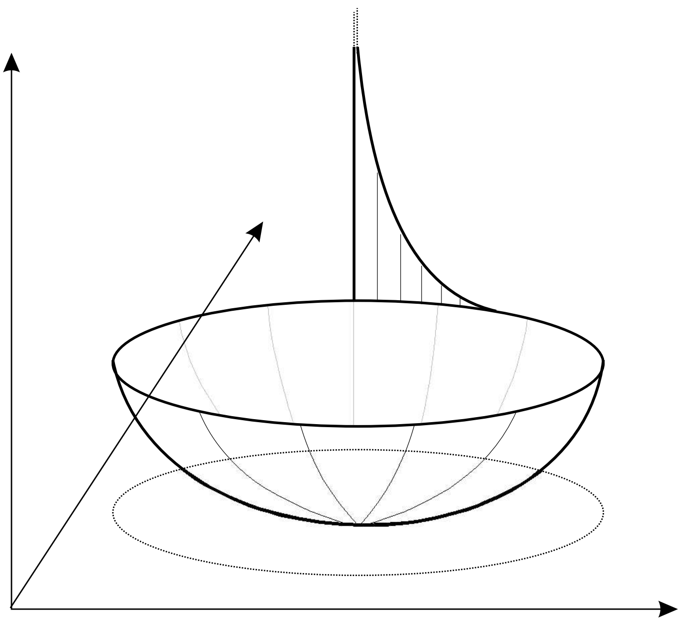


Figure 3.3: An unbounded convex function on a compact set

*This can be rewritten as*

$$(\boldsymbol{q}, r)(\bar{\boldsymbol{x}}, f(\bar{\boldsymbol{x}})) \leq (\boldsymbol{q}, r)(\boldsymbol{x}, t) \quad (\boldsymbol{x}, t) \in E_C^f. \tag{10}$$

*We conclude that $r \geq 0$ holds true as $(\bar{\boldsymbol{x}}, t) \in E_C^f$ is true for large positive $t \in \mathbb{R}$. Next,* (10) *implies in particular (choose $t = f(x)$) the inequalities*

$$r\,(f(\bar{\boldsymbol{x}}) - f(\boldsymbol{x})) \leq \boldsymbol{q}(\boldsymbol{x} - \bar{\boldsymbol{x}}) \ (\boldsymbol{x} \in C). \tag{11}$$

*This in turn implies $r > 0$. Indeed, if $r = 0$, then we have $\boldsymbol{q} \neq \boldsymbol{0}$. Choose $i$ such that $q_i \neq 0$ is the case and take*

$$\boldsymbol{x}^{\pm\varepsilon} \ := \ \bar{\boldsymbol{x}} \pm \varepsilon \boldsymbol{e}^i$$

*which is an element of $C$ for small $\varepsilon > 0$. This implies*

$$0 \leq \pm q_i \varepsilon,$$

*a contradiction which verifies the fact that $r > 0$ is true. Now we may divide by $r > 0$ in equation* (11) *obtaining*

$$(f(\bar{\boldsymbol{x}}) - f(\boldsymbol{x})) \leq \frac{\boldsymbol{q}}{r}(\boldsymbol{x} - \bar{\boldsymbol{x}}) \quad (\boldsymbol{x} \in C). \tag{12}$$

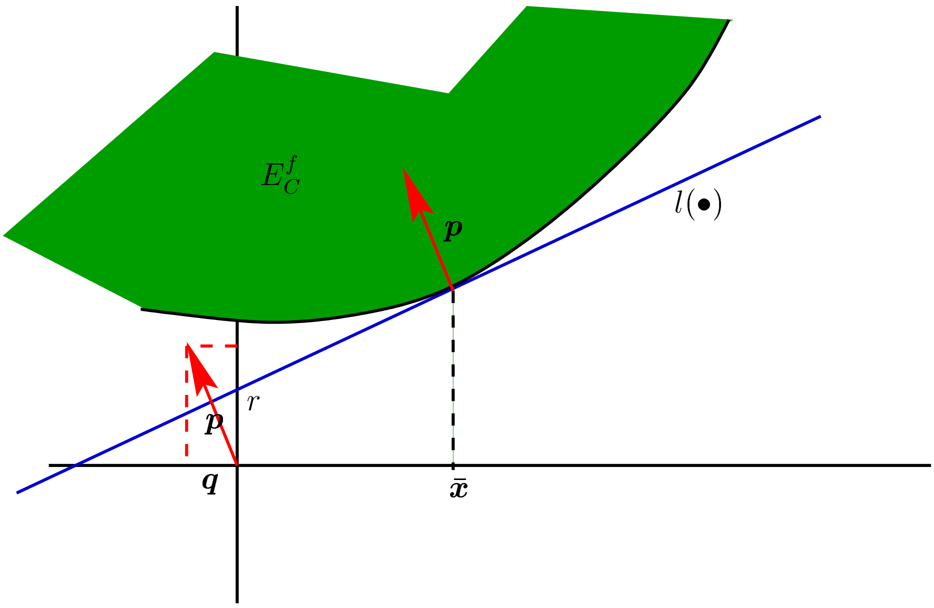



Figure 3.4: Separation and the Gradient

or

$$f(\bar{\boldsymbol{x}}) - \frac{\boldsymbol{q}}{r}(\boldsymbol{x} - \bar{\boldsymbol{x}}) \leq f(\boldsymbol{x}) \quad (\boldsymbol{x} \in C). \tag{13}$$

The inequalities (13) allow for a nice interpretation: it is seen that the linear function

$$l(\bullet) := f(\bar{\boldsymbol{x}}) - \frac{\boldsymbol{q}}{r}(\bullet - \bar{\boldsymbol{x}})$$

has a graph located below the graph of $f$ and satisfies $l(\bar{\boldsymbol{x}}) = f(\bar{\boldsymbol{x}})$. That is, we represented the supporting hyperplane as a supporting linear function. Figure 3.4 reflects the situation in $\mathbb{R}$, here the gradient equals the derivative of $f$.

Of course the gradient of this function (i.e.,$-\frac{\boldsymbol{q}}{r}$) equals the gradient of $f$ if the latter one exists. For in this case, we take again $\boldsymbol{x} = \bar{\boldsymbol{x}} \pm \varepsilon \boldsymbol{e}^i \quad (i = 1, \ldots, n)$ with small $\varepsilon > 0$. Then (13) reads also

$$f(\bar{\boldsymbol{x}} \pm \varepsilon \boldsymbol{e}^i) - f(\bar{\boldsymbol{x}}) \geq -\frac{q_i}{r}(\pm\varepsilon) \quad (i = 1, \ldots, n). \tag{14}$$

Division by $\pm\varepsilon$ changes the sign once, thus

$$\frac{f(\bar{\boldsymbol{x}} + \varepsilon \boldsymbol{e}^i) - f(\bar{\boldsymbol{x}})}{\varepsilon} \geq -\frac{q_i}{r} \geq \frac{f(\bar{\boldsymbol{x}} - \varepsilon \boldsymbol{e}^i) - f(\bar{\boldsymbol{x}})}{-\varepsilon} \quad (i = 1, \ldots, n). \tag{15}$$

Letting $\varepsilon$ approach 0, we obtain

$$\frac{\partial f}{\partial x_i}(\bar{\boldsymbol{x}}) = -\frac{q_i}{r} \quad (i = 1, \dots, n). \tag{16}$$

Thus, the function $f$ and the linear function $l$ have the same gradient.

If partial derivatives do not exist, then it can be derived from (15) that right and left hand derivatives exist. The slopes of the supporting hyperplane (i.e., the numbers $-\frac{q_i}{r}$ $(i = 1, \dots, n)$ )are dominated by the left hand derivatives and the right hand derivatives in turn are dominated by these slopes.

# Chapter 2

# The Simplex Algorithm

This chapter describes the simplex algorithm. By the previous section we know that a linear function defined on a convex polyhedron attains its maximum, if a t all, then et some vertex of the polyhedron. Therefore, it is desirable to find a procedure which represents a successive visit of vertices with the aim of increasing the value of the linear function to be maximized.

# 1 Exchanging vertices

The procedure explained in Remark 2.6 (based on Theorem 2.5)of CHAPTER 1 yields all the extremal points of a certain convex polyhedron. However, with growing numbers of dimensions and restrictions (or rows and columns of the describing matrix $\boldsymbol{A}$), the number of equations to be solved grows considerably. In most cases the procedure is practically not feasible.

However, we want to compute the extremal points in view of the fact that linear or convex functions achieve their maximum, if at all, then on the vertices. It is our aim to describe a 'practical' procedure in order to compute all vertices. This will eventually be done by moving from vertex to vertex until a maximizing value of a (linear) function is reached.

Within this section we consider a partial problem: Assuming we know a vertex $\bar{\boldsymbol{x}}$ - how do we reach the 'next' or 'adjacent' vertex?

**Remark 1.1.** *For a start it is preferable to discuss this question in the framework provided by Theorem 2.8 (not by Theorem 2.5). Thus, given an $m \times n$-matrix $\boldsymbol{A}$ and a vector $\boldsymbol{b} \in \mathbb{R}^m$, we consider a convex polyhedron of the type*

$$D = D^0_{\boldsymbol{A},\boldsymbol{b}} = \{\boldsymbol{x} \in \mathbb{R}^n \,\big|\, \boldsymbol{x} \geq 0, \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$$

*- and we frequently just write $D$ and omit the subscripts. As previously, we use*

$$I := \{1, \cdots, m\} \quad , \quad J := \{1, \cdots, n\} \; ,$$

*and whenever $\boldsymbol{x} \in \mathbb{R}^n$ (or more specific $\boldsymbol{x} \in D = D^0_{\boldsymbol{A},\boldsymbol{b}}$), then we shall write*

$$J^+(\boldsymbol{x}) = J^+ = \{j \in J \,|\, \bar{x}_j > 0\}$$

*Recall that, using this notation, $\bar{\boldsymbol{x}} \in D$ is a vertex of $D$ if and only if*

$$(\boldsymbol{A}_{\bullet j})_{j \in J^+}$$

*are linearly independent (Theorem 2.8). Hence, every vertex of $D$ has* ***at most*** *$m$ positive coordinates.*

**Definition 1.2.** 1. *A vertex $\bar{\boldsymbol{x}}$ of $D$ is said to be* ***degenerate*** *if it has less then $m$ positive coordinates, i.e., if*

$$(1) \qquad \mid J^+ \mid = \mid J^+(\bar{\boldsymbol{x}}) \mid < m$$

*holds true.*

2. *Otherwise $\bar{\boldsymbol{x}}$ is said to be* ***non-degenerate (n.d.)****.*

3. *The pair* $(\boldsymbol{A}, \boldsymbol{b})$ *is said to be* ***non-degenerate (n.d.)****, if all vertices of* $D = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geq 0,\ \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$ *are non-degenerate.*

**Example 1.3.** Let $n = 4$ , $m = 2$ and

$$\begin{aligned} \boldsymbol{A} &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix} \\ \boldsymbol{b} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

Then

$$D = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \,\middle|\, \boldsymbol{x} \geq 0 \ , \ \sum_{i=1}^{4} x_i = 1 \ , \ x_2 + 2x_4 = 1 \right\} \ ,$$

thus $D$ can be viewed as the intersection of the hyperplane $\{\boldsymbol{x} \in \mathbb{R}^4 \mid x_2 + 2x_4 = 1\}$ and the standard simplex $\underline{\overline{\mathbf{X}}}^4 = \left\{\boldsymbol{x} \in \mathbb{R}^4 \,\middle|\, \boldsymbol{x} \geq 0, \sum_{i=1}^{4} x_i = 1\right\}$ . This intersection is two dimensional and we can identify the points $\boldsymbol{e}^2 = (0, 1, 0, 0)$ , $(\frac{1}{2}, 0, 0, \frac{1}{2})$, and $(0, 0, \frac{1}{2}, \frac{1}{2})$. Thus, we obtain the the sketch represented in Figure 1.1. Clearly, $\boldsymbol{e}^2$ is degenerate since there is just one positive

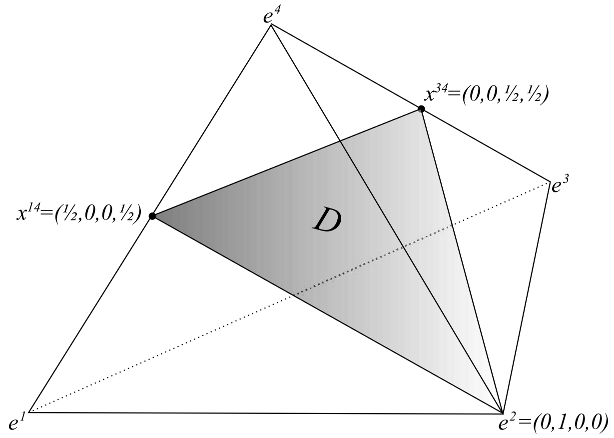



Figure 1.1: $D = \left\{\boldsymbol{x} \in \mathbb{R}^4 \,\middle|\, \boldsymbol{x} \geq 0 \ , \ \sum_{i=1}^{4} x_i = 1 \ , \ x_2 + 2x_4 = 1\right\}$

coordinate (and 1 linearly independent vector corresponding to it).

**Example 1.4.** Let $n$ be arbitrary and $m = 1$, that is

$$D = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geq 0 \ , \ \boldsymbol{a}\boldsymbol{x} = b_0\}$$

for a suitable vector $\boldsymbol{a} \in \mathbb{R}^3$ and $b_0 \in \mathbb{R}$. Thus $D$ is the intersection of a hyperplane and the positive orthant of $\mathbb{R}^3$. In order to have non-degeneracy

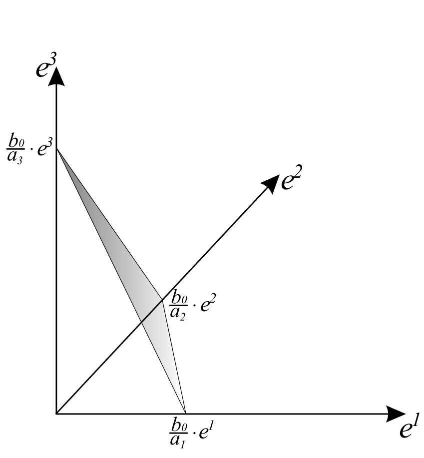



Figure 1.2: The case $m = 1$. (Example 1.4 )

ensured we see at once that $b_0 \neq 0$ is necessary and sufficient. If so, the vertices of $D$ are given by $\frac{b_0}{a_i}\ \boldsymbol{e}^i$ for those $i \in J$ that satisfy $a_i \neq 0$ and $\frac{b_0}{a_i} > 0$ . See Figure 1.2.

**Example 1.5.** Let $n = 3$ and $m = 2$, then

$$D = \left\{\boldsymbol{x} \in \mathbb{R}^3 \mid \boldsymbol{x} \geq 0\ ,\ x_2 = 1\ ,\ x_1 + x_3 = 1\right\}$$

is n.d. (Figure 1.3)

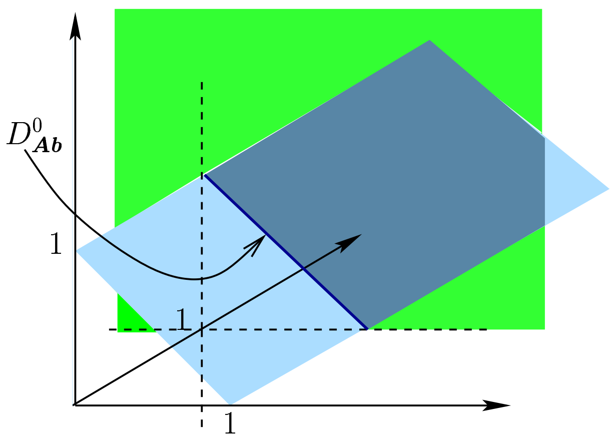



Figure 1.3: The line segment is n.d. (Example 1.5 )

while $D = \{\boldsymbol{x} \in \mathbb{R}^3 \mid \boldsymbol{x} \geq 0\ ,\ x_1 + x_3 = 1\ ,\ x_2 + x_3 = 1\}$ is not. (Figure 1.4)

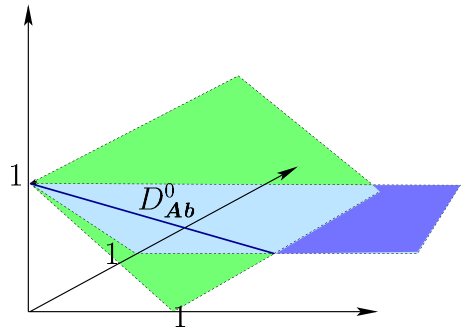



Figure 1.4: A degenerate line segment. (Example 1.5 )

**Remark 1.6.** *If $(\boldsymbol{A}, \boldsymbol{b})$ is n.d. and $D$ is compact, then every $\boldsymbol{x} \in D$ has at least $m$ positive coordinates. For, in this case we can appeal to Theorem 2.11 of* CHAPTER *1 (the Krein-Milman-Theorem). That is, given $\boldsymbol{x} \in D$, pick $k \in \mathbb{N}$ , $\boldsymbol{\alpha} \in \underline{\overline{\mathbf{X}}}^k$, and $\boldsymbol{x}^1, \cdots, \boldsymbol{x}^k \in EXT\ (D)$ such that*

$$\boldsymbol{x} = \boldsymbol{x}^{\boldsymbol{\alpha}} = \sum_{l=1}^{k} \alpha_l x^l \ .$$

*As at least one $\alpha_l$ has to be positive, $\boldsymbol{x}$ will have positive coordinates at least with the positive coordinates of $\boldsymbol{x}^l$ .*

*As it turns out, we can prove more without assuming compactness (and thus without appealing to the Krein-Milman-Theorem).*

**Theorem 1.7.** *Let $(\boldsymbol{A}, \boldsymbol{b})$ be n.d. and let $\bar{\boldsymbol{x}} \in D$. Then $\bar{\boldsymbol{x}} \in \boldsymbol{Ext}\, D$ if and only if $\bar{\boldsymbol{x}}$ has exactly $m$ positive coordinates.*

**Proof:**

**1$^{\text{st}}$STEP** :

If $\bar{\boldsymbol{x}}$ is a vertex, then $\bar{\boldsymbol{x}}$ has $m$ positive coordinates by Theorem 2.8 of CHAPTER 1 and by the very Definition 1.2.

**2$^{\text{nd}}$STEP** :

Assume now that $\bar{\boldsymbol{x}}$ has $m$ positive coordinates; write $J^+(\bar{\boldsymbol{x}}) = \{i \mid \bar{x}_i > 0\}$; thus

$$\mid J^+(\bar{\boldsymbol{x}}) \mid = m \ .$$

Now assume *per absurdum* that $\bar{\boldsymbol{x}}$ is not a vertex. Then we find $\boldsymbol{x}^0, \boldsymbol{x}^1 \in D$ , $\boldsymbol{x}^0 \neq \boldsymbol{x}^1$ and $0 < \bar{\lambda} < 1$ such that

$$\bar{\boldsymbol{x}} = \boldsymbol{x}^{\bar{\lambda}} = \bar{\lambda}\boldsymbol{x}^1 + (1 - \bar{\lambda})\boldsymbol{x}^0 \ .$$

As usual, for $i \in J - J^+(\bar{\boldsymbol{x}})$ we have

$$0 = \bar{x}_i = \bar{\lambda} x_i^1 + (1-\bar{\lambda}) x_i^0 \geq \bar{\lambda} 0 + (1-\bar{\lambda}) 0 = 0$$

- and as all inequalities employed must be equations we observe that

$$x_i^0 = x_i^1 = 0 \quad (i \in J - J^+).$$

Now we consider the case that $\boldsymbol{x}^1 - \boldsymbol{x}^0$ has positive coordinates (otherwise take $\boldsymbol{x}^0 - \boldsymbol{x}^1$). In this case the number

$$\begin{aligned} \hat{\lambda} &= min\{\lambda \in \mathbb{R} \mid \boldsymbol{x}^\lambda \geq 0\} \\ &= min\{\lambda \in \mathbb{R} \mid \boldsymbol{x}^0 + \lambda(\boldsymbol{x}^1 - \boldsymbol{x}^0) \geq 0\} \end{aligned}$$

is well defined and nonpositive.

Also $\hat{\boldsymbol{x}} := \boldsymbol{x}^{\hat{\lambda}}$ has at least one zero coordinate $i_0 \in J^+$ - for otherwise $\hat{\lambda}$ could be decreased, thus

$$(2) \qquad \mid J^+(\hat{\boldsymbol{x}}) \mid \leq m - 1$$

Clearly $\hat{\boldsymbol{x}} \geq 0$ and as $\boldsymbol{x}^0, \boldsymbol{x}^1$ satisfy $\boldsymbol{A}\hat{\boldsymbol{x}}^0 = \boldsymbol{A}\boldsymbol{x}^1 = \boldsymbol{b}$ it follows that

$$\boldsymbol{A}\hat{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}^0 + \hat{\lambda}(\boldsymbol{A}\boldsymbol{x}^1 - \boldsymbol{A}\boldsymbol{x}^0) = \boldsymbol{b} \ ,$$

that is $\hat{\boldsymbol{x}} \in D$ .

Since $(\boldsymbol{A}, \boldsymbol{b})$ is n.d., $\hat{\boldsymbol{x}}$ cannot be extremal, this would contradict Definition 1.2 (compare (1) and (2)).

Thus we may repeat the above procedure, obtaining $\hat{\hat{\boldsymbol{x}}} \in D$ such that

$$J^+(\hat{\hat{\boldsymbol{x}}}) \leq m - 2 \ .$$

This way we finally prove that $\mathbf{0} = (0, \ldots, 0) \in D$ holds true. This constitutes a contradiction as $\mathbf{0}$ is certainly a degenerate vertex, **q.e.d.**

**Remark 1.8.** *1. If $D \neq \emptyset$, then $\boldsymbol{Ext}\, D \neq \emptyset$. For, the procedure employed in the $2^{nd}STEP$ of the proof of Theorem 1.7 will produce a sequence of elements $\bar{\boldsymbol{x}}, \hat{\boldsymbol{x}}, \hat{\hat{\boldsymbol{x}}}, \ldots \tilde{\boldsymbol{x}}$ of $D$. The last of these vectors is an extremal point - possibly $(0, \ldots, 0)$ .*

2. *Therefore, if $D \neq \emptyset$ and $(\boldsymbol{A}, \boldsymbol{b})$ is n.d., then rank $\boldsymbol{A} = m$, hence $n \geq m$.*

3. *If $D \neq \emptyset$ and $(\boldsymbol{A}, \boldsymbol{b})$ is nondegenerate, then the procedure employed in the $2^{nd}STEP$ of the proof of Theorem 1.7 shows, that any $\boldsymbol{x} \in D$ has at least $m$ positive coordinates.*

4. *If rank $\boldsymbol{A} = m$ (and nondegeneracy does not necessarily prevail), then, for any vertex $\bar{\boldsymbol{x}}$ of $D$, there is an index set $\bar{J} \subseteq J$ such that the following holds true:*

$$J^{+}(\bar{\boldsymbol{x}}) \subseteq \bar{J} \subseteq J \tag{3}$$

$$|\bar{J}| = m \tag{4}$$

$$(\boldsymbol{A}_{\bullet j})_{j \in \bar{J}} \quad \text{are linearly independent.} \tag{5}$$

*If $\bar{\boldsymbol{x}}$ is n.d., then $\bar{J} = J^{+}(\bar{\boldsymbol{x}})$ .*

**Definition 1.9.** *Let $\boldsymbol{A}$ be an $m \times n$ matrix and let $\boldsymbol{b} \in \mathbb{R}^n$ be a vector. Let $\bar{\boldsymbol{x}}$ be a vertex of $D^0_{\boldsymbol{A},\boldsymbol{b}}$ and let $\bar{J} \subseteq J$ be an index set satisfying equations* (3), (4)*, and* (5) *of Remark 1.8. Then $(A_{\bullet j})_{j \in \bar{J}}$ is called a* ***basis corresponding to $\bar{\boldsymbol{x}}$*** *(**the** basis if $\bar{\boldsymbol{x}}$ is n.d.).*

*On the other hand, let $\hat{J} \subseteq J$, $\mid \hat{J} \mid = m$, such that the vectors $(\boldsymbol{A}_{\bullet j})_{j \in \hat{J}}$ are linearly independent. The unique solution $\bar{\boldsymbol{x}} \in \mathbb{R}^n$ of the system of linear equations*

$$\begin{array}{rcll} \sum_{j \in \hat{J}} x_j \; \boldsymbol{A}_{\bullet j} & = & \boldsymbol{b} & \\ x_j & = & 0 & (j \in J - \hat{J}) \end{array} \tag{6}$$

*is called the* ***basis-solution*** *(corresponding to $\hat{J}$ or to $(A_{\bullet j})_{j \in \hat{J}}$). A basis solution $\bar{\boldsymbol{x}}$ is said to be* ***feasible*** *if $\bar{\boldsymbol{x}} \geq 0$ holds true, i.e., if $\bar{\boldsymbol{x}} \in D$ is satisfied.*

**Corollary 1.10.** *If a matrix $\boldsymbol{A}$ has rank $m$, then* the feasible basis-solutions are precisely the vertices of $D = D_{\boldsymbol{A},\boldsymbol{b}}$.

This way we have seen that vertices and bases consisting of columns of $\boldsymbol{A}$ are closely connected. Therefore, the procedure of moving from a given vertex to another one most likely can be understood as exchangement of the corresponding bases.

The most natural way to change a basis is given by just throwing out *one member* of a basis and exchanging it by a column of $\boldsymbol{A}$ that so far was *not* a member.

Conveniently, it turns out that this is not only the straight forward way but also admits for a nice geometrical interpretation: exchanging *just one* basis vector amounts to moving to *adjacent* vertex - in the intuitive sense that the movement takes place along the edge of $D$ joining the two vertices.

We have as yet no formal definition of the terms *adjacent*, *edge*, etc. - but the following example enlightens the situation.

**Example 1.11.** (cf. Example 2.7 of Chapter 1. ) Let $m = 2, n = 4$ and consider $\boldsymbol{A} = \left(\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right)$ , $\boldsymbol{b} = (1, \frac{1}{2})$, this way we obtain

$$D = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \boldsymbol{x} \geq 0, \sum_{i=1}^{4} x_i = 1, x_4 = \frac{1}{2} \right\}$$

The basis corresponding to the vertex $x^k$ $(k = 1, 2, 3)$ is $\boldsymbol{A}_{\bullet k}, \boldsymbol{A}_{\bullet 4}$. All three

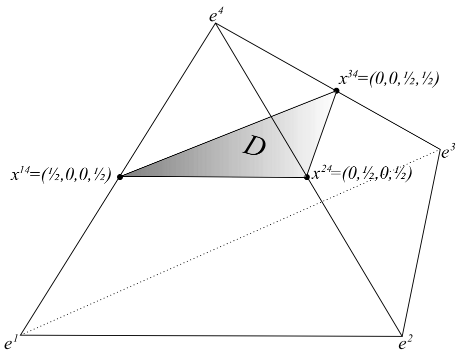



Figure 1.5: Example 1.11

are adjacent and moving from $x^k$ to $x^l$ amounts to exchanging $\boldsymbol{A}_{\bullet k}$ $\boldsymbol{A}_{\bullet l}$ . See Figure 1.5

**Example 1.12.** Next, consider again for $m = 2, n = 4$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{3} \end{pmatrix} \;,\; \boldsymbol{b} = \begin{pmatrix} 1 \\ \frac{1}{4} \end{pmatrix},$$

and

$$D = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \;\middle|\; \boldsymbol{x} \geq 0, \sum_{i=1}^{4} x_i = 1 \;, x_3 + \frac{x_4}{3} = \frac{1}{4} \right\}$$

The vertices are $\boldsymbol{x}^{13}, \boldsymbol{x}^{23}$ and $\boldsymbol{x}^{41}, \boldsymbol{x}^{42}$ with $\boldsymbol{x}^{ij} = \frac{3}{4}\boldsymbol{e}^i + \frac{1}{4}\boldsymbol{e}^j$ . Note that $\boldsymbol{x}^{41}$ and $\boldsymbol{x}^{42}$ are adjacent (switch from $1, 4$ to $2, 4$ by exchanging 1 and 2) - but $\boldsymbol{x}^{41}$ and $\boldsymbol{x}^{23}$ are not. Figure 1.6 shows that $D^0_{\boldsymbol{Ab}}$ geometrically has the form of a trapeziod.

**Definition 1.13.** *1. Let $(\boldsymbol{A}, \boldsymbol{b})$ be nondegenerate such that $D = D^0_{\boldsymbol{A},\boldsymbol{b}} \neq \emptyset$ holds true. Let $\bar{\boldsymbol{x}}$ be a vertex of $D$. We shall say that $(\boldsymbol{A}, \boldsymbol{b}, \bar{\boldsymbol{x}})$ constitutes a **standard vertex configuration**. Within this context, we write*

$$\bar{J} = J^+(\bar{\boldsymbol{x}}) = \{i \mid \bar{x}_i > 0\}$$

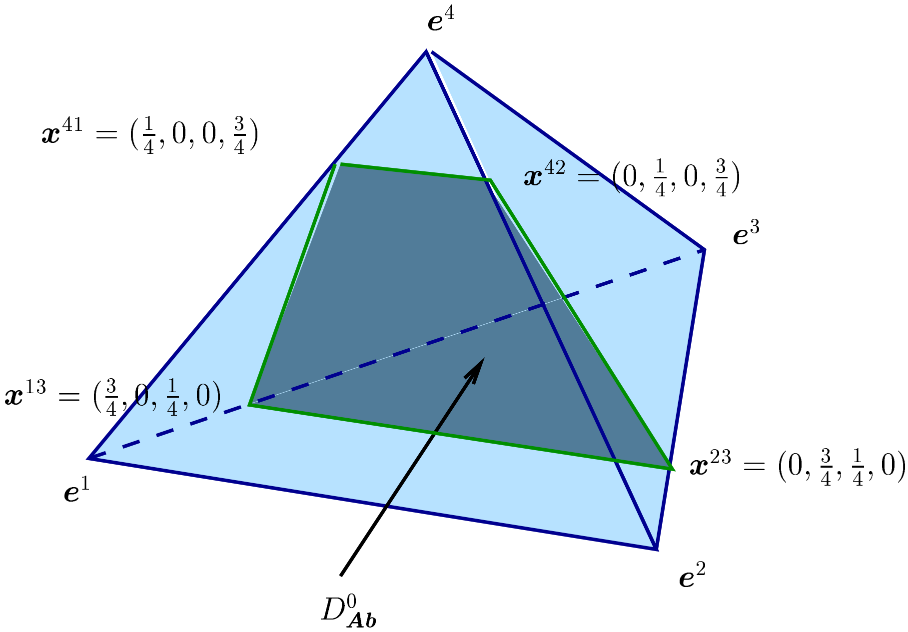



Figure 1.6: The Trapezoid (Example 1.12)

*and denote by*

$$\bar{\boldsymbol{A}} = (\boldsymbol{A}_{\bullet j})_{j \in \bar{J}}$$

*the $m \times m$ matrix consisting of column vectors which form the basis corresponding to $\bar{\boldsymbol{x}}$ .*

2. *Let $(\boldsymbol{A}, \boldsymbol{b}, \bar{\boldsymbol{x}})$ be a standard vertex configuration. For any $k \in J - \bar{J}$ we define real numbers*

$$(\lambda^k_j)_{j \in \bar{J}}$$

*via the unique representation*

$$\boldsymbol{A}_{\bullet k} = \sum_{j \in \bar{J}} \lambda^k_j \boldsymbol{A}_{\bullet j} \tag{7}$$

*of column $k$ by means of $\bar{\boldsymbol{A}}$ . Then the numbers $(\lambda^k_j)^{k \in J - \bar{J}}_{j \in \bar{J}}$ are called the **tableau elements** (corresponding to $\bar{\boldsymbol{x}}$).*

3. *Let $(\boldsymbol{A}, \boldsymbol{b}, \bar{\boldsymbol{x}})$ be a standard vertex configuration. Define for any $k \notin \bar{J}$ a linear subspace of $\mathbb{R}^n$ by*

$$\bar{L} = L^{-\bar{J}-k} := \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ x_j = 0 \quad (j \in J - (\bar{J} + k))\}.$$

*(Our notational convention is to use $J - k$ for $J - \{k\}$ and $J + j$ for $J \cup \{j\}$ $(k \in J,\ j \notin J)$)*

4. *Let* $(\boldsymbol{A}, \boldsymbol{b}, \bar{\boldsymbol{x}})$ *be a standard vertex configuration and let, for some* $k \notin \bar{J}$ *the linear subspace* $\bar{L}$ *be defined as in item 3. The mapping*

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R}^n \\ \theta & \longrightarrow & \boldsymbol{x}^\theta \end{array}$$

*defined by*

$$x_j^\theta = \begin{cases} \bar{x}_j - \theta\lambda_j^k & j \in \bar{J} \\ \theta & j = k \\ 0 & j \in J - (\bar{J} + k) \end{cases} \tag{8}$$

*is said to constitute the* ***canonical parametrization*** *of* $\bar{L}$ .

5. *Let* $(\boldsymbol{A}, \boldsymbol{b}, \bar{\boldsymbol{x}})$ *be a standard vertex configuration and let, for some* $k \notin \bar{J}$ *the linear subspace* $\bar{L}$ *be defined as in item 3. The convex set* $D \cap \bar{L}$ *is called the* ***edge*** *(of* $D$*)* ***emerging at*** $\bar{\boldsymbol{x}}$ ***in direction*** $k$.

We hasten to demonstrate that our nomenclature is justified.

**Theorem 1.14.** *Let* $(\boldsymbol{A}, \boldsymbol{b}, \bar{\boldsymbol{x}})$ *be a standard vertex configuration.*

1. $\bar{L} = L^{-\bar{J}-k}$ *is an affine subspace of* $\mathbb{R}^n$ *with dimension 1.*

2. *The canonical parametrization is a bijective mapping* $\mathbb{R} \to \bar{L}$.

3. *For* $\theta = 0$ *the canonical parametrization, yields* $\boldsymbol{x}^0 = \bar{\boldsymbol{x}}$.

4. *For small* $\theta > 0$ *we have* $\boldsymbol{x}^\theta \in D$ *(thus* $\bar{L} \cap D \neq \emptyset$ *holds always true).*

**Proof:**

1. The dimension of $\bar{L}$ is 1 since

$$rank \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{e}^j \end{pmatrix}_{j \in J - (\bar{J}+k)}$$

$$\begin{aligned} &= n - (m+1) + \text{ rank } (\boldsymbol{A}_{\bullet j})_{j \in \bar{J}+k} \\ &= n - (m+1) + m = n - 1 \end{aligned}$$

2. The image of $\mathbb{R}$ via the canonical parametrization clearly is a linear subspace of dimension 1 and the mapping $\theta \to \boldsymbol{x}^\theta$ is bijective. Therefore it suffices to show that $\boldsymbol{x}^\theta \in \bar{L} \quad (\theta \in \mathbb{R})$ holds true.

   However, for $\theta \in \mathbb{R}$ we obtain

$$\begin{aligned}\boldsymbol{b} &= \boldsymbol{A}\bar{\boldsymbol{x}} \\ &= \sum_{j\in\bar{J}} \bar{x}_j \boldsymbol{A}_{\bullet j} \\ &= \sum_{j\in\bar{J}} \bar{x}_j \boldsymbol{A}_{\bullet j} + \theta\underbrace{(\boldsymbol{A}_{\bullet k} - \sum_{j\in\bar{J}} \lambda_j^k \boldsymbol{A}_{\bullet j})}_{0} \\ &= \sum_{j\in\bar{J}} (\bar{x}_j - \theta\lambda_j^k)\boldsymbol{A}_{\bullet j} + \theta\boldsymbol{A}_{\bullet k} \\ &= \boldsymbol{A}\boldsymbol{x}^\theta .\end{aligned}$$

   Now, as $\boldsymbol{x}^\theta \geq 0$ for sufficiently small $\theta$, it follows that $\boldsymbol{x}^\theta \in D$ holds true for such $\theta$ ,

**q.e.d.**

**Remark 1.15.** *Clearly, $\boldsymbol{x}^\theta \notin D$ for $\theta < 0$ . Thus, $\bar{L} \cap D$ is either an interval or else a half line. Which case prevails will depend on the sign of $\lambda_j^k$. E.g., if $\lambda_j^k \leq 0$ for all $j$, then $\boldsymbol{x}^\theta \in D$ for all $\theta \geq 0$, thus the edge $L \cap D$ is unbounded.*

**Theorem 1.16.** *Let $(\boldsymbol{A}, \boldsymbol{b}, \bar{\boldsymbol{x}})$ constitute a standard situation. Let $k \in I - \bar{I}$ and consider $\bar{L} = L^{-\bar{I}-k}$ and the canonical parametrization given by Definition 1.14.*

1. *The following statements are equivalent*
   - *(a) $\bar{L} \cap D$ is bounded.*
   - *(b) $\Theta := \{\theta \geq 0 \mid x_j^\theta = 0$ for some $j \in \bar{J}\} \neq \emptyset$.*
   - *(c) $\{j \in \bar{J} \mid \lambda_j^k > 0\} \neq \emptyset$ .*
   - *(d) The quantity*

$$\begin{aligned}\hat{\theta} &:= \min \{\theta \mid \theta \in \Theta\} \\ &= \min \left\{ \frac{\bar{x}_j}{\lambda_j^k} \,\middle|\, j \in \bar{J} \, , \lambda_j^k > 0 \right\}\end{aligned}$$

     *is well defined, that is, a real number.*

2. *If $\hat{\theta}$ exists, then*

$$\hat{\boldsymbol{x}} := \boldsymbol{x}^{\hat{\theta}}$$

*is a vertex of $D$ (the "second vertex at $\bar{L} \cap D$").*

3. *If $\hat{\theta}$ exists, then there is a* ***unique*** *$j_0 \in \bar{J}$ such that*

$$\hat{x}_{j_0} = x^{\hat{\theta}}_{j_0} = 0 \ .$$

4. *The basis corresponding to $\hat{x}$ is, therefore*

$$(\boldsymbol{A}_{\bullet j})_{j \in \bar{J} - j_0 + k}$$

5. *The restriction of the canonical parametrization*

$$\begin{array}{ccc} [0, \hat{\theta}] & \longrightarrow & [\bar{\boldsymbol{x}}, \hat{\boldsymbol{x}}] \\ \theta & \longrightarrow & \boldsymbol{x}^{\theta} \end{array}$$

*parametrizes the edge $\bar{L} \cap D = [\bar{\boldsymbol{x}}, \hat{\boldsymbol{x}}]$, i.e., constitutes a bijective mapping.*

**Proof:** Clearly $\Theta$ is nonempty if and only if there exists a positive $\lambda^k_j$. In this case

$$\begin{aligned} \min\{\theta \in \Theta\} &= \min\left\{\theta \geq 0 \,\middle|\, \theta = \tfrac{\bar{x}_j}{\lambda^k_j} \text{ for some } j \in \bar{J} \ , \ \lambda^k_j > 0\right\} \\ &= \min\left\{\tfrac{\bar{x}_j}{\lambda^k_j} \,\middle|\, j \in \bar{J} \ , \ \lambda^k_j > 0\right\} \end{aligned}$$

Moreover, $\hat{\boldsymbol{x}} = \boldsymbol{x}^{\hat{\theta}}$ has exactly $m$ positive coordinates (that is, the coordinates $j \in \bar{J} - j_0 + k$), hence is a vertex by the n.d. requirement in view of Theorem 1.7

Further positive coordinates of $\bar{\boldsymbol{x}}$ (i.e., coordinates $j \in \bar{J}$) must not vanish since, in view of 1.13, $\boldsymbol{x} \in D$ must have at least $m$ positive coordinates. **q.e.d.**

**Example 1.17.** We continue with the discussion of Example 1.11. Recall the defining quantities for $D_{\boldsymbol{Ab}}$ which are

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{3} \end{pmatrix} \ , \ \boldsymbol{b} = \begin{pmatrix} 1 \\ \frac{1}{4} \end{pmatrix} ,$$

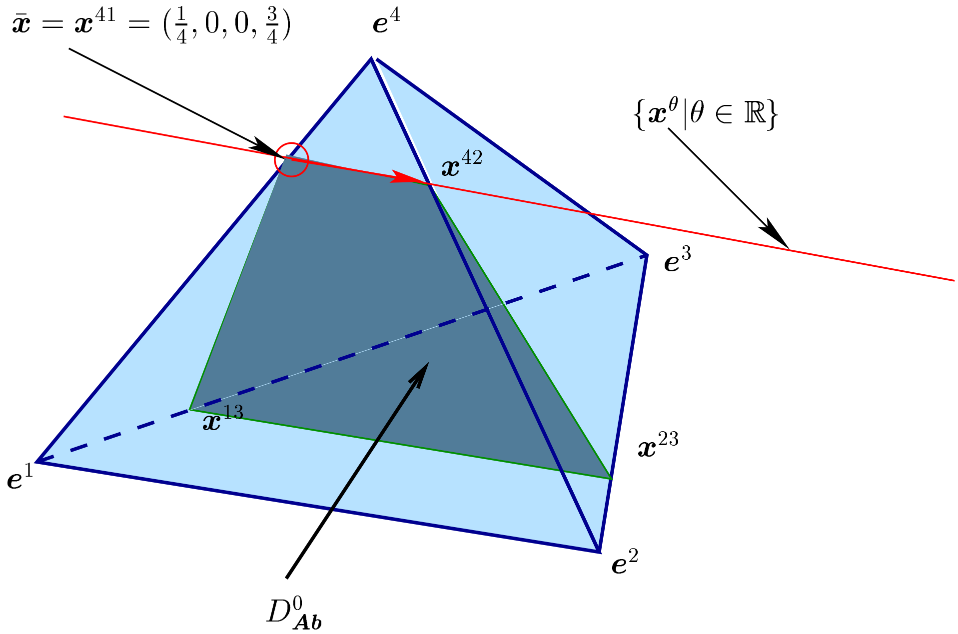



Figure 1.7: Parametrizing an edge of the Trapezoid (Example 1.17)

Consider the extremal point $\bar{\boldsymbol{x}} = \boldsymbol{x}^{41} = (\frac{1}{4}, 0, 0, \frac{3}{4})$, then $(\boldsymbol{A}, \boldsymbol{b}, \bar{\boldsymbol{x}})$ constitute a standard situation as we have seen in the context of Example 1.11. The basis corresponding to $\bar{\boldsymbol{x}}$ is given by $A_{\bullet 1} = (1, 0)$ and $A_{\bullet 4} = (1, 1/3)$.

Take $k = 2$ (i.e., we want the second coordinate to become positive). The equation

$$\boldsymbol{A}_{\bullet 2} = \lambda_1^2 \boldsymbol{A}_{\bullet 1} + \lambda_4^2 \boldsymbol{A}_{\bullet 4}$$

requires $\lambda_1^2 = 1, \lambda_4^2 = 0$ since $\boldsymbol{A}_{\bullet 2} = \boldsymbol{A}_{\bullet 1}$.

Therefore,

$$\begin{aligned} \boldsymbol{x}^\theta &= (\tfrac{1}{4} - \theta\lambda_1^2, \theta, 0, \tfrac{3}{4} - \theta\lambda_4^2) \\ &= (\tfrac{1}{4} - \theta, \theta, 0, \tfrac{3}{4}) \end{aligned}$$

Clearly $\hat{\theta} = \frac{1}{4}$, thus $\hat{\boldsymbol{x}} = (0, \frac{1}{4}, 0, \frac{3}{4}) = \boldsymbol{x}^{42}$.

This way we have exchanged the basis vector $\boldsymbol{A}_{\bullet 1}$ against $\boldsymbol{A}_{\bullet 2}$ .

**Remark 1.18.** 1. *Consider a standard situation $(\boldsymbol{A}, \boldsymbol{b}, \bar{\boldsymbol{x}})$. Let $\bar{J} = J^+(\bar{\boldsymbol{x}})$ as previously. As $D$ is nondegenerate, we know that $n \geq m$ indices in $J - \bar{J}$ are possible candidates in order to play the role of $k$ in 1.13, 1.14, 1.15, and 1.16. Thus, at every vertex $\bar{\boldsymbol{x}} \in D$ there are exactly $n - m$ edges in $D$.*

2. *Consider a standard situation* $(\boldsymbol{A}, \boldsymbol{b}, \bar{\boldsymbol{x}})$ *and let* $\boldsymbol{c} \in \mathbb{R}^n$. *Define the linear function* $f : D \to \mathbb{R}$ *via*

$$f(\boldsymbol{x}) = \boldsymbol{c}\boldsymbol{x} = \sum_{j=1}^{n} c_j \, x_j \quad (\boldsymbol{x} \in D)$$

*(the "objective function"). As a preliminary exercise, let us compare the values*

$$f(\bar{\boldsymbol{x}}) = \boldsymbol{c}\bar{\boldsymbol{x}} \quad \text{and} \quad f(\boldsymbol{x}^\theta) = \boldsymbol{c}\boldsymbol{x}^\theta$$

for $\theta > 0$ .

*We come up with*

$$(9) \qquad \begin{aligned} \boldsymbol{c}\boldsymbol{x}^\theta &= \textstyle\sum_{j\in\bar{J}} c_j x_j^\theta + c_k\theta \\ &= \textstyle\sum_{j\in\bar{J}} c_j(\bar{x}_j - \theta\lambda_j^k) + c_k\theta \\ &= \textstyle\sum_{j\in\bar{J}} c_j\bar{x}_j + \theta \left(c_k - \sum_{\bar{J}} c_j\lambda_j^k\right) \\ &=: \; \boldsymbol{c}\bar{\boldsymbol{x}} + \theta\,(c_k - z_k) \end{aligned}$$

*If we define*

$$z_k := \sum_{j\in\bar{J}} c_j\lambda_j^k = c_{\bar{J}}\lambda_{\bar{J}}^k,$$

*then it is obvious that* $f$ *increases (strictly) along the edge* $[\bar{\boldsymbol{x}}, \hat{\boldsymbol{x}}]$ *if and only if* $c_k > z_k$ *holds true. This motivates the following*

**Theorem 1.19.** *Let* $(\boldsymbol{A}, \boldsymbol{b}, \bar{\boldsymbol{x}})$ *be a standard situation and let*

$$\bar{J} = \{j \mid \bar{x}_j > 0\} = J^+(\bar{\boldsymbol{x}}) \; .$$

*Let* $\boldsymbol{c} \in \mathbb{R}^n$ *and* $f : D \to \mathbb{R}$ , $f(\boldsymbol{x}) = \boldsymbol{c}\boldsymbol{x}$ $(\boldsymbol{x} \in \mathbb{R}^n)$ .

*For every* $k \in J - \bar{J}$ *let* $\lambda_j^k$ $(j \in \bar{J})$ *denote the corresponding tableau elements and define*

$$z_k = c_{\bar{J}}\lambda_{\bar{J}}^k := \sum_{j\in\bar{J}} c_j\lambda_j^k.$$

*Then the following holds true.*

*(a) If* $z_k \geq c_k \quad (k \in J - \bar{J})$, *then* $\bar{\boldsymbol{x}}$ *maximizes* $f$ *on* $D$, *i.e.,*

$$f(\bar{\boldsymbol{x}}) = \max\{f(\boldsymbol{x}) \mid \boldsymbol{x} \in D\}$$

*holds true.*

*(b) If there is* $k \in J - \bar{J}$ *and* $j \in \bar{J}$ *such that*

$$z_k < c_k, \quad \lambda_j^k > 0$$

*holds true, then there exists* $\hat{\boldsymbol{x}} \in D$ *as given by Theorem 1.16 and* $\hat{\boldsymbol{x}}$ *satisfies*

$$\boldsymbol{c}\hat{\boldsymbol{x}} > \boldsymbol{c}\bar{\boldsymbol{x}} \ .$$

*That is, the movement from* $\bar{\boldsymbol{x}}$ *to* $\hat{\boldsymbol{x}}$ *increases* $f$ *strictly.*

*(c) Finally, if there is* $k \in J - \bar{J}$ *satisfying*

$$z_k < c_k \text{ and } \lambda_j^k \leq 0 \quad (j \in \bar{J}),$$

*then* $f$ *is unbounded on* $D$ *and* $\max\{f(\boldsymbol{x}) \mid \boldsymbol{x} \in D\}$ *does not exist.*

**Proof:** In view of our previous discussion, *b)* and *c)* should now be rather obvious. By Remark 1.18 we know that $f$ increases with positive $\theta$ and by Theorem 1.16 the two alternatives occurring are represented by an unbounded or bounded edge at $\bar{\boldsymbol{x}}$, thus our claim is a direct consequence of these results.

Hence it remains to prove *a)* - and clearly the problem is that we have a 'local' maximum but we do not know whether it is a global one. More precisely, Remark 1.18 tells us that $f$ decreases in the direction of each one of the $n-m$ edges joining at $\bar{\boldsymbol{x}}$. But this does not tell us immediately that $f$ could not have a larger value somewhere else.

The answer to this intuitive problem is, of course, that $f$ is a linear function. If it decreases in every direction form $\bar{\boldsymbol{x}}$ then it decreases also in every direction pointing into $D$ - and can hardly increase again to assume a large value at some further edge...

Formally, let $\boldsymbol{x} \in D$ be arbitrary. Then we obtain the following equations

$$\begin{aligned}
\sum_{\bar{J}} \bar{x}_j \boldsymbol{A}_{\bullet j} &= \boldsymbol{b} \\
&= \sum_{j \in J} x_j \boldsymbol{A}_{\bullet j} \\
&= \sum_{j \in \bar{J}} x_j \boldsymbol{A}_{\bullet j} + \sum_{l \in J - \bar{J}} x_l \boldsymbol{A}_{\bullet l} \\
&= \sum_{j \in \bar{J}} x_j \boldsymbol{A}_{\bullet j} + \sum_{l \in J - \bar{J}} x_l \left( \sum_{j \in \bar{J}} \lambda_j^l \boldsymbol{A}_{\bullet j} \right) \\
&= \sum_{j \in \bar{J}} \left( x_j + \sum_{l \in J - \bar{J}} x_l \lambda_j^l \right) \boldsymbol{A}_{\bullet j}.
\end{aligned}$$

as $(\boldsymbol{A}_{\bullet j})_{j \in \bar{J}}$ are linearly independent, we obtain

$$\bar{x}_j = x_j + \sum_{l \in J - \bar{J}} x_l \lambda_j^l. \tag{10}$$

This establishes a relation between $\bar{\boldsymbol{x}}$ and $\boldsymbol{x}$ which permits us to compare the values $\boldsymbol{c}\bar{\boldsymbol{x}}$ and $\boldsymbol{cx}$. Indeed, we now compute $\boldsymbol{cx}$ and $\boldsymbol{c}\bar{\boldsymbol{x}}$, estimating them as follows

$$\begin{aligned}
\boldsymbol{cx} &= \sum_{\bar{J}} c_j x_j + \sum_{J - \bar{J}} c_l x_l \\
&\leq \sum_{\bar{J}} c_j x_j + \sum_{J - \bar{J}} z_l x_l \\
&= \sum_{\bar{J}} c_j x_j + \sum_{l \in J - \bar{J}} \left( \sum_{j \in \bar{J}} c_j \lambda_j^l \right) x_l \\
&= \sum_{j \in \bar{J}} c_j \left( x_j + \sum_{l \in J - \bar{J}} x_l \lambda_j^l \right) \\
&= \sum_{j \in \bar{J}} c_j \bar{x}_j \quad \text{(cf.(10))} \\
&= c\bar{\boldsymbol{x}} \ ,
\end{aligned}$$

**q.e.d.**

**Example 1.20.** We return to Examples 1.12 and 1.17. Within this context, the polyhedron $D^0_{\boldsymbol{Ab}}$ was given by

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{3} \end{pmatrix} \ , \ \boldsymbol{b} = \begin{pmatrix} 1 \\ \frac{1}{4} \end{pmatrix} .$$

The geometrical picture is repeated in Figure 1.8 With respect to the vertex $\bar{\boldsymbol{x}} = x^{41} = (\frac{1}{4}, 0, 0\frac{3}{4})$ we have computed for $k = 2$ the tableau elements $\lambda^2_1 = 1$

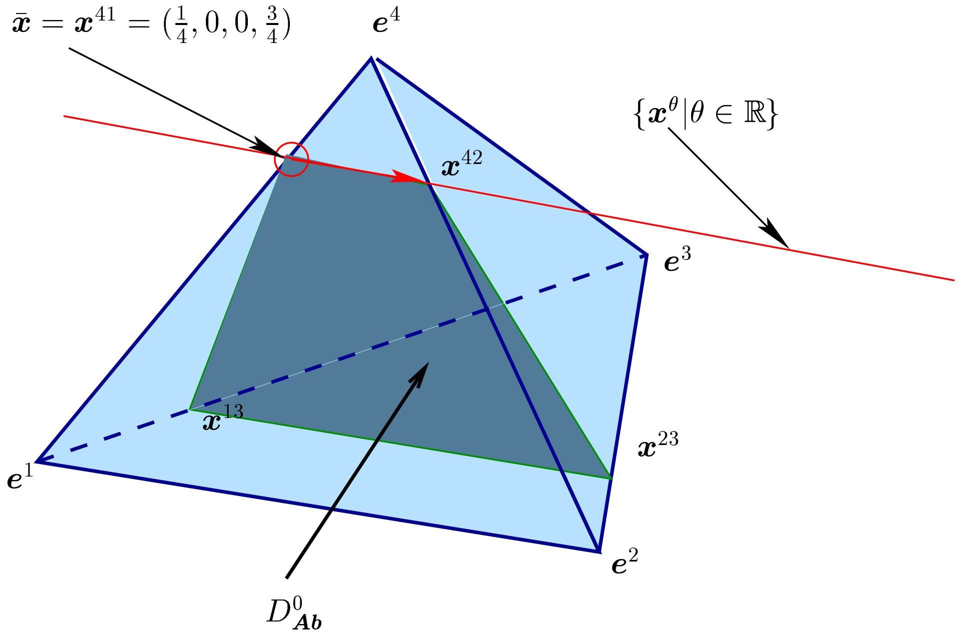



Figure 1.8: The Trapezoid revisited

)

and $\lambda^2_4 = 0$. Thus, $z_k = z_2 = \sum c_j\lambda^2_j = c_1$ for every $\boldsymbol{c} \in \mathbb{R}^n$. Taking $\boldsymbol{x}^\theta$ as well from Example 1.17, i.e.

$$\boldsymbol{x}^\theta = \left(\frac{1}{4} - \theta, \theta, 0, \frac{3}{4}\right),$$

we find

$$\begin{aligned} \boldsymbol{cx}^\theta &= c\bar{\boldsymbol{x}} + \theta(c_2 - z_2) \\ &= c\bar{\boldsymbol{x}} + \theta(c_2 - c_1) \end{aligned}$$

and $f$ is increasing from $\bar{\boldsymbol{x}}$ to $\hat{\boldsymbol{x}} = \boldsymbol{x}^{42} = \left(0, \frac{1}{4}, 0, \frac{3}{4}\right)$ if and only if $c_2 > c_1$ .

**Remark 1.21.** *Consider the situation that occurs within the framework of Theorem 1.19 if item b) prevails, i.e.,assume that we have the existence of at least one*

*$k \in J - \bar{J}$ and $j \in \bar{J}$ satisfying with*

$$z_k < c_k \ , \quad \lambda_j^k > 0.$$

*Clearly, the choice of $k \in J - \bar{J}$ is not necessarily unique.*

*In view of formulae (8) and (9), that is*

$$cx^\theta = c\bar{\boldsymbol{x}} + \theta(c_k - z_k),$$

*it seems plausible to choose $k$ under the circumstances such that $c_k - z_k$ is **maximal** (or $z_k - c_k$ **minimal**) in order to increse the slope of $f$ as much as possible. Nothing formally exact can be presented in order to support this choice, as it is not only the slope of the function but also the distance to travel which eventally determines the value of the function at the far vertex.*

# 2 The Simplex Tableau

Within this section we present the formal algorithm that serves for an exchange of vertices. Given some vertex – or basis solution – we exchange one basis vector against another one, thus moving to an adjacent vertex. Among various adjacent vertices we choose the one which yields the largest increment with respect to the linear function under consideration (unless the objective function is unbounded).

The parametrization presented previously turns out to be a vehicle only. A formal algorithm allows to compute all the details in order to specify the next vertex. This algorithm is given by a simple procedure (the rectangle rule) which is performed on a certain type of matrix, called a tableau.

Note that a matrix can be seen as a function defined on the Cartesian product of two finite sets. Most of the time we have used the sets $I = \{1, \ldots, m\}$ and $J = \{1, \ldots, n\}$. Whenever we use these particular index sets there is a tendency to view the rows and columns of the matrix in the same ordering as is indicated by the ordering of the natural numbers imposed on $I$ and $J$. In what follows, this implicit assumption may be violated; more precisely, a row with index $j$ might occur at a 'position' different from $j$ at some matrix constructed for a certain purpose.

We start out with the definition of the rectangle rule.

**Definition 2.1.** *Let $J' \subseteq J$ and let*

$$\boldsymbol{S} := J' \times (J - J') \to \mathbb{R} \tag{1}$$

*be a matrix. Let $j_0 \in J', k_0 \in J - J'$ be such that*

$$\alpha := s_{j_0}^{k_0} = s(j_0, k_0) > 0 \tag{2}$$

*holds true. We shall say that the matrix*

$$\boldsymbol{S}^0 : (J' - j_0 + k_0) \times (J - J' + j_0 - k_0) \to \mathbb{R} \tag{3}$$

*is generated by $\boldsymbol{S}$ via the* ***rectangle rule*** *(with $(j_0, k_0)$ serving as the* ***pivot****), if the relation as indicated by the following scetch is established between $\boldsymbol{S}$*

*and* $\boldsymbol{S}^0$*. We write*

$$
(4)\qquad \boldsymbol{S} = \begin{array}{c} \\ \\ \\ j_0 \\ \\ \\ \\ \\ \\ \\ \\ \end{array}
\underbrace{\overset{\qquad\quad k_0}{\left.\begin{array}{|ccc|c|ccccc|}
\hline
\cdot & \cdot & \cdot & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
* & * & * & \boldsymbol{\alpha} & * & * & * & \boldsymbol{\beta} & * \\
\hline
\cdot & \cdot & \cdot & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \boldsymbol{\gamma} & \cdot & \cdot & \cdot & \boldsymbol{\delta} & \cdot \\
\cdot & \cdot & \cdot & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\end{array}\right\}}}_{J-J'} J' \quad .
$$

*and we want* $\boldsymbol{S}^0$ *to look as follows:*

$$
(5)\qquad \boldsymbol{S}^0 = \begin{array}{c} \\ \\ \\ k_0 \\ \\ \\ \\ \\ \\ \\ \\ \end{array}
\underbrace{\overset{\qquad\quad j_0}{\left.\begin{array}{|ccc|c|ccccc|}
\hline
\cdot & \cdot & \cdot & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
* & * & * & \frac{1}{\boldsymbol{\alpha}} & * & * & * & \frac{\boldsymbol{\beta}}{\boldsymbol{\alpha}} & * \\
\hline
\cdot & \cdot & \cdot & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -\frac{\boldsymbol{\gamma}}{\boldsymbol{\alpha}} & \cdot & \cdot & \cdot & \boldsymbol{\delta} - \frac{\boldsymbol{\beta}}{\boldsymbol{\alpha}}\boldsymbol{\gamma} & \cdot \\
\cdot & \cdot & \cdot & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & * & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\end{array}\right\}}}_{J-J'+j_0-k_0} J' - j_0 + k_0 \quad .
$$

*We write*

$$\boldsymbol{S}^0 = \boldsymbol{R}_{j_0}^{k_0}\boldsymbol{S} \tag{6}$$

*in order to describe that* $\boldsymbol{S}^0$ *is obtained from* $\boldsymbol{S}$ *by the rectangle rule.*

Next we define the type of matrix (the ***tableau***) we want to deal with under the rectangle rule.

**Definition 2.2.** *1. Let* $(\boldsymbol{A}, \boldsymbol{b}, \bar{\boldsymbol{x}})$ *be a standard vertex configuration (cf. Definition 1.13) and let*

$$f : \mathbb{R}^n \to \mathbb{R}$$

*be a linear function, i.e.,* $f(\boldsymbol{x}) = \boldsymbol{cx}$ $(\boldsymbol{x} \in \mathbb{R}^n)$ *with suitable* $\boldsymbol{c} \in \mathbb{R}^n$*.*
*For* $j \in \bar{J}$ *and* $k \in J - \bar{J}$ *define real numbers* $\lambda_j^k$ *via*

$$\boldsymbol{A}_{\bullet k} = \sum_{j \in \bar{J}} \lambda_j^k \boldsymbol{A}_{\bullet j} \tag{7}$$

(these numbers are uniquely defined)*. Also, define*

$$z_k = \sum_{\bar{J}} c_j \lambda_j^k \quad (k \in J - \bar{J}) \tag{8}$$

*and*

$$\zeta_k = z_k - c_k \quad (k \in J - \bar{J}). \tag{9}$$

*Then we call the matrix*

$$\mathsf{T}(\bar{\boldsymbol{x}}) = \begin{pmatrix} \boldsymbol{\Lambda} & \bar{\boldsymbol{x}}_{\bar{J}} \\ \boldsymbol{\zeta} & \boldsymbol{c}\bar{\boldsymbol{x}} \end{pmatrix}$$

$$= \quad \bar{J}\left\{ \overbrace{\begin{pmatrix} & & & \bar{x}_{j'} \\ & \vdots & & \vdots \\ \cdots & \lambda_j^k & \cdots & \bar{x}_j \\ & \vdots & & \vdots \\ & & & \bar{x}_{j''} \\ \hline \cdots & \zeta_k & \cdots & \boldsymbol{c}\bar{\boldsymbol{x}} \end{pmatrix}}^{J-\bar{J}} \right.$$

*the* ***tableau*** *corresponding to* $\bar{\boldsymbol{x}}$*.*

2. *Let $k_0 \in J - \bar{J}$ be such that $\zeta_{k_0} = \min_{J-\bar{J}} \zeta_k < 0$ holds true then $k_0$ defines a **pivot column** (of $\mathbf{T}(\bar{\boldsymbol{x}})$).*

   (Recall Remark 1.21; the index $k_0$ represents an edge with maximal increment of the function $f$.)

3. *Let $k_0$ be a pivot column and let $j_0 \in \bar{J}$ be such that*

$$\frac{\bar{x}_{j_0}}{\lambda_{j_0}^{k_0}} = \min\left\{ \frac{\bar{x}_j}{\lambda_j^{k_0}} \,\middle|\, j \in \bar{J}, \lambda_j^{k_0} > 0 \right\} \tag{10}$$

   *holds true. Then $j_0$ defines a **pivot row**.*

   (In the context of Theorem 1.16, this quantity coincides with $\hat{\theta}$ and if it is finite, then the edge corresponding to $k_0$ is bounded, hence has at least one further endpoint. At this endpoint the coordinate $j_0$ will vanish).

4. *If $k_0$ and $j_0$ as defined within the above items are well defined, then $\mathbf{T}(\bar{\boldsymbol{x}})$ is called **pivotable** and $(j_0, k_0)$ (or sometimes $\lambda_{j_0}^{k_0}$) is called the **pivot element**.*

   (The condition describes the case that there is an edge $k_0$ on which $f$ increases and which admits for a second vertex).

**Remark 2.3.** *Let $(\boldsymbol{A}, \boldsymbol{b}, \bar{\boldsymbol{x}})$ be a standard situation and let $\mathbf{T}(\bar{\boldsymbol{x}})$ be defined by Definition 2.2, thus*

$$\mathbf{T}(\bar{\boldsymbol{x}}) = \begin{pmatrix} \boldsymbol{\Lambda} & \bar{\boldsymbol{x}}_{\bar{J}} \\ \boldsymbol{\zeta} & c\bar{\boldsymbol{x}} \end{pmatrix} \tag{11}$$

*Suppose $k_0$ is a pivoting column of $\mathbf{T}(\bar{\boldsymbol{x}})$. We may list the quantities $\frac{\bar{x}_j}{\lambda_j^{k_0}}$ $(j \in \bar{J})$ – as far as they are defined – by adding an additional column to the right of $\mathbf{T}(\bar{\boldsymbol{x}})$. Also, we list the indices $j \in J^+$ and $k \in J - J^+$ as the natural ordering of rows and columns may be disturbed.*

*This we obtain the following scheme (we do not call it a matrix by obvious reasons). The numbers **(1)** ... **(7)** indicating various regions to be explained below:*

(12)
$$
\mathcal{T}(\bar{\boldsymbol{x}}) \;:=\;
\begin{array}{|c|ccc|c|c|}
\hline
* & \textbf{(1)} & k \in J - \bar{J} & & \multicolumn{2}{c|}{*} \\
\hline
\textbf{(2)} & \textbf{(3)} & \vdots & & \textbf{(4)} & \textbf{(7)} \\
 & & \vdots & & & \\
j \in \bar{J} & \cdots & \lambda_j^k & \cdots & \bar{x}_j & \frac{\bar{x}_j}{\lambda_j^{k_0}} \\
 & & \vdots & & & \\
\hline
 & \textbf{(5)} & & & \textbf{(6)} & \\
* & \cdots & \zeta_k & \cdots & \boldsymbol{c}\bar{\boldsymbol{x}} & * \\
\hline
\end{array}
$$

*The contents of regions* **(1)** *and* **(2)** *is obvious; these regions assign the correct indices to the quantities appearing in the center.*

*In region* **(3)**...**(6)** *the elements of* $\mathbf{T}(\bar{\boldsymbol{x}})$ *appear. The term 'tableau' will be used in connection with the above scheme* $\mathcal{T}(\bar{\boldsymbol{x}})$ *as well as with the matrix* $\mathbf{T}(\bar{\boldsymbol{x}})$.

**Theorem 2.4.** *Let* $(\boldsymbol{A}, \boldsymbol{b}, \bar{\boldsymbol{x}})$ *be a standard situation. Suppose that* $\mathbf{T}(\bar{\boldsymbol{x}})$ *is pivotable with pivot element* $(j_0, k_0)$.

*Suppose* $\hat{\boldsymbol{x}} = \boldsymbol{x}^{\hat{\theta}}$ *is obtained via Theorem 1.16 (Case b) as the neighboring edge to* $\bar{\boldsymbol{x}}$. *Then, with* $k = k_0$ *we obtain*

(13)
$$\mathbf{T}(\hat{\boldsymbol{x}}) = \boldsymbol{R}_{j_0}^{k_0}\mathbf{T}(\bar{\boldsymbol{x}}).$$

*That is,* $\mathbf{T}(\hat{\boldsymbol{x}})$ *is obtained from* $\mathbf{T}(\bar{\boldsymbol{x}})$ *by applying the rectangle rule.*

**Proof:** The rectangle rule has to be verified for the elements notated within the groups **(3)**...**(6)** in the tableau $\mathcal{T}(\bar{\boldsymbol{x}})$. For **(1)** and **(2)** we exchange the indices $k_0$ and $j_0$ while **(7)** is obtained by a simple division.

Thus, we first want to deal with **(3)**, i.e., with the tableau elements $\lambda_j^k$. Now, the elements $\hat{\lambda}_j^k$ of $\mathbf{T}(\hat{\boldsymbol{x}})$ are determined by

(14)
$$\boldsymbol{A}_{\bullet k} = \sum_{j \in \bar{J} - j_0} \hat{\lambda}_j^k \boldsymbol{A}_{\bullet j} + \hat{\lambda}_{k_0}^k \boldsymbol{A}_{\bullet k_0} \quad (k \in J - \bar{J} + j_0 - k_0)$$

On the other hand we had from $T(\bar{\boldsymbol{x}})$

(15)
$$\boldsymbol{A}_{\bullet k_0} = \sum_{j \in \bar{J}} \lambda_j^{k_0} \boldsymbol{A}_{\bullet j}$$

Now we have assumed $\lambda_{j_0}^{k_0} > 0$ since this is the pivot-element in $\mathbf{T}(\bar{\boldsymbol{x}})$, hence (15) can be rewritten to

$$\boldsymbol{A}_{\bullet j_0} = -\sum_{j \in \bar{J} - j_0} \frac{\lambda_j^{k_0}}{\lambda_{j_0}^{k_0}} \boldsymbol{A}_{\bullet j} + \frac{1}{\lambda_{j_0}^{k_0}} \boldsymbol{A}_{\bullet k_0} \tag{16}$$

Next, consider the the indices $k \in J - \bar{J} - k_0$ with respect to $\mathbf{T}(\bar{\boldsymbol{x}})$. We have

$$\begin{aligned} \boldsymbol{A}_{\bullet k} &= \sum_{j \in \bar{J}} \lambda_j^k \boldsymbol{A}_{\bullet j} \\ &= \sum_{j \in \bar{J} - j_0} \lambda_j^k \boldsymbol{A}_{\bullet j} + \lambda_{j_0}^k \left( - \sum_{j \in \bar{J} - j_0} \frac{\lambda_j^{k_0}}{\lambda_{j_0}^{k_0}} \boldsymbol{A}_{\bullet j} + \frac{1}{\lambda_{j_0}^{k_0}} \boldsymbol{A}_{\bullet k_0} \right) \\ &= \sum_{j \in \bar{J} - j_0} \left( \lambda_j^k - \lambda_{j_0}^k \frac{\lambda_j^{k_0}}{\lambda_{j_0}^{k_0}} \right) \boldsymbol{A}_{\bullet j} + \frac{\lambda_{j_0}^k}{\lambda_{j_0}^{k_0}} \boldsymbol{A}_{\bullet k_0} \end{aligned} \tag{17}$$

We can now compare (14) and (17), thus obtain, for $k \in J - \bar{J} - k_0$:

$$\begin{aligned} \hat{\lambda}_{k_0}^k &= \frac{\lambda_{j_0}^k}{\lambda_{j_0}^{k_0}} \\ \hat{\lambda}_j^k &= \lambda_j^k - \lambda_{j_0}^k \frac{\lambda_{j_0}^k}{\lambda_{j_0}^{k_0}} \quad (j \in \bar{J} - j_0). \end{aligned} \tag{18}$$

Moreoever, for $k = j_0$ a comparison of (16) and (14) yields

$$\begin{aligned} \hat{\lambda}_{k_0}^{j_0} &= \frac{1}{\lambda_{j_0}^{k_0}} \\ \hat{\lambda}_j^{j_0} &= -\frac{\lambda_j^{k_0}}{\lambda_{j_0}^{k_0}} \quad (j \in \bar{J} - j_0). \end{aligned} \tag{19}$$

Now, after some inspection it is indeed seen that (18) and (19) constitute the rectangle rule for group **(3)** in (12).

As to group**(4)**, i.e., the coordinates of $\bar{\boldsymbol{x}}$ we have to compare them with those of $\hat{\boldsymbol{x}}$. However, in view of Theorem 1.16 we know that

$$\hat{\boldsymbol{x}}_{k_0} = \hat{\theta} = \frac{\bar{\boldsymbol{x}}_{j_0}}{\lambda_{j_0}^{k_0}}$$

and

$$\hat{\boldsymbol{x}}_j = \bar{\boldsymbol{x}} - \hat{\theta}\lambda_j^{k_0} = \bar{\boldsymbol{x}} - \frac{\bar{\boldsymbol{x}}_{j_0}}{\lambda_{j_0}^{k_0}}\lambda_j^{k_0} \qquad (j \in \bar{J} - j_0),$$

are satisfied, thus we recognize the rectangle rule.

Group **(6)** is particularly easy to treat as the equations

$$\begin{aligned} c\hat{\boldsymbol{x}} &= cx^{\hat{\theta}} & &= c\bar{\boldsymbol{x}} + \hat{\theta}\,(c_{k_0} - z_{k_0}) \\ &= c\bar{\boldsymbol{x}} - \hat{\theta}\zeta_{k_0} & &= c\bar{\boldsymbol{x}} - \tfrac{\bar{x}_{j_0}}{\lambda_{j_0}^{k_0}}\ \zeta_{k_0} \end{aligned}$$

immediately show the rectangle rule.

Finally, group **(5)** we shall not deal with explicitely; we have to consider

$$\hat{\zeta}_k = \hat{z}_k - c_k = \sum_{j \in \bar{J} - j_0 + k_0} c_j \hat{\lambda}_j^k - c_k.$$

By plugging in $\hat{\lambda}_j^k$ (which we know from (3)) and some reshuffling the rectangle rule is confirmed, **q.e.d.**

**Example 2.5.** We continue with the discussion of the trapezoid example which we left at Example 1.20. Recall the data which were given by

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1/3 \end{pmatrix}, \quad \boldsymbol{b} \ = \ (1, \frac{1}{4}) \text{ and } \bar{\boldsymbol{x}} = (1/4, 0, 0, 3/4),$$

we reproduce the corresponding picture in Figure 2.1. In addition we choose $\boldsymbol{c} = (10, 0, 2, 0)$. We know already the tableau elements

$$\lambda_1^2 = 1, \ \lambda_4^2 = 0$$

and as

$$\boldsymbol{A}_{\bullet 3} = -2\boldsymbol{A}_{\bullet 1} + 3\boldsymbol{A}_{\bullet 4}$$

we can compute further tableau elements to be

$$\lambda_1^3 = -2 \qquad \lambda_4^3 = 3$$

Now, as $\boldsymbol{c} = (10, 0, 2, 0)$ we have

$$\begin{aligned} z_2 &= \textstyle\sum_{j \in \{1,4\}} \ \lambda_j^2 c_j &&= c_1 = 10; \ \ \zeta_2 = z_2 - c_2 = 10 \\ z_3 &= \textstyle\sum_{j \in \{1,4\}} \ \lambda_j^3 c_j &&= -2c_1 + 3c_4 = -20; \ \ \zeta_3 = z_3 - c_3 = -22 \end{aligned}$$

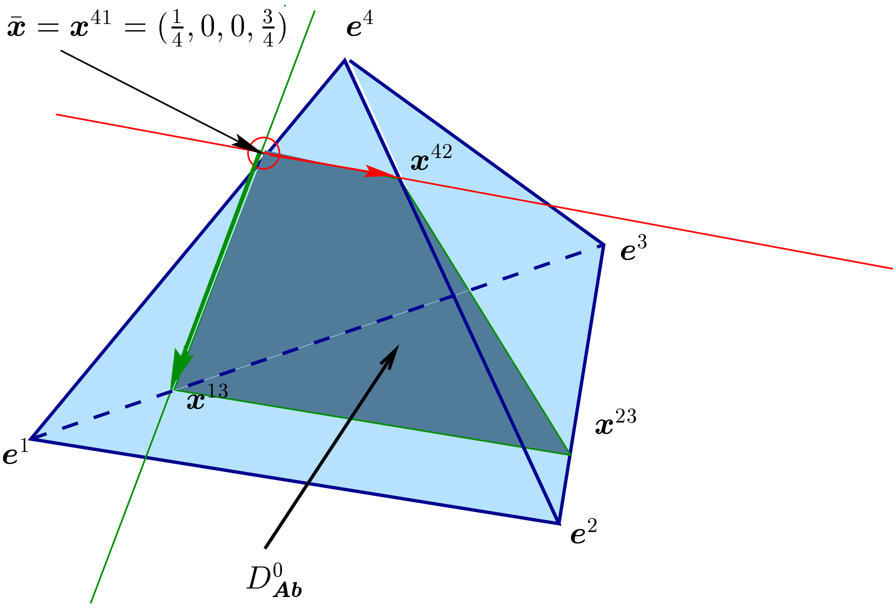



Figure 2.1: The Trapezoid again

Next, $c\bar{x} = \frac{10}{4}$ and the tableau is

(20)

| $*$ | 2 | 3 | $*$ | |
|---|---|---|---|---|
| 1 | 1 | $-2$ | $\frac{1}{4}$ | ? |
| 4 | 0 | 3 | $\frac{3}{4}$ | ? |
| $*$ | 10 | $-22$ | $\frac{10}{4}$ | $*$ |

Clearly, $k_0 = 3$ yields a pivoting column as $\zeta_3 = -22 < 0$; thus we add the quotients of group **(4)** (i.e. $\bar{x}$) and the $\lambda$'s in column 3 of group **(3)**, thus obtaining

(21)

| $*$ | 2 | 3 | $*$ | |
|---|---|---|---|---|
| 1 | 1 | $-2$ | $\frac{1}{4}$ | $-\frac{1}{8}$ |
| 4 | 0 | **3** | $\frac{3}{4}$ | $\frac{1}{4}$ |
| $*$ | 10 | $-22$ | $\frac{10}{4}$ | $*$ |

The minimal positive element in group **(7)** is $\frac{1}{4} > 0$, this identifies $(4, 3)$ as the pivoting entry and 3 as the pivot. Applying the rectangle rule yields

(22)

| | | | | |
|---|---|---|---|---|
| $*$ | 2 | 4 | $*$ | |
| 1 | 1 | $\frac{2}{3}$ | $\frac{3}{4}$ | ? |
| 3 | 0 | $\frac{1}{3}$ | $\frac{1}{4}$ | ? |
| $*$ | 10 | $\frac{22}{3}$ | $\frac{32}{4}$ | $*$ |

Now all $\zeta_k$ are positive, thus we have all the information for a maximum of $\boldsymbol{cx}$ over $D$. The maximizing element is $\hat{\boldsymbol{x}} = (3/4, 0, 1/4, 0)(= \boldsymbol{x}^{13})$ and the value of the maximum is $\boldsymbol{c}\hat{\boldsymbol{x}} = \frac{32}{4}$. (Besides, the rows should indicate the representation of $\boldsymbol{A}_{\bullet 2}$ and $\boldsymbol{A}_{\bullet 4}$ by $\boldsymbol{A}_{\bullet 1}$ and $\boldsymbol{A}_{\bullet 3}$).

# 3 The Two Phase Method

Within the previous sections we have discussed convex polyhedra which were defined by a set of inequalities. By contrast there is also the mixed version which involves equations and inequalities. We now return to the version discussing inequalities. An additional advantage will be that we are in the position to develop a procedure for the initial tableau.

As previously we consider data given by a triple $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ such that $\boldsymbol{A}$ is an $m \times n$-Matrix while $\boldsymbol{b} \in \mathbb{R}^m$ and $\boldsymbol{c} \in \mathbb{R}^n$ are vectors representing the constraints and the objective function.

Within this section we will always assume that

$$\boldsymbol{b} > 0 \quad \textit{or} \quad \boldsymbol{b} \geq 0$$

is satisfied according to whether we want to insist on nondegeneracy or not. Recall our notation for a certain type of polyhedron, we write

$$C := C^0_{\boldsymbol{A},\boldsymbol{b}} := \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geq 0 \ , \ \boldsymbol{A}\boldsymbol{x} \leq b\}.$$

We also consider a linear function

$$f : \mathbb{R}^n \to \mathbb{R} \ , \ f(\boldsymbol{x}) = \boldsymbol{c}\boldsymbol{x} \quad (\boldsymbol{x} \in \mathbb{R}^n).$$

which is specified once a vector $\boldsymbol{c} \in \mathbb{R}^n$ is given. Our aim is to compute the quantity

$$\max_C f = \max\{\boldsymbol{c}\boldsymbol{x} \mid \boldsymbol{x} \in C\}$$

as well as a maximizer of this function, that is, an element of the set

$$M_C f := \operatorname{argmax}\{\boldsymbol{c}\boldsymbol{x} \mid \boldsymbol{x} \in C\} := \{\boldsymbol{x} \in C \mid \boldsymbol{c}\boldsymbol{x} = \max_C f\}.$$

In order to establish a connection between the "C-problem" as indicated above and the "D-Problem" that we have discussed in Sections 1 and 2, we introduce the following notation. Given some $m \times n$–matrix $\boldsymbol{A}$ we introduce the $m \times (n+m)$–matrix $\widetilde{\boldsymbol{A}}$ via

$$(1) \qquad \widetilde{\boldsymbol{A}} := (\boldsymbol{A}, \boldsymbol{I}_m) = \left( \begin{array}{c|ccc} \overbrace{\qquad \boldsymbol{A} \qquad}^{n} & \multicolumn{3}{c}{\overbrace{\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}}^{m}} \end{array} \right).$$

To this matrix we associate a convex polyhedron (of "D version") by

$$\widetilde{D} = D^0_{\widetilde{\boldsymbol{A}},\boldsymbol{b}} := \{\boldsymbol{z} \in \mathbb{R}^{n+m} \mid \boldsymbol{z} \geq 0, \widetilde{\boldsymbol{A}}\boldsymbol{z} = \boldsymbol{b}\} \tag{2}$$

The latter type can be treated with the simplex algorithm as explained in the last two sections. As it turns out, the two polyhedra can bijectively be mapped on each other.

**Lemma 3.1.** *Consider the two linear mappings*

$$\begin{aligned} &p \ : \ \mathbb{R}^n \to \mathbb{R}^{n+m} \ , \\ &p(\boldsymbol{x}) = (\boldsymbol{x}, \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) \quad (\boldsymbol{x} \in \mathbb{R}^n) \end{aligned} \tag{3}$$

*and*

$$\begin{aligned} &q \ : \ \mathbb{R}^{n+m} \to \mathbb{R}^n \ , \\ &q(\boldsymbol{z}) = q(z_1, \cdots, z_n, z_{n+1}, \cdots, z_{n+m}) = (z_1, \cdots, z_n) \quad (\boldsymbol{z} \in \mathbb{R}^{n+m}). \end{aligned} \tag{4}$$

*Then*

$$p : C \to \widetilde{D} \ , \ q : \widetilde{D} \to C \tag{5}$$

*holds true. Moreover, these mappings are bijective and inverse to each other.*

**Proof:** Clearly, we have

$$p(\boldsymbol{x}) \geq 0 \ \ (\boldsymbol{x} \in C)$$

and

$$\widetilde{\boldsymbol{A}}p(\boldsymbol{x}) \ = \ (\boldsymbol{A}, \boldsymbol{I}_m)(\boldsymbol{x}, \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) \ = \ \boldsymbol{A}\boldsymbol{x} + (\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) \ = \ \boldsymbol{b},$$

hence

$$p(\boldsymbol{x}) \in \widetilde{D} \quad (\boldsymbol{x} \in C).$$

On the other hand, suppose we have $\boldsymbol{z} \in \widetilde{D}$ (that is in particular $\boldsymbol{z} \geq 0$), then we come up with

$$\boldsymbol{b} \ = \ \widetilde{\boldsymbol{A}}\boldsymbol{z} \ = \ \boldsymbol{A}\underbrace{(z_1, \cdots, z_n)}_{q(\boldsymbol{z})} + \underbrace{(z_{n+1}, \cdots, z_{n+m})}_{\geq 0} \geq \boldsymbol{A}q(\boldsymbol{z}) \tag{6}$$

that is $q(\boldsymbol{z}) \in C$ . Thus we have shown that (5) is indeed true. In particular we infer from (6) that

$$\boldsymbol{b} - \boldsymbol{A}(z_1, \ldots, z_n) = (z_{n+1}, \ldots, z_{n+m}) \tag{7}$$

is valid for $\boldsymbol{z} \in \widetilde{D}$.

It is obvious that $p$ is an injective mapping satisfying $q \circ p = id_C$. Therefore, it remains to show that $p \circ q = id_{\widetilde{D}}$ holds true.

This follows immediately in view of

$$\begin{aligned} p \circ q(\boldsymbol{z}) &= p(z_1, \cdots, z_n) \\ &= (z_1, \cdots, z_n, \boldsymbol{b} - \boldsymbol{A}(z_1, \cdots, z_n)) \\ &= (z_1, \cdots, z_n, z_{n+1}, \cdots, z_{n+m}) \quad (\boldsymbol{z} \in \widetilde{D}) \ , \end{aligned}$$

where the last equation follows from (7),

**q.e.d.**

The following remark refers to a widespread terminology concerning *slack variables.*

**Remark 3.2.** *In a less formal context, the variables*

$$y_i := z_{n+i} = b_i - \boldsymbol{A}_{i\bullet}\boldsymbol{x}$$

*are sometimes denoted as* ***slack variables*** *. The intuitive meaning is that an inequality 'allows for slack' and this slack is 'eliminated' by introducing an additional variable which changes the inequality to an equation. As frequently, the term variable is not well defined, the precise notion is to speak about bijective mappings. Observe that regarding the polyhedron*

$$C = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geq 0 \ , \ \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$$

*there is one slack variable introduced per equation, that is, we obtain the polyhedron*

$$\widetilde{D} = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{n+m} \mid (\boldsymbol{x}, \boldsymbol{y}) \geq 0, \ \boldsymbol{A}_{i\bullet}\boldsymbol{x} + y_i = b_i \quad (i = 1, \cdots, m)\}.$$

**Example 3.3.** Figure 3.1 shows a situation for $n = 2$ and $m = 1$. Observe that the triangular polyhedron $C^0_{\boldsymbol{Ab}} \subseteq \mathbb{R}^2$ is "lifted" into $\mathbb{R}^{n+m} = \mathbb{R}^3$ by the mapping $p$ and, in turn the projection $q$ acts as the inverse mapping. The vertices $\mathbf{0} \in C^0_{\boldsymbol{Ab}}$ and $(0, 0, b_0) \in D^0_{\widetilde{\boldsymbol{A}}\boldsymbol{b}}$ correspond to each other.

**Example 3.4.** More specifically, let $n = 2$ and $m = 1$ and consider the polyhedron

$$C \ = \ \{\boldsymbol{x} \in \mathbb{R}^2 \mid \boldsymbol{x} \geq 0 \ , \ x_1 + 2x_2 \leq 1\}$$

(see Figure 3.2), then we obtain (see Figure 3.3)

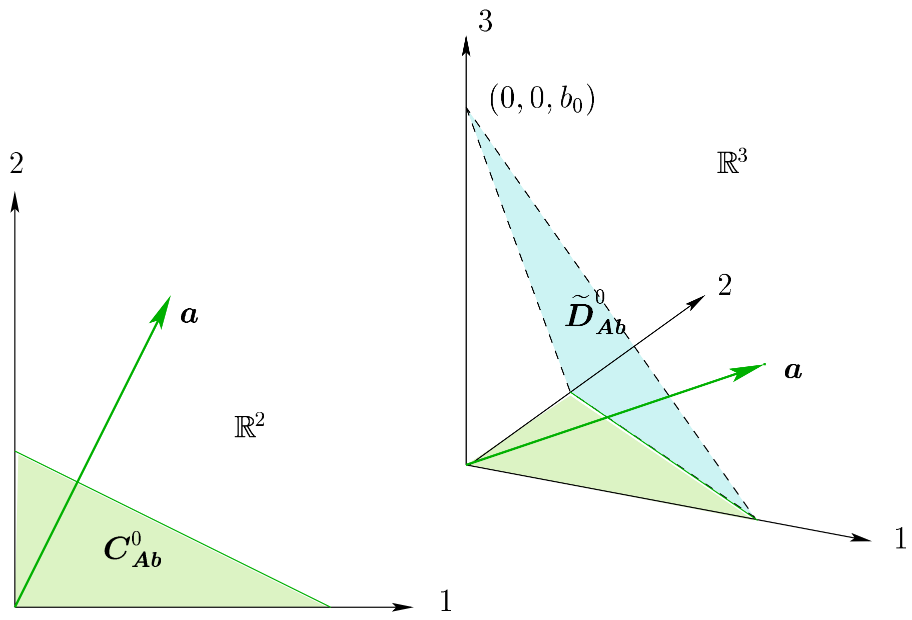



Figure 3.1: Identification of $C$ and $D$

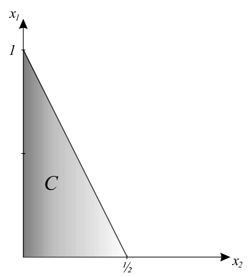



Figure 3.2: $C \;=\; \{\boldsymbol{x} \in \mathbb{R}^2 \mid \boldsymbol{x} \geq 0\ ,\ x_1 + 2x_2 \leq 1\}$

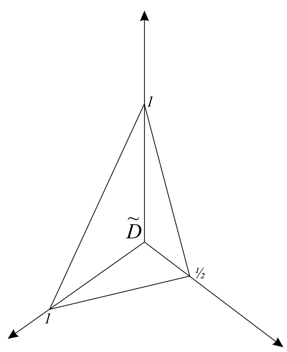



Figure 3.3: $\widetilde{D} = \{(\boldsymbol{x}, y_1) \in \mathbb{R}^3 \mid (\boldsymbol{x}, y_1) \geq 0, x_1 + 2x_2 + y_1 = 1\}$

$$\widetilde{D} = \{(\boldsymbol{x}, y_1) \in \mathbb{R}^3 \mid (\boldsymbol{x}, y_1) \geq 0, x_1 + 2x_2 + y_1 = 1\}.$$

The action of the mappings $p$ and $q$ can be nicely visualized.

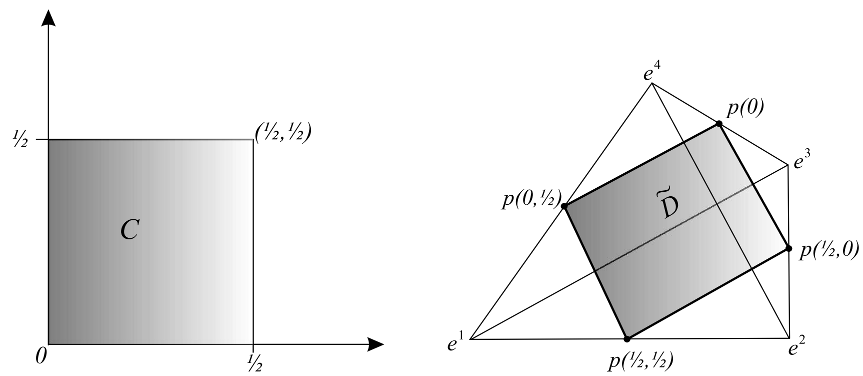
**Example 3.5.** An example with $m = n = 2$ can also be geometricaly represented (Figure 3.4). Put

$$C = \{\boldsymbol{x} \in \mathbb{R}^2 \mid \boldsymbol{x} \geq 0, x_1 \leq 1/2, x_2 \leq 1/2\}.$$

Then we have

$$\widetilde{D} = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^4 \mid (\boldsymbol{x}, \boldsymbol{y}) \geq 0, \quad x_1 + y_1 = 1/2, \quad x_2 + y_2 = 1/2\}$$

$$= \{\boldsymbol{z} \in \mathbb{R}^4 \mid \boldsymbol{z} \geq 0, \quad z_1 + z_3 = 1/2, \quad z_2 + z_4 = 1/2\}$$




(a) $C$ (b) $\widetilde{D}$

Figure 3.4: A square in $\mathbb{R}^2$ is mapped onto a square in $\mathbb{R}^4$

This example shows that a suitable square in $\mathbb{R}^2$ is mapped into a variant of our notorious trapezoid. Because the geometric representation misses one dimension of $\mathbb{R}^4$, the projection property of the mapping $q$ cannot be represented.

**Remark 3.6.** *1. As $p$ is affine and $q$ is linear, both being bijective mappings, the vertices of $C$ and $\widetilde{D}$ will be mapped onto each other respectively. That is, $p$ and $q$ establish bijective mapping between the sets of extremal points as well.*

*Indeed, for $\boldsymbol{x}^0, \boldsymbol{x}^1 \in C$ , $\boldsymbol{z}^0, \boldsymbol{z}^1 \in \widetilde{D}$ and $\lambda \in [0, 1]$. we have*

$$p(\lambda \boldsymbol{x}^1 + (1 - \lambda)\boldsymbol{x}^0) = \lambda p(\boldsymbol{x}^1) + (1 - \lambda)p(\boldsymbol{x}^0)$$

$$q(\lambda \boldsymbol{z}^1 + (1 - \lambda)\boldsymbol{z}^0) = \lambda q(\boldsymbol{x}^1) + (1 - \lambda)q(\boldsymbol{z}^0).$$

*It follows immediately that any $x \in C$ admits of a convex combination if and only if $q(\boldsymbol{x})$ does. The same holds true for $p$.*

2. *In particular, the vertex $\mathbf{0} = (0, \cdots, 0) \in C$ and the vertex $q(\mathbf{0}) = (\mathbf{0}, \boldsymbol{b}) \in \widetilde{D}$ are mapped onto each other.*

3. *Now consider the function $f : \mathbb{R}^n \to \mathbb{R}$ , $f(\boldsymbol{x}) = \boldsymbol{cx}$ $(\boldsymbol{x} \in \mathbb{R}^n)$ which is in particular defined on the convex polyhedron $C$. We put*

$$\widetilde{\boldsymbol{c}} \in \mathbb{R}^{n+m}, \widetilde{\boldsymbol{c}} = (c_1, \cdots, c_n, 0, \cdots, 0)$$

*as well as*

$$\widetilde{f} : \mathbb{R}^{n+m} \to \mathbb{R} \ , \ \widetilde{f}(\boldsymbol{z}) = \widetilde{c}\boldsymbol{z} = \boldsymbol{c}(z_1, \cdots, z_n) = f(q(\boldsymbol{z}))$$

*Clearly, we have*

$$\widetilde{f} = f \circ q \ , \ f = \widetilde{f} \circ p \ .$$

*From this it follows immeditately that $q : M_C f \to M_{\widetilde{D}} \widetilde{f}$ and $p : M_{\widetilde{D}} \widetilde{f} \to M_C f$ are bijective mappings and invers to each other as well. That is, $\bar{\boldsymbol{x}} \in C$ is a maximizer of $f$ if and only if $(\boldsymbol{x}, \boldsymbol{b} - \boldsymbol{Ax})$ is a maximizer of $\widetilde{f}$ in $\widetilde{D}$.*

4. *Therefore, whenever we wish to find elements $\bar{\boldsymbol{x}} \in M_C f$, we can as well try to find elements $\widetilde{\boldsymbol{z}} \in M_D \widetilde{f}$. The advantage is that the vector $(0, \ldots, 0, \boldsymbol{b})$ yields at once an initial vertex of $\widetilde{D}$. Hence we can employ the simplex algorithm explained in* SECTIONS *1 and 2 using this extremal point for a start.*

   *The corresponding basis is provided by the vectors $(\widetilde{A}_{\bullet n+1} \cdots, \widetilde{A}_{\bullet n+m}) = I_n$.*

   *Note that even if the requirement $b > 0$ is substituted by $b \geq 0$ nevertheless the vectors $(\widetilde{A}_{\bullet n+1} \cdots, \widetilde{A}_{\bullet n+m}) = I_n$ constitute a basis.*

The following theorem describes the initial tableau with respect to the vertex $(0, \boldsymbol{b}) \in \widetilde{D}$ in the sense of Definition 2.2 and Remark 2.3. True, within the framework of SECTION 2 everything was formulated assuming non degeneracy. Nevertheless, in order to construct the initial tableau all we need is a basis consisting of the columns corresponding to $(0, \boldsymbol{b})$. These are the columns taken from $\widetilde{\boldsymbol{A}} = (\boldsymbol{A}, \boldsymbol{I}_m)$ that is, the column constructed by means of the unit vectors.

**Theorem 3.7.** *Let $\widetilde{\boldsymbol{A}} = (\boldsymbol{A}, \boldsymbol{I}_m)$. The tableau $\mathcal{T}(0, \boldsymbol{b})$ corresponding to $\widetilde{\boldsymbol{A}}, \boldsymbol{b}$ and $\bar{\boldsymbol{z}} = (0, \boldsymbol{b})$ (see Remark 2.3) is given as follows:*

$$
(8) \qquad \mathcal{T}(0,\boldsymbol{b}) = \begin{array}{|c|ccc|c|c|}
\hline
* & \textbf{(1)} & & & \multicolumn{2}{c|}{*} \\
 & 1 & \cdots\cdots & n & \multicolumn{2}{c|}{} \\
\hline
\textbf{(2)} & \textbf{(3)} & \vdots & & \textbf{(4)} & \textbf{(7)} \\
n+1 & & & & b_1 & \\
\vdots & \cdots & \boldsymbol{A} & \cdots & \vdots & \vdots \\
n+m & & \vdots & & b_m & \\
\hline
 & \textbf{(5)} & & & \textbf{(6)} & \\
* & -c_1 & \cdots & -c_n & 0 & * \\
\hline
\end{array}
$$

**Proof: 1$^{\text{st}}$STEP** and **2$^{\text{nd}}$STEP** : The first and second areas are defined in an obvious manner; clearly we have $\bar{J} = \{n+1, \cdots, n+m\} \subseteq J = \{1, \cdots, n, n+1, \cdots, n+m\}$.

**3$^{\text{rd}}$STEP** : The columns $\widetilde{\boldsymbol{A}}_{\bullet j}$ $(j = n+1, \cdots, n+m)$ are the unit vectors by definition, hence we obtain for $k = 1 \ldots n$:

$$
\begin{aligned}
\boldsymbol{A}_{\bullet k} = \widetilde{\boldsymbol{A}}_{\bullet k} &= \sum_{j=n+1}^{n+m} \lambda_j^k \widetilde{\boldsymbol{A}}_{\bullet j} \\
&= \sum_{j=1}^{m} \lambda_{j+n}^k e^j = \begin{pmatrix} \lambda_{n+1}^k \\ \vdots \\ \lambda_{n+m}^k \end{pmatrix} .
\end{aligned}
$$

This *defines* the tableau elements $\lambda_\bullet^k$ as to be the columns $\boldsymbol{A}_{\bullet k}$.

**4$^{\text{th}}$STEP** : The shape of the fourth region is obviously the one indicated in view of $\bar{\boldsymbol{z}} = (0, \cdots, 0, b_1, \cdots, b_m)$.

**5$^{\text{th}}$STEP** : For $k = 1, \cdots, n$ (that is $k \in J - \bar{J}$) we obtain

$$
z_k = \sum_{j=n+1}^{n+m} \widetilde{c}_j \lambda_j^k = 0 \ ,
$$

and consequently we have

$$
\zeta_k = z_k - c_k = -c_k \ .
$$

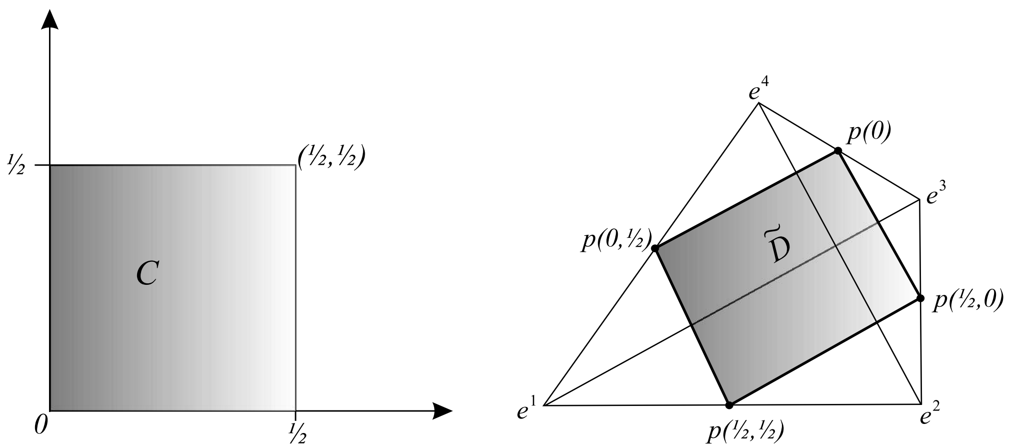
**6$^{\text{th}}$STEP** : Finally, the equation $\widetilde{c}\bar{\boldsymbol{z}} = 0$ determines the value of the objective function in region **(6)**. **q.e.d.**

**Example 3.8.** In Example 3.4 we have

$$C = \{\boldsymbol{x} \in \mathbb{R}^2 \mid \boldsymbol{x} \geq 0 \ , \ x_1 \leq 1/2, x_2 \leq 1/2\}$$

and as a consequence

$$\widetilde{D} = \{\boldsymbol{z} \in \mathbb{R}^4 \mid \sum_{i=1}^{4} z_i = 1 \ , \ z_1 + z_3 = 1/2 \ , \ z_2 + z_4 = 1/2\}$$




(a) $C$ (b) $\widetilde{D}$

Figure 3.5: example: $n = 2, \ m = 1 \qquad a \in \mathbb{R}^2_+, \ b_0 > 0$

The matrix corresponding to $C$ is given by $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the vector $\boldsymbol{b}$ is provided by $\boldsymbol{b} = (\frac{1}{2}, \frac{1}{2})$. We put

$$\boldsymbol{c} = (2, \ \ -1) \ .$$

The initial tableau $\mathcal{T}(0, b)$ is given as follows:

(9)

| | | | | |
|---|---|---|---|---|
| $*$ | 1 | 2 | $*$ | |
| 3 | **1** | 0 | $\frac{1}{2}$ | $\boldsymbol{\frac{1}{2}}$ |
| 4 | 0 | 1 | $\frac{1}{2}$ | $-$ |
| $*$ | $-2$ | 1 | 0 | $*$ |

The tableau is pivotable around the index pair $(3,1)$ with tableau element 3 in the upper left corner of region **(3)**, the result is

$$
\text{(10)} \qquad
\begin{array}{|c|cc|c|c|}
\hline
* & 3 & 2 & \multicolumn{2}{c|}{*} \\
\hline
1 & 1 & 0 & \frac{1}{2} & - \\
 & & & & \\
4 & 0 & 1 & \frac{1}{2} & - \\
\hline
* & 2 & 1 & 1 & * \\
\hline
\end{array}\,.
$$

Since $\zeta_3, \zeta_2 > 0$ is true, we have as a consequence

$$\bar{\boldsymbol{z}} = (1/2, 0, 0, 1/2)$$

as a maximizer in $\widetilde{D}$. The corresponding maximizer in $C$ is obtained by projection (that is, by an application of the mapping $q$). This yields

$$\bar{\boldsymbol{x}} = (1/2, 0).$$

The value of the objective function in this point is given by $\widetilde{c}\bar{\boldsymbol{z}} = c\bar{\boldsymbol{x}} = 1$.

Now we return to the version of a polyhedron in the form $D^0_{\boldsymbol{A},\boldsymbol{b}}$. This version we have used in SECTIONS 2 and 3 in order to develop the exchange of vertices and the corresponding action of the rectangle rule on tableaus.

Consider the familiar version given by

$$D = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geq 0, \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\} = D^0_{\boldsymbol{A},\boldsymbol{b}}$$

Again, we assume $\boldsymbol{b} \geq 0$ in advance and we are looking for the quantity $\max_D f$ and elements of the set $M_C f$, where $f$ is given by

$$f : \mathbb{R}^n \to \mathbb{R}\ ,\ f(\boldsymbol{x}) = \boldsymbol{c}\boldsymbol{x} \quad (\boldsymbol{x}, \boldsymbol{c} \in \mathbb{R}^n)$$

Our procedure will be slightly different from the previous ones. We do not consider bijective mappings. Instead it is our aim to point out an initial vertex. Therefore, let us define the following quantities:

$$
\begin{aligned}
\widehat{\boldsymbol{A}} &:= (\boldsymbol{A}, I_m) = \begin{pmatrix} & 1 & & 0 \\ \boldsymbol{A} & & \ddots & \\ & 0 & & 1 \end{pmatrix} \\
\widehat{\boldsymbol{c}} &:= (0, \cdots, 0, -1, \cdots, -1) \in \mathbb{R}^{n+m} \\
\widehat{D} &= \{\boldsymbol{z} \in \mathbb{R}^{n+m} \mid \boldsymbol{z} \geq 0, \widehat{\boldsymbol{A}}\boldsymbol{z} = \boldsymbol{b}\} = D^0_{\widehat{\boldsymbol{A}},\boldsymbol{b}}, \\
\widehat{f} &: \mathbb{R}^{n+m} \to \mathbb{R}, \widehat{f}(\boldsymbol{z}) = \widehat{\boldsymbol{c}}\boldsymbol{z} \quad (\boldsymbol{z} \in \mathbb{R}^{n+m})
\end{aligned} \tag{11}
$$

On a first glance nothing has been changed with respect to the structure of the mappings induced by the above quantities. However, our method in order to aquire an initial vertex is slightly different. This is explained by means of the following remark.

**Remark 3.9.** *1.* $\bar{\boldsymbol{z}} = (0, \cdots, 0, b) \in \mathbb{R}^{n+m}$ *is a vertex of* $\widehat{D}$ *and hence can be used as the initial vertex for the* $\widehat{D}$*-problem.*

2. *For all* $\boldsymbol{z} \in \widehat{D}$ *we have* $\widehat{\boldsymbol{c}}\boldsymbol{z} \leq 0$*. If* $D \neq \emptyset$ *holds true, then the following statements are equivalent:*

$$\widehat{z}_{n+1} = \cdots = \widehat{z}_{n+m} = 0 \tag{12}$$

$$\widehat{\boldsymbol{z}} \in M_{\widehat{D}}\widehat{f} \tag{13}$$

$$\widehat{c}\widehat{\boldsymbol{z}} = 0 \tag{14}$$

$$(\widehat{z}_1, \cdots, \widehat{z}_n) \in D. \tag{15}$$

*Observe that the simplex procedure with* $\bar{z}$ *as an initial vertex ends up at some point* $\widehat{z}$ *of the above type. This* $\widehat{z}$ *yields an element of* $D$ *by just taking the first coordinates.*

3. *Let the pair* $(\boldsymbol{A}, \boldsymbol{b})$ *be non degenerate and let* $\widehat{\boldsymbol{z}}$ *be a vector as described in item (2) which results from the simplex procedure. That is, assume that we have found a vertex of* $\widehat{D}$ *with vanishing last coordinates. Then* $\widehat{\boldsymbol{x}} = (\widehat{z}_1, \cdots, \widehat{z}_n)$ *is a vertex of* $D$*. Every convex combination of elements of the type of* $\widehat{\boldsymbol{x}}$ *implies a similar convex combination of elements of the type of* $\widehat{\boldsymbol{z}}$*.*

Now we have the following lemma.

**Lemma 3.10.** *Let* $\bar{\boldsymbol{z}} = (0, \cdots, 0, b) \in \mathbb{R}^{n+m}$ *be a vertex of* $\widehat{D}$ *in the sense of item (1) of Remark 3.9 and let* $\widehat{f}$ *be the function defined accordingly. The simplex tableau* $\mathcal{T}(\bar{\boldsymbol{z}})$ *corresponding to* $\bar{\boldsymbol{z}}$ *(see Remark 2.3) is given by:*

(16) $\mathcal{T}(\bar{\boldsymbol{z}}) =$

| | | | |
|---|---|---|---|
| $*$ | **(1)** $1 \quad \cdots\cdots \quad n$ | $*$ | |
| **(2)** $n+1$ $\vdots$ $n+m$ | **(3)** $\vdots$ $\cdots \quad \boldsymbol{A} \quad \cdots$ $\vdots$ | **(4)** $b_1$ $\vdots$ $b_m$ | **(7)** $\vdots$ |
| $*$ | **(5)** $-\sum_{i\in I} a_{i1} \quad \cdots \quad -\sum_{i\in I} a_{in}$ | **(6)** $-\sum_{i\in I} b_i$ | $*$ |

**Proof:** The areas **(3)** and **(4)** are obtained exactly as in the proof of Theorem 3.7. In addition we have for $k \in \{1, \cdots, n\} = J$ :

$$\begin{aligned} \widehat{\zeta}_k = z_k - \widehat{c}_k = z_k \quad &= \sum_{j=n+1}^{n+m} \lambda_j^k \widehat{c}_j \\ = \sum_{i=1}^{m} a_{ik}(-1) \quad &= - \sum_{i=1}^{m} a_{jk}. \end{aligned}$$

Similarly we have

$$\widehat{\boldsymbol{c}}\bar{\boldsymbol{z}} = (0, \cdots, 0, -1, \cdots, -1)(0, \boldsymbol{b}) = -\sum_{i\in I} b_i.$$

**q.e.d.**

**Remark 3.11.** *Consider the tableau as indicated in Lemma 3.10.*

*Following each step in the first phase, such that an index* $n+l$ *from the left group (i.e., from* $\bar{J}$*) has been transformed to the above group (i.e., to* $J - \bar{J}$*), we can cancel the corresponding columns. Hence, all indices exceeding* $n$ *can successively be omitted.*

(17)

| $*$ | $1 \quad\quad n+l \quad\quad n$ | $*$ | |
|---|---|---|---|
| $n+1$ | $*$ | $-$ | $-$ |
| $\vdots$ | $* \quad \vdots$ | | |
| $j$ | $*$ | | |
| $\vdots$ | $* \quad \vdots$ | | |
| $n+m$ | $*$ | $-$ | $-$ |
| $*$ | | | $*$ |
| | $\uparrow$ *cancel* | | |

.

*This is so because corresponding to the index $n+l$ there is a basis vector $\widehat{\boldsymbol{A}}_{\bullet n+l} = e^l$. The representation of this basis vector with the aid of tableau elements will never be employed because the corresponding index will never be transformed to the group below. Observe that the movement in direction of a coordinate $n+l$ would never increase the value $\widehat{c}z$, on the contrary, it would strictly decrease as $\widehat{c}_{n+l} = -1$.*

*At the end of this procedure we will obtain a tableau such that the indices exceeding $n$ will not occur. As a consequence this tableau yields immediately a representation of the vectors $\widehat{\boldsymbol{A}}_{\bullet k} = \boldsymbol{A}_{\bullet k} \quad (k \in J - \bar{J})$ by means of $\widehat{\boldsymbol{A}}_{\bullet j} = \boldsymbol{A}_{\bullet j} \quad (j \in \bar{J})$. In addition the coordinates $\bar{z}_j \quad (j \in \bar{J})$ are exactly the coordinates of the new initial vertex for the second phase.*

| | | | |
|---|---|---|---|
| $*$ | $J - \bar{J}$ | $*$ | |
| $\bar{J}$ | $\boldsymbol{\Lambda}$ | $\bar{\boldsymbol{z}}$ | $-$ |
| $*$ | $\bar{\boldsymbol{\zeta}}$ | $0$ | $*$ |

(18)

| | | | |
|---|---|---|---|
| $*$ | $J - \bar{J}$ | $*$ | |
| $\bar{J}$ | $\boldsymbol{\Lambda}$ | $\bar{\boldsymbol{x}}$ | $-$ |
| $*$ | | | $*$ |
| | $\uparrow$ *replace* | $\uparrow$ | |

*Note, however, that the vector* $\widehat{\boldsymbol{c}}$ *has been constructed for the purposes of the new phase. Therefore, the lowest row of the tableau contains the wrong data, the data* $\boldsymbol{\zeta}$ *and* $\boldsymbol{c}\bar{\boldsymbol{x}}$ *have to be computed from scratch.*

**Remark 3.12.** *Throughout this section we have always used the condition* $\boldsymbol{b} \geq 0$*. There is also a method of geting rid of this condition, however we will not pursue this idea.*

# Chapter 3

# Duality

The previous section describes a computational method, the simplex procedure, which serves to compute optimal elements of a linear programming problem if they exist at all. The question of existence of optimal elements will now be considered from a more theoretical point ov view. We start out by attaching a twin to each linear program: the dual program. Apart from the structural consideration (understanding the relation of the twins helps understanding the nature of optimal solutions), this procedure will also be helpful in order to consider the existence problem.

# 1 Dual pairs of LP.'s

There are two aims within this section we are heading at. On one hand the increasingly intricate connections between systems of inequalities and equations (represented by different versions of convex polyhedra like $C_{a,b}$ and $D^0_{a,b}$) should be cleared. On the other hand, the simplex algorithm as discussed in the previous sections is a practical or applied method. What is presently missing are general existence theorems concerning the solutions of linear programs. All of these problems should be dealt with in a unified framework. The joined topic combining these various areas is the idea of duality. Within this context it is possible to treat all the questions mentioned above sinultaneously.

We start out with a formal definition of duality.

**Definition 1.1.** *1. For $\rho, \sigma = 1, 2$ let $m_\rho, n_\rho \in \mathbb{N}$ and let*

$$\begin{array}{ll} \boldsymbol{A}^{\rho\sigma} & \textit{be an } m_\rho \times n_\sigma\textit{-Matrix,} \\ \boldsymbol{b}^\rho & \in \mathbb{R}^{m_\rho}, \textit{ and} \\ \boldsymbol{c}^\rho & \in \mathbb{R}^{n_\rho}, \end{array}$$

*then the $(m_1 + m_2 + 1) \times (n_1 + n_2 + 1)$-matrix*

$$\Gamma = \begin{array}{r} m_1 \{ \\ m_2 \{ \\ 1 \{ \end{array} \left\{ \begin{array}{ccc} \boldsymbol{A}^{11} & \boldsymbol{A}^{12} & \boldsymbol{b}^1 \\ \boldsymbol{A}^{12} & \boldsymbol{A}^{22} & \boldsymbol{b}^2 \\ \boldsymbol{c}^1 & \boldsymbol{c}^2 & \boldsymbol{0} \end{array} \right\} \tag{1}$$
$$\qquad\qquad \underbrace{\quad}_{n_1} \underbrace{\quad}_{n_2} \underbrace{\quad}_{1}$$

*is called a (mixed)* ***linear program (LP.).***

*2. The elements of the set*

$$E = \{ (\boldsymbol{x}^1, \boldsymbol{x}^2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \mid \\ \mid \boldsymbol{A}^{11}\boldsymbol{x}^1 + \boldsymbol{A}^{12}\boldsymbol{x}^2 \leq \boldsymbol{b}^1, \quad \boldsymbol{A}^{21}\boldsymbol{x}^1 + \boldsymbol{A}^{22}\boldsymbol{x}^2 = \boldsymbol{b}^2, \quad \boldsymbol{x}^2 \geq 0 \} \tag{2}$$

*are called the* ***feasible solutions*** *of the linear program $\Gamma$.*

*3.* $(\bar{\boldsymbol{x}}^1, \bar{\boldsymbol{x}}^2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ *are called* ***optimal solutions*** *of the linear program* $\Gamma$ *if*

$$(\bar{\boldsymbol{x}}^1, \bar{\boldsymbol{x}}^2) \in M_E f \tag{3}$$

*holds true. This definition involves the function*

$$\begin{aligned} f &: \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R} \\ f(\boldsymbol{x}^1, \boldsymbol{x}^2) &= \boldsymbol{c}^1\boldsymbol{x}^1 + \boldsymbol{c}^2\boldsymbol{x}^2 \quad ((\boldsymbol{x}^1, \boldsymbol{x}^2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \end{aligned} \tag{4}$$

**Remark 1.2.** *The above definition encourages the reader to look for a solution of the following problem:*

*'find* $\max_E\ f$ *and* $(\bar{\boldsymbol{x}}^1, \bar{\boldsymbol{x}}^2) \in M_E f$*'.*

*As a consequence many text books choose to consider this problem or task as 'the linear program'. Within such a framework, an apropriate description is supposed to be given by the system of inequalities and equations following below.*

$$\textbf{(0)} \qquad \begin{array}{rcl} \boldsymbol{A}^{11}\boldsymbol{x}^1 + \boldsymbol{A}^{12}\boldsymbol{x}^2 & \leq & \boldsymbol{b}^1 \\ \boldsymbol{A}^{21}\boldsymbol{x}^1 + \boldsymbol{A}^{22}\boldsymbol{x}^2 & = & \boldsymbol{b}^2 \\ \boldsymbol{x}^2 & \geq & 0 \\ & & \\ e^1\boldsymbol{x}^1 + \boldsymbol{c}^2\boldsymbol{x}^2 & \to & \max \end{array} \tag{5}$$

*This kind of description, while mathematically not quite correct, has its advantages. For instance, the notion is much more suggestive and the reader can more easily come to grasp with the above formulation compared to the mathematical correct one.*

*We will therefore accept* **(0)** *as a shorthand notation which suggests that the data, the feasible, and the optimal solutions of a linear program are specified by a correct definition as provided in Definition 1.1*

Nevertheless, Definition 1.1 has a great advantage as well: it allows for the precise handling of the idea of *duality.*

**Definition 1.3.** *1. Let* $\Gamma$ *be a linear program. Define* $\Gamma^*$ *by the mapping which exchanges* $\rho \to 3 - \rho \quad (\rho = 1, 2)$*, more precisely* $\Gamma^*$ *is the linear program defined by*

$$\Gamma^* := \begin{pmatrix} \boldsymbol{A}^{22} & \boldsymbol{A}^{21} & \boldsymbol{b}^2 \\ \boldsymbol{A}^{12} & \boldsymbol{A}^{11} & \boldsymbol{b}^1 \\ \boldsymbol{c}^2 & \boldsymbol{c}^1 & \boldsymbol{0} \end{pmatrix} \tag{6}$$

2. *Also, we define*

$$\widehat{\Gamma} := -\Gamma^{*\top} \tag{7}$$

*here the upper index* $^\top$ *reflects the transposed version of a matrix.*

3. *A pair* $(\Gamma, \Delta)$ *of linear programs is called* ***dual*** *if*

$$\Delta = \widehat{\Gamma} \tag{8}$$

*holds true.*

**Remark 1.4.** *It is at once verified that the following relations hold true:*

$$\Gamma^{*\top} = \Gamma^{\top *}\ ,\ \Gamma^{**} = \Gamma\ ,\ \Gamma^{\top\top} = \Gamma. \tag{9}$$

*Therefore, if* $(\Gamma, \Delta)$ *is a dual pair, then we have*

$$\widehat{\Delta} = -\Delta^{*\top} = (\Gamma^{*\top})^{*\top} = \Gamma^{*\top\top *} = \Gamma\ . \tag{10}$$

*This means that* $(\Delta, \Gamma)$ *is a dual pair as well; the message is that the ordering of the elements of the dual pair is not important. That is, a dual pair can be written as*

$$(\Gamma, \widehat{\Gamma}) \quad \textit{or} \quad (\widehat{\Delta}, \Delta)$$

*in any case.*

*Nevertheless, we frequently find formulations of the type that* $\widehat{\Gamma}$ *is the program 'dual to* $\Gamma$*' or* $\Gamma$ *is the program 'primal to* $\widehat{\Gamma}$*'. These notations are useful in order to distinguish the two candidates for consideration. However, the attachment of the qualifiers primal and dual is quite arbitrary: any element of a pair* $(\Gamma, \Delta)$ *can play the role of the primal or the dual program respectively.*

Again let us turn to the shorthand notation, this has consequences with respect to duality as well. The 'problem' implied by a linear program $\Gamma$ is represented by **(0)**. Which problem, using this notation, corresponds to $\widehat{\Gamma}$? Obviously we have

$$\widehat{\Gamma} = -\Gamma^{*\top} = - \begin{pmatrix} \boldsymbol{A}^{22\top} & \boldsymbol{A}^{12\top} & \boldsymbol{c}^2 \\ \boldsymbol{A}^{21\top} & \boldsymbol{A}^{11\top} & \boldsymbol{c}^1 \\ \boldsymbol{b}^2 & \boldsymbol{b}^1 & 0 \end{pmatrix} \tag{11}$$

and the corresponding problem as in the sense of Remark 1.2 is presented by the following system of equations and inequalities:

$$
(12) \qquad \begin{array}{rcl}
-\boldsymbol{A}^{22\top} z^1 - \boldsymbol{A}^{12\top} z^2 & \leq & -\boldsymbol{c}^2 \\
-\boldsymbol{A}^{21\top} z^1 - \boldsymbol{A}^{11\top} z^2 & = & -\boldsymbol{c}^1 \\
z^2 & \geq & 0 \\
\\
-\boldsymbol{b}^2 z^1 - \boldsymbol{b}^1 z^2 & \to & \max
\end{array}
$$

We now put

$$u^1 = z^2 \ , \ u^2 = z^1 \ ,$$

and multiply each of the equations (12) by –1. Thereafter we reorder the equations such that the matrix $\boldsymbol{A}^{11}$ appears on the left upper corner and we write the equations and inequalities without using the transposed versions of the matrices. This way we obtain

$$
(13) \qquad (\widehat{\mathbf{0}}) \qquad \begin{array}{rcl}
u^1 \boldsymbol{A}^{11} + u^2 \boldsymbol{A}^{21} & = & \boldsymbol{c}^1 \\
u^1 \boldsymbol{A}^{12} + u^2 \boldsymbol{A}^{22} & \geq & \boldsymbol{c}^2 \\
u^1 & \geq & 0 \\
\\
\boldsymbol{b}^1 u^1 + \boldsymbol{b}^2 u^2 & \to & \min
\end{array}
$$

We call $(\widehat{\mathbf{0}})$ in a slightly sloppy way the 'dual problem' to $(\mathbf{0})$. A precise definition corresponds to this formulation of the dual problem in the sense of Remark 1.2 and Definition 1.3.

**Remark 1.5.** *The various variables which appear in the primal and dual representation of types* $(\mathbf{0})$ *and* $(\widehat{\mathbf{0}})$ *are as well subject to an interpretation. This interpretation corresponds to the dimensions of the vectors or variables involved. The interpretation can be formulated by means of the following list. Note that the dimensions appear explicitely in the matrices when we represent the programs as in* (1).

$n_1$: *number of unrestricted variables in* $(\mathbf{0})$, *number of equations in* $(\widehat{\mathbf{0}})$
$n_2$: *number of restricted variables in* $(\mathbf{0})$, *number of inequalities in* $(\widehat{\mathbf{0}})$
$m_1$: *number of inequalities in* $(\mathbf{0})$, *number of restricted variables in* $(\widehat{\mathbf{0}})$
$m_2$: *number of equations in* $(\mathbf{0})$, *number of unrestricted variables in* $(\widehat{\mathbf{0}})$

*This way it is seen that to some extend and with liberal interpretation unrestricted variables in the primal version correspond to equations in the dual version and vice versa, while restricted variables correspond to inequalities and vice versa.*

We now discuss certain special versions of the general version represented by formula 1. These versions correspond to a particularly easy shape of the corresponding 'popular' problem as represented by **(0)** or $(\widehat{\mathbf{0}})$. We list these problems accordingly in order to provide easy reference.

**Example 1.6.** Let

$$\Gamma = \begin{pmatrix} 0 & \boldsymbol{A} & \boldsymbol{b} \\ 0 & \mathbf{0} & 0 \\ 0 & \boldsymbol{c} & 0 \end{pmatrix} \tag{14}$$

The only relevant variable is $\boldsymbol{x} = \boldsymbol{x}^2$ and the 'popular problem' (i.e., the corresponding version of **(0)**) is given by

$$(\boldsymbol{I}) \qquad \begin{array}{rcl} \boldsymbol{Ax} & \leq & \boldsymbol{b} \\ \boldsymbol{x} & \geq & \mathbf{0} \\ \boldsymbol{cx} & \to & \max \end{array} \tag{15}$$

Accordingly the dual version is

$$\widehat{\Gamma} = - \begin{pmatrix} 0 & \boldsymbol{A}^\top & \boldsymbol{c} \\ 0 & \mathbf{0} & 0 \\ 0 & \boldsymbol{b} & 0 \end{pmatrix} \tag{16}$$

which corresponds to the problem

$$(\widehat{\boldsymbol{I}}) \qquad \begin{array}{lcl} \boldsymbol{uA} & \geq & \boldsymbol{c} \\ \boldsymbol{u} & \geq & \mathbf{0} \\ \boldsymbol{ub} & \to & \min\ . \end{array} \tag{17}$$

This way we have defined a pair of ***standard programs*** $(\boldsymbol{I})$ and $(\widehat{\boldsymbol{I}})$ which, at this stage, are seen as particular versions of **(0)** and $(\widehat{\mathbf{0}})$.

In fact the feasible set of the version indicated by $(\boldsymbol{I})$ is the polyhedron

$$C^0_{A,b} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{Ax} \leq \boldsymbol{b}, \boldsymbol{x} \geq \mathbf{0}\}$$

which has been discussed in previous sections.

**Example 1.7.** Put

$$\Gamma = \begin{pmatrix} 0 & \mathbf{0} & 0 \\ 0 & \boldsymbol{A} & \boldsymbol{b} \\ 0 & \boldsymbol{c} & 0 \end{pmatrix} \tag{18}$$

the shorthand version is

$$(\boldsymbol{II}) \qquad \begin{array}{rcl} \boldsymbol{Ax} & = & \boldsymbol{b} \\ \boldsymbol{x} & \geq & \mathbf{0} \\ \boldsymbol{cx} & \to & max \end{array} \tag{19}$$

The dual program is

$$\widehat{\Gamma} = \quad - \begin{pmatrix} \boldsymbol{A}^\top & 0 & \boldsymbol{c} \\ \mathbf{0} & 0 & 0 \\ \boldsymbol{b} & 0 & 0 \end{pmatrix} \tag{20}$$

which, in shorthand, is given by

$$(\widehat{\boldsymbol{II}}) \qquad \begin{array}{rcl} \boldsymbol{uA} & \geq & \boldsymbol{c} \\ \boldsymbol{ub} & \to & min \end{array} \tag{21}$$

Observe that the 'correspondence' between equations / inequalities in the primal and unrestricted / restricted variables in the dual (and vice versa) is always reflected: $(\boldsymbol{II})$ features equations only - hence we have only unrestricted variables $\boldsymbol{u}$ in $(\widehat{\boldsymbol{II}})$. As $\boldsymbol{x}$ is restricted (that is, $\boldsymbol{x} \geq \mathbf{0}$ in $(\boldsymbol{II})$,) we find that $(\widehat{\boldsymbol{II}})$ has inequalities only etc.

The feasible set of $(\boldsymbol{II})$ is

$$D^0_{\boldsymbol{A},\boldsymbol{b}} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geq \mathbf{0},\ \boldsymbol{Ax} = \boldsymbol{b}\}$$

which is also familiar from previous discussions.

**Example 1.8.** If we choose

$$\Gamma = \begin{pmatrix} \boldsymbol{A} & 0 & \boldsymbol{b} \\ \mathbf{0} & 0 & 0 \\ \boldsymbol{c} & 0 & 0 \end{pmatrix} \tag{22}$$

then we may write

$$(\boldsymbol{III}) \qquad \begin{array}{rcl} \boldsymbol{Ax} & \leq & \boldsymbol{b} \\ \boldsymbol{cx} & \to & max \end{array} \tag{23}$$

in order to indicate the informal version. The dual program is

$$(24) \qquad \widehat{\Gamma} = - \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{0} & \boldsymbol{A}^\top & \boldsymbol{c} \\ 0 & \boldsymbol{b} & 0 \end{pmatrix}$$

or for short

$$(25) \qquad (\widehat{\boldsymbol{III}}) \quad \begin{array}{rcl} \boldsymbol{uA} & = & \boldsymbol{c} \\ \boldsymbol{u} & \geq & \mathbf{0} \\ \boldsymbol{ub} & \to & min \end{array}$$

Not all versions are relevant. E.g., the one given by

$$(26) \qquad \Gamma = \begin{pmatrix} \mathbf{0} & 0 & 0 \\ \boldsymbol{A} & 0 & \boldsymbol{b} \\ \boldsymbol{c} & 0 & 0 \end{pmatrix}$$

i.e.

$$(27) \qquad \begin{array}{rcl} \boldsymbol{Ax} & = & \boldsymbol{b} \\ \boldsymbol{cx} & \to & max \end{array}$$

with dual version

$$(28) \qquad \widehat{\Gamma} = - \begin{pmatrix} \mathbf{0} & 0 & 0 \\ \boldsymbol{A}^\top & 0 & \boldsymbol{c} \\ \boldsymbol{b} & 0 & 0 \end{pmatrix}$$

or

$$(29) \qquad \begin{array}{rcl} \boldsymbol{uA} & = & \boldsymbol{c} \\ \boldsymbol{ub} & \to & min, \end{array}$$

leads to unbounded or constant objective functions unless the set of feasible solutions is trivial.

**Example 1.9.** Consider the LP. suggested by

$$(30) \qquad \begin{array}{rcl} x_1 & \leq & 1/2 \\ x_2 & \leq & 1/2 \\ \boldsymbol{x} & \geq & 0 \\ 2x_1 - x_2 & \to & max \end{array}$$

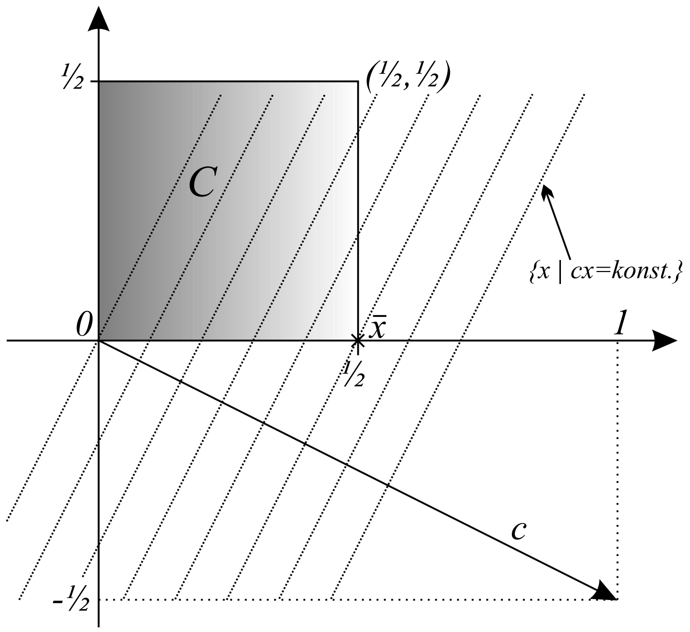



Figure 1.1: The feasible set and $\boldsymbol{c}$ in Exampe 1.9

The feasible set is

$$\{\boldsymbol{x} \in \mathbb{R}^2 \mid \boldsymbol{x} \geq 0 \ , \ \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$$

with $\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , $\boldsymbol{b} = (\frac{1}{2}, \frac{1}{2})$, and $\boldsymbol{c} = (2, -1)$. See Figure 1.1.

As the linear program is of type $(\boldsymbol{I})$, the dual is of type $(\widehat{\boldsymbol{I}})$, i.e., we have

$$\begin{array}{rcl} \boldsymbol{u}\boldsymbol{A} & \geq & \boldsymbol{c} \\ \boldsymbol{u} & \geq & 0 \\ \boldsymbol{u}\boldsymbol{b} & \to & min \end{array}$$

or

$$\begin{array}{rclcl} u_1 & \geq & 2 & & \\ u_2 & \geq & 0 & \geq & -1 \quad . \\ \frac{1}{2}(u_1 + u_2) & \to & min & & \end{array}$$

The feasible set is depicted in Figure 1.2.

Both programs admit of optimal solutions which can immediately be taken from the figures. The one for $(\boldsymbol{I})$ is $\bar{\boldsymbol{x}} = (\frac{1}{2}, 0)$ and the one for $(\widehat{\boldsymbol{I}})$ is $\bar{\boldsymbol{u}} = (2, 0)$. Note that the optimal values of the objective functions are given by

$$\boldsymbol{c}\bar{\boldsymbol{x}} = 1 = \bar{\boldsymbol{u}}b \ .$$

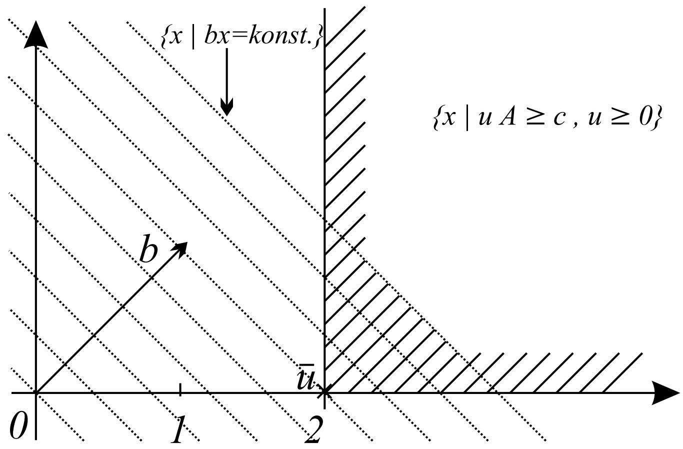



Figure 1.2: The feasible set in of the dual problem in Example 1.9

**Remark 1.10.** *In a previous section (Section 3 of* CHAPTER *2) we have seen that the problem involving a polyhedron of type* $C_{\boldsymbol{A},\boldsymbol{b}}$ *(the 'C-problem') in and the one involving a polyhedron of type* $D^0_{\boldsymbol{A},\boldsymbol{b}}$ *( the 'D-problem') are equivalent in a well defined sense which is established by a pair of bijective mappings or, verbally, by the introduction of 'slack variables'. There is a certain analogy in the present context: the operations which have been considered so far (represented by the mappings * and* $\widehat{\ }$*) can be augmented by further operations (introduction of slack variables or the like) which we first describe verbally. We may change or transform linear programs by means of the following procedures:*

1. *We may replace an equation by two inequalities (this way it seems obvious that version* $(\boldsymbol{II})$ *can be transformed into version* $(\boldsymbol{I})$*).*
2. *We may replace a free variable* $x_i$ *by two restricted variables. This is verbally described by introducing* $x_i = y_i - z_i,\ y_i \geq 0,\ z_i \geq 0$*. (This way it is possible to transform version* $(\boldsymbol{III})$ *into version* $(\boldsymbol{I})$*).*
3. *We may introduce slack variables. (This way we can transform version* $(\boldsymbol{I})$ *into version* $(\boldsymbol{II})$*, compare Section 3 of* CHAPTER *2).*
4. *We may treat a restricted variable as a free variable by considering the restriction as an inequality to be incorporated in the corresponding matrix. (This way we can transform version* $(\boldsymbol{I})$ *into version* $(\boldsymbol{III})$*).*

*By applying these transformations it is obviously possible to transform the versions* $(\mathbf{0})$, $(\boldsymbol{I})$, $(\boldsymbol{II})$, *and* $(\boldsymbol{III})$ *into each other and the same is true for the versions* $(\widehat{\mathbf{0}})$, $(\widehat{\boldsymbol{I}})$, $(\widehat{\boldsymbol{II}})$, $(\widehat{\boldsymbol{III}})$.

In order to be precise for at least one version, consider the version **(0)** which is represented in equation (1). Obviously the corresponding system of equations and inequalities, that is version **(0)**, is equivalent to the following set in which we have inequalities and restricted variables only:

$$(31)\qquad \begin{array}{rcl} \boldsymbol{A}^{11}(\boldsymbol{y}-\boldsymbol{z})+\boldsymbol{A}^{12}\boldsymbol{x}^2 & \leq & \boldsymbol{b}^1 \\ \boldsymbol{A}^{21}(\boldsymbol{y}-\boldsymbol{z})+\boldsymbol{A}^{22}\boldsymbol{x}^2 & \leq & \boldsymbol{b}^2 \\ -\boldsymbol{A}^{21}(\boldsymbol{y}-\boldsymbol{z})-\boldsymbol{A}^{22}\boldsymbol{x}^2 & \leq & -\boldsymbol{b}^2 \\ \boldsymbol{x}^2 & \geq & \boldsymbol{0} \\ \boldsymbol{y} & \geq & \boldsymbol{0} \\ \boldsymbol{z} & \geq & \boldsymbol{0} \\ \boldsymbol{c}^1(\boldsymbol{y}-\boldsymbol{z})+\boldsymbol{c}^2\boldsymbol{x}^2 & \to & max \end{array}$$

This version is obviously of the form

$$(32)\qquad \begin{array}{rcl} \boldsymbol{Aw} & \leq & \boldsymbol{0} \\ \boldsymbol{w} & \geq & \boldsymbol{0} \\ \boldsymbol{wc} & \to & max\ , \end{array}$$

that is, it is equivalent to a program of the form $(\boldsymbol{I})$.

**Remark 1.11.** *From the viewpoint of a concise representation it must be emphasized that the exact versions of the operations indicated by describing the items of the previous remark can only be specified by matrix operations. More precisely, it should be possible to represent linear programs by matrices as is done in formula* (1) *and then apply a matrix operation which results in the new version to be obtained.*

For example, if we consider the above rewriting of version **(0)** as described by the transition from equation (31) to equation (32) then, given the corresponding matrix $\Gamma$ we obtain a matrix $S(\Gamma)$ which is of the form

$$(33)\quad \Gamma' = S(\Gamma) = \begin{pmatrix} 0 & \boldsymbol{A}' & \boldsymbol{b}' \\ 0 & \boldsymbol{0} & 0 \\ 0 & \boldsymbol{c}' & 0 \end{pmatrix} = \begin{pmatrix} 0 & \boldsymbol{A}^{11} & -\boldsymbol{A}^{11} & \boldsymbol{A}^{12} & \boldsymbol{b}^1 \\ 0 & \boldsymbol{A}^{21} & -\boldsymbol{A}^{21} & \boldsymbol{A}^{22} & \boldsymbol{b}^2 \\ 0 & \boldsymbol{A}^{21} & \boldsymbol{A}^{21} & -\boldsymbol{A}^{22} & -\boldsymbol{b}^2 \\ 0 & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & 0 \\ 0 & \boldsymbol{c}^1 & -\boldsymbol{c}^1 & \boldsymbol{c}^2 & 0 \end{pmatrix}$$

In general we will not elaborate on this topic,as the matrix operations involved in this discussion are of no further practical importance.

**Remark 1.12.** *The reader may convince himself that the operations as discussed above and the operation* $\widehat{\ } = -^{*T}$ *commute. This can be expressed by the following diagram.*

$$\begin{array}{ccccc} (\mathbf{0}) & \Gamma & \longrightarrow & \widehat{\Gamma} & (\widehat{\mathbf{0}}) \\ & \downarrow & & \downarrow & \\ (\boldsymbol{I}) & \Gamma' & \longrightarrow & \widehat{\Gamma}' & (\widehat{\boldsymbol{I}}) \end{array}$$

*Verbally we may express this fact by arguing that the introduction of slack variables in the primal program induces the introduction of inequalities in the dual program. These new inequalities together with the existing ones may lead to certain equations etc. In other words, in can be seen that adding slack variables and transforming various versions into each others constitutes consistent operations.*

*Again we shall not exhibit a more formal mechanism as it is of no further interest. However, the reader should be aware that all of these operations can be made precise.*

The next question is obviously the one relating the solutions of primal and dual programs in particular the optimal solutions if any. Of course there may also be a relation of optimal and feasible solutions when transforming programs by admitting slack variables. In fact we have discussed these relations to some extent in SECTION 3 of CHAPTER 2.

In order to clear up the connection between solutions (optimal and feasible ones) of a dual pair we recall Farkas' Lemma or the Theorem of the alternative (cf. Theorem 1.18) of CHAPTER 1. Indeed the Theorem of the Alternative also provides the impression that there is something like a dual statement being formulated, albeit in a negative version. Similarly, as we have discussed possibilities of transforming linear programs and taking the dual, it is possible to formulate analogous versions of alternative theorems. The procedure is quite familiar (using slack variables etc.) and we shall exhibit the relations between alternative theorems and linear programs in what follows.

In particular a theorem of the Alternative is the appropriate method in order to solve existence problems for pairs of dual programs. Note that Farkas' Lemma or a a Theorem of the Alternative is essentially derived from a separation theorem as discussed in Section 1 of Chapter 1.

# 2 The Main Theorem of Linear Programming

Within this section we discuss the Main Theorem of Linear Programming or Duality Theorem. It claims the simultaneous existence of solutions of a dual pair of LP.'s and states that the optimal values of both objective functions are equal. We achieve this important result by means of a Theorem of the Alternative (Farkas' Lemma), hence it is essentially based on a separation theorem.

In view of the results obtained in the previous section, it does not matter which version of a dual pair is under consideration. Because of the nice symmetry properties we choose to deal with a pair represented in version $(\boldsymbol{I}, \widehat{\boldsymbol{I}})$. However, it should be clear that any other version can be substituted.

**Theorem 2.1.** *Let* $(\Gamma, \widehat{\Gamma})$ *be a dual pair of Linear Programs that appear in versions* $(\boldsymbol{I}, \widehat{\boldsymbol{I}})$.

1. *If both LP.'s admit of feasible solutions, then both admit of optimal solutions (and vice versa, of course).*
2. *If* $(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{u}})$ *is a pair of feasible solutions for* $\Gamma$ *and* $\widehat{\Gamma}$ *then*
$$\boldsymbol{c}\widehat{\boldsymbol{x}} \leq \widehat{\boldsymbol{u}}\boldsymbol{b}$$
*holds true.*
3. *A pair of feasible solutions* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})$ *for* $\Gamma$ *and* $\widehat{\Gamma}$ *respectively is a pair of optimal solutions if*
$$\boldsymbol{c}\bar{\boldsymbol{x}} = \bar{\boldsymbol{u}}\boldsymbol{b}$$
*holds true.*

**Proof:** If $\boldsymbol{x}$ is feasible for $\Gamma$ and $\boldsymbol{u}$ is feasible for $\widehat{\Gamma}$, then we have

$$\begin{aligned} \boldsymbol{Ax} &\leq \boldsymbol{b} \\ \boldsymbol{x} &\geq \boldsymbol{0} \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{uA} &\geq \boldsymbol{c} \\ \boldsymbol{u} &\geq \boldsymbol{0} \end{aligned}$$

hence

$$\boldsymbol{cx} \leq \boldsymbol{uAx} \leq \boldsymbol{ub} \; .$$

Consequently, both objective functions are bounded; the second statement is obvious. **q.e.d.**

**Theorem 2.2. (Main Theorem of Linear Programming, Duality Theorem)**
*Let $(\Gamma, \widehat{\Gamma})$ be a dual pair of Linear Programs. Then either both programs admit of optimal solutions or none of them. In the first case any pair of optimal solutions $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})$ for $\Gamma$ and $\widehat{\Gamma}$ respectively yields the same value of the objective functions of both programs.*

**Proof:** Without loss of generality, we may assume that $\Gamma$ and $\widehat{\Gamma}$ appear in versions $(\boldsymbol{I})$ and $(\widehat{\boldsymbol{I}})$ .

**1<sup>st</sup>STEP** :

Consider the linear system of inequalities in variables $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ given by

$$\begin{aligned} \boldsymbol{Ax} &\le \boldsymbol{b} \\ -\boldsymbol{x} &\le \boldsymbol{0} \\ -\boldsymbol{A}^\top \boldsymbol{u} &\le -\boldsymbol{c} \\ -\boldsymbol{u} &\le \boldsymbol{0} \\ -\boldsymbol{cx} + \boldsymbol{ub} &\le 0 \end{aligned}$$

which is also written

$$(1) \qquad \begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ -\boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{A}^\top \\ \boldsymbol{0} & -\boldsymbol{I}_m \\ -\boldsymbol{c} & \boldsymbol{b} \end{pmatrix} (\boldsymbol{x}, \boldsymbol{u}) \le \begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{0} \\ -\boldsymbol{c} \\ \boldsymbol{0} \\ 0 \end{pmatrix} .$$

If this system admits of a solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})$, then $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{u}}$ are feasible for $\Gamma$ and $\widehat{\Gamma}$. Moreover $\bar{\boldsymbol{u}}\boldsymbol{b} \le \boldsymbol{c}\bar{\boldsymbol{x}}$ is satisfied. In view of Theorem 2.1, $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{u}}$ constitute optimal solutions satisfying

$$\bar{\boldsymbol{u}}\boldsymbol{b} = \boldsymbol{c}\bar{\boldsymbol{x}} .$$

**2<sup>nd</sup>STEP** :

Alternatively, assume that the inequalities (1) admit of no solution.

In view of a Theorem of the Alternative (Farkas Lemma, Theorem 1.19 of Chapter 1 ), there is a solution

$$\boldsymbol{0} \le \bar{\boldsymbol{z}} = (\bar{\boldsymbol{u}}, \bar{\boldsymbol{x}}', \bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}', \bar{t}) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ \times \mathbb{R}^m_+ \times \mathbb{R}_+$$

such that

$$\bar{\boldsymbol{z}}\begin{pmatrix} \boldsymbol{A} & \boldsymbol{0} \\ -\boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{A}^\top \\ \boldsymbol{0} & -\boldsymbol{I}_m \\ -\boldsymbol{c} & \boldsymbol{b} \end{pmatrix} = \boldsymbol{0}$$

and

$$\bar{\boldsymbol{z}}\begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{0} \\ -\boldsymbol{c} \\ \boldsymbol{0} \\ 0 \end{pmatrix} < 0$$

holds true.

The above set of equations and inequalities are rewritten as follows

$$\begin{array}{rcl} \bar{\boldsymbol{u}}\boldsymbol{A} - \bar{\boldsymbol{x}}' - \bar{t}\boldsymbol{c} & = & \boldsymbol{0} \\ -\boldsymbol{A}\bar{\boldsymbol{x}} - \bar{\boldsymbol{u}}' + \bar{t}\boldsymbol{b} & = & \boldsymbol{0} \\ \bar{\boldsymbol{u}}\boldsymbol{b} & < & \boldsymbol{c}\bar{\boldsymbol{x}} \\ \bar{\boldsymbol{u}}, \bar{\boldsymbol{x}}', \bar{\boldsymbol{x}}, \bar{\boldsymbol{u}}' & \geq & \boldsymbol{0} \\ \bar{t} & \geq & 0 \end{array} \tag{2}$$

**3rdSTEP** :

We want to prove that $\bar{t} = 0$ holds necessarily true. Assume *per absurdum* that $\bar{t} > 0$ is the case.

Because of

$$\begin{array}{rcl} \bar{\boldsymbol{u}}\boldsymbol{A}\bar{\boldsymbol{x}} & = & \bar{\boldsymbol{x}}'\boldsymbol{x} + \bar{t}\boldsymbol{c}\bar{\boldsymbol{x}} \\ & \geq & \bar{t}\boldsymbol{c}\bar{\boldsymbol{x}} \\ & > & \bar{t}\bar{\boldsymbol{u}}\boldsymbol{b} \\ & = & \bar{\boldsymbol{u}}\boldsymbol{A}\bar{\boldsymbol{x}} + \bar{\boldsymbol{u}}\bar{\boldsymbol{u}}' \\ & \geq & \bar{\boldsymbol{u}}\boldsymbol{A}\bar{\boldsymbol{x}} \end{array}$$

we have a contradiction, hence $\bar{t} = 0$ is indeed true.

Consequently, the inequalities (2) can be rewritten as follows:

$$\begin{array}{lcl} \bar{\boldsymbol{u}}\boldsymbol{A} & = & \bar{\boldsymbol{x}}' \geq \boldsymbol{0} \\ \boldsymbol{A}\bar{\boldsymbol{x}} & = & -\bar{\boldsymbol{u}}' \leq \boldsymbol{0} \\ \bar{\boldsymbol{u}}\boldsymbol{b} & < & \boldsymbol{c}\bar{\boldsymbol{x}} \\ \bar{\boldsymbol{u}} & \geq & \boldsymbol{0} \\ \bar{\boldsymbol{x}} & \geq & \boldsymbol{0} \end{array} \tag{3}$$

**$4^{\text{th}}$STEP :**

Now we can come to our final conclusion: if $\Gamma$ has feasible solutions, say $\boldsymbol{x}^0$ satisfying

$$(4) \qquad \begin{array}{rcl} \boldsymbol{A}\boldsymbol{x}^0 & \leq & \boldsymbol{b} \\ \boldsymbol{x}^0 & \geq & \boldsymbol{0} \end{array}$$

then, for $t \geq 0$, consider

$$\boldsymbol{x}^t := \boldsymbol{x}^0 + t\bar{\boldsymbol{x}} \ .$$

It turns out immediately that $\boldsymbol{x}^t$ is feasible as well since $\boldsymbol{x}^t \geq \boldsymbol{0}$ is obvious and

$$\boldsymbol{A}\boldsymbol{x}^t = \underbrace{\boldsymbol{A}\boldsymbol{x}^0}_{\leq\ b} + t\underbrace{\boldsymbol{A}\bar{\boldsymbol{x}}}_{\leq\ 0} \quad \leq \boldsymbol{b}$$

shows feasibility (cf. (3)).

Similarly, if $\boldsymbol{u}^0$ is a feasible solution for $\widehat{\Gamma}$, then

$$\boldsymbol{u}^t := \boldsymbol{u}^0 + t\bar{\boldsymbol{u}}$$

is feasible for $\widehat{\Gamma}$ as well. But in view of (3) we have, for sufficiently large real $t$:

$$t(\boldsymbol{c}\bar{\boldsymbol{x}} - \bar{\boldsymbol{u}}\boldsymbol{b}) > \boldsymbol{u}^0\boldsymbol{b} - \boldsymbol{c}\boldsymbol{x}^0,$$

that is,

$$\boldsymbol{c}(\boldsymbol{x}^0 + t\bar{\boldsymbol{x}}) > (\boldsymbol{u}^0 + t\bar{\boldsymbol{u}})\boldsymbol{b}$$

or

$$\boldsymbol{c}\boldsymbol{x}^t > \boldsymbol{u}^t\boldsymbol{b} \ ;$$

this, however, constitutes a contradiction to Theorem 2.1, which requires $\boldsymbol{c}\boldsymbol{x}^t \leq \boldsymbol{u}^t\boldsymbol{b}$ .

This means that not both LP.'s admit of a feasible solution.

**$5^{\text{th}}$STEP :**

Moreover, if $\Gamma$ admits of a feasible solution $\boldsymbol{x}^0$ (and $\widehat{\Gamma}$ not), then, combining (3) and (4) we obtain

$$\begin{array}{rcl} \boldsymbol{c}\bar{\boldsymbol{x}} & > & \bar{\boldsymbol{u}}\boldsymbol{b} \\ & \geq & \bar{\boldsymbol{u}}(\boldsymbol{A}\boldsymbol{x}^0) \\ & = & (\bar{\boldsymbol{u}}\boldsymbol{A})\boldsymbol{x}^0 \\ & \geq & \boldsymbol{0}\cdot\boldsymbol{0} = 0 \end{array}$$

Hence

$$\begin{aligned} \boldsymbol{c}\boldsymbol{x}^t &= \boldsymbol{c}(\boldsymbol{x}^0 + t\bar{\boldsymbol{x}}) \\ &= \boldsymbol{c}\boldsymbol{x}^0 + t\boldsymbol{c}\bar{\boldsymbol{x}} \to \infty \quad (t \to \infty) \ , \end{aligned}$$

i.e., the objective function of $\Gamma$ is unbounded. **q.e.d.**

Obviously the following version of the Main Theorem is equivalent:

**Theorem 2.3 (Main Theorem of Linear Programming, $2^{nd}$ version).** *Let $(\Gamma, \widehat{\Gamma})$ be a dual pair of LP.'s. If $\Gamma$ has optimal solutions, then so has $\widehat{\Gamma}$ and vice versa. In this case, assuming the version $(\boldsymbol{I}, \widehat{\boldsymbol{I}})$, we obtain*

$$\max\{\boldsymbol{c}\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} = \min\{\boldsymbol{u}\boldsymbol{b} \mid \boldsymbol{u}\boldsymbol{A} \geq \boldsymbol{c}, \boldsymbol{u} \geq \boldsymbol{0}\}. \tag{5}$$

Finally, we add a third formulation to be referred to in due course:

**Theorem 2.4 (Main Theorem, $3^{rd}$ version).** *Let $(\Gamma, \widehat{\Gamma})$ be a dual pair of LP.'s. Then both programs have optimal solutions if and only if they both have feasible solutions.*

*If $(\Gamma, \widehat{\Gamma})$ appear in version $((\boldsymbol{I}), (\widehat{\boldsymbol{I}}))$, then a pair of feasible solutions $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})$ is optimal if and only if*

$$\boldsymbol{c}\bar{\boldsymbol{x}} = \bar{\boldsymbol{u}}\boldsymbol{b}$$

*holds true.*

The following optimality criterion is an immediate consequence of the Main Theorem. It is most usefull as a test for a candidate to be verified as an optimal solution.

**Theorem 2.5 (Optimality Criterion).** *Let $(\Gamma, \widehat{\Gamma})$ be a dual pair of linear programs which appear in version $(\boldsymbol{I}, \widehat{\boldsymbol{I}})$. A feasible solution $\bar{\boldsymbol{x}}$ of $\Gamma$ is optimal if and only if there exists a feasible solution $\bar{\boldsymbol{u}}$ of $\widehat{\Gamma}$ such that the following holds true:*

$$\begin{array}{lcll} \bar{x}_j > 0 & \Longrightarrow & \bar{\boldsymbol{u}}\boldsymbol{A}_{\bullet j} = c_j & (j \in J) \\ \bar{u}_i > 0 & \Longrightarrow & \boldsymbol{A}_{i\bullet}\bar{\boldsymbol{x}} = b_i & (i \in I) \end{array} \tag{6}$$

*Whenever $\bar{\boldsymbol{u}}$ satisfies formula (6) then $\bar{\boldsymbol{u}}$ is an optimal solution of $\widehat{\Gamma}$.*

**Proof:**
**$1^{st}$STEP** : Let $\bar{\boldsymbol{x}}$ be an optimal solution of $\Gamma$. In view of the main theorem

of linear programming (Theorem 2.2), there exists an optimal solution $\bar{\boldsymbol{u}}$ of $\widehat{\Gamma}$ such that the following holds true.

$$
\begin{array}{llll}
 & \boldsymbol{c}\bar{\boldsymbol{x}} = \sum_{j\in J} c_j\bar{x}_j & \leq \sum_{j\in J} \bar{\boldsymbol{u}}\boldsymbol{A}_{\bullet j}\bar{x}_j & = \sum_{j\in J^+} \bar{\boldsymbol{u}}\boldsymbol{A}_{\bullet j}\bar{x}_j \\
(7) & = \bar{\boldsymbol{u}}\boldsymbol{A}\bar{\boldsymbol{x}} & = \sum_{i\in I} \bar{u}_i\boldsymbol{A}_{i\bullet}\bar{\boldsymbol{x}} & = \sum_{i\in I^+} \bar{u}_i\boldsymbol{A}_{i\bullet}\bar{\boldsymbol{x}} \\
 & \leq \sum_{i\in I} \bar{u}_i b_i & = \bar{\boldsymbol{u}}\boldsymbol{b} & = \boldsymbol{c}\bar{\boldsymbol{x}}
\end{array}
$$

Here $J^+$ denotes that set of coordinates at which $\bar{\boldsymbol{x}}$ is positive and, similarly $I^+$ denotes the set of coordinates at which $\bar{\boldsymbol{u}}$ is positive. Now suppose (6) is violated for some $i \in I$ or some $j \in J$. Then the above set of inequalities would be a strict one which is not possible in view of the main theorem of linear programming.

**2$^{\text{nd}}$STEP** : Suppose on the other hand that (6) is satisfied. Then there are feasible solutions of both programs which obviously satisfy

$$
\begin{array}{llll}
(8) & \boldsymbol{c}\bar{\boldsymbol{x}} = \sum_{j\in J} c_j\bar{x}_j & = \sum_{j\in J} \bar{\boldsymbol{u}}\boldsymbol{A}_{\bullet j}\bar{x}_j & = \bar{\boldsymbol{u}}\boldsymbol{A}\bar{x} \\
 & = \sum_{i\in I} \bar{u}_i\boldsymbol{A}_{i\bullet}\bar{\boldsymbol{x}} & = \sum_{i\in I} \bar{u}_i b_i & = \bar{\boldsymbol{u}}\boldsymbol{b}
\end{array}
$$

However, this obviously means that $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})$ is a pair of optimal solutions for both programs respectively, **q.e.d.**

**Example 2.6.** Let

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \boldsymbol{b} = (1, 1, \frac{3}{2}) \quad \boldsymbol{c} = (2, 1)$$

Consider the version

$$
(\widehat{\boldsymbol{I}}) \quad
\begin{array}{lcl}
\boldsymbol{uA} & \geq & \boldsymbol{c} \\
\boldsymbol{u} & \geq & \boldsymbol{0} \\
\boldsymbol{ub} & \to & \min
\end{array}
$$

This means also that we have to look for

$$\min\{u_1 + u_2 + \frac{3}{2}u_3 \mid \boldsymbol{u} \geq 0, \quad u_1 + u_3 \geq 2, \quad u_2 + u_3 \geq 1\}$$

The dual program is represented by version (1) as follows:

$$
(\boldsymbol{I}) \quad
\begin{array}{lcl}
\boldsymbol{Ax} & \leq & \boldsymbol{b} \\
\boldsymbol{x} & \geq & 0 \\
\boldsymbol{cx} & \to & \max
\end{array}
$$

which asks for the computation of

$$\max\{2x_1 + x_2 \mid \boldsymbol{x} \geq 0, \quad x_1 \leq 1, \quad x_2 \leq 1, \quad x_1 + x_2 \leq \frac{3}{2}\}$$

This problem obviously is solved in a rather trivial way by taking

$$\begin{array}{ll} \bar{\boldsymbol{x}} & = (1, \frac{1}{2}) \\ \boldsymbol{c}\bar{\boldsymbol{x}} & = \frac{5}{2} \end{array}$$

Of course, one can also go through the simplex algorithm in order to solve this problem. Obviously we have

$$\begin{array}{rcl} \boldsymbol{A}_{1\bullet}\bar{\boldsymbol{x}} & = & b_1 \\ \boldsymbol{A}_{2\bullet}\bar{\boldsymbol{x}} & < & b_2 \\ \boldsymbol{A}_{3\bullet}\bar{\boldsymbol{x}} & = & b_2 \end{array}$$

Therefore we are looking for an optimal solution of $(\widehat{\boldsymbol{I}})$ among the solutions of the form

$$\boldsymbol{u} = (u_1, 0, u_3).$$

As $\bar{\boldsymbol{x}} > 0$ we have necessarily

$$\bar{\boldsymbol{u}}\boldsymbol{A}_{\bullet 1} = c_1 \ , \ \bar{\boldsymbol{u}}\boldsymbol{A}_{\bullet 2} = c_2 \ ,$$

that is,

$$\begin{array}{rcl} \bar{u}_1 + \bar{u}_3 & = & 2 \\ \bar{u}_2 + \bar{u}_3 & = & 1 \\ \bar{u}_2 & = & 0 \end{array}$$

As a consequence we have

$$\bar{\boldsymbol{u}} = (1, 0, 1) \ .$$

This is a feasible solution which in addition satisfies

$$\bar{\boldsymbol{u}}\boldsymbol{b} = \frac{5}{2} = \boldsymbol{c}\bar{\boldsymbol{x}} \ ,$$

hence $\bar{\boldsymbol{u}}$ is optimal.

# 3 Shadow Prices

We provide some economic interpretation of the solutions of a dual pair. Recall the standard interpretation of an LP. as a linear production process. The incentive of the decision maker is provided by the sales revenue/profit he can obtain by selling the products on a hypothetical market. In order to generate products one has to make use of the production process (the input output matrix), there is no way to directly sell quantities of a factor on this market.

Nevertheless, one would expect that some worth or value is thereby implicitely attached to the factors: as factors can be convertet to goods, they are valuable. This idea can be made precise, it is possible to assign what is called "shadow prices" to production factors and it turns out that shadow prices are provides by *optimal* dual solutions.

To make this more precise, let us consider a pair of two dual linear programs in the form $(\boldsymbol{I}, \widehat{\boldsymbol{I}})$. That is we are given the two problems in shorthand notation as follows:

$$(1) \qquad (\boldsymbol{I}) \qquad \begin{array}{rcl} \boldsymbol{Ax} & \leq & \boldsymbol{b} \\ \boldsymbol{x} & \geq & \boldsymbol{0} \\ \boldsymbol{cx} & \to & \max \end{array}$$

and

$$(2) \qquad (\widehat{\boldsymbol{I}}) \qquad \begin{array}{rcl} \boldsymbol{uA} & \geq & \boldsymbol{c} \\ \boldsymbol{u} & \geq & \boldsymbol{0} \\ \boldsymbol{ub} & \to & \min \ . \end{array}$$

Now we change the constraint vector $\boldsymbol{b}$ and (for some $i_0 \in I$ and $\varepsilon > 0$) replace it by a slightly modified version which is

$$(3) \qquad \boldsymbol{b}^\star \ := \ \boldsymbol{b} + \varepsilon \boldsymbol{e}^{i_0}.$$

The new "primal" problem is thus suggested by

$$(4) \qquad (\boldsymbol{I}^\star) \qquad \begin{array}{rcl} \boldsymbol{Ax} & \leq & \boldsymbol{b}^\star \\ \boldsymbol{x} & \geq & \boldsymbol{0} \\ \boldsymbol{cx} & \to & \max \end{array}$$

and the "dual version" is

$$(5) \qquad (\widehat{\boldsymbol{I}}^\star) \qquad \begin{array}{rcl} \boldsymbol{uA} & \geq & \boldsymbol{c} \\ \boldsymbol{u} & \geq & \boldsymbol{0} \\ \boldsymbol{ub}^\star & \to & \min \ , \end{array}$$

which has the same feasible solutions as $(\widehat{\boldsymbol{I}})$

Suppose now, we are given a pair of optimal solutions $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})$ for the original (unchanged) problem, i.e., for $(\boldsymbol{I})$ and $(\widehat{\boldsymbol{I}})$ (see (1) and (2)) Also, suppose that there is an optimal solution $\boldsymbol{x}^\star$ for the modified primal problem $(\widehat{\boldsymbol{I}})$ in (4) which satisfies the following

**Shadow Price Assumption:**

$$(6)\qquad \begin{array}{rcl} \boldsymbol{x}^\star_j > 0 & \Rightarrow & \bar{\boldsymbol{x}}_j > 0 \quad (j \in J) \\ \boldsymbol{A}_i\bar{\boldsymbol{x}} \; = \; b_i & \Rightarrow & \boldsymbol{A}_i\boldsymbol{x}^\star_i \; = \; b^\star_i \quad (i \in I) \end{array}$$

It will turn out that this condition is not too restrictive, but we will postpone the discussion in order to continue with our economic interpretation.

In view of the Optimality Criterion (Theorem 2.5), the above Assumption implies the following conclusions that establish a relation between $\bar{\boldsymbol{x}}$ and $\boldsymbol{u}^\star$:

$$(7)\qquad \begin{array}{lclcl} \boldsymbol{x}^\star_j > 0 & \Rightarrow & \bar{\boldsymbol{x}}_j > 0 & \Rightarrow & \bar{\boldsymbol{u}}\boldsymbol{A}_{\bullet j} \; = \; c_j \quad (j \in J) \\ \bar{\boldsymbol{u}}_i > 0 & \Rightarrow & \boldsymbol{A}_{i\bullet}\bar{\boldsymbol{x}} \; = \; b_i & \Rightarrow & \boldsymbol{A}_{i\bullet}\boldsymbol{x}^\star_i \; = \; b^\star_i \quad (i \in I). \end{array}$$

We may, therefore, argue that the pair $(\boldsymbol{x}^\star, \bar{\boldsymbol{u}})$ as well satisfies the Optimality Criterion. However, the vector $\bar{\boldsymbol{u}}$ satisfies the constraints

$$\begin{array}{lcl} \boldsymbol{uA} & \geq & \boldsymbol{c} \\ \boldsymbol{u} & \geq & \boldsymbol{0} \end{array}$$

which are the same for $\widehat{\boldsymbol{I}}$ and $\widehat{\boldsymbol{I}}^\star$. Therefore, the pair $(\boldsymbol{x}^\star, \bar{\boldsymbol{u}})$ constitutes a pair of optimal solutions for (2) and its dual, i.e., for $(\boldsymbol{I}^\star, \widehat{\boldsymbol{I}}^\star)$.

From this we obtain

$$(8)\qquad \begin{array}{rcl} \max\{\boldsymbol{cx} \mid \boldsymbol{x} \geq \boldsymbol{0},\; \boldsymbol{Ax} \leq \boldsymbol{b}^\star\} & = & \boldsymbol{cx}^\star \; = \; \boldsymbol{b}^\star\bar{\boldsymbol{u}} \\ & = & \boldsymbol{b}\bar{\boldsymbol{u}} + \varepsilon\bar{u}_{i_0} \; = \; \boldsymbol{c}\bar{\boldsymbol{x}} + \varepsilon\bar{u}_{i_0} \\ & = & \max\{\boldsymbol{cx} \mid \boldsymbol{x} \geq \boldsymbol{0},\; \boldsymbol{Ax} \leq \boldsymbol{b}\} \\ & & + \varepsilon\bar{u}_{i_0}. \end{array}$$

Verbally: if the quantity of factor $i_0$ available is increased by some $\varepsilon > 0$, then the profit/sales revenue is increased by $\varepsilon\bar{u}_{i_0}$. Hence, $\bar{u}_{i_0}$ can indeed be seen as some sort of "price" attached to a unit of factor $i_0$. This justifies the term ***shadow prices*** for the coordinates of the optimal dual variable.

It must be stressed that "prices" of this kind exist only in context with *optimal solutions.* The decision maker or entrepeneur can assign a "price" to a factor only if production is arranged in an optimal way and stays in an optimum when the additional quantity of a factor is added. This idea of a "price" is more clearly exhibited in the general context of Equilibrium Theory and Game Theory, where "prices" (formaly dual variables) appear only "in equilibrium".

Mathematically the above derivation obviously shows the existence of certain partial derivatives provided the "Shadow Price Assumption" holds true. More precisely, define for some $(\boldsymbol{A}, \boldsymbol{c}) \in \mathbb{R}^{m\times n} \times \mathbb{R}^n$ a function $f = f^{\boldsymbol{Ac}} : \mathbb{R}^m \to \mathbb{R}$ via

$$f(\boldsymbol{b}) = f^{\boldsymbol{Ac}}(\boldsymbol{b}) \ := \ \max\{\boldsymbol{cx} | \boldsymbol{x} \geq 0, \boldsymbol{Ax} \leq \boldsymbol{b}\} \tag{9}$$

we have shown the following.

**Lemma 3.1 (The Shadow Price Lemma).** *Let $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m_+ \times \mathbb{R}^n$ be such that $(\boldsymbol{I})$ and $(\boldsymbol{I}^\star)$ admit of feasible (and hence optimal) solutions. Let $\bar{\boldsymbol{u}}$ be an optimal solution of $(\boldsymbol{I}^\star)$. Furthermore, let $i_0 \in I$ and suppose there is $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ and $\boldsymbol{b}^\star = \boldsymbol{b} + \varepsilon \boldsymbol{e}^{i_0}$ the Shadow Price Assumption* (6) *is satisfied. Then the function $f^{\boldsymbol{Ac}}$ admits of a partial derivative in direction $i_0$ at $\boldsymbol{b}$ which is given by*

$$\frac{\partial f^{\boldsymbol{Ac}}}{\partial \boldsymbol{b}_{i_0}}(\boldsymbol{b}) \ = \ \bar{u}_{i_0} \ . \tag{10}$$

This will now be extended to a large class of Linear Programs. To this end, we will first of all exhibit a class of LP.'s such that the Shadow Price Assumption is satisfied.

**Lemma 3.2.** *Let $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m_+ \times \mathbb{R}^n$ be such that $(\boldsymbol{I})$ and $(\boldsymbol{I}^\star)$ admit of optimal solutions $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{u}})$. Let $\bar{\boldsymbol{x}}$ be a vertex of*

$$C^0_{\boldsymbol{Ab}} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geq 0, \ \boldsymbol{Ax} \leq \boldsymbol{b}\}$$

*and let $\widetilde{\boldsymbol{z}} = p(\bar{\boldsymbol{x}})$ be the image of $\bar{\boldsymbol{x}}$ under the "slack variable mapping " $p$ (cf. Lemma 3.1, $p(\boldsymbol{x}) = (\boldsymbol{x}, \boldsymbol{b} - \boldsymbol{Ax})$ $(\boldsymbol{x} \in \mathbb{R}^n)$). Also, let $\widetilde{\boldsymbol{A}} := (\boldsymbol{A}, \boldsymbol{I}_m)$ be the corresponding matrix for the "D–problem" as in* CHAPTER *2,* SECTION *3. If $(\widetilde{\boldsymbol{A}}, \boldsymbol{b}, \widetilde{\boldsymbol{z}})$ is non–degenerate, then, for $i_0 \in I$ and sufficiently small $\varepsilon > 0$, there exists*

$$\boldsymbol{x}^\star \in C^0_{\boldsymbol{Ab}^\star} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geq 0, \ \boldsymbol{Ax} \leq \boldsymbol{b}^\star\}$$

*which is optimal with respect to $(\boldsymbol{I}^\star)$ such that* (6) *is satisfied.*

**Proof:** Let $\bar{J} := \{1, \ldots, n, n+1, \ldots n+m\}$ denote the index set for the columns of $\widetilde{\boldsymbol{A}}$ and for the coordinates of $\widetilde{\boldsymbol{z}}$. Let $\widetilde{J} := \{j \in \bar{J} | \widetilde{z}_j > 0\}$ denote the index set for the positive coordinates of $\widetilde{\boldsymbol{z}}$. The size of this set is $|\widetilde{J}| = m$ in view of the assumption concerning nondegeneracy. The square matrix

$$\boldsymbol{A}^0 := (\boldsymbol{A}_{\bullet j})_{j \in \bar{J}}$$

is nonsingular by the same assumption.

Consequently, the system of linear equations given by

$$\boldsymbol{A}^0 \boldsymbol{x} = \boldsymbol{e}^{i_0} \tag{11}$$

has a unique solition, say $\boldsymbol{w}^0 \in \mathbb{R}^{\widetilde{J}}$. The restriction $\widetilde{\boldsymbol{z}}_{\widetilde{J}}$ of $\widetilde{\boldsymbol{z}}$ satisfies, therefore,

$$\begin{aligned} \boldsymbol{A}^0(\widetilde{\boldsymbol{x}}_{\widetilde{J}} + \varepsilon \widetilde{\boldsymbol{w}}^0) &= \boldsymbol{A}^0 \widetilde{\boldsymbol{x}}_{\widetilde{J}} + \varepsilon \boldsymbol{A}^0 \widetilde{\boldsymbol{w}}^0 x \\ &= \widetilde{\boldsymbol{A}} \widetilde{\boldsymbol{z}} + \varepsilon \boldsymbol{e}^{i_0} \\ &= \boldsymbol{b} + \varepsilon \boldsymbol{e}^{i_0} = \boldsymbol{b}^\star . \end{aligned} \tag{12}$$

In other words, $\boldsymbol{z}^\varepsilon := \widetilde{\boldsymbol{z}} + \varepsilon(\boldsymbol{w}^0 \bigoplus \boldsymbol{0})$ is the unique solution of the linear system of equations

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} + \varepsilon \boldsymbol{e}^{i_0} = \boldsymbol{b}^\star . \tag{13}$$

For sufficiently small $\varepsilon > 0$ it will happen that $\boldsymbol{z}^\varepsilon \geq 0$ holds true and that $\{j \in \bar{J} | x^\varepsilon_j > 0\} = \widetilde{J}$ is the case. That is, the basis corresponding to the vertex $\boldsymbol{z}^\varepsilon$ is the same as the one corresponding to $\widetilde{\boldsymbol{z}}$. Then the first part of the Assumption as formulated in (6) is obviously satisfied for the projections $\bar{\boldsymbol{x}} = q(\widetilde{\boldsymbol{z}})$ and $\boldsymbol{x}^\star := q(\boldsymbol{z}^\varepsilon)$.

We can see immediately that the second part of (6) is satisfied as well. For, if $\boldsymbol{A}_{k\bullet}\bar{\boldsymbol{x}} = b_k$ holds true for some $k \in J$, then $\widetilde{\boldsymbol{z}} = p(\bar{\boldsymbol{x}}) = (\boldsymbol{x}, \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x})$ yields $\widetilde{z}_{n+k} = 0$. From this it follows that $\boldsymbol{z}^\varepsilon_{n+k} = 0$ holds true which in turn implies $\boldsymbol{A}_{k\bullet}\boldsymbol{x}^\star = b_k$.

It remains to be shown that $\boldsymbol{x}^\star$ is optimal with respect to the LP. indicated by $(\boldsymbol{I}^\star)$. To this end, we compare the tableaus induced, i.e., $\mathbf{T}(\widetilde{\boldsymbol{z}})$ and $\mathbf{T}(\boldsymbol{z}^\varepsilon)$ (see Definition 2.2). Clearly, the tableau elements collected in $\boldsymbol{\Lambda}$ are the same as the matrix $\widetilde{\boldsymbol{A}}$ has not changed. The same is true for the vector $\boldsymbol{c}$ and hence for the coordinates of $\boldsymbol{\zeta}$, as is easily verified. The optimality of $\widetilde{\boldsymbol{z}}$ is reflected by $\boldsymbol{\Lambda}$ and $\boldsymbol{\zeta}$ and depends on these quantities only. As the quantities are the same, we know that $\boldsymbol{z}^\varepsilon$ is optimal as well. The argument is at once carried over to the projections, as optimality is preserved by the mappings $p$ and $q$ (Remark 3.6),

**q.e.d.**

**Lemma 3.3.** *Let $n \geq m$ and $(\boldsymbol{A}, \boldsymbol{b}) \in \mathbb{R}^{m\times n} \times \mathbb{R}_+$. If every square $m \times m$ submatrix of $\boldsymbol{A}$ is nonsingular and $\boldsymbol{b}$ is not contained in any linear subspace spanned by less than $m$ columns of $\boldsymbol{A}$, then $(\boldsymbol{A}, \boldsymbol{b})$ is nondegenerate.*

**Proof:** This is rather obvious, as the condition means that any basis of column vectors corresponding to some vertex has to contain $m$ such column vectors. Otherwise the vector $\boldsymbol{b}$ would permit a representation by means of less that $m$ columns which span a smaller linear subspace.

**q.e.d.**

**Corollary 3.4.** *The set of linear programs $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) \in \mathbb{R}^{m\times n} \times \mathbb{R}_+ \times \mathbb{R}^n$ such that $(\boldsymbol{A}, \boldsymbol{b})$ is n.d. is open and dense in the topology of $\mathbb{R}^{m\times n} \times \mathbb{R}_+ \times \mathbb{R}^n$.*

**Proof:** First of all, one has to take care of the fact that $\boldsymbol{b}$ avoids finitely many subspaces spanned by the columns. However, the subspace of triplets such that $\boldsymbol{b}$ is contained in a subspace spanned by $n-1$ columns is a lower dimensional subspace itself. Therefore it is sufficient to avoid finitely many subspaces – which defines an open and dense subset.

Similarly, the subspace of triplets such that $n$ columns of $\boldsymbol{A}$ are linearly dependant is a linear subspce.

One obtaines an open set since the determinant is a continuous function.

**q.e.d.**

Combining all this we may now explain the shadow price vector as the almost everywhere gradient of the function $f^{\boldsymbol{Ac}}$ which assigns the value of the Linear Program to the constraint vector. To simplify matters, we restrict the discussion to nonnegative matrices $\boldsymbol{A}$. For, if $\boldsymbol{A}$ and $\boldsymbol{b}$ are positive, then clearly $(\boldsymbol{I})$ and $(\boldsymbol{I}^\star)$ admit of feasible solutions, hence optimal soltions exist.

That is, we obtain

**Corollary 3.5.** *There is an open and dense set of LP.'s $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c})$ (in the topology of $\mathbb{R}_+^{m\times n} \times \mathbb{R}_+^m \times \mathbb{R}^n$) with the following property: the maximizers $\bar{\boldsymbol{x}}$ of $(\boldsymbol{I})$ and $\bar{\boldsymbol{u}}$ of $(\boldsymbol{I}^\star)$ are uniquely defined. The function $f^{\boldsymbol{Ac}}$ is differentiable at $\boldsymbol{b}$ with gradient*

$$\frac{\partial f^{\boldsymbol{Ac}}}{\partial \boldsymbol{b}}(\boldsymbol{b}) = \bar{\boldsymbol{u}} \; . \tag{14}$$

**Proof: 1[st]STEP** : Consider the set $\mathcal{A}$ of triples $(\boldsymbol{A}, \boldsymbol{b}, \boldsymbol{c}) \in \mathbb{R}_+^{m\times n} \times \mathbb{R}_+^m \times \mathbb{R}^n$ satisfying the following conditions:

1. $\boldsymbol{A} > \boldsymbol{0}, \quad \boldsymbol{b} > \boldsymbol{0}$.
2. $(\boldsymbol{I})$ and $(\boldsymbol{I}^\star)$ admit of unique optimal solutions.
3. Every square submatrix of $\boldsymbol{A}$ is nonsingular.
4. For every $I_0 \subseteq I$ the restriction $\boldsymbol{b}_{I_0}$ of $\boldsymbol{b}$ is not contained in a subspace spanned by less that $|I_0|$ o f the restricted columns $(\boldsymbol{A}_{i\bullet})_{i\in I_0} = \boldsymbol{A}_{I_0\bullet}$.

It is not hard to see that the set $\mathcal{A}$ is an open and dense set.

**2^nd^STEP** : We show that for all triplets in $\mathcal{A}$ it follows that $(\widetilde{\boldsymbol{A}}, \boldsymbol{b}) = (\boldsymbol{A}, I_m, \boldsymbol{b})$ is nondegenerate.

To this end, we appeal to Lemma 3.3. An $m \times m$ submatrix of $\widetilde{\boldsymbol{A}}$ consists of some $l$ columns of unit vectors, say $(\boldsymbol{e}^{n+i})_{i\in I_0}$ and some $m-l$ columns of $\boldsymbol{A}$, say $(\boldsymbol{A}_{\bullet j})_{j\in J_0}$ (with $|I_0| + |J_0| = m$). This submatrix is nonsigular if the rank of the submatrix of $\boldsymbol{A}$ involved, say $\boldsymbol{A}_{\bullet J_0}$, is $m-l$. This is implied by the conditions for the set $\mathcal{A}$.

Similarly, $\boldsymbol{b}$ avoids any subspace spanned by less than $m$ columns of $\widetilde{\boldsymbol{A}}$ if, in the above situation, the restriction $\boldsymbol{b}_{I_0^c}$ is not contained in a subspace spanned by less than $|J_0|$ columns of the square submatrix $(a_{ij})_{i\notin I_0, j\in J_0}$. This again is guaranteed by the conditions defining $\mathcal{A}$. This finishes the second step in view of Lemma 3.3.

**3^rd^STEP** : Now consider Lemma 3.2. The conditions of this Lemma are satisfied for the elements of $\mathcal{A}$, hence the Shadow Price Assumption (6) is fulfilled for every $i_0 \in I$. Therefore, the partial derivatives exist in every direction and equal the coordinates of the dual optimal solution by the Shadow Price Lemma 3.1,

**q.e.d.**

We close with some remark concerning the computation of shadow pricesin the “D–problem”.

**Theorem 3.6.** *Let*

$$(\boldsymbol{II}) \qquad \begin{aligned} \boldsymbol{Ax} &= \boldsymbol{b} \\ \boldsymbol{x} &\geq \boldsymbol{0} \\ \boldsymbol{cx} &\to max \end{aligned} \tag{15}$$

*and*

$$(\widehat{\boldsymbol{II}}) \qquad \begin{aligned} \boldsymbol{uA} &\geq \boldsymbol{c} \\ \boldsymbol{ub} &\to min \end{aligned} \tag{16}$$

*indicate a dual pair of LP.'s and assume that* $(\boldsymbol{A}, \boldsymbol{b}, \bar{\boldsymbol{x}})$ *is a nd. standard configuration. Also, let*

$$\bar{\boldsymbol{A}} \;=\; (\boldsymbol{A}_{\bullet j})_{j \in \bar{J}}$$

*be the square submatrix consisting of the rows of the basis of* $\bar{\boldsymbol{x}}$*. If* $\bar{\boldsymbol{x}}$ *is optimal, then*

$$\bar{\boldsymbol{u}} \;:=\; \boldsymbol{c}_{\bar{J}} \bar{\boldsymbol{A}}^{-1} \tag{17}$$

*is an optimal solution of the dual problem.*

**Proof:** Write

$$\boldsymbol{A} \;=\; \left(\bar{\boldsymbol{A}}, \widehat{\boldsymbol{A}})\right)$$

in order to separate the basis and non–basis columns of $\boldsymbol{A}$. Also let $\Lambda$ denote the matrix of the tableau elements. Then we know that for any non–basis column we have

$$\boldsymbol{A}_{\bullet k} \;=\; \sum_{j \in \bar{J}} \lambda_j^k \boldsymbol{A}_{\bullet j} \;=\; \left( \sum_{j \in \bar{J}} \boldsymbol{A}_{\bullet j} \Lambda_{j \bullet} \right)_{\bullet k} \;=\; \left(\bar{\boldsymbol{A}} \Lambda\right)_{\bullet k}$$

or, for short

$$\widehat{\boldsymbol{A}} \;=\; \bar{\boldsymbol{A}} \Lambda \;.$$

We conclude that

$$\begin{aligned} \bar{\boldsymbol{u}} \boldsymbol{A} \;&=\; \boldsymbol{c}_{\bar{J}} \bar{\boldsymbol{A}}^{-1} (\bar{\boldsymbol{A}}, \widehat{\boldsymbol{A}}) \;=\; \boldsymbol{c}_{\bar{J}} \bar{\boldsymbol{A}}^{-1} (\bar{\boldsymbol{A}}, \bar{\boldsymbol{A}} \Lambda) \\ &=\; \boldsymbol{c}_{\bar{J}} (\boldsymbol{I}_m, \Lambda) \;=\; \boldsymbol{z} \;\geq\; \boldsymbol{c}, \end{aligned} \tag{18}$$

which shows that $\bar{\boldsymbol{u}}$ as defined in (17) is feasible for the dual program $\widehat{(\boldsymbol{II})}$. Moreover, we have

$$\bar{\boldsymbol{u}} \boldsymbol{b} \;=\; \left( \boldsymbol{c}_{\bar{J}} \bar{\boldsymbol{A}}^{-1} \right) \boldsymbol{b} \;=\; \boldsymbol{c}_{\bar{J}} \bar{\boldsymbol{x}} \;=\; \boldsymbol{c} \bar{\boldsymbol{x}} \;, \tag{19}$$

which shows that $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{x}})$ constitute a dual pair of optimal solutions,

**q.e.d.**

# Chapter 4

# Games and Equilibria

We now leave the topic of Linear Programming and focus our attention to the theory of Bimatrix Game. This topic provides an introduction to Non-cooperative Game Theory. Within this context we discuss decision problems for *more* than just one person, that is, *multipersonal decision problems* or *games*. There is an abundance of new concepts and ideas that distinguishes Game Theory proper from Optimization or, for that matter, Bimatrix Games from Linear Programming.

# 1 Bimatrix Games

We start out with a general definition of a noncooperative N-person game in normal form. The procedure is similar to the one presented in Section 1 of Chapter 3 with respect to the topic of a linear program. We have to provide three definitions. The first one is dealing with the game as a well defined mathematical object. The second one will refer to the feasible set which, in this case, is the set of strategies. The third one concerns the solution concept. Within the framework of Linear Programming we are aiming at optimal solutions. Within the framework of Game Theory the solution concept is given by the idea of the Nash equilibrium.

**Definition 1.1.** *A noncooperative **N-person game** (in normal form) is given by a $2N-$tuple*

$$(1) \qquad \Gamma = (S_1, \ldots, S_N; F_1, \ldots, F_N),$$

*such that $S_1, \ldots, S_N$ are arbitrary sets and*

$$(2) \qquad F_k : S_1 \times \ldots \times S_N \longrightarrow \mathbb{R} \quad (k = 1, \ldots, N)$$

*is a real valued function. The set $S_k \quad (k = 1, \ldots, N)$ is called the **strategy space** of player $k$ and the function $F_k \quad (k = 1, \ldots, N)$ is called player $k's$ **payoff function**.*

At this stage the term "strategy" has no serious foundation, a strategy is a mere index from a certain set. Each player has the choices available listed in his strategy set.

The intuitive idea is that players draw a strategy more or less simultaneously and, given a strategy n-tupel $s \in S = S_1 \times \ldots \times S_N$, player $k$ receives the payoff $F_k(s)$. We assume that no player has any information concerning the choice of his opponents. They may submit their strategy to a referee, independently on the choice of everybody else or they may introduce the strategy into some mechanism or black box which evaluates the payoff for all players.

We do not rule out that there is communication before the choice of strategies. Players may discuss the situation, suggest appropriate choices or threat another player with hurting him by some uncomfortable choice. However, no commitment is possible and whatever the promises for good conduct or the threats with evil consequences might have been, none of these can be enforced when the actual choice of strategy takes place.

In the first approach, the game is played just once, we are dealing with the "one shot game". Repeated play will be discussed in a larger framework, however, it changes the strategic possibilities, it may allow for threats or punishment. These intricacies will omitted in our first and introductory presentation.

This is the general version for the $N-$person game. Within our present context we focus on two particular elementary types of games.

**Definition 1.2.** *Let* $I = \{1, \ldots, m\}$ *and* $J = \{1, \ldots, n\}$ *and let*

$$\boldsymbol{A} : I \times J \longrightarrow \mathbb{R} \text{ and } \boldsymbol{B} : I \times J \longrightarrow \mathbb{R} \tag{3}$$

*be functions or matrices defines on* $I \times J$. *Then*

$$\Gamma_0 = (I, J; \boldsymbol{A}, \boldsymbol{B}) \tag{4}$$

*is called a* ***bimatrix game*** *('in pure strategies'). The rows (or rather the elements of* $I$*) are the* ***stratetgies of player*** **1** *and the columns (respectively, the elements of* $J$*) are called the* ***strategies of player*** **2**.

A bimatrix game is a two-person noncooperative game with finite strategy spaces. The interpretation of the model as presented above obviously specifies as follows: player 1 chooses a row and player 2 chooses a column. By these choices a unique element in both the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is definied; the first one specifies the payoff to player 1, the second one to player 2.

Bimatrix games are the raw material of elementary noncooperative theory. Supposedly they represent rudimentary strategic situations in which the players face decision problems the outcome of which depends on one's own choices and those of an opponent.

There is a host of such elementary examples. The stories told in the context of such an example are rather elaborate and maybe too fancy. However, these stories are most useful in keeping in mind certain paradigms of basic strategic multipersonal decision problems.

**Example 1.3.** 1. ***The Prisoners Dilemma.*** Let $m = n = 2$ and hence $I = J = \{1, 2\}$. Define the matrices

$$\boldsymbol{A} = \begin{pmatrix} 3 & 0 \\ 5 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} 3 & 5 \\ 0 & 1 \end{pmatrix} , \tag{5}$$

then $\Gamma = (I, J; \boldsymbol{A}, \boldsymbol{B})$ is called the prisoners dilemma.

The story behind this example is as follows. Two tramps are being cought by the sheriff and there is some but not sufficient evidence that they committed a theft in the neighborhood. Both are locked in jail and the attorney of state offers both to act as a witness against the other one (in which case there will be no penalty at all for the witness and a heavy one for the other guy). On the other hand they agreed in advance not to confess anything - in which case there is no sufficient proof and a mild penalty for trespassing will apply only. Hence, each player has two strategies: To cooperate with his buddy (first row and first column) or to defect and confess to the attorney of state (second row and second column). Cooperation between the two convicts yields a rather good payoff of three units, but there is a strong drive for each player to deviate and collect five units under the assumption that his opponent will stick to the agreement. The danger of being betrayed and receiving zero payment may cause both players to confess and end up with a payoff of one unit for each.

2. ***Chicken.*** Let $m = n = 2$ and hence $I = J = \{1, 2\}$. Define the matrices

$$\boldsymbol{A} = \begin{pmatrix} 3 & 2 \\ 5 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} 3 & 5 \\ 2 & 1 \end{pmatrix} , \tag{6}$$

then $\Gamma = (I, J; \boldsymbol{A}, \boldsymbol{B})$ is called the chicken game.

This game represents the dubious game that may have been played by youngsters with old cars in a lonely spot. Two drivers are racing towards each other, if someone swerves, the other one is considered a hero and obtains considerable social prestige. If both stick to the final goal, the game results in desaster. Each player has two strategies: To eventually swerve or to hold on and risk a crash. Swerving is the first strategy and to hold on is the second strategy for each player. To risk a crash yields a high payoff of five units if the opponent 'chickens out'. But if both players decide to stick to the second strategy, then the outcome is the desastrous one.

3. ***2 Finger Morra.*** Let $m = n = 2$ and hence $I = J = \{1, 2\}$. Define the matrices

$$\boldsymbol{A} = \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix} , \tag{7}$$

then $\Gamma = (I, J; \boldsymbol{A}, \boldsymbol{B})$ is called a 2-finger morra.

This is a game played in Sicily. Each player can either show one finger (first strategy) or two fingers (second strategy). If the total number of fingers is odd (which is necessarily 3), then player 1 receives the number of fingers as payment from player 2. If the number of fingers is even player 2 receives the same payment from player 1.

Note that in this case $\boldsymbol{B} = -\boldsymbol{A}$ or $\boldsymbol{A} + \boldsymbol{B} = 0$ is the case, we speak of a ***zero sum game***. The sum of the payoffs in always zero; this we can interprete to the extent that one player pays what his opponent receives and vice versa. This kind of game reflects a direct conflict of interests.

4. ***The Battle of Sexes.*** Let $m = n = 2$ and hence $I = J = \{1, 2\}$. Define the matrices

$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} , \tag{8}$$

then $\Gamma = (I, J; \boldsymbol{A}, \boldsymbol{B})$ is called the battle of sexes.

This game is by no means martial but represents an almost cooperative situation in which, however, both players slightly prefer different outcomes. There are two choices for a married couple: To either go to the boxing event ($1^{st}$ row and $1^{st}$ column) or to the opera ($2^{nd}$ row and $2^{nd}$ column). In the traditional version the male prefers the boxing event and the female would like to go to the opera. But none of them is in favor of going without his or her partner. Therefore if they agree on the first strategy (first row and column, i.e. the boxing event) then player 1 receives two units and player 2 receives one unit. If they both decide in favor of the opera player 2 receives the better payoff (second row and second column). If they fail to coordinate their choices they both get a zero payoff.

This type of game can at once be generalized (Box, Bar, Dinner, Opera)

$$\boldsymbol{A} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \tag{9}$$

5. ***Stone, Scissors, Paper.*** Let $m = n = 3$ and hence $I = J = \{1, 2, 3\}$. Define the matrices

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, \tag{10}$$

then $\Gamma = (I, J; \boldsymbol{A}, \boldsymbol{B})$ is called the stone-scissors-and-paper game.

Here the interpretation is obvious. The three strategies for each of the players are labelled stone, scissors, paper and the payoff to player 1 is according to whether stone hits scissors (–1) etc.

**Example 1.4.** Consider the game ***Safe and Risky*** which is again defined by $m = n = 2$ and by matrices

$$(11) \qquad \boldsymbol{A} = \begin{pmatrix} 9 & 0 \\ x+1 & x \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} 9 & x+1 \\ 0 & x \end{pmatrix} ,$$

for real $x$ with $0 < x < 8$. We may choose $x = 7$ obtaining

$$(12) \qquad \boldsymbol{A} = \begin{pmatrix} 9 & 0 \\ 8 & 7 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} 9 & 8 \\ 0 & 7 \end{pmatrix} .$$

Or, for the sake of the following argument, let us choose $x$ close to 8, with $x < 8$. Then it seems rather obvious that players 1 and 2 will someway or other *coordinate* their choice of strategies. The goal should be that both receive a payoff of 9.

However, after some consideration both players may note that it is a safe option to use the second row and second column since a payoff of 7 is guaranteed and a slightly better payoff results if the opponent sticks to the coordinating strategy..

On the other hand if one sticks to the coordination strategy aiming for 9 units the slight danger of the opponent playing the safe strategy is a little bit disturbing. Suppose he prefers to play save. Then all of the sudden one is held down to zero units. And, given this considerationd should one expect that the opponent has the same doubts? Should I expect that my opponent believes me to be sufficiently considerate to expect him to deviate? Eventually it may not be safe to take chances by playing the profitable first strategy.

The following definition introduces new strategy spaces for both players. We imagine that they extend their strategic possibilities in a particular way which is basic to game theoretical considerations. The procedure is to introduce mixed strategies. A mixed strategy is a probability distribution over the original strategy space; in this case a mixed strategy for player 1 is a probability over the rows and a mixed strategy for player 2 is a probability over the columns. The corresponding payoffs are then obtained by the expectation of the original payoffs taken with respect to the product probability over the rows and columns.

Intuitively both players employ a random device or lottery in order to choose a pure strategy. The motivation for this procedure may be given in various ways. For instance in the 2-finger morra, if played repeatedly, a player will change his choice of one or two fingers. He may be afraid of this opponent detecting a pattern in his behavior. Therefore there is an incentive to delegate the choice of one or two fingers to a random device (a coin, a dice, a lottery ...). This way a lottery is performed each time which provides no clue for the opponent to detect a pattern of behavior.

Of course the strategic choices of a player employing a random device are not deminished. On the contrary, he has now the choice of selecting the probability distribution governing his experiment. In the 2-finger morra the question arises with which kind of probabilities one or two fingers should be chosen.

Note also that we assume the random devices to be independent in the technical sense. This is reflected by employing the product probability when computing the expected payoff.

Here is the formal definition.

**Definition 1.5.** *Let $m, n$ be integers and let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $m \times n-$matrices. Define*

$$\underline{\overline{\mathbf{X}}} \;:=\; \underline{\overline{\mathbf{X}}}^m \;=\; \left\{ x \in \mathbb{R}^m \mid x \geq 0, \sum_{i \in I} x_i = 1 \right\} \tag{13}$$

*and*

$$\underline{\overline{\mathbf{Y}}} \;:=\; \underline{\overline{\mathbf{Y}}}^n \;=\; \left\{ y \in \mathbb{R}^n \mid y \geq 0, \sum_{j \in J} y_j = 1 \right\}. \tag{14}$$

*Extend the matrices to the effect that they define functions*

$$\begin{aligned} \boldsymbol{A} \;&:\; \underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}} \longrightarrow \mathbb{R} \\ \boldsymbol{A}(\boldsymbol{x}, \boldsymbol{y}) \;&=\; \boldsymbol{xAy} \;:=\; \sum_{i \in I, j \in J} x_i a_{ij} y_j \end{aligned} \tag{15}$$

*and*

$$\begin{aligned} \boldsymbol{B} &: \underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}} \longrightarrow \mathbb{R} \\ \boldsymbol{B}(\boldsymbol{x},\boldsymbol{y}) &= \boldsymbol{xBy} := \sum_{i\in I, j\in J} x_i a_{ij} y_j \ , \end{aligned} \tag{16}$$

*then*

$$\Gamma = (\underline{\overline{\mathbf{X}}}, \underline{\overline{\mathbf{Y}}}; \boldsymbol{A}, \boldsymbol{B}) \tag{17}$$

*is a* ***bimatrix game in mixed strategies.*** $\underline{\overline{\mathbf{X}}}$ *is the space of player 1's mixed strategies and similarly,* $\underline{\overline{\mathbf{Y}}}$ *is the space of player 2's mixed strategies. Frequently* $\Gamma$ *is referred to as to the* ***mixed extension*** *of* $\Gamma_0 = (I, J; \boldsymbol{A}, \boldsymbol{B})$.

The mixed extension again is a noncooperative $N-$person game in the sense of Definition 1.1. The game is being played exactly as the pure game: each player choses a mixed strategy and each player receives the payoff resulting from the joined choice in terms of expectation. Note that we can also consider this to be the second step of the program mentioned initially: We have now defined the equivalent of the feasible set, that is all possible choices of strategies. As yet we have not defined the solution concept for a game.

**Remark 1.6.** *The unit vectors* $\boldsymbol{e}^i \in \mathbb{R}^m$ *and* $\boldsymbol{e}^j \in \mathbb{R}^n$ *define particular mixed strategies: they reflect a lottery choosing* $i \in I$ *and* $j \in J$ *with probability* 1 *respectively. The corresponding payoff e.g. to player* 1 *is obviously*

$$\boldsymbol{e}^i \boldsymbol{A} \boldsymbol{e}^j = a_{ij} \tag{18}$$

*which is as well obtained in the pure game* $\Gamma_0$ *by playing* $i \in I$ *and* $j \in J$ *respectively. Analogously, player 2's payoffs* $b_{ij}$ *are available in the mixed extension. This way all possibilities of choice and all payoffs resulting thereoff available in* $\Gamma_0$ *are available in* $\Gamma$ *as well. Losely speaking we have described an* ***embedding*** *of the pure game into the mixed one.*

*Note that the payoffs if one player plays pure and the other mixed can be written*

$$\boldsymbol{e}^i \boldsymbol{A} \boldsymbol{y} = \boldsymbol{A}_{i\bullet} \boldsymbol{y} \text{ and } \boldsymbol{xBe}^j = \boldsymbol{xB}_{\bullet j} \quad (i \in I,\ j \in J). \tag{19}$$

The following definition is introduces the basic solution concept for Noncooperative Game Theory.

**Definition 1.7.** *Let* $\Gamma = (S_1, \ldots, S_N; F_1, \ldots, F_N)$ *be a noncooperative N-person game. A strategy N-tuple* $\bar{s} = (\bar{s}^1, \ldots, \bar{s}^N) \in S := S^1 \times \cdots \times S^N$

*is called a* ***Nash equilibrium*** *if, for all* $k \in \{1, \ldots, N\}$ *and all* $s^k \in S^k$ *the inequality*

$$F^k(\bar{s}^1, \ldots, \bar{s}^k, \ldots, \bar{s}^N) \geq F^k(\bar{s}^1, \ldots, s^k, \ldots, \bar{s}^N) \tag{20}$$

*holds true.*

There are many interpretations assigned to the concept of Nash equilibrium. The most obvious one is that of a stable situation: if at some stage (of preplay discussion) players find themselves in the situation of a Nash equilibrium, then it is not profitable for anyone to deviate provided the opponents stick to the equilibrium in question. Another interpretation is that a player estimates, guesses, or beliefs that his opponents will play according to the $N-1$tuple represented and optimizes his payoff given this belief. Then, if all players believe consistently in the actions of their opponents an equilibrium is at hand.

The notion of equilibrium turns out to be much more involved compared to the simple idea of optimum or maximum which is persued in linear programming or, more generally, in optimization. It must be stressed that the idea of optimum is not applicable to the context of multi-personal decisions. A good decision for one player facing a certain strategy of his opponents might be a bad one if they change their behavior. To persue some appealing payoff regardless of the other players interest is obviously folly. One should try to anticipate the opponents behavior, but maybe they will anticipate this and so on. And, in fact, ideas like this are captured by the Nash equilibrium.

**Remark 1.8.** *For two players the inequalities defining a Nash equilibrium*

$$\bar{s} = (\bar{s}^1, \bar{s}^2) \in S = S^1 \times S^2$$

*are written as follows:*

$$\begin{aligned} F^1(\bar{s}^1, \bar{s}^2) &\geq F^1(s^1, \bar{s}^2) \quad (s^1 \in S^1) \\ F^2(\bar{s}^1, \bar{s}^2) &\geq F^2(\bar{s}^1, s^2) \quad (s^2 \in S^2). \end{aligned} \tag{21}$$

*Thus, we imagine the two payoff functions such that, whenever one coordinate is fixed at the equilibrium value, then with respect to the other one the corresponding payoff function yields a maximum at the equilibrium payoff.*

Let us focus on bimatrix games. Then the payoff function is given by an $m \times n$ matrix. We consider the pure game and the mixed game and just write down the definition of equilibrium as follows.

**Corollary 1.9.** *1. Let $\Gamma_0 = (I, J; \boldsymbol{A}, \boldsymbol{B})$ be a bimatrix game in pure strategies. Then a Nash equilibrium is pair $(\bar{\imath}, \bar{\jmath}) \in I \times J$ indicating a row and a column such that the inequalities*

$$a_{\bar{\imath}\bar{\jmath}} \geq a_{i\bar{\jmath}} \quad (i \in I), \quad b_{\bar{\imath}\bar{\jmath}} \geq b_{\bar{\imath}j} \quad (j \in J) \tag{22}$$

*are satisfied. That is, row $\bar{\imath}$ maximizes the payoff elements in matrix $\boldsymbol{A}$ when column $\bar{\jmath}$ is fixed and vice versa for $\boldsymbol{B}$.*

*2. Let $\Gamma = (\underline{\overline{\mathbf{X}}}, \underline{\overline{\mathbf{Y}}}; \boldsymbol{A}, \boldsymbol{B})$ be a bimatrix game in mixed strategies. Then a Nash equilibrium is a pair of vectors (probabilities, mixed strategies) $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}}$ such that the inequalities*

$$\begin{aligned} \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} &\geq \boldsymbol{x}\boldsymbol{A}\bar{\boldsymbol{y}} \quad (\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}) \\ \bar{\boldsymbol{x}}\boldsymbol{B}\bar{\boldsymbol{y}} &\geq \bar{\boldsymbol{x}}\boldsymbol{B}\boldsymbol{y} \quad (\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}) \end{aligned} \tag{23}$$

*hold true.*

**Example 1.10.** Let us return to the examples collected in 1.3. Within the 'prisoners dilemma' it is easily seen that there is a Nash equilibrium indicated by $(\bar{\imath}, \bar{\jmath}) = (2, 2)$ (the second row and column), which yields a payoff of 1 to each player.

An important message is conveyed by this simple example. Not only is the Nash equilibrium no immediate relative of 'optimal behavior' of any kind but on the contrary it may lead to a rather unsatisfactory stable situation. In this example both of our tramps would prefer to keep their contract and not make a confession to the attorney of state. What they might prefer even more is to act as a witness and let the other guy keep the contract. None of these situations is stable and this way they both end up in confessing.

The 'battle of sexes' has two equilibria, written in terms of rows and columns $(1, 1)$ and $(2, 2)$.

Here is another important message. Clearly, equilibrium is by no means unique. Moreover, both equilibria provide different payoffs to each player. Clearly, player 1 would prefer the first one while player 2 would prefer the second one. Hence we recognize that the situation is basically diferent from the one encountered in optimization problems (e.g., in Linear Programming): equilibria are not unique and their 'worth' to various players is at variance. As a consequence, there is no of maximizing the payoff or finding "optimal solutions". In "optimization" problems a maximizer might not be unique, but at least the maximum is (if it exists). Thus the "value of a Linear Program" is a well defined concept.

Not so with a game. Both equilibria in the "battle of sexes" example are rather appealing. They represent the success of coordination and the game has a certain drift towards cooperative behavior quite in contrast to the prisoner's dilemma. However, equilibrium is not uniquely defined, optimal payoff is not a suitable concept, and there are situations which yield symmetric yet different equilibrium payoffs in such a disturbing degree that selecting an equilibrium seems to be impossible.

As it turns out, the 'battle of sexes' in addition has an equilibrium in mixed strategies that does not occur in the pure version. Consider the pair of mixed strategies given by

$$\bar{\boldsymbol{x}} = (\frac{2}{3}, \frac{1}{3}) \quad \bar{\boldsymbol{y}} = (\frac{1}{3}, \frac{2}{3}). \tag{24}$$

It is not hard to see that

$$\boldsymbol{A}_{1\bullet}\bar{\boldsymbol{y}} = \boldsymbol{A}_{2\bullet}\bar{\boldsymbol{y}} = \frac{2}{3}$$

as well as

$$\bar{\boldsymbol{x}}\boldsymbol{B}_{\bullet 1} = \bar{\boldsymbol{x}}\boldsymbol{B}_{\bullet 2} = \frac{2}{3}$$

holds true. From this it follows that we have

$$\boldsymbol{x}\boldsymbol{A}\bar{\boldsymbol{y}} = x_1\boldsymbol{A}_{1\bullet}\bar{\boldsymbol{y}} + x_2\boldsymbol{A}_{2\bullet}\bar{\boldsymbol{y}} = (x_1 + x_2)\frac{2}{3} = \frac{2}{3} = \ldots = \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}}.$$

The analogous relation is true for the corresponding payoffs of player 2, hence $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ is an equilibrium.

A third message is obtained from the considration of other types of games. There is a host of examples that yield no Nash equilibrium in pure strategies at all. E.g., the 'stone,scissors,paper' example is obviously of this kind. On the other hand, the 'stone,scissors,paper' example does have an equilibrium in mixed stratgies. Indeed, the pair of mixed strategies $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) = ((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}))$ is easily seen to satisfy

$$\boldsymbol{A}_{1\bullet}\bar{\boldsymbol{y}} = \boldsymbol{A}_{2\bullet}\bar{\boldsymbol{y}} = \boldsymbol{A}_{3\bullet}\bar{\boldsymbol{y}} = 0.$$

Therefore we have

$$\boldsymbol{x}\boldsymbol{A}\bar{\boldsymbol{y}} = \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} = 0 \quad (x \in \underline{\overline{\mathbf{X}}}) .$$

by the same reasoning as above. Again the analogue for $\boldsymbol{B}$ is obvious, hence we have an equilibrium at hand.

With respect to the mixed extension,the situation is indeed far more advantageous. Equilibria do always exist and the equilibrium concept eventually turns out to be a most fruitfull one. In order to elaborate on this statement, we will start out with a simple Lemma. It shows that it suffices to verify a finite number of inequalities in order to check for the equilibrium property.

**Lemma 1.11 (The Finite Test).** *Let* $\Gamma = (\underline{\overline{\mathbf{X}}}, \underline{\overline{\mathbf{Y}}}; \boldsymbol{A}, \boldsymbol{B})$ *be a bimatrix game in mixed strategies. A pair of mixed strategies* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ *is a Nash equilibrium if and only if the following inequalities hold true:*

$$(25)\qquad \begin{aligned} \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} &\geq \boldsymbol{A}_{i\bullet}\bar{\boldsymbol{y}} \quad (i \in I), \\ \bar{\boldsymbol{x}}\boldsymbol{B}\bar{\boldsymbol{y}} &\geq \bar{\boldsymbol{x}}\boldsymbol{B}_{\bullet j} \quad (j \in J). \end{aligned}$$

*Thus, it suffices to check* $m+n$ *inequalities in order to make sure that a given pair constitutes an equilibrium.*

**Proof:** Because of

$$\boldsymbol{A}_{i\bullet}\bar{\boldsymbol{y}} = \boldsymbol{e}^i\boldsymbol{A}\bar{\boldsymbol{y}} \quad (i \in I), \quad \bar{\boldsymbol{x}}\boldsymbol{B}_{\bullet j} = \bar{\boldsymbol{x}}\boldsymbol{B}\boldsymbol{e}^j \quad (j \in J)$$

the inequalities (25) are certainly satisfied by any equilibrium $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$.

Assume on the other hand that (25) holds true, then, for any $x \in \underline{\overline{\mathbf{X}}}$ we find

$$\bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} = \left(\sum_{i\in I} x_i\right)\bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} \geq \sum_{i\in I} x_i\boldsymbol{A}_{i\bullet}\bar{\boldsymbol{y}} = \boldsymbol{x}\boldsymbol{A}\bar{\boldsymbol{y}}$$

and the analogue equation for player 2's payoff. Consequently, $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ is an equilibrium.

**q.e.d.**

**Corollary 1.12.** *1. If* $(\bar{i}, \bar{j})$ *is an equilibrium of the pure game* $\Gamma_0$, *then the corresponding unit vectors* $\boldsymbol{e}^{\bar{i}}$ *and* $\boldsymbol{e}^{\bar{j}}$ *constitute an equilibrium in the mixed extension* $\Gamma$. *Thus, the embedding of the pure game into to the mixed extension does not diminish the set of equilibria.*

*2. For any* $(\boldsymbol{x}, \boldsymbol{y}) \in \underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}}$ *the inequalities*

$$(26)\quad \min_{k\in I} \boldsymbol{A}_{k\bullet}\boldsymbol{y} \leq \min_{k\in I, \bar{x}_k>0} \boldsymbol{A}_{k\bullet}\boldsymbol{y} \leq \boldsymbol{x}\boldsymbol{A}\boldsymbol{y} \leq \max_{k\in I, \bar{x}_k>0} \boldsymbol{A}_{k\bullet}\boldsymbol{y} \leq \max_{k\in I} \boldsymbol{A}_{k\bullet}\boldsymbol{y}$$

*are satisfied, the same holds analogously for the matrix* $\boldsymbol{B}$.

**Proof:** The proof of the *first item* is at once based on (25).

The one of the *second item* follows from the fact that

$$\sum_{i \in I} x_i \boldsymbol{A}_{i\bullet}\boldsymbol{y} \le \left( \sum_{i \in I} x_i \right) \max_{k \in I} \boldsymbol{A}_{k\bullet}\boldsymbol{y}$$

etc. is true. That is, player 1's payoff at any of his mixed strategies against a fixed strategy of his opponent is between his worst and best payoff at pure strategies. Clearly, this results from the fact that the mixture is obtained by a convex combination of all the payoff at pure strategies.

**q.e.d.**

**Definition 1.13.** *A pure strategy $i \in I$ is called a* ***best response*** *against a mixed strategy $\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}$ if it satisfies*

$$\boldsymbol{A}_{i\bullet}\boldsymbol{y} \;=\; \max_{k \in I} \boldsymbol{A}_{k\bullet}\boldsymbol{y} \quad . \tag{27}$$

*An analogous definition holds true for $j \in J$ w.r.t. being a best response against $\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}$ .*

This way, the result of Lemma 1.11 $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ states that a pair of strategies constitutes an equilibrium if and only if the payoff for player 1 at $\bar{\boldsymbol{x}}$ against $\bar{\boldsymbol{y}}$ is as good as at any of his best responses against $\bar{\boldsymbol{y}}$.

Now we have

**Theorem 1.14 (The Optimality Criterion).** *Let $\Gamma = (\underline{\overline{\mathbf{X}}}, \underline{\overline{\mathbf{Y}}}, \boldsymbol{A}, \boldsymbol{B})$ be a bimatrix game. A pair of strategies $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ is a Nash equilibrium if and only if the following criterion holds true:*

$$\begin{aligned} \bar{x}_i > 0 \implies \boldsymbol{A}_{i\bullet}\bar{\boldsymbol{y}} \;&=\; \max_{k \in I} \boldsymbol{A}_{k\bullet}\bar{\boldsymbol{y}} \quad (i \in I) \\ \bar{y}_j > 0 \implies \bar{\boldsymbol{y}}\boldsymbol{B}_{\bullet j} \;&=\; \max_{l \in J} \bar{\boldsymbol{x}}\boldsymbol{B}_{\bullet l} \quad (j \in J). \end{aligned} \tag{28}$$

That is, at equilibrium each player puts positive probability only on best responses against the opponents mixed strategy.

**Proof:** **1$^{\text{st}}$STEP** : Let $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ be an equilibrium, then, using Lemma 1.11, we obtain

$$\begin{aligned} \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} \;&= \max_{k \in I} \boldsymbol{A}_{k\bullet}\bar{\boldsymbol{y}} = \left( \sum_{i \in I, \bar{x}_i > 0} \bar{x}_i \right) \max_{k \in I} \boldsymbol{A}_{k\bullet}\bar{\boldsymbol{y}} \\ &\ge \sum_{i \in I, \bar{x}_i > 0} \bar{x}_i \boldsymbol{A}_{i\bullet}\bar{\boldsymbol{y}} \;=\; \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}}. \end{aligned} \tag{29}$$

Non of the inequalities involved can be a strict one, this proves the first line of (28). The analogous version is true for the matrix $\boldsymbol{B}$, this way we realize that the criterion (28) is satisfied.

**2$^{\text{nd}}$STEP** : On the other hand, assume that (28) is satisfied for some pair of strategies $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$. Then, for any $i \in I$, we have

$$
\begin{aligned}
\bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} &= \sum_{k\in I, \bar{x}_k > 0} \bar{x}_k \boldsymbol{A}_{k\bullet}\bar{\boldsymbol{y}} \\
&= \left( \sum_{k\in I, \bar{x}_k > 0} \bar{x}_k \right) \max_{k\in I} \boldsymbol{A}_{k\bullet}\bar{\boldsymbol{y}} \\
&= \max_{k\in I} \boldsymbol{A}_{k\bullet}\bar{\boldsymbol{y}} \\
&\geq \boldsymbol{A}_{i\bullet}\bar{\boldsymbol{y}} \quad (i \in I) \ .
\end{aligned} \tag{30}
$$

The analogue holds true for player $2's$ payoff, hence $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ is an equilibrium in view of the 'finite test' Lemma 1.11. **q.e.d.**

**Example 1.15. 1.** Recall the 'stone, scissors, paper' example which was presented in Examples 1.3. The payoff matrix for the first player is

$$
\boldsymbol{A} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} , \tag{31}
$$

and the one for the second player is given by $\boldsymbol{B} = -\boldsymbol{A}$. We argue that the pair of mixed strategies $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) = ((\frac{1}{3}, \frac{1}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}))$ constitutes an equilibrium because

$$
\bar{x}_i > 0 \Longrightarrow \boldsymbol{A}_{i\bullet}\bar{\boldsymbol{y}} = \max_{k\in I} \boldsymbol{A}_{k\bullet}\bar{\boldsymbol{y}} = 0 \quad (i \in I)
$$

holds true and the same is true for the matrix $\boldsymbol{B}$.

**2.** A second example is provided by the matrices

$$
\boldsymbol{A} = \begin{pmatrix} 5 & 3 & -4 & -1 \\ -6 & -3 & 5 & 3 \end{pmatrix} , \quad \boldsymbol{B} = \begin{pmatrix} -1 & -4 & 7 & 11 \\ 3 & 4 & -9 & -19 \end{pmatrix} . \tag{32}
$$

Consider the mixed strategies given by

$$
\bar{\boldsymbol{x}} = (\frac{3}{5}, \frac{2}{5}) \ , \quad \bar{\boldsymbol{y}} = (\frac{9}{20}, 0, \frac{11}{20}, 0). \tag{33}
$$

We want to check that the conditions of the Optimality Criterion are satisfied. Indeed, we have both coordinates of $\bar{\boldsymbol{x}}$ positive, i.e., $\bar{x}_1 > 0,\ \bar{x}_2 > 0$ and the conditions (28), i.e.,

$$\boldsymbol{A}_{1\bullet}\bar{\boldsymbol{y}} = \frac{45}{20} - \frac{44}{20} = \frac{1}{20}$$

$$\boldsymbol{A}_{2\bullet}\bar{\boldsymbol{y}} = -\frac{54}{20} + \frac{55}{20} = \frac{1}{20}$$

are satisfied. On the other hand, the positive coordinates of $\bar{\boldsymbol{y}}$ are $\bar{y}_1 > 0,\ \bar{y}_3 > 0$, hence we have to make sure that

$$\bar{\boldsymbol{x}}\boldsymbol{B}_{\bullet 1} = \bar{\boldsymbol{x}}\boldsymbol{B}_{\bullet 3} \geq \bar{\boldsymbol{x}}\boldsymbol{B}_{\bullet i} \quad (i \in I)$$

is satisfied. The data are

$$\begin{array}{lcl} \bar{\boldsymbol{x}}\boldsymbol{B}_{\bullet 1} = \frac{3}{5} & , & \bar{\boldsymbol{x}}\boldsymbol{B}_{\bullet 2} = -\frac{4}{5} \\ \bar{\boldsymbol{x}}\boldsymbol{B}_{\bullet 3} = \frac{3}{5} & , & \bar{\boldsymbol{x}}\boldsymbol{B}_{\bullet 4} = -\frac{5}{5} \end{array},$$

which verifies that $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ is indeed a Nash equilibium. The payoffs at this equilibrium are

$$\bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} = \frac{1}{20}, \quad \bar{\boldsymbol{x}}\boldsymbol{B}\bar{\boldsymbol{y}} = \frac{3}{5} .$$

**Theorem 1.16 (Nash's Theorem).** *Let $\Gamma = (\underline{\mathbf{X}}, \underline{\mathbf{Y}}, \boldsymbol{A}, \boldsymbol{B})$ be a mixed bimatrix game. Then $\Gamma$ has a Nash equilibrium.*

**Proof: 1$^{\text{st}}$STEP** : Within the context of this proof we will heavily depend on an important result of topological nature, the ***Brouwer Fixed Point Theorem***. Within the territory of Mathematical Economics this theorem plays a role as important as the separation theorems that has been discussed in the first chapter. Existence theorems of all kind of equilibria depend on this (or the related *Kakutani-*) fixed point theorem.

As a proof is beyond the scope of this volume we shall only cite the appropriate version:

**Theorem 1.17 (Brouwer's Fixed Point Theorem).**
*Let $C \subseteq \mathbb{R}^n$ be a compact and convex set. A continuous function $f : C \to C$ defined on $C$ and ranging within this set has a fixed point. That is, $f$ admits of $z \in C$ such that $f(z) = z$ holds true.*

There is also a short piece of notation we shall use within the present proof: we write $^+$ in order to indicate the *positive part* of a real number, i.e., we write $\alpha^+ := \max\{\alpha, 0\}$.

**2ndSTEP** :

In order to apply this theorem to our present problem, i.e., the existence of Nash equilibria, we consider the following function

$$\begin{aligned} f \quad &: \overline{\underline{\mathbf{X}}} \times \overline{\underline{\mathbf{Y}}} \quad \to \overline{\underline{\mathbf{X}}} \times \overline{\underline{\mathbf{Y}}} \\ f(\boldsymbol{x}, \boldsymbol{y}) &= (\boldsymbol{x}', \boldsymbol{y}') \end{aligned}$$

(34)
$$\begin{aligned} x'_i &= \frac{x_i + (\boldsymbol{A}_{i\bullet}\boldsymbol{y} - \boldsymbol{x}\boldsymbol{A}\boldsymbol{y})^+}{1 + \sum_{k\in I}(\boldsymbol{A}_{k\bullet}\boldsymbol{y} - \boldsymbol{x}\boldsymbol{A}\boldsymbol{y})^+} \quad (i \in I), \\ y'_j &= \frac{y_j + (\boldsymbol{x}\boldsymbol{B}_{\bullet j} - \boldsymbol{x}\boldsymbol{B}\boldsymbol{y})^+}{1 + \sum_{l\in J}(\boldsymbol{x}\boldsymbol{B}_{\bullet l} - \boldsymbol{x}\boldsymbol{B}\boldsymbol{y})^+} \quad (j \in J). \end{aligned}$$

Clearly, the set $\overline{\underline{\mathbf{X}}} \times \overline{\underline{\mathbf{Y}}}$ is convex and compact, the fact that $f$ maps this set into itself is obvious by the construction. Also, all ingredients used to build $f$ can be seen to reflect continuous functions, hence $f$ is continuous and satisfies all the requirement of Brouwer's Fixed Point Theorem.

Note that $f$ admits also a nice interpretation: given $\boldsymbol{x}$ and $\boldsymbol{y}$ we consider the transition from $\boldsymbol{x}$ to $\boldsymbol{x}'$. If $\boldsymbol{x}$ is not an equilibrium, then some responses $i$ are better then $\boldsymbol{x}$, in particular those $i \in I$ that constitute best responses. We put more weight on these best (or good) responses, hoping that eventually the $(\ )^+$-term on the enumerator will vanish – in which case we expect an equilibrium.

Note that any equilibrium is easily verified to constitute a fixed point of $f$ via the Finite Test Lemma 1.11. What we want to verify is, of course, the reverse statement. Indeed, this is the content of the formal development to follow.

**3rdSTEP** : Now, let $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ a fixed point of the mapping $f$ to be obtained by Brouwer's Theorem. We know that

(35)
$$f(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) = (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$$

holds true. Let us insert this into the definition, that is, into (34). Exploiting for the moment the first part only, we obtain

$$\bar{x}_i \left(1 + \sum_{k\in I}(\boldsymbol{A}_{k\bullet}\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}})^+\right) = \bar{x}_i + (\boldsymbol{A}_{i\bullet}\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}})^+ \quad (i \in I),$$

or

$$(36) \qquad \bar{x}_i \left( \sum_{k \in I} (\boldsymbol{A}_{k\bullet}\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}})^+ \right) = (\boldsymbol{A}_{i\bullet}\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}})^+ \quad (i \in I).$$

Now, in view of the second *item* in Corollary 1.12 we can find $i_0 \in I$ such that $x_{i_0} > 0$ and

$$\boldsymbol{A}_{i_0}\bar{\boldsymbol{y}} \leq \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}}$$

holds true. Obviously this implies

$$(37) \qquad (\boldsymbol{A}_{i_0}\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}})^+ = 0$$

Therefore, evaluating (36) for $i = i_0$ and observing $\bar{x}_{i_0} > 0$ and (37), we obtain

$$(38) \qquad \sum_{k \in I} (\boldsymbol{A}_{k\bullet}\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}})^+ = 0.$$

This can only happen when all terms under the summation sign do indeed vanish, that is, we have

$$(39) \qquad \boldsymbol{A}_{i\bullet}\bar{\boldsymbol{y}} \leq \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} \quad (i \in I) .$$

The analogue for the matrix $\boldsymbol{B}$ (derived from the second part of the defining equations in (34)) reads of course

$$(40) \qquad \bar{\boldsymbol{x}}\boldsymbol{B}_{\bullet j} \leq \bar{\boldsymbol{x}}\boldsymbol{B}\bar{\boldsymbol{y}} \quad (j \in J) .$$

Both equations show that $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ is an equilibrium by means of the Finite Test Lemma 1.11,

**q.e.d.**

The Brouwer Fixed Point Theorem is not constructive, it provides an existence theorem for equilibria but no direct procedure to actually compute an equilibrium or all equilibria. There are some immediate observations resting on the Optimality Criterion which constitute a crude method for the computation of equilibria. However, on the long run, we will have to deal with a more involved algorithm.

Let us first discuss some easy conclusions.

**Lemma 1.18.** *Let $\Gamma = (\underline{\overline{\mathbf{X}}}, \underline{\overline{\mathbf{Y}}}; \boldsymbol{A}, \boldsymbol{B})$ be a bimatrix game in mixed strategies and let $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ be an equilibrium. Then there exists index sets $\bar{I} \subseteq I$ and $\bar{J} \subseteq J$ as well as real numbers $\bar{\lambda}$ and $\bar{\mu}$ such that*

$$(41) \qquad (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{\lambda}, \bar{\mu})$$

*is a solution of the linear system of equations* $m+n+2$ *in variables* $(\boldsymbol{x}, \boldsymbol{y}, \lambda, \mu)$ *given by*

$$\begin{array}{rcll} \boldsymbol{A}_{i\bullet}\boldsymbol{y} - \lambda &=& 0 & i \in \bar{I} \\ y_j &=& 0 & j \in \bar{J}^c \\ \sum_{j\in\bar{J}} y_j &=& 1 & \\ & & & \\ \boldsymbol{x}\boldsymbol{B}_{\bullet j} - \mu &=& 0 & j \in \bar{J} \\ x_i &=& 0 & i \in \bar{I}^c \\ \sum_{i\in\bar{I}} x_i &=& 1 & . \end{array} \tag{42}$$

**Proof:** Define then index set $\bar{I}$ by

$$\bar{I} := \{i \in I \mid \bar{x}_i > 0\} \tag{43}$$

and let $\bar{\lambda}$ be given by

$$\bar{\lambda} := \max_{k\in I} \boldsymbol{A}_{k\bullet}\bar{\boldsymbol{y}}. \tag{44}$$

Then, in view of the Optimality Criterion, we have

$$i \in \bar{I} \Rightarrow \bar{x}_i > 0 \Rightarrow \boldsymbol{A}_{i\bullet}\bar{\boldsymbol{y}} = \bar{\lambda}. \tag{45}$$

Hence $\bar{\boldsymbol{y}}$ satisfies the first group of equations (42). The analogous argument yields the same statement with respect to $\bar{\boldsymbol{x}}$,

**q.e.d.**

**Remark 1.19.** *Given a bimatrix game in mixed strategies* $\Gamma = (\underline{\overline{\mathbf{X}}}, \underline{\overline{\mathbf{Y}}}; \boldsymbol{A}, \boldsymbol{B})$ *consider the matrix*

$$\mathcal{A} = \begin{pmatrix} & & & -1 \\ & \boldsymbol{A} & & \vdots \\ & & & -1 \\ 1 & \dots & 1 & 0 \end{pmatrix}. \tag{46}$$

*If* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ *is an equilibrium, then we may consider the linear system in variables* $(x_i)_{i\in\bar{I}}$ *and* $(y_j)_{j\in\bar{J}}$ *as well as* $\lambda, \mu$ *obtained from (42) by canceling all the zero coordinates, that is the system*

$$\begin{array}{rcll} \boldsymbol{A}_{i\bullet}\boldsymbol{y} - \lambda &=& 0 & i \in \bar{I} \\ \sum_{j\in\bar{J}} y_j &=& 1 & \\ & & & \\ \boldsymbol{x}\boldsymbol{B}_{\bullet j} - \mu &=& 0 & j \in \bar{J} \\ \sum_{i\in\bar{I}} x_i &=& 1 & . \end{array} \tag{47}$$

*The coefficient matrix of the first part of this system is a submatrix of $\mathcal{A}$ corresponding to the row in $\bar{I}$ and the columns in $\bar{J}$ and including the elements of the last row and column. This matrix may be written*

$$\mathcal{A}' = \begin{pmatrix} & & & -1 \\ & \boldsymbol{A}' & & \vdots \\ & & & -1 \\ 1 & \dots & 1 & 0 \end{pmatrix}. \tag{48}$$

*Consequently, defining the suitable matrix $\mathcal{B}'$ in a similar way, we may represent the system* (47) *for short as*

$$\mathcal{A}'(\bar{\boldsymbol{y}}, \lambda) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad , \quad (\bar{\boldsymbol{x}}, \mu)\mathcal{B}' = (0, \cdots, 0, 1) \ . \tag{49}$$

*To every equilibrium there corresponds a pair of submatrices $\boldsymbol{A}', \boldsymbol{B}'$ or rather a suitable pair of submatrices $\mathcal{A}', \mathcal{B}'$ (including the last row and column elements) such that $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \lambda, \mu)$ is a solution of* (49).

*As a consequence, one can specify a primitive method of computing all equilibria. This procedure is a relative of the one presented for the computation of the extremal points of a convex polyhedron as indicated In Remark 2.6.*

*The procedure runs as follows:*

1. *Pick all possible submatrices $\mathcal{A}'$ and $\mathcal{B}'$ (these are finitely many!) and compute all the corresponding solutions of systems* (49) *or* (42) *respectively.*

2. *Check that the solutions $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ obtained are nonnegative vectors.*

3. *Check that the solutions satisfy the Optimality Criterion (Theorem 1.14). I.e., make sure that the values $\bar{\lambda}$, $\bar{\mu}$ given by the solution of* (42) *are the maximal ones among the values $\boldsymbol{A}_{i\bullet}\bar{\boldsymbol{y}}$ and $\bar{\boldsymbol{x}}\boldsymbol{B}_{\bullet j}$.*

*The procedure is most inefficient. Note that the solutions of the linear systems of equations are not necessarily unique.*

**Remark 1.20.** *In a well defined sense, it is sufficient to consider square nondegenerate submatrices of $\mathcal{A}$ and $\mathcal{B}$. More precisely, consider the polyhedra of type*

$$\begin{aligned} \mathcal{H}_{T,U} := \quad & \{(\boldsymbol{y}, \lambda) \in \underline{\overline{\mathbf{Y}}} \times \mathbb{R} \mid \\ & \boldsymbol{A}_{i\bullet}\boldsymbol{y} = \lambda \geq \boldsymbol{A}_{k\bullet}\boldsymbol{y}, \ (i \in T, k \in T^c), \ y_l = 0 \ (l \in U)\} \end{aligned} \tag{50}$$

*for* $T \subseteq I,\ U \subseteq J$ *and*

$$\mathcal{L}_{R,V} := \big\{(\boldsymbol{x},\mu) \in \underline{\overline{\mathbf{X}}} \times \mathbb{R} \,\big|\, \boldsymbol{x}\boldsymbol{B}_{\bullet j} = \mu \geq \boldsymbol{x}\boldsymbol{B}_{\bullet l}\ (j \in R, l \in R^c),\ x_k = 0\ (k \in V)\big\} \tag{51}$$

*for* $R \subseteq J,\ V \subseteq I$. *Similarly to the above construction, it is seen that any equilibrium point is located within some convex compact polyhedron*

$$\mathcal{H}_{T,R^c} \times \mathcal{L}_{R,T^c}. \tag{52}$$

*There are finitely many of these polyhedra (corresponding to the finitely many index sets we can choose). It is sufficient, to compute the* extremal points *of these polyhedra, again, these are finitely many. By a standard argument it is seen that the extremal points correpond to square and nonsingular matrices* $\mathcal{A}'$ *as indicated in* (49) *and the analogous square and nonsingular matrices* $\mathcal{B}'$. *Hence by computing the extremal points and forming suitable (!) convex combinations one obtains all equilibria.*

**Remark 1.21.** *Remark 1.19 and Remark 1.20 provide a (in general very tedious) procedure that yields all the equilibrium points of a bimatrix game. The corresponding algorithm for the case of one player/optimizer – Linear Programming – is indicated in* CHAPTER *1 by Remark 2.6. There is an obvious relationship between both procedures. Nevertheless, a warning is apt at this point: the set of equilibria of a bimatrix game is by no means a convex one (other than the set of feasible solutions or the set of optimal solutions of a linear program).*

*For instance, the example 'Battle of Sexes' as introduced in the 4. item of Example 1.3 has exactly the equilibria*

$$(\boldsymbol{e}^1,\boldsymbol{e}^1)\ ,\ (\boldsymbol{e}^2,\boldsymbol{e}^2)\ ,\ \text{and}\ \ ((\frac{2}{3},\frac{1}{3}),(\frac{1}{3},\frac{2}{3})).$$

In order to indicate the *modus operandi* of the above mentioned procedure, let us discauss a somewhat more extensive example.

**Example 1.22 (The Fish Pond).** This example depicts a general supervision or inspection problem. Player 1, for illustrative purpose, is called the fish thief or poacher and player 2 is the inspector.

The matrices are

$$A = \begin{pmatrix} -1 & a_1 & . & \cdots & . & a_1 \\ a_2 & -1 & a_2 & \cdots & . & a_2 \\ \vdots & & & & & \vdots \\ a_m & a_m & . & \cdots & . & -1 \end{pmatrix}$$

(53) and

$$B = \begin{pmatrix} 1 & 0 & . & \cdots & . & 0 \\ 0 & 1 & 0 & \cdots & . & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & . & \cdots & . & 1 \end{pmatrix}$$

and the interpretation is as follows:

There are $m = n$ fishponds in the neighborhood, the number $a_i$ reflects the expected 'yield' of pond *No.* $i$ when a medium skilled fisherman exercises his abilities at this pond. In the beginning dust of the evening, player 1 advances towards one of these ponds. In order to confuse player 2 and since he likes mixed strategies, he has chosen the pond to fish in for today by tossing a (manipulated, what else to expect from a fish poacher) dice during the long hours of the lazy afternoon.

The inspector, on the other hand, is aware that poaching will take place this evening. Alas, the usual cuts in the budget have deprived him of his hords of deputies, now he is on his own and can just visit one pond (he does not like to extend his hours too much and ponds are not too close in distance...). Fortunately, the fish office features a PC and the inspector, believing that a PC can produce random numbers (an applied view ...) has chosen the pond to visit also at random (in the hectic turmoil of his afternoons office hours).

If the inspector happens to catch the poacher, he obtains a unit payoff (a small reward by the landlord) and the thief looses a unit (again, the laws have become lenient). Otherwise, the inspecter walkes home slightly frustrated (payoff 0), and the poacher enjoys a quiet evening at his pond, receiving an ample reward of $a_i$ in terms of (expected?) fish units.

Now, from the dry viewpoint of the theorist, we follow the primitive procedure as indicated by Remark 1.21 to produce equilibria:

The approach

$$\boldsymbol{A}_{i\bullet}\boldsymbol{y} \;=\; \boldsymbol{A}_{i'\bullet}\boldsymbol{y} \quad (i, i' \in I)$$

leads to

$$\begin{aligned} \boldsymbol{A}_{i\bullet}\boldsymbol{y} &= -y_i + a_i\left(\textstyle\sum_{k\neq i} y_k\right) \\ &= -y_i + a_i(1 - y_i) \\ &= a_i - (1 + a_i)y_i \;:=\; \bar{\lambda} \quad (i \in I) \end{aligned} \tag{54}$$

from which we compute

$$y_i \;=\; \frac{a_i - \bar{\lambda}}{1 + a_i} \quad (i \in I) \tag{55}$$

(the denominator is positive !). The constant $\bar{\lambda}$ is obtained from the requirement

$$\begin{aligned} 1 \;=\; \sum_{i\in I} y_i &= \sum_{i\in I} \frac{a_i}{1+a_i} \;-\; \bar{\lambda}\sum_{i\in I}\frac{1}{1+a_i} \\ &=: \alpha - \bar{\lambda}\beta. \end{aligned} \tag{56}$$

which means

$$\bar{\lambda} \;=\; \frac{\alpha - 1}{\beta}. \tag{57}$$

The quantities

$$\alpha = \sum_{i\in I}\frac{a_i}{1+a_i} \quad \text{and} \quad \beta = \sum_{i\in I}\frac{1}{1+a_i} \tag{58}$$

are given with the data of the game, hence $\bar{\lambda}$ can be determined and $\boldsymbol{y}$ follows with (55).

The corresponding computation with respect to the matrix $\boldsymbol{B}$ is much easier. The unique solution of the linear system of equations

$$\begin{aligned} \boldsymbol{x}\boldsymbol{B}_{\bullet 1} \;=\; \ldots \;&=\; \boldsymbol{x}\boldsymbol{B}_{\bullet n} \\ \sum_{i\in I} x_i \;&=\; 1 \end{aligned} \tag{59}$$

is obviously

$$\bar{\boldsymbol{x}} \;=\; (\frac{1}{n}, \ldots, \frac{1}{n}) \;=\; (\frac{1}{m}, \ldots, \frac{1}{m}). \tag{60}$$

Now we are in a not uncommon situation: we solved a system of equations, hence obtained a candidate for an equilibrium. But solving a certain set of equations is a necessary condition for equilibrium, not a sufficient one. Possibly some additional inequalities are not fulfilled. In the present case, we have to check whether the solutions result in vectors with nonnegative coordinates. Thereafter, it can be verified that $(\boldsymbol{x}, \boldsymbol{y})$ as determined by (55) and (60) are probabilties and satisfy the Optimality Criterion : all coordinates are positive and all responses yield the same payoff.

Now the requirement

$$0 \leq y_i \;=\; \frac{a_i - \bar{\lambda}}{1 + a_i} \quad (i \in I) \tag{61}$$

is obviously satisfied for $a_i \geq \bar{\lambda}$, that is for

$$\beta a_i \geq \alpha - 1 \quad (i \in I),$$

or

$$a_i \sum_{k \in I} \frac{1}{1 + a_k} \geq \sum_{k \in I} \frac{a_k}{1 + a_k} - 1 \quad (i \in I). \tag{62}$$

This equation determines a set of data $a \in \mathbb{R}^m$ for which (55) and (60) provide an equilibrium. This set is notempty. For, if

$$a_i \;=\; c \;=\; const \quad (i \in I)$$

happens to be the case, then we find (62) to be reduced to

$$\frac{nc}{1 + c} \geq \frac{nc}{1 + c} - 1 \quad ,$$

which is obviously true. Morover, we can argue that in, view of this result, (62) is satisfied within a neighborhood of the diagonal of $\mathbb{R}^m$.

A somewhat more elaborate consideration provides us with a larger set of data (included in the neighborhood mentioned above). If

$$|a_i - a_k| \leq \frac{1}{m} \quad (i, k \in I)$$

is true, then we have

$$\sum_{k \in I} \frac{a_k - a_i}{1 + a_k} \leq \sum_{k \in I} \frac{|a_k - a_i|}{1 + a_k} \leq \frac{1}{m} \sum_{k \in I} \frac{1}{1 + a_k} \leq 1 \quad (i \in I),$$

which again implies (62).

We come up with a nice result: if the yield of all fishponds is approxiamately equal, then there is a mixed equilibrium putting positive probability on fishing and watching at each pond.

At this equilibrium, the violator will visit each pond with equal probability (oh yes, no manipulating of coins necessary), while the inspector has to carefully choose his probabilities by using (55) (fortunately, he can use his PC. Unfortunately, the generation of random numbers by a PC, dubious a procedure as it is, is not discussed within this volume).

The payoffs can be computed at once: for player 2 as a civil servant, it is immediately seen that his payoff at the equilibrium at hand is given by

$$\text{(63)} \qquad \bar{\boldsymbol{x}}\boldsymbol{B}\bar{\boldsymbol{y}} = \bar{\boldsymbol{x}}\boldsymbol{B}_{\bullet j} = \frac{1}{n} \quad (j \in J) \ .$$

Hence the inspector, having a small chance of catching the violator, gets a small $\frac{1}{n}$ in expectation (perhaps slightly baffling as he was so busy with his PC).

For player 1 (who invested little in his computations, unless you believe that he had to do all the computations of the inspector as well), it follows from (54) and (57) that his payoff is

$$\text{(64)} \qquad \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} = \boldsymbol{A}_{i\bullet}\boldsymbol{y} = \bar{\lambda} = \frac{\alpha - 1}{\beta}. \quad (i \in I, \ i \geq 3).$$

Generally, the violators payoff increases with increasing yield of the fishponds. a growing incentive to engage in the game... The poor inspector (a civil servant) doesn't care for fish and for the quality of the ponds – he gets a fixed salary and (hopefully) has a sufficient incentive to do his duty (the payoff to him does not reflect the hardships of a cold and rainy night at the ponds ...)

We are not finished as yet: how about landscapes with fishponds of greatly varying yield?

Consider the situation that $a_1$ and $a_2$ exceed all other yields $a_i$ substantially. More precisely, let us require a condition

$$\text{(65)} \qquad \frac{a_1 a_2 - 1}{a_1 + a_2} \geq a_i \quad (i \in I, \ i \geq 3) \ ,$$

which can be satisfied for the first two quantities being relatively large with respect to the other ones.

We look for a solution of the system

$$(66)\qquad \begin{array}{rcl} \boldsymbol{A}_{1\bullet}\boldsymbol{y} &=& \boldsymbol{A}_{2\bullet}\boldsymbol{y} \geq \boldsymbol{A}_{i\bullet}\boldsymbol{y} \quad (i \geq 3) \;\; ; \quad y_j \;=\; 0 \quad (j \geq 3) \\ \boldsymbol{x}\boldsymbol{B}_{\bullet 1} &=& \boldsymbol{x}\boldsymbol{B}_{\bullet 2} \geq \boldsymbol{x}\boldsymbol{B}_{\bullet j} \quad (j \geq 3) \;\; ; \quad x_i \;=\; 0 \quad (i \geq 3). \end{array}$$

the computation is quite analogous to the first case: we have to solve for just 2 variables $y_1, y_2$: the system reduces to

$$(67)\qquad \begin{array}{rl} a_1 - (1 + a_1)y_1 = & a_2 - (1 + a_2)y_2; \\ y_1 + y_2 = 1 & . \end{array}$$

The solution is

$$(68)\qquad \bar{\boldsymbol{y}} \;=\; \frac{1}{a_1 + a_2 + 2}(a_1 + 1, a_2 + 1).$$

The solution satisfies

$$\boldsymbol{A}_{1\bullet}\boldsymbol{y} \;=\; a_1 - (1 + a_1)y_1 \geq a_i - (1 + a_i)0 \;=\; \boldsymbol{A}_{i\bullet}\boldsymbol{y} \quad (i \in I)$$

because this is just equivalent with (65). The second part of (66) results in the solution

$$(69)\qquad \bar{\boldsymbol{x}} \;=\; (\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0)$$

and again one can check that the Optimality Criterion is satisfied.

The interpretation of this result is close at hand: as the ponds *No.* 1 and 2 are much more promising, the violator as well as the inspector concentrate their activities on theese two ponds; the probabilities are given by equations (68) and (69).

The ponds yielding little prey are left out to rest quietly in the beginning dusk when the inspector marches around one of those favorite ponds of the landlord while the poacher sneaks around the other one – or maybe not, whatever the result of tossing a coin and running the inspectors powerfull PC.

A host of further alternatives describes the full variety of possible landscapes of fishponds. We do not wish to enter this territory as the theory is tedious and our PC provides only a small *APL* workspace with limited $\square$ *WA*.

# 2 Zero-Sum Matrix Games

We consider a particular class of bimatrix games in mixed strategies. This class historically was the first to be analyzed extensively and it is also a class with extremely smooth equilibrium properties.

**Definition 2.1.** *1. A 2-person game $\Gamma = (S_1, S_2; F_1, F_2)$ is said to be a* ***zero-sum game*** *if*

$$F_2 \;=\; -\,F_1$$

*holds true. We write $\Gamma = (S_1, S_2; F_1)$ or rather $\Gamma = (S_1, S_2; F)$ in order to indicate this kind of game, the player index at the payoff function can be omitted. It is always understood that the payoff $F$ as indicated is the one for player 1 and that player 2 receives the payoff described by $-F$.*

*2. A zero-sum bimatrix game is just called a* ***matrix game****; we write*

$$\Gamma_0 = (I, J; \boldsymbol{A})$$

*instead of $\Gamma_0 = (I, J; \boldsymbol{A}, -\boldsymbol{A})$ for the version in pure strategies and*

$$\Gamma = (\underline{\overline{\mathbf{X}}}, \underline{\overline{\mathbf{Y}}}; \boldsymbol{A})$$

*for the mixed extension $\Gamma = (\underline{\overline{\mathbf{X}}}, \underline{\overline{\mathbf{Y}}}; \boldsymbol{A}, -\boldsymbol{A})$.*

The important property is not just that player 1 receives what player 2 pays and *vice versa*. Essential is the direct conflict of interest reflected by the model. As a consequence, implicit cooperation, coordination, and joint ventures that may possibly appear in the general bimatrix game, will fade into the background.

Thus, matrix games represent conflicts most understandably, maybe naively, direct: each player wants to get as much payoff as possible and, therefore, hurt his opponent as much as he can. This is a small sector of the many possible ways of conflict and cooperation that may be suggested by a bimatrix game.

It turns out that the actual restriction imposed on the model by the zero-sum assumption is so powerful that it it leads to a rather streamlined theory. One should accept that this theory concernes only a small part of game theoretical modelling.

**Remark 2.2.** *If* $\Gamma = (S_1, S_2; F)$ *is a matrix game, then* $\bar{s} \in S_1 \times S_2$ *is a Nash equilibrium if and only if*

$$F(\bar{s}^1, s^2) \geq F(\bar{s}^1, \bar{s}^2) \geq F(s^1, \bar{s}^2) \quad (s^1 \in S_1,\ s^2 \in S_2) \tag{1}$$

*hold true. Therefore,* $\bar{s}$ *is also called a* ***saddle point*** *(see Figure 2.1) .*

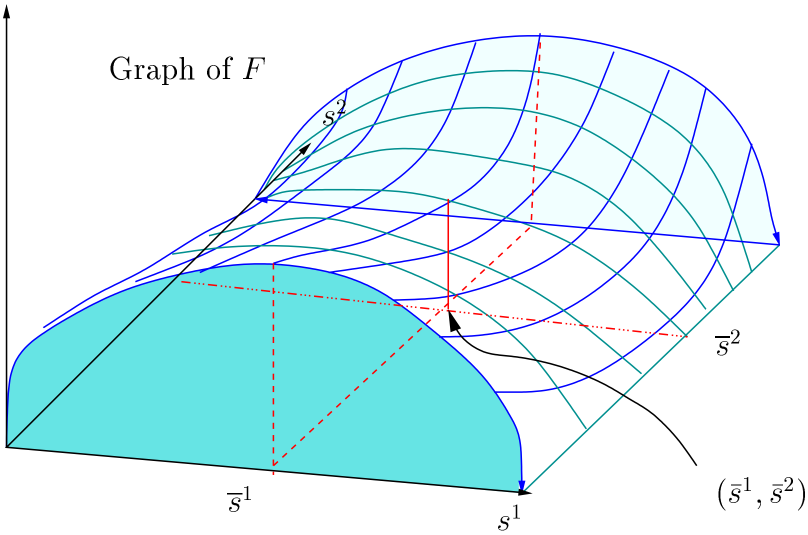



Figure 2.1: A Payoff–function with a Saddle Point

**Lemma 2.3.** *Let* $\Gamma = (\underline{\overline{\mathbf{X}}}, \underline{\overline{\mathbf{Y}}}; \boldsymbol{A})$ *be matrix game.*

1. *For every* $\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}$ *let*

$$\varphi(\boldsymbol{y}) \ := \ \max_{x \in \underline{\overline{\mathbf{X}}}} \boldsymbol{x}\boldsymbol{A}\boldsymbol{y} \ .$$

   *Then* $\varphi : \underline{\overline{\mathbf{Y}}} \to \mathbb{R}$ *is a continuous function.*

2. *The quantities*

$$\max_{\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}} \min_{\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}} \boldsymbol{x}\boldsymbol{A}\boldsymbol{y}$$

   *and*

$$\min_{\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}} \max_{\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}} \boldsymbol{x}\boldsymbol{A}\boldsymbol{y}$$

   *are well defined.*

3. *The above quantities satisfy the inequality*

$$\min_{\boldsymbol{y}\in\underline{\overline{\mathbf{Y}}}} \max_{\boldsymbol{x}\in\underline{\overline{\mathbf{X}}}} \boldsymbol{xAy} \geq \max_{\boldsymbol{x}\in\underline{\overline{\mathbf{X}}}} \min_{\boldsymbol{y}\in\underline{\overline{\mathbf{Y}}}} \boldsymbol{xAy} \ .$$

**Proof: 1^st^STEP** : The mapping

$$(\boldsymbol{x},\boldsymbol{y}) \to \boldsymbol{xAy}$$

is continuous and hence *uniformly* continuous on the compact set $\underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}}$.

That is, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|(\boldsymbol{x},\boldsymbol{y}) - (\boldsymbol{x}',\boldsymbol{y}')| < \delta \Longrightarrow |\boldsymbol{xAy} - \boldsymbol{x}'\boldsymbol{Ay}'| < \varepsilon \tag{2}$$

holds true for $(\boldsymbol{x},\boldsymbol{y}), (\boldsymbol{x}',\boldsymbol{y}') \in \underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}}$ .

(Note that this statement can be derived directly from the bilinearity of the function involved, one does not need the *Heine–Borel Theorem.*)

**2^nd^STEP** : Now let $\varepsilon > 0$ be given. Choose $\delta$ as prescribed in the 1st STEP and consider a pair $\bar{\boldsymbol{y}}, \widehat{\boldsymbol{y}}$ satisfying

$$|\bar{\boldsymbol{y}} - \widehat{\boldsymbol{y}}| < \delta.$$

Also, choose two corresponding maximizers, say $\bar{\boldsymbol{x}}$ and $\widehat{\boldsymbol{x}}$ such that

$$\begin{aligned} \varphi(\bar{\boldsymbol{y}}) &= \max_{\boldsymbol{x}\in\underline{\overline{\mathbf{X}}}} \boldsymbol{xA}\bar{\boldsymbol{y}} = \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} \\ \varphi(\widehat{\boldsymbol{y}}) &= \max_{\boldsymbol{x}\in\underline{\overline{\mathbf{X}}}} \boldsymbol{xA}\widehat{\boldsymbol{y}} = \widehat{\boldsymbol{x}}\boldsymbol{A}\widehat{\boldsymbol{y}} \end{aligned} \tag{3}$$

is fulfilled.

Then we have

$$\begin{aligned} \varphi(\bar{\boldsymbol{y}}) &= \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} \\ &\geq \widehat{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} \\ &> \widehat{\boldsymbol{x}}\boldsymbol{A}\widehat{\boldsymbol{y}} - \varepsilon \\ &= \varphi(\widehat{\boldsymbol{y}}) - \varepsilon \ . \end{aligned} \tag{4}$$

Here, the second inequality uses (3) and the third one employs (2) as

$$|(\widehat{\boldsymbol{x}},\bar{\boldsymbol{y}}) - (\widehat{\boldsymbol{x}},\widehat{\boldsymbol{y}})| < \delta$$

holds true indeed. Symmetrically, we obtain

$$
\begin{aligned}
\varphi(\widehat{\boldsymbol{y}}) &= \widehat{\boldsymbol{x}}\boldsymbol{A}\widehat{\boldsymbol{y}} \\
&\geq \bar{\boldsymbol{x}}\boldsymbol{A}\widehat{\boldsymbol{y}} \\
&> \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} - \varepsilon \\
&= \varphi(\bar{\boldsymbol{y}}) - \varepsilon \ .
\end{aligned} \tag{5}
$$

Obviously we have now

$$
|\varphi(\bar{\boldsymbol{y}}) \ - \ \varphi(\widehat{\boldsymbol{y}})| < \varepsilon \ ,
$$

which shows that $\varphi$ is continuous (actually we have proved uniform continuity).

**3rdSTEP** : The second statement of our present lemma is now obvious.

**4thSTEP** : The third statement is quite easy to verify. It is, however, the first part of an important theorem in zero-sum theory, the *Min-Max Theorem* to be made precise later on.

For any $(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}) \in \overline{\underline{\mathbf{X}}} \times \overline{\underline{\mathbf{Y}}}$ we have

$$
\widehat{\boldsymbol{x}}\boldsymbol{A}\widehat{\boldsymbol{y}} \geq \min_{\boldsymbol{y}\in\overline{\underline{\mathbf{Y}}}} \widehat{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y}
$$

and hence

$$
\max_{\boldsymbol{x}\in\overline{\underline{\mathbf{X}}}} \boldsymbol{x}\boldsymbol{A}\widehat{\boldsymbol{y}} \geq \max_{\boldsymbol{x}\in\overline{\underline{\mathbf{X}}}} \min_{\boldsymbol{y}\in\overline{\underline{\mathbf{Y}}}} \boldsymbol{x}\boldsymbol{A}\boldsymbol{y}
$$

is true. Therefore, taking min on the left side (the right side is a real number) we obtain indeed

$$
\min_{\boldsymbol{y}\in\overline{\underline{\mathbf{Y}}}} \max_{\boldsymbol{x}\in\overline{\underline{\mathbf{X}}}} \boldsymbol{x}\boldsymbol{A}\boldsymbol{y} \geq \max_{\boldsymbol{x}\in\overline{\underline{\mathbf{X}}}} \min_{\boldsymbol{y}\in\overline{\underline{\mathbf{Y}}}} \boldsymbol{x}\boldsymbol{A}\boldsymbol{y},
$$

**q.e.d.**

**Remark 2.4.** *The following Remark concernes all zero sum games provided the quantites*

$$
\max_{\boldsymbol{x}\in\overline{\underline{\mathbf{X}}}} \min_{\boldsymbol{y}\in\overline{\underline{\mathbf{Y}}}} \boldsymbol{x}\boldsymbol{A}\boldsymbol{y} \qquad \textit{and} \qquad \min_{\boldsymbol{y}\in\overline{\underline{\mathbf{Y}}}} \max_{\boldsymbol{x}\in\overline{\underline{\mathbf{X}}}} \boldsymbol{x}\boldsymbol{A}\boldsymbol{y}
$$

*do exist. However, we presently focus on bimatrix games in mixed strategies and our interpretations are particulaly usefull within this framework.*

*Consider the quantity*

$$
\underline{v}_\Gamma := \max_{\boldsymbol{x}\in\overline{\underline{\mathbf{X}}}} \min_{\boldsymbol{y}\in\overline{\underline{\mathbf{Y}}}} \boldsymbol{x}\boldsymbol{A}\boldsymbol{y}, \tag{6}
$$

*we claim that player 1, by a suitable choice of a strategy, can ensure himself to receive at least this much. Indeed, for every* $\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}$*, the worst that player 2 can do to player 1 is*

$$\min_{\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}} \boldsymbol{xAy}.$$

*Therefore, player 1 can guarantee himself the quantity he obtains by maximizing among all these gloomy prospects, this maximum is just* $\underline{v}_\Gamma$*. And what player 1 should do in order to make sure that he gets at least this much is choosing* $\bar{\boldsymbol{x}} \in \underline{\overline{\mathbf{X}}}$ *such that*

$$\min_{\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}} \bar{\boldsymbol{x}}\boldsymbol{Ay} \;=\; \max_{\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}} \min_{\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}} \boldsymbol{xAy} \;=\; \underline{v}_\Gamma. \tag{7}$$

*To aim for this payoff, as far as player 1 is concerned, means to play safe and not to take chances; there may be more in the game to be reached by player 1 but this is a cautious and risk averse strategy. Therefore, the quantity* $\underline{v}_\Gamma$ *is sometimes called the* ***lower value*** *of the game* $\Gamma$.

*On the other hand, player 2 can prevent player 1 from getting more than*

$$\bar{v}_\Gamma \;:=\; \min_{\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}} \max_{\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}} \boldsymbol{xAy}. \tag{8}$$

*Indeed, for each* $\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}$*, the maximal payoff player 1 can achieve is* $\max_{\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}} \boldsymbol{xAy}$*. Hence, by suitable choice of* $\bar{\boldsymbol{y}}$ *player 2 can hold down player 1 to*

$$\max_{\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}} \boldsymbol{xA}\bar{\boldsymbol{y}} \;=\; \min_{\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}} \max_{\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}} \boldsymbol{xAy} \;=\; \bar{v}_\Gamma. \tag{9}$$

*So far the discussion was lead from the viewpoint of player 1. But of course, as we are in a zero-sum game, the quantity* $\bar{v}_\Gamma$ *is up to the* $-$*-sign the value player 2 can guarantee himself etc.*

*Now it is* not *true that the above two versions of what one can achieve for sure and what one can be prevented from will generally coincide. On the contrary, with respect to the pure game it is seen at once that both versions may differ.*

*For, in the pure game of 'stone, scissors, paper' represented as* $\Gamma_0 = (\underline{\overline{\mathbf{X}}}, \underline{\overline{\mathbf{Y}}}; \boldsymbol{A})$ *with*

$$A \;=\; \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \tag{10}$$

*the two quantities are*

$$\max_{i \in I} \min_{j \in J} a_{ij} = -1 \;<\; 1 = \min_{j \in J} \max_{i \in I} a_{ij} \;,$$

*they do not coincide. This means that there is an "indefinite" domain of payoffs that neither player can achieve for sure. Nor can either player be prevented by whatever kind of skillful choice of strategies to achieve a payoff within this domain.*

The Main Theorem of Matrix Game Theory states that the upper and lower value coincide for the mixed extension: there is no gap between what a player can guarantee himself and the maximum he can achieve.

**Theorem 2.5 (MIN–MAX Theorem, J. v. Neumann).**
*Let* $\Gamma = (\underline{\mathbf{X}}, \underline{\mathbf{Y}}; \boldsymbol{A})$ *be a matrix game in mixed strategies. Then*

$$\underline{v}_\Gamma = \max_{\boldsymbol{x}\in\underline{\mathbf{X}}} \min_{\boldsymbol{y}\in\underline{\mathbf{Y}}} \boldsymbol{x}\boldsymbol{A}\boldsymbol{y} = \min_{\boldsymbol{y}\in\underline{\mathbf{Y}}} \max_{\boldsymbol{x}\in\underline{\mathbf{X}}} \boldsymbol{x}\boldsymbol{A}\boldsymbol{y} = \bar{v}_\Gamma \tag{11}$$

*holds true.*

**Proof:** In view of Lemma 2.3 it suffices to prove the $\geq -$inequality. To this end we shall (at this stage) make use of the existence theorem concerning equilibria in bimatrix games, i.e., of Nash's Theorem (1.16), which in turn rests on the Brouwer Fixed Point Theorem.

Accordingly, there exists an equilibrium, say $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$, which satisfies (as we have a zero sum game)

$$\bar{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y} \geq \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} \geq \boldsymbol{x}\boldsymbol{A}\bar{\boldsymbol{y}} \quad (\boldsymbol{x}, \boldsymbol{y}) \in \underline{\mathbf{X}} \times \underline{\mathbf{Y}}. \tag{12}$$

Hence we conclude:

$$\begin{aligned} \max_{\boldsymbol{x}\in\underline{\mathbf{X}}} \min_{\boldsymbol{y}\in\underline{\mathbf{Y}}} \boldsymbol{x}\boldsymbol{A}\boldsymbol{y} &\geq \min_{\boldsymbol{y}\in\underline{\mathbf{Y}}} \bar{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y} \\ &= \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} \\ &= \max_{\boldsymbol{x}\in\underline{\mathbf{X}}} \boldsymbol{x}\boldsymbol{A}\bar{\boldsymbol{y}} \\ &\geq \min_{\boldsymbol{y}\in\underline{\mathbf{Y}}} \max_{\boldsymbol{x}\in\underline{\mathbf{X}}} \boldsymbol{x}\boldsymbol{A}\boldsymbol{y} \end{aligned} \tag{13}$$

**q.e.d.**

**Definition 2.6.** *Let* $\Gamma = (\underline{\mathbf{X}}, \underline{\mathbf{Y}}; \boldsymbol{A})$ *be a matrix game. Then*

$$v_\Gamma := \max_{\boldsymbol{x}\in\underline{\mathbf{X}}} \min_{\boldsymbol{y}\in\underline{\mathbf{Y}}} \boldsymbol{x}\boldsymbol{A}\boldsymbol{y} = \min_{\boldsymbol{y}\in\underline{\mathbf{Y}}} \max_{\boldsymbol{x}\in\underline{\mathbf{X}}} \boldsymbol{x}\boldsymbol{A}\boldsymbol{y} \quad ( = \bar{v}_\Gamma = \underline{v}_\Gamma) \tag{14}$$

*is the **value** of* $\Gamma$.

The value of the game is what player 1 can definitely achieve by playing a suitable mixed strategy and what player 2 can (up to the sign) ensure as his maximal payment. The question as to what is meant by playing suitably in order to ensure the value is readily answered:

**Theorem 2.7.** *Let $\Gamma$ be a matrix game in mixed strategies.*

1. *For every equilibrium $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ $\Gamma$ the payoff is the value of the game, i.e.,*

$$\bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} = v_\Gamma. \tag{15}$$

2. *If $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ and $(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}})$ are equilibria of $\Gamma$, then so are $(\widehat{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ and $(\bar{\boldsymbol{x}}, \widehat{\boldsymbol{y}})$ (of course again with the same payoff, i.e., the value).*

**Proof:** As to *item 1* , this has already been proved in (13). So we turn to *item 2* : Because we may use the equilibrium inequalities (i.e., (12)) both with respect to $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ and with respect to $(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}})$, we obtain

$$\bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} \geq \widehat{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} \geq \widehat{\boldsymbol{x}}\boldsymbol{A}\widehat{\boldsymbol{y}} \geq \bar{\boldsymbol{x}}\boldsymbol{A}\widehat{\boldsymbol{y}} \geq \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}}. \tag{16}$$

This shows that all the expressions involved are equal, hence we continue with

$$\bar{\boldsymbol{x}}\boldsymbol{A}\widehat{\boldsymbol{y}} = \widehat{\boldsymbol{x}}\boldsymbol{A}\widehat{\boldsymbol{y}} \geq \boldsymbol{x}\boldsymbol{A}\widehat{\boldsymbol{y}} \quad (\boldsymbol{x} \in \overline{\underline{\mathbf{X}}}) \tag{17}$$

and similarly

$$\bar{\boldsymbol{x}}\boldsymbol{A}\widehat{\boldsymbol{y}} = \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} \leq \bar{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y} \quad (\boldsymbol{y} \in \overline{\underline{\mathbf{Y}}}). \tag{18}$$

This shows that $(\bar{\boldsymbol{x}}, \widehat{\boldsymbol{y}})$ is an equilibrium as well.

**q.e.d.**

**Example 2.8.** The third *item* in Example 1.3 is the ***2 Finger Morra.*** which for $m = n = 2$ is given by the matrix

$$\boldsymbol{A} = \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}. \tag{19}$$

If we solve the linear system of equations given by

$$\begin{aligned} \boldsymbol{A}_{1\bullet}\boldsymbol{y} &= \boldsymbol{A}_{2\bullet}\boldsymbol{y} \\ y_1 + y_2 &= 1 \end{aligned}$$

then we come up with the unique solution

$$\bar{\boldsymbol{y}} = \left(\frac{7}{12}, \frac{5}{12}\right)$$

and the anolgous system for $\bar{\boldsymbol{x}}$ yields

$$\bar{\boldsymbol{x}} = (\frac{7}{12}, \frac{5}{12})$$

as well. By the Optimality Criterion this constitutes an equilibrium; the value is the payoff at this equilibrium and turns out to be

$$v_\Gamma = \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} = \boldsymbol{A}_{1\bullet}\bar{\boldsymbol{y}} = \boldsymbol{A}_{2\bullet}\bar{\boldsymbol{y}} = \frac{1}{12}.$$

the seeming symmetry of the game is deceptive, player 1 is in an advantageous position and will (in expectation) gain $\frac{1}{12}$ at each play.

As we have seen, the answer provided to the above question (how to play in order to achieve the value) is a straightforward one: play an equilibrium strategy, it does not matter which one. More than that: while an equilibrium is a *pair* of strategies, the above theorem (and the MIN-MAX Theorem) show that *each player* can just choose an arbitrary part of an equilibrium pair in order to achieve the value of the game. This is in marked contrast to the general situation in bimatrix games, where equilibria represent a *joint* decision to opt for a pair of strategies. In the context of a matrix game it is therefore possible to speak of optimal strategies.

This motivates the following definition.

**Definition 2.9.** *Let $\Gamma$ be a matrix game and let $\bar{\boldsymbol{x}} \in \underline{\overline{\mathbf{X}}}$ be a (mixed) strategy such that there is $\bar{\boldsymbol{y}} \in \underline{\overline{\mathbf{Y}}}$ so that $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ is an equilibrium. Then $\bar{\boldsymbol{x}}$ is called an **optimal strategy**. The analogous definition holds true for player 2.*

**Theorem 2.10.** *Let $\Gamma = (\underline{\overline{\mathbf{X}}}, \underline{\overline{\mathbf{Y}}}; \boldsymbol{A})$ be a matrix game.*

1. *The value of the game satisfies*

$$v_\Gamma = \max_{\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}} \min_{j \in J} \boldsymbol{x}\boldsymbol{A}_{\bullet j} = \min_{\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}} \max_{i \in I} \boldsymbol{A}_{i\bullet}\boldsymbol{y}. \tag{20}$$

2. *Let*

$$C := \big\{(\boldsymbol{x}, \boldsymbol{y}, \lambda) \in \underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}} \times \mathbb{R} \;\big| \;\big|\; \boldsymbol{x}\boldsymbol{A}_{\bullet j} \geq \lambda \geq \boldsymbol{A}_{i\bullet}\boldsymbol{y} \quad (i \in I, j \in J)\big\} \tag{21}$$

*Then $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{\lambda}) \in C$ holds true if and only if $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{y}}$ are optimal strategies for player 1 and player 2 respectively and $\bar{\lambda} = v_\Gamma$ holds true.*

3. *Define now*

$$D = \left\{ \boldsymbol{x} \in \underline{\overline{\mathbf{X}}} \,\middle|\, \boldsymbol{x}\boldsymbol{A}_{\bullet j} \geq v_\Gamma \quad (j \in J) \right\}. \tag{22}$$

*Then $D$ is precisely the set of optimal strategies of player* 1.

4. *The optimal strategies of each player constitute a convex compact polyhedron (in $\underline{\overline{\mathbf{X}}}$ and $\underline{\overline{\mathbf{Y}}}$ respectively).*

5. *The Nash equilibria constitute a convex compact polyhedron (in $\underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}}$).*

**Proof:** As for *item 1,* we know that for $\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}$ the equation

$$\min_{j \in J} \boldsymbol{x}\boldsymbol{A}_{\bullet j} = \min_{\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}} \boldsymbol{x}\boldsymbol{A}\boldsymbol{y} \tag{23}$$

holds true (cf. Corollary 1.12).

The proof of *item 2* is the only one that is not straightforward.

**1$^{\text{st}}$STEP** : If $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ is an equilibrium, then we have

$$v_\Gamma = \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} \leq \bar{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y} \quad (\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}) \tag{24}$$

and in particular

$$v_\Gamma \leq \bar{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{e}^j = \bar{\boldsymbol{x}}\boldsymbol{A}_{\bullet j} \quad (j \in J); \tag{25}$$

analogously we obtain

$$v_\Gamma \geq \boldsymbol{A}_{i\bullet}\bar{\boldsymbol{y}}. \tag{26}$$

That is, we find indeed that $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, v_\Gamma) \in C$ holds true.

**2$^{\text{nd}}$STEP** :

On the other hand, consider a triple $(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}, \widehat{\lambda}) \in C$. We have

$$\begin{aligned} v_\Gamma &= \min_{\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}} \max_{i \in I} \boldsymbol{A}_{i\bullet}\boldsymbol{y} \\ &\leq \max_{i \in I} \boldsymbol{A}_{i\bullet}\widehat{\boldsymbol{y}} \quad \leq \widehat{\lambda} \quad \leq \min_{j \in J} \widehat{\boldsymbol{x}}\boldsymbol{A}_{\bullet j} \\ &\leq \max_{\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}} \min_{j \in J} \boldsymbol{x}\boldsymbol{A}_{\bullet j} \quad = v_\Gamma \end{aligned} \tag{27}$$

and we conclude that the equations

$$\widehat{\lambda} = v_\Gamma = \min_{j \in J} \widehat{\boldsymbol{x}}\boldsymbol{A}_{\bullet j} = \min_{\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}} \widehat{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y} \tag{28}$$

holds true. We show quite analogously that we have as well

$$\widehat{\lambda} \;=\; v_\Gamma \;=\; \max_{\boldsymbol{x}\in\underline{\overline{\mathbf{X}}}} \boldsymbol{x}\boldsymbol{A}\widehat{\boldsymbol{y}}. \tag{29}$$

Consequently we obtain

$$\boldsymbol{x}\boldsymbol{A}\widehat{\boldsymbol{y}} \;\leq\; v_\Gamma \;\leq\; \widehat{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y}. \tag{30}$$

This way we see that indeed $(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}})$ is an equilibrium (a pair of optimal strategies) and $\widehat{\lambda}$ is the value of the game.

The remaining statements as formulated in *items 3,4,* and *5* are now quite easy to prove.

**q.e.d.**

**Corollary 2.11.** *Let* $\boldsymbol{d} = (0,\dots,0,1) \in \mathbb{R}^{m+1}$. *Then*

$$v_\Gamma \;=\; \max\left\{ \boldsymbol{d}(\boldsymbol{x},\lambda) \;\middle|\; \begin{array}{l} (\boldsymbol{x},\lambda) \in \mathbb{R}^{m+1}, \\ \boldsymbol{x}\boldsymbol{A}_{\bullet j} - \lambda \geq 0 \quad (j \in J) \\ \boldsymbol{x} \geq 0\ , \quad \sum_{i=1}^{m} x_i = 1 \end{array} \right\} \tag{31}$$

*holds true. Any optimal solution of the Linear Program suggested by* (31) *supplies an optimal strategy* $\bar{\boldsymbol{x}}$ *of the matrix game* $\Gamma = (\underline{\overline{\mathbf{X}}}, \underline{\overline{\mathbf{Y}}}; \boldsymbol{A})$.

In other words, if we introduce a matrix

$$\bar{\boldsymbol{A}} := \begin{pmatrix} & 1 & -1 & \\ \boldsymbol{A} & \vdots & \vdots & \boldsymbol{I}_m \\ & 1 & -1 & \\ -1\dots-1 & 0 & 0 & \mathbf{0} \end{pmatrix} \tag{32}$$

as well as a restricting vector

$$\boldsymbol{c} \;:=\; (0,\dots,0,1,-1,\mathbf{0}) \tag{33}$$

and a vector representing the objective function, say

$$\boldsymbol{b} \;=\; (0,\dots,0,-1) \tag{34}$$

then using the sloppy (but suggestive) way of writing LP.'s introduced in SECTION 2 of CHAPTER 2 we obtain

$$(35) \qquad (\widehat{\boldsymbol{II}}) \qquad \begin{array}{rcl} (\boldsymbol{x},\lambda)\bar{\boldsymbol{A}} & \geq & \boldsymbol{c} \\ (\boldsymbol{x},\lambda)\boldsymbol{b} & \to & min \end{array}$$

as the LP. we have in mind. Using the Simplex Algorithm (SECTIONS2 - 2 of CHAPTER 2) we may compute the value and optimal strategies – provided we can ensure nondegeneracy.

**Proof of Corollary 2.11**

By *item 3* of Theorem 2.10 it follows clearly, that a pair $(\bar{\boldsymbol{x}}, v_\Gamma)$ is feasible whenever $\bar{\boldsymbol{x}}$ is an optimal strategy of player 1. That is, $(\bar{\boldsymbol{x}}, v_\Gamma)$ is an element of the feasible set indicated in formula (31). Consequently, the max exceeds the value of the objective function at this particular pair, i.e., we have

$$(36) \qquad \max\{\boldsymbol{d}(\boldsymbol{x},\lambda) \mid \ldots\} \geq \boldsymbol{d}(\bar{\boldsymbol{x}}, v_\Gamma) \;=\; v_\Gamma .$$

On the other hand, let us assume that a pair $(\widehat{\boldsymbol{x}}, \widehat{\lambda})$ is feasible so that it satisfies

$$(37) \qquad \widehat{\boldsymbol{x}}\boldsymbol{A}_{\bullet j} \geq \widehat{\lambda}\ , \quad \widehat{\boldsymbol{x}} \geq 0\ , \quad \sum_{i\in I} \widehat{x}_i \;=\; 1 \ .$$

Now pick an optimal strategy $\bar{\boldsymbol{y}}$ for player 2, then we have

$$(38) \qquad v_\Gamma \geq \boldsymbol{A}_{i\bullet}\bar{\boldsymbol{y}} \quad (i \in I)$$

(by the $3^{rd}$ *item* of Theorem 2.10). Now we find

$$(39) \qquad v_\Gamma \geq \sum_{i\in I} \widehat{x}_i \boldsymbol{A}_{i\bullet}\bar{\boldsymbol{y}} \;=\; \sum_{j\in J} \widehat{\boldsymbol{x}}\boldsymbol{A}_{\bullet j}\bar{y}_j \geq \widehat{\lambda}\sum_{j\in J}\bar{y}_j \;=\; \widehat{\lambda} \;=\; \boldsymbol{d}(\widehat{\boldsymbol{x}}, \widehat{\lambda}).$$

This implies all statements claimed. **q.e.d.**

The above theorem links Linear Programming techniques with the computation of optimal strategies in matrix games. Note that everything we have done until now, as far as the existence of such optimal strategegeis is concerned, rests on Nash's Theorem, which in turn uses the Brouwer Fixed Point Theorem. The latter one is a rather heavy tool. As the Duality Theorem of Linear Programming was essentially based on a separation theorem (which is a less involved result compared to Brouwers Theorem), it may be useful to exploit the results from Linear Programming for a simplified existence proof concerning optimal strategies. Indeed, this project is easily performed, we can base an existence proof on the Duality Theorem of Linear Programming. The clue is to construct the suitable LP.

To prepare this, we start out with a simple lemma:

**Lemma 2.12.** *Let*

$$\boldsymbol{E} \;=\; \begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix} \tag{40}$$

*as well as* $\boldsymbol{A}$ *be* $m \times n$*-matrices and let* $\alpha > 0$ *and* $\beta$ *be reals. Consider the familiy of matrices*

$$\boldsymbol{A}^{\alpha}_{\beta} \;:=\; \alpha\boldsymbol{A} + \beta\boldsymbol{E} \quad (\alpha \in \mathbb{R}_{++},\ \beta \in \mathbb{R}) \tag{41}$$

*and the corresponding family of matrix games*

$$\Gamma^{\alpha}_{\beta} = (\underline{\overline{\mathbf{X}}}, \underline{\overline{\mathbf{Y}}}; \boldsymbol{A}^{\alpha}_{\beta}) \quad (\alpha \in \mathbb{R}_{++},\ \beta \in \mathbb{R}). \tag{42}$$

*Then* $\Gamma \;=\; \Gamma^{1}_{0}$ *and* $\Gamma^{\alpha}_{\beta}$ $(\alpha \in \mathbb{R}_{++},\ \beta \in \mathbb{R})$ *have the same optimal strategies for both players. Moreover, the family of values satisfies*

$$v_{\Gamma^{\alpha}_{\beta}} \;=\; \alpha v_{\Gamma} + \beta \quad (\alpha \in \mathbb{R}_{++},\ \beta \in \mathbb{R}). \tag{43}$$

**Proof:** Because of

$$\boldsymbol{x}\boldsymbol{A}^{\alpha}_{\beta}\boldsymbol{y} \;=\; \boldsymbol{x}(\alpha\boldsymbol{A} + \beta\boldsymbol{E})\boldsymbol{y} \;=\; \alpha\boldsymbol{x}\boldsymbol{A}\boldsymbol{y} + \beta \quad (\boldsymbol{x} \in \underline{\overline{\mathbf{X}}},\ \boldsymbol{y} \in \underline{\overline{\mathbf{Y}}},\ \alpha > 0,\ \beta \in \mathbb{R}) \tag{44}$$

the inequalities concerning Nash equilibrium can be immediately established, the remaining statements are easy. **q.e.d.**

Frequently a game $\Gamma^{\alpha}_{\beta}$ is said to be *strategically equivalent* to $\Gamma \;=\; \Gamma^{1}_{0}$ as the strategies and the equilibria are the same.

**Remark 2.13 (Second proof of the MinMax Theorem).**

*The linear program as suggested by Corollary 2.11 is certainly not the only one that serves for the computation of* $v_{\Gamma}$ *and of optimal strategies. If we consider* *the following pair of dual LP.'s, then we obtain simultaneously a proof for the existence of optimal strategies which is based on the Duality Theorem of Linear Programming – and hence on a separation theorem.*

**Proof (of Theorem 2.5):**

We assume that the matrix involved satisfies $\boldsymbol{A} > 0$; this is not a serious restriction in view of our previous lemma. Tentatively let us use the notation

$$\boldsymbol{e}^{(m)} := (1, \ldots, 1) \in \mathbb{R} \tag{45}$$

in order to be able to distinguish between vectors $\boldsymbol{e}^{(m)}$ and $\boldsymbol{e}^{(n)}$.

Now consider the LP.'s indicated as follows:

$$(\boldsymbol{I}) \qquad \begin{array}{rcl} \boldsymbol{A}\boldsymbol{y} & \leq & \boldsymbol{e}^{(m)} \\ \boldsymbol{y} & \geq & \mathbf{0} \\ \boldsymbol{e}^{(n)}\boldsymbol{y} & \to & \max \end{array} \tag{46}$$

and

$$(\widehat{\boldsymbol{I}}) \qquad \begin{array}{lcl} \boldsymbol{x}\boldsymbol{A} & \geq & \boldsymbol{e}^{(n)} \\ \boldsymbol{x} & \geq & \mathbf{0} \\ \boldsymbol{x}\boldsymbol{e}^{(m)} & \to & \min \ , \end{array} \tag{47}$$

both of them constitute a dual pair.

Clearly, $\boldsymbol{y} = \mathbf{0}$ is a feasible solution for $(\boldsymbol{I})$.

Because we assume $\boldsymbol{A} > 0$, it is seen that, for sufficiently large $t \in \mathbb{R}_+$, the vector $\boldsymbol{x}^t := (t, \ldots, t) = t\boldsymbol{e}^{(m)}$ is a feasible solution for $(\widehat{\boldsymbol{I}})$. Hence, in view of the Duality Theorem of Linear Programming (we use the version in CHAPTER 2, SECTION 2, given as Theorem 2.4), we know that both programs admit of optimal solutions, say $\widehat{\boldsymbol{x}}$ and $\widehat{\boldsymbol{y}}$ such that

$$\widehat{\boldsymbol{x}}\boldsymbol{e}^{(m)} = \boldsymbol{e}^{(n)}\widehat{\boldsymbol{y}} =: c \tag{48}$$

prevails. Necessarily we must have $\widehat{\boldsymbol{x}} \neq \mathbf{0}$ as otherwise $\widehat{\boldsymbol{x}}$ cannot be feasible for program $(\widehat{\boldsymbol{I}})$. Therefore the constant c introduced is $c > 0$. Now define

$$\bar{\boldsymbol{x}} := \frac{\widehat{\boldsymbol{x}}}{c}, \quad \bar{\boldsymbol{y}} := \frac{\widehat{\boldsymbol{y}}}{c}, \tag{49}$$

such that we obtain $\bar{\boldsymbol{x}} \in \underline{\overline{\mathbf{X}}}$, $\bar{\boldsymbol{y}} \in \underline{\overline{\mathbf{Y}}}$.

Then, for arbitrary $\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}$ and $\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}$, we obtain the following string of equations and inequalities:

$$
\begin{aligned}
\boldsymbol{x}\boldsymbol{A}\bar{\boldsymbol{y}} &= \frac{\boldsymbol{x}\boldsymbol{A}\widehat{\boldsymbol{y}}}{c} \leq \frac{\boldsymbol{x}\boldsymbol{e}^{(m)}}{c} = \frac{1}{c} \\
&= \frac{\boldsymbol{e}^{(n)}\boldsymbol{y}}{c} \leq \frac{\widehat{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y}}{c} = \bar{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y}.
\end{aligned} \tag{50}
$$

This means that $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ is an equilibrium and $\frac{1}{c} = v_\Gamma$.

**q.e.d.**

We will now describe a *graphical procedure* that works for small numbers of rows $m$ or columns $n$ of the matrix $\boldsymbol{A}$ involved in a matrix game.

Let

$$\Gamma = (\underline{\overline{\mathbf{X}}}, \underline{\overline{\mathbf{Y}}}; \boldsymbol{A})$$

be a bimatrix game with $m = 2$. Consider, for each $j \in J$, the function

$$f_j : \underline{\overline{\mathbf{X}}} \to \mathbb{R}, \quad f_j(\boldsymbol{x}) := \boldsymbol{x}\boldsymbol{A}_{\bullet j} \quad (\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}) \tag{51}$$

and also the function

$$f : \underline{\overline{\mathbf{X}}} \to \mathbb{R}, \quad f(\boldsymbol{x}) := \min_{j \in J} \boldsymbol{x}\boldsymbol{A}_{\bullet j} = \min_{j \in J} f_j(\boldsymbol{x}) \quad (\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}). \tag{52}$$

Then we have

$$\max_{\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}} f(\boldsymbol{x}) = \max_{\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}} \min_{j \in J} \boldsymbol{x}\boldsymbol{A}_{\bullet j} = \max_{\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}} \min_{\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}} \boldsymbol{x}\boldsymbol{A}\boldsymbol{y} = v_\Gamma$$

and any $\bar{\boldsymbol{x}}$ satisfying

$$f(\bar{\boldsymbol{x}}) = \max_{\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}} f(\boldsymbol{x})$$

clearly yields

$$
\begin{aligned}
\bar{\boldsymbol{x}}\boldsymbol{A}_{\bullet l} &\geq \min_{j \in J} \bar{\boldsymbol{x}}\boldsymbol{A}_{\bullet j} \\
&= f(\bar{\boldsymbol{x}}) \\
&= \max_{\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}} \min_{j \in J} \bar{\boldsymbol{x}}\boldsymbol{A}_{\bullet j} = v_\Gamma \quad (l \in J),
\end{aligned}
$$

which indicates in view of Theorem 2.10 that $\bar{\boldsymbol{x}}$ is an optimal strategy for player 1 and $f(\bar{\boldsymbol{x}})$ is the value of the game.

Now, as we assume tentatively that $m = 2$, the graph of any of the functions $f_j$ can easily be sketched:

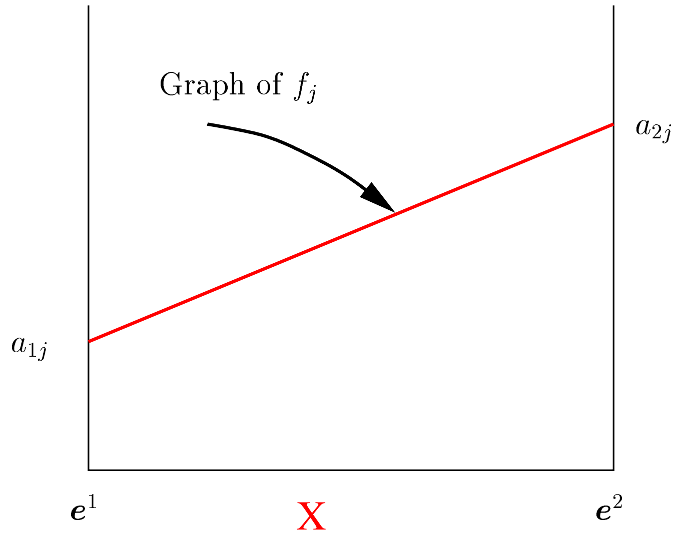



Figure 2.2: Graph of a function $f_j$

We can represent the simplex $\underline{\overline{\mathbf{X}}}$ by a line segment or intervall with endpoints $\boldsymbol{e}^1$ and $\boldsymbol{e}^2$. The graph of some $f_j$ is a line segment as $f_j$ is a linear function. In fact, the value of this function at the endpoints is obviously

$$f_j(\boldsymbol{e}^i) \;=\; \boldsymbol{A}_{i\bullet}\boldsymbol{e}^i \;=\; a_{ij} \quad (i \;=\; 1,2).$$

That is, the two elements of column $j$ which are the matrix entries $a_{1j}$ and $a_{2j}$, determine the height of the graph at the endpoints and hence the complete behavior of the graph.

By drawing the graphs of *all* functions $f_j$ (corresponding to all columns of the matrix $\boldsymbol{A}$) we can, as a next step, easily identify the graph of the minimum function $f$. The maximizer(s) of this function as well as the maximal value now yield a graphical solution for the problem of finding value and optimal strategies.

Generally, the determination of the optimal strategies may not be precise when performed graphically. The graphical representation yields, however, the columns involved in the systems of linear equations the solutions of which provides the desired quantities.

More precisely, the set

$$\begin{aligned} \{\boldsymbol{x} \in \underline{\overline{\mathbf{X}}} \,|\, f(\boldsymbol{x}) = v_\Gamma\} \;&=\; \{\boldsymbol{x} \in \underline{\overline{\mathbf{X}}} \,\Big|\, \min_j f_j(\boldsymbol{x}) = v_\Gamma\} \\ \{\boldsymbol{x} \in \underline{\overline{\mathbf{X}}} \,|\, \boldsymbol{x}\boldsymbol{A}_{\bullet j} = v_\Gamma\} \;&=\; \{\boldsymbol{x} \in \underline{\overline{\mathbf{X}}} \,|\, \boldsymbol{x}\boldsymbol{A}_{\bullet j} \geq v_\Gamma\} \end{aligned} \tag{53}$$

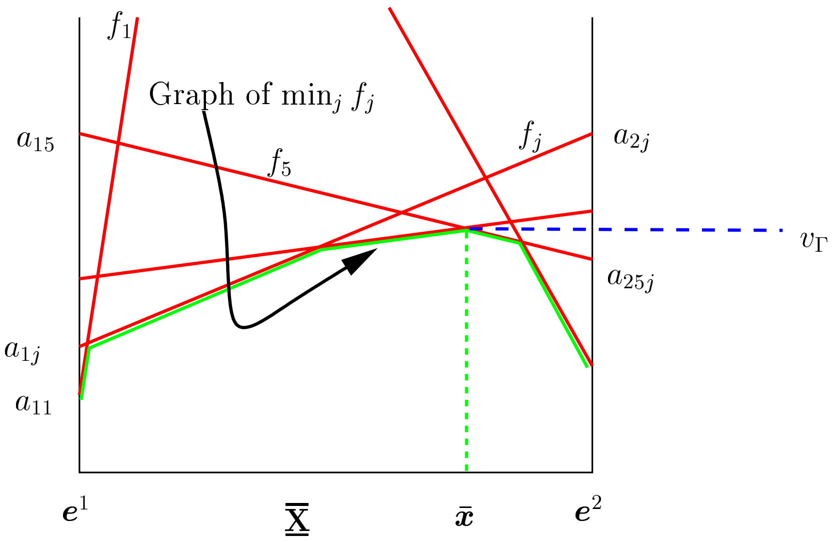



Figure 2.3: The Minimum of all functions $f_j$

is the set of optimal strategies of player 1. This set is an interval or a point in $\underline{\overline{\mathbf{X}}}$ and the extremals can be found by solving two equations of the form

$$\boldsymbol{x}\boldsymbol{A}_{\bullet j} = \boldsymbol{x}\boldsymbol{A}_{\bullet k}, \quad \boldsymbol{x}_1 + \boldsymbol{x}_2 = 1,$$

employing typically two of the functions $f_j, f_k$ that are involved.

We can also determine the optimal strategies of player 2. To this end, let $\bar{\boldsymbol{x}}$ be an optimal strategy of player 1. Define

$$\bar{J} \ := \ \{j \in J \mid \bar{\boldsymbol{x}}\boldsymbol{A}_{\bullet j} \ = \ f(\bar{\boldsymbol{x}})\} \tag{54}$$

which graphically indicates the $f_j$ involved in finding the value $v_\Gamma$. The columns $j \in \bar{J}$ indicate the best replies of player 2 against $\bar{\boldsymbol{x}}$. Hence, the optimal strategies of player 2 have to satisfy $\bar{y}_j \ = \ 0 \ \ (j \notin \bar{J})$. Therefore, an aproach to find the optimal strategies for player 1 is described by an attempt to solve the system

$$\begin{aligned} \boldsymbol{A}_{1\bullet}\boldsymbol{y} &= \boldsymbol{A}_{2\bullet}\boldsymbol{y} \\ y_j &= 0 \qquad (j \in \bar{J}^c) \\ \sum_{j\in J} y_j &= 1. \end{aligned} \tag{55}$$

E.g., if it so happens that we have $|\bar{J}| \ = \ 2, \ J \ = \ \{j,k\}$, and

$$\boldsymbol{x}\boldsymbol{A}_{\bullet j} \ = \ \boldsymbol{x}\boldsymbol{A}_{\bullet k}$$

determines $\bar{\boldsymbol{x}}$ uniquely, then the submatrix of $\boldsymbol{A}$ given by

$$\boldsymbol{A}^0 := \begin{pmatrix} a_{1j} & a_{1k} \\ a_{2j} & a_{2k} \end{pmatrix}$$

determines $\bar{\boldsymbol{y}}$ uniquely via

$$\begin{aligned} \boldsymbol{A}^0_{1\bullet}(y_j, y_k) &= \boldsymbol{A}^0_{2\bullet}(y_j, y_k) \\ y_j + y_k &= 1 \\ y_l &= 0 \quad (l \in J,\ l \neq j, k). \end{aligned}$$

The following example illustrates the procedure.

**Example 2.14.** Let

$$\boldsymbol{A} = \begin{pmatrix} 0 & 4 & 1 & 1000 & 3 \\ 5 & 1 & 2 & -2 & 3 \end{pmatrix} \tag{56}$$

The scetch resulting from this example represents the five columns of $\boldsymbol{A}$ by the corresponding straight lines, which depict the graphs of the functions $f_j$ $(j = 1, \ldots, 5)$. The endpoints of these lines are easily determined by inserting the two entries of the corresponding row.

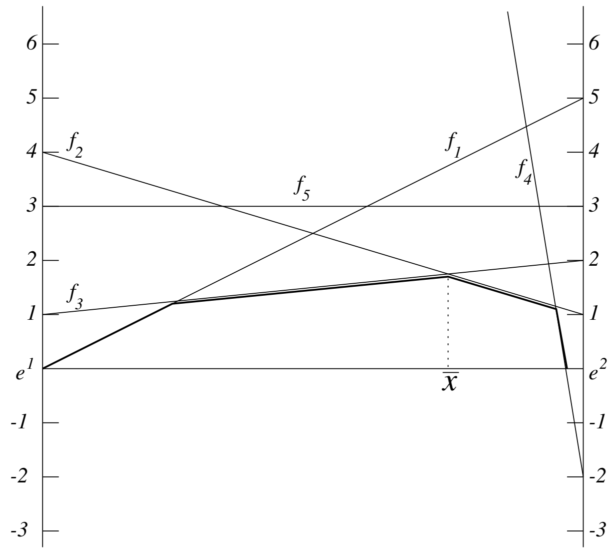



Figure 2.4: Example 2.14

The minimum function $f$ shows a unique maximizer $\bar{\boldsymbol{x}}$ which involves column 2 and column 3. Accordingly, the solution of the system

$$\boldsymbol{x}\boldsymbol{A}_{\bullet 2} = \boldsymbol{x}\boldsymbol{A}_{\bullet 3}\,, \quad x_1 + x_2 = 1 \tag{57}$$

yields

$$4x_1 + x_2 \;=\; x_1 + 2x_2 \;, \quad x_1 + x_2 \;=\; 1$$

and hence

$$\bar{\boldsymbol{x}} \;=\; \left(\frac{1}{4}, \frac{3}{4}\right)$$

which is the optimal strategy of player 1. As none of the pure strategies $i = 1, 4, 5$ is a best reply at $\bar{\boldsymbol{x}}$, we know that the optimal strategies of player 2 will have to satisfy

$$y_1 \;=\; y_4 \;=\; y_5 \;=\; 0.$$

Therefore the solution of

$$\boldsymbol{A}^0_{1\bullet}\boldsymbol{y} \;=\; \boldsymbol{A}^0_{2\bullet}\boldsymbol{y} \;, \quad y_2 + y_3 \;=\; 1, \tag{58}$$

i.e., of

$$4y_2 + y_3 \;=\; y_2 + 2y_3 \;, \quad y_2 + y_3 \;=\; 1,$$

is $\left(\frac{1}{4}, \frac{3}{4}\right)$ which yields $\bar{\boldsymbol{y}} = \left(0, \frac{1}{4}, \frac{3}{4}, 0, 0\right)$ for the optimal strategy of player 2. The value of the game is

$$v_\Gamma \;=\; \boldsymbol{A}_{1\bullet}\bar{\boldsymbol{y}} \;=\; \boldsymbol{A}_{2\bullet}\bar{\boldsymbol{y}} \;=\; \frac{7}{4}.$$

Note that we do not actually use the graphical representation for the practical computation of the relevant data. Rather, the sketch tells us which systems of linear equations we have to solve.

# Chapter 5

# The Lemke–Howson Algorithm

With this chapter we present a version of the algorithm due to LEMKE and HOWSON. This algorithm is a relative of the simplex procedure, yet it moves alternatingly in both the simplices $\underline{\overline{\mathbf{X}}}$ and $\underline{\overline{\mathbf{Y}}}$ of mixed strategies. The algorithm yields at least one Nash equilibrium point. In addition, it provides a surprising insight into the structure of the equilibrium set: given a version of nondegeneracy, this set is nonempty,discrete, and consists of an odd number of equilibria. All these facts are exhibited by the procedure. Thus, in particular, we obtain a further proof for the existence of equilibria.

# 1 Nondegenerate Games

The procedure presented within this chapter is a relative of the simplex algorithm. It is based on the Optimality Criterion (Theorem 1.14) which connects the positive coordinates of an equilibrium strategy with best response properties against the opponents strategy. As we have demonstrated, equilibria can be found as the solutions of certain systems of linear equations, compare e.g. Lemma 1.18 and Remark 1.19. The naive procedure growing out of this idea, i.e., the computation of equilibria by means of the evaluation of all square submatrices of the payoff matrices is an extremely tedious one. Similar to the simplex algorithm, the Lemke–Howson algorithm follows a certain path built up from the edges and vertices of appropriate polyhedra.

These polyhedra, not so unexpected, result from the Optimality Criterion, they are the polyhedra of "best reply". We start out with a formal definition of these polyhedra. Simultaneously, we consider the subsimplices $\overline{\underline{\mathbf{X}}}$ and $\overline{\underline{\mathbf{Y}}}$ defined by the requirement that some coordinates vanish. Again, this seems not so unnatural in view of the Optimality Criterion.

**Definition 1.1.** *Let $\Gamma$ be a bimatrix game. The polyhedra*

$$(1) \qquad \mathbf{K}_i := \left\{ \boldsymbol{y} \in \overline{\underline{\mathbf{Y}}} \,\middle|\, \boldsymbol{A}_{i\bullet}\boldsymbol{y} \geq \boldsymbol{A}_{k\bullet}\boldsymbol{y} \ (k \in I) \right\} \quad (i \in I),$$

*as well as the polyhedra*

$$(2) \qquad \mathbf{K}_T := \bigcap_{i \in T} \mathbf{K}_i \quad (\emptyset \neq T \subseteq I),$$

*are called* ***polyhedra of best reply****. The polyhedra*

$$(3) \qquad \mathbf{Y}_U := \left\{ \boldsymbol{y} \in \overline{\underline{\mathbf{Y}}} \mid y_j = 0 \ \ (j \in U) \right\} \quad (U \subseteq J,\ U \neq J);$$

*are called* ***faces*** *of $\overline{\underline{\mathbf{Y}}}$.*

*Analogously we use $\mathbf{L}_j$ , $\mathbf{L}_R$ , $\mathbf{X}_V$ etc. for $j \in J$ , $\emptyset \neq R \subseteq J$ , $V \subseteq I$, $V \neq I$; these are subsets of $\overline{\underline{\mathbf{X}}}$ . E.g. we have*

$$(4) \qquad L_j \ := \ \left\{ \boldsymbol{x} \in \overline{\underline{\mathbf{X}}} \,\middle|\, \boldsymbol{x}\boldsymbol{B}_{\bullet j} \geq \boldsymbol{x}\boldsymbol{B}_{\bullet l} \quad (l \in J) \right\} \ .$$

**Example 1.2.** In order to clarify the geometric shape of the objects as defined above, let $n = m = 3$ and consider the matrices

$$(5) \qquad \boldsymbol{A} \ = \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{B} \ = \ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \ .$$

We consider the polyhedra of best reply given by

$$\mathbf{L}_1 = \left\{ \boldsymbol{x} \in \underline{\overline{\mathbf{X}}} \,\middle|\, \boldsymbol{x}\boldsymbol{B}_{\bullet 1} \geq \boldsymbol{x}\boldsymbol{B}_{\bullet l} \ (l \in J) \right\} = \left\{ \boldsymbol{x} \in \underline{\overline{\mathbf{X}}} \mid x_3 \geq x_1 \ , \ x_3 \geq x_2 \right\},$$

and

$$\boldsymbol{K}_2 = \left\{ \boldsymbol{y} \in \underline{\overline{\mathbf{Y}}} \,\middle|\, \boldsymbol{A}_{2\bullet}\boldsymbol{y} \geq \boldsymbol{A}_{k\bullet}\boldsymbol{y} \ (k \in I) \right\} = \left\{ \boldsymbol{y} \in \underline{\overline{\mathbf{Y}}} \,\middle|\, y_2 \geq y_1, y_2 \geq y_3 \right\}$$

as well as

$$\mathbf{L}_{23} = \left\{ \boldsymbol{x} \in \underline{\overline{\mathbf{X}}} \mid x_1 = x_2 \geq x_3 \right\}.$$

Also we note that

$$\mathbf{L}_{123} = \mathbf{K}_{123} = \left\{ \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \right\}$$

is rather obvious. It turns out that we obtain the simplex $\underline{\overline{\mathbf{Y}}}$ as the union of all polyhedra of best reply which, whenever we are willing to neglect the boundaries, almost looks like a decomposition. Of course, the boundaries of some $\mathbf{K}_i$ are given by certain $\mathbf{K}_{ij}$ or by the intersection of $\mathbf{K}_i$ with certain $\mathbf{Y}_j$ etc., the geometrical appearance as a decomposition is only superficial. Nevertheless it is most usefull to view the following sketch of the "decomposition" obtained in $\underline{\overline{\mathbf{Y}}}$ and $\underline{\overline{\mathbf{X}}}$ respectively.

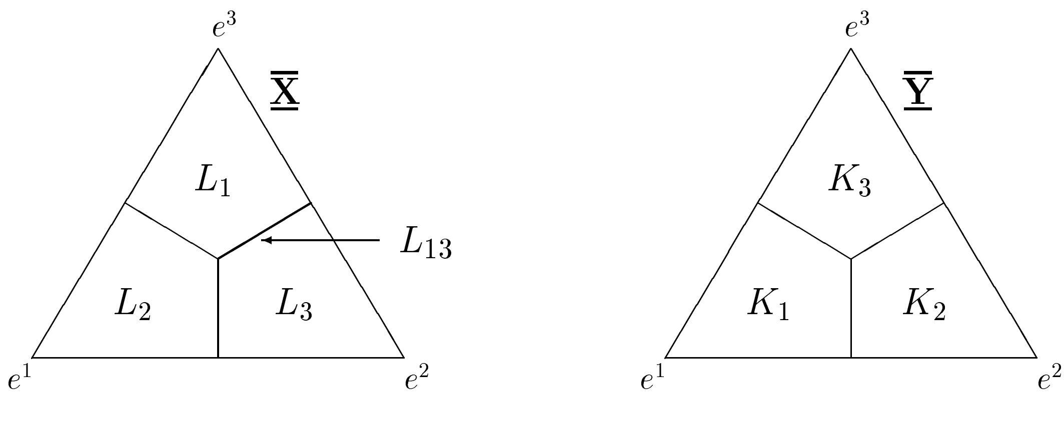



Figure 1.1: The Polyhedra of Best Response in Example 1.2

In view of these geometrical interpretations of the polyhedra of best reply we attempt do reformulate the Optimality Criterion as follows:

**Remark 1.3.** *Let* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}}$. *Define index sets*

$$T := \{i \in I \mid \bar{x}_i > 0\}\ ,\ V := \{i \in I \mid \bar{x}_i = 0\} \tag{6}$$

*as well as*

$$R := \{j \in J \mid \bar{y}_j > 0\}\ ,\ U := \{j \in J \mid \bar{y}_j = 0\} \tag{7}$$

*Then* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ *is an equilibrium if and only if*

$$(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in (\mathbf{L}_R \bigcap \mathbf{X}_V) \times (\mathbf{K}_T \bigcap \mathbf{Y}_U) \tag{8}$$

*holds true.*

*The proof is trivial in view of the Optimality Criterion 1.14. For instance, if* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ *is an equilibrium, then* $\bar{\boldsymbol{y}} \in \mathbf{Y}_U$ *is true by definition, moreover for* $i \in T$ *we have always* $\bar{x}_i > 0$ *and hence* $\boldsymbol{A}_{i\bullet}\boldsymbol{y} = \max_k \boldsymbol{A}_{k\bullet}\boldsymbol{y}$. *That is,* $\bar{\boldsymbol{y}} \in \mathbf{K}_i$ $(i \in T)$ *or* $\boldsymbol{y} \in \mathbf{K}_T$. *And vice versa.*

**Example 1.4.** Let $m = 2, n = 4$ and pick the matrices as follows :

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ -1 & -2 & 1 & 2 \end{pmatrix},\quad \boldsymbol{B} = \begin{pmatrix} -1 & -4 & 5 & 6 \\ 3 & 4 & -3 & -6 \end{pmatrix}. \tag{9}$$

Recall the graphical method employed at the end of SECTION 2. Again we consider the affine function $\boldsymbol{x} \longrightarrow \boldsymbol{x}\boldsymbol{B}_{\bullet j}$ defined on $\underline{\overline{\mathbf{X}}}$; the values of this function at the endpoints of $\underline{\overline{\mathbf{X}}}$ (that is, at $\boldsymbol{e}^1$ and $\boldsymbol{e}^2$) are:

$$\boldsymbol{e}^1\boldsymbol{B}_{\bullet j} = b_{1j} \quad \text{and} \quad \boldsymbol{e}^2\boldsymbol{B}_{\bullet j} = b_{2j}$$

This procedure results in four graphs of the corresponding functions defined on $\underline{\overline{\mathbf{X}}}$ and hence implies the "decomposition" of $\underline{\overline{\mathbf{X}}}$ into the polyhedra of best reply as follows.

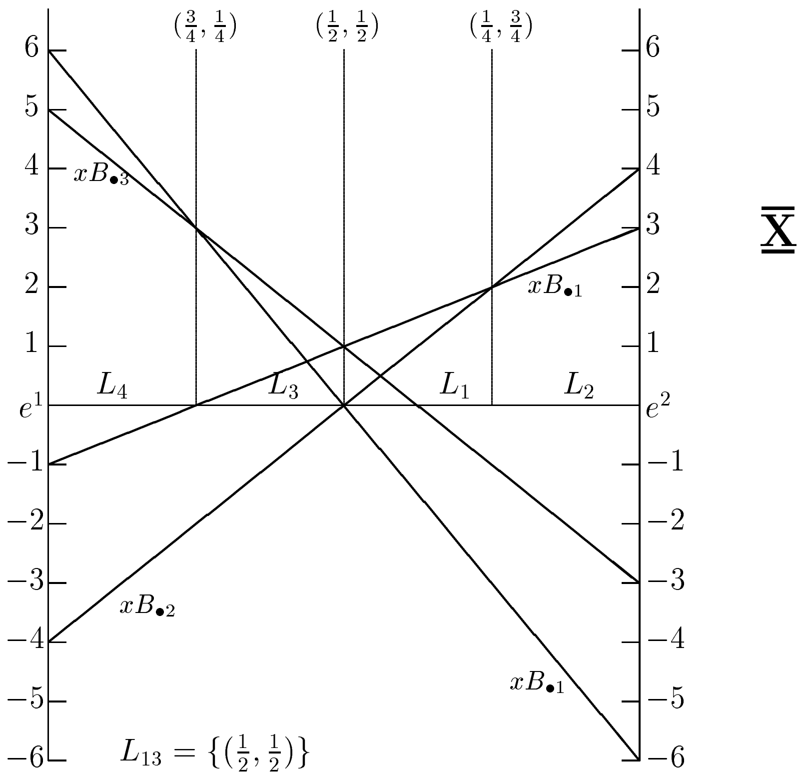



Figure 1.2: The Polyhedra of Best Response; Example 1.4

The simplex $\underline{\overline{\mathbf{Y}}} \subseteq \mathbb{R}^4$ is three dimensional (see Example 1.3) It is easily seen that the points

$$\boldsymbol{y}^1 = (0, 1/2, 0, 1/2) \qquad \boldsymbol{y}^2 = (1/2, 0, 0, 1/2)$$

$$\boldsymbol{y}^3 = (1/2, 0, 1/2, 0) \qquad \boldsymbol{y}^4 = (0, 1/2, 1/2, 0)$$

are located on the hyperplane

$$\{\boldsymbol{y} \in \mathbb{R}^4 \mid \boldsymbol{A}_{1\bullet}\boldsymbol{y} = \boldsymbol{A}_{2\bullet}\boldsymbol{y}\} \ ,$$

the (two-dimensional) intersection with $\underline{\overline{\mathbf{Y}}}$ of which can be represented graphically. We obtain the following sketch:

The polyhedra $\mathbf{K}_i$ "decompose" the simplex $\underline{\overline{\mathbf{Y}}}$ (not properly); the "edges" constitute a "*net*" in $\underline{\overline{\mathbf{Y}}}$. An equilibrium is given by

$$(\bar{\boldsymbol{x}}\ ,\ \bar{\boldsymbol{y}}) = \left( (\frac{1}{2}\ , \frac{1}{2}), \boldsymbol{y}^3 \right) \ .$$

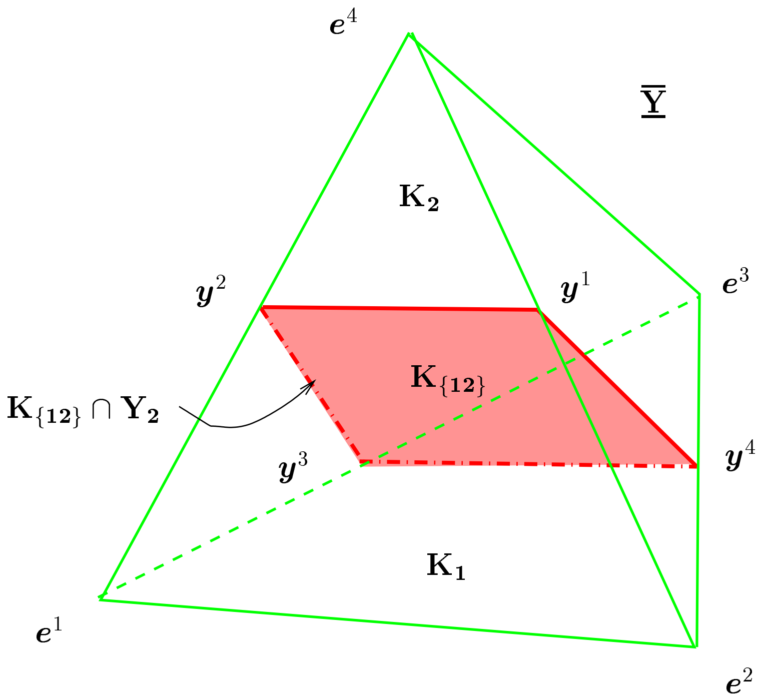



Figure 1.3: The Decomposition in Example 1.4

This is easily verified in view of the Optimality Criterion - or else geometrically in view of Remark 1.3. For indeed, we have

$$
\begin{aligned}
(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) &\in (\mathbf{L}_{\{1,3\}} \bigcap \mathbf{X}_{\emptyset}) \times (\mathbf{K}_{\{1,2\}} \bigcap \mathbf{Y}_{\{24\}}) \\
&= (\mathbf{L}_{T^c} \bigcap \mathbf{X}_{\emptyset}) \times (\mathbf{K}_{J} \bigcap \mathbf{Y}_{T}) \,.
\end{aligned}
$$

Observe that, in addition, in this case the polyhedra of best reply and the faces generated by $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ do not contain any further points. That is, we obtain

$$\left(\mathbf{L}_{\{1,3\}} \bigcap \mathbf{X}_{\emptyset}\right) \times \left(\mathbf{K}_{\{1,2\}} \bigcap \mathbf{Y}_{\{2,4\}}\right) = \{\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}\}$$

This property is (among others) die main object of our next definition. It is a version of *nondegeneracy* that is established by this kind of consideration. Again it is important to observe that the index sets at $\mathbf{Y}_{\bullet}$ and $\mathbf{L}_{\bullet}$ (at $\mathbf{X}_{\bullet}$ and $\mathbf{K}_{\bullet}$) are complementary when we are dealing with an equilibrium.

Visually, we observe a kind of "net" which (if we focus on $\underline{\overline{\mathbf{Y}}}$) is generated by the corresponding equations. E.g.,

$$\mathbf{K}_2 \bigcap \mathbf{Y}_{24} \ \text{ by } \ y_2 = 0, \ y_4 = 0$$

and

$$\mathbf{K}_{12} \bigcap \mathbf{Y}_2 \text{ by } \boldsymbol{A}_{1\bullet}\boldsymbol{y} = \boldsymbol{A}_{2\bullet}\boldsymbol{y},\ y_2 = 0\ .$$

Here the index sets are complimentary up to one index. Indeed, it is a further aim of the next definition to make sure that the polyhedra we have mentioned always represent one dimensional lines or points as is indicated.

**Definition 1.5.** *1. We would like to associate dimensions with polyhedra. The dimension of a convex polyhedron is defined to be the dimension of the smallest linear subspace containing this polyhedron.*

*2. Negative dimensions will be associated with the empty set.*

*3. For $\emptyset \neq T \subseteq I$ and $U \subseteq J,\ U \neq J$ let*

$$\mathbf{H}_{T,U} := \mathbf{K}_T \bigcap \mathbf{Y}_U \subseteq \underline{\overline{\mathbf{Y}}}$$

*4. Analogously we define $\mathbf{G}_{R,V} := \mathbf{L}_R \bigcap \mathbf{X}_V \subseteq \underline{\overline{\mathbf{X}}}$ .*

We are now in the position to formulate the version of nondegeneracy that is appropriate within the context of mixed bimatrix games.

**Definition 1.6.** *1. A bimatrix game $\Gamma$ is said to be **nondegenerate** if, for all $\emptyset \neq T \subseteq I$ and $U \subseteq J,\ U \neq J$ the following holds true:*

$$\mathbf{H}_{T,U} \neq \emptyset \Rightarrow \ dim\ \mathbf{H}_{T,U} = n- \mid T \mid - \mid U \mid \tag{10}$$

*and analogously for all $\emptyset \neq R \subseteq J$ and $V \subseteq I,\ V \neq I$,*

$$\mathbf{G}_{R,V} \neq \emptyset \Rightarrow \ dim\ \mathbf{G}_{R,V} = m- \mid R \mid - \mid V \mid \ . \tag{11}$$

*2. If $\mathbf{H}_{T,U} \neq \emptyset$ and $\mid T \mid + \mid U \mid = n$ holds true, then the unique element $\bar{\boldsymbol{y}} \in \mathbf{H}_{T,U}$ (and the set $\mathbf{H}_{T,U}$ as well) is called a **vertex** ("of the $\underline{\overline{\mathbf{Y}}}$-net"). If $\mid T \mid + \mid U \mid = n-1$ then $\mathbf{H}_{T,U}$ is said to be an **edge** ("of the $\underline{\overline{\mathbf{Y}}}$-net"). (The analogous definitions hold true with respect to $\underline{\overline{\mathbf{X}}}$ or, more figuratively, for the $\underline{\overline{\mathbf{X}}}$-net).*

**Remark 1.7.** *1. An easy computation reveals that the intersection of two poyhedra of the above type is obtained as follows:*

$$\begin{aligned} & \mathbf{H}_{T,U} \bigcap \mathbf{H}_{T',U'} \\ = \ & (\mathbf{K}_T \bigcap \mathbf{Y}_U) \bigcap (\mathbf{K}_{T'} \bigcap \mathbf{Y}_{U'}) \\ = \ & (\mathbf{K}_T \bigcap \mathbf{K}_{T'}) \bigcap (\mathbf{Y}_U \bigcap \mathbf{Y}_{U'}) \\ = \ & \mathbf{K}_{T \cup T'} \bigcap \mathbf{Y}_{U \cup U'} \\ = \ & \mathbf{H}_{T \cup T'\ ,\ U \cup U'} \end{aligned}$$

2. *Let* $\mid T \mid + \mid U \mid = n$ *and consider the corresponding system of linear equations satisfied by* $y \in \mathbf{H}_{T,U}$*:*

$$\begin{aligned} \boldsymbol{A}_{i\bullet}\boldsymbol{y} &= \max_{k\in I} \boldsymbol{A}_{k\bullet}\boldsymbol{y} && (i \in T) \\ y_j &= 0 && (j \in U) \\ \sum_{j\in J} y_j &= 1 \end{aligned}$$

*As previously (compare* SECTION *1) it is advantagous to introduce a new variable* $\lambda$ *and to focus in the system of linar equations given by*

$$(12) \qquad \begin{aligned} \boldsymbol{A}_{i\bullet}\boldsymbol{y} - \lambda &= 0 && (i \in T) \\ y_j &= 0 && (j \in U) \\ \sum_{j\in J} y_j &= 1 && . \end{aligned}$$

*The coefficient matrix is square as equation* (12) *consists of*

$$\mid T \mid + \mid U \mid + 1 = n + 1$$

*equations in* $n+1$ *variables. If it is nonsingular, then* $(\bar{\boldsymbol{y}}, \bar{\lambda})$ *is the only solution of the above system, that is, the nondegeneracy condition* (10) *is satisfied with respect to this particular set* $\mathbf{H}_{T,U}$*. Obviously, nondegeneracy means that the dimension of the subpolyhedra of best reply is obtained by just counting the equations involved.*

**Example 1.8.** The following sketches show situations that do *not* occur when nondegeneracy prevails (Figure 1.4). E.g. regarding the left side sketch, the unit vector $\boldsymbol{e}^1$ is contained in $\mathbf{K_{23}}$, i.e., $\emptyset \neq \mathbf{K}_{23} \cap \mathbf{Y}_{23} = \mathbf{H}_{\{\mathbf{23}\}\{\mathbf{23}\}}$. Nondegeneracy would require that the dimension of this polyhedron is $-1$. More generally: a unit vector can be located in exactly one polyhedron $\mathbf{K}_i$ only.

Similarly, the right hand figure suggests that $\emptyset \neq \mathbf{K}_{123} \cap \mathbf{Y}_3 = \mathbf{H}_{\{\mathbf{123}\}\{\mathbf{3}\}}$ is true, which again contradicts nonegeneracy.

Here are some first simple properties that hold true for nd. games.

**Corollary 1.9.** *Suppose* $\Gamma$ *is a nondegenerate game. Then the following holds true:*

1. *The number of equilibria is finite.*

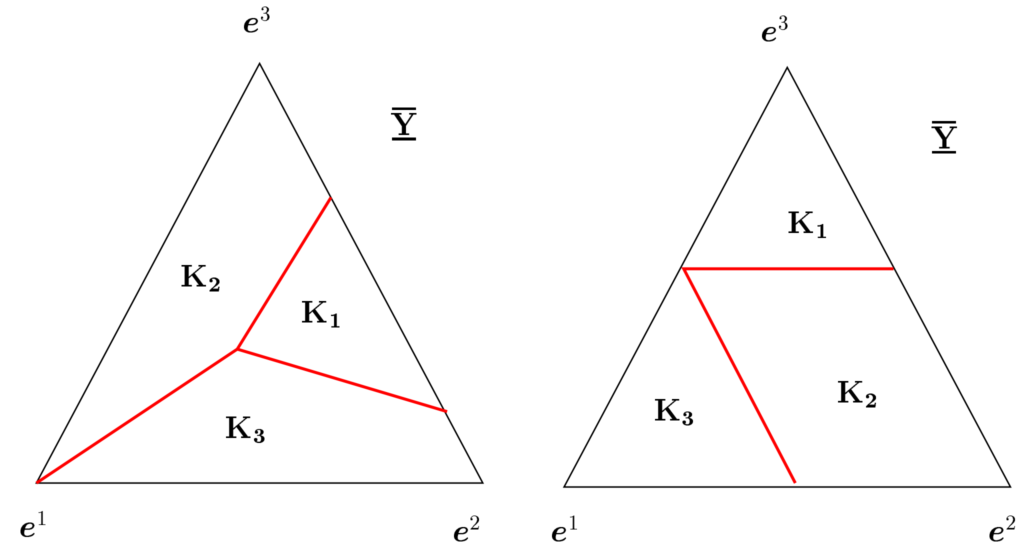



Figure 1.4: Best Response Polyhedra with Degeneracy

2. *If* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}}$ *is an equilibrium and*

$$T = \{i \mid \bar{x}_i > 0\} \quad R = \{j \mid \bar{y}_j > 0\}$$

*denotes the corresponding sets of positive coordinates, then* $\mid T \mid = \mid R \mid$ *and*

$$(\mathbf{L}_R \cap \mathbf{X}_{T^c}) \times (\mathbf{K}_T \cap \mathbf{Y}_{R^c}) = \{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})\}$$

*holds true.*

3. $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}}$ *is an equilibrium if and only if the following (stronger) version of the Optimality Criterion holds true:*

$$\begin{aligned} \bar{x}_i > 0 &\iff \boldsymbol{A}_{i\bullet}\bar{\boldsymbol{y}} = \max_{k \in I} \boldsymbol{A}_{k\bullet}\bar{\boldsymbol{y}} \quad (i \in I) \\ \bar{y}_j > 0 &\iff \bar{\boldsymbol{y}}\boldsymbol{B}_{\bullet j} = \max_{l \in J} \bar{\boldsymbol{x}}\boldsymbol{B}_{\bullet l} \quad (j \in J). \end{aligned} \tag{13}$$

*That is, at equilibrium each player puts positive probability exactly on the best responses against the opponents mixed strategy.*

**Proof:**

**1st STEP** :

First of all, we prove the second statement. Let $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ be an equilibrium. In view of Remark 1.3 we have (using $T := \{i \in I | \bar{\boldsymbol{x}}_i > 0\}$, $R := \{j \in J | \bar{\boldsymbol{y}}_j > 0\}$)

$$(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \mathbf{G}_{R,T^c} \times \mathbf{H}_{T,R^c}.$$

As we have assumed nondegeneracy, we conclude that

$$\begin{array}{lcccc} (14) & 0 \;\leq & \boldsymbol{dim}\ \mathbf{H}_{T,R^c} & = & n- \mid T \mid - \mid R^c \mid \\ & = & n- \mid T \mid -(n-|R|) & = & \mid R \mid - \mid T \mid, \end{array}$$

holds true. That is, we obtain the inequality $\mid R \mid \geq \mid T \mid$ .

By reasons of symmetry we have as well

$$0 \;\leq\; \boldsymbol{dim}\ \mathbf{G}_{R,T^c} \;=\; \mid T \mid - \mid R \mid \;,$$

and hence $\mid T \mid = \mid R \mid$. Obviously equation (14) implies now $\boldsymbol{dim}\ \mathbf{H}_{T,R^c} = 0$.

**2ndSTEP** : The last statement of our Theorem is now obvious: If $i \in \mathbf{K}_i$ and $\bar{x}_i = 0$ holds true for some $i \in \boldsymbol{I}$, then $i \notin T$, hence

$$\bar{\boldsymbol{y}} \in \mathbf{K}_{T+i} \cap \mathbf{Y}_R$$

but

$$\boldsymbol{dim}\ \mathbf{H}_{T+i,R^c} = n - (|T|+1) - |R^c| = -1 \;,$$

contradicting nondegeneracy.

**3rdSTEP** :

It remains to prove the first statement: the number of equilibria is finite.

Given any equilibrium $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$, define $R$ and $T$ as previously and consider the linear system of equations in variables $(\boldsymbol{y}, \lambda) \in \mathbb{R}^{n+1}$ given by

$$\begin{array}{llll} (15) & \boldsymbol{A}_{i\bullet}\boldsymbol{y} - \lambda & = \; 0 & (i \in T) \\ & y_i & = \; 0 & (j \in R^c) \\ & \sum_{j \in J} y_j & = \; 1 & . \end{array}$$

Define

$$\bar{\lambda} \;:=\; \max_k \boldsymbol{A}_{k\bullet}\bar{\boldsymbol{y}} \;,$$

then the pair $(\bar{\boldsymbol{y}}, \bar{\lambda})$ is a solution of the linear system 15. We claim that it is the only solution.

Indeed, assume that there is a second solution $(\hat{\boldsymbol{y}}, \lambda)$ . Then, for sufficiently small $\varepsilon > 0$, define

$$\boldsymbol{y}^{\varepsilon} := (1-\varepsilon)\bar{\boldsymbol{y}} + \varepsilon\hat{\boldsymbol{y}}.$$

It turns out that, for small $\varepsilon$, $\boldsymbol{y}^{\varepsilon}$ is an element of $\underline{\overline{\mathbf{Y}}}$. For given $j \in R^c$, we have $\boldsymbol{y}^{\varepsilon}_j = 0$, and for $j \in R$ we have $\bar{y}_j > 0$, therefore $\bar{y}^{\varepsilon}_j > 0$ if $\varepsilon$ is sufficiently small.

In addition we must have $\boldsymbol{A}_{k\bullet}\bar{\boldsymbol{y}} < \bar{\lambda} \quad (k \notin T)$, for otherwise the non degeneracy assumption is violated. Therefore, if $\varepsilon$ is sufficiently small, it follows that

$$\boldsymbol{A}_{i\bullet}\boldsymbol{y}^{\varepsilon} = (1-\varepsilon)\bar{\lambda} + \varepsilon\widehat{\lambda} > \boldsymbol{A}_{k\bullet}\boldsymbol{y}^{\varepsilon}$$

is satisfied for $i \in T$ , $k \in T^c$ .

Hence, we conclude that $\boldsymbol{y}^{\varepsilon} \in \mathbf{H}_{T,R^c}$ is true indeed. But then

$$\boldsymbol{dim}\ \mathbf{H}_{T,R^c} \geq 1$$

follows, which is impossible in view of the nondegeneracy condition.

Therefore, $(\bar{\boldsymbol{y}}, \bar{\lambda})$ is the unique solution of (15). Now as there are only finitely many systems of the nature indicated in (15) (each one corresponding to a pair $R, T$), it follows that there are only finitely many equilibria, so our first statement is verified, **q.e.d.**

Now we begin to collect the first interesting facts concerning the structure of the "net" appearing in $\underline{\overline{\mathbf{Y}}}$.

**Lemma 1.10.** *Let $\Gamma$ be non degenerate and let $\bar{\boldsymbol{y}}$ be a vertex in $\underline{\overline{\mathbf{Y}}}$. If $\bar{\boldsymbol{y}}$ is a basis vector then there are exactly $n-1$ edges containing $\bar{\boldsymbol{y}}$. Otherwise, if $\bar{\boldsymbol{y}}$ is not a basisvector, there are exactly $n$ edges containing $\bar{\boldsymbol{y}}$.*

**Proof: 1<sup></sup>**

**Proof: $\mathbf{1^{st}}$STEP** : Let us assume that $\bar{\boldsymbol{y}} = \boldsymbol{e}^1 \in \mathbf{Y}_{\{2,\ldots,n\}}$ holds true. We cannot find $i, k \in I$ such that

$$\bar{\boldsymbol{y}} \in \mathbf{K}_i \cap \mathbf{K}_k$$

is satisfied (cf. Example 1.8, Figure 1.4 and Figure 1.5). For otherwise we would have $\bar{\boldsymbol{y}} \in \mathbf{H}_{\{i,k\},\{2,\ldots,n\}}$ and $\boldsymbol{dim}\ \mathbf{H}_{\{i,k\},\{2,\ldots,n\}} = n-(n+1) = -1$, hence $\mathbf{H}_{\{i,k\},\{2,\ldots,n\}} = \emptyset$. Consequently, the index $i$ satisfying $\bar{\boldsymbol{y}} \in \mathbf{K}_i$ is uniquely determined.

Therefore we find a unique $i \in I$ satisfying $\bar{\boldsymbol{y}} \in \mathbf{H}_{\{i\},\{2,\ldots n\}}$. It follows all the more that $\bar{\boldsymbol{y}} \in \mathbf{H}_{\{i\},\{2,\ldots,j-1,j+1,\ldots,n\}} := \mathbf{H}^{(j)}$ holds true for $j = 2, \ldots, n$. The polyhedra $\mathbf{H}^{(j)}$ are exactly the $n-1$ edges containing $\bar{\boldsymbol{y}}$.

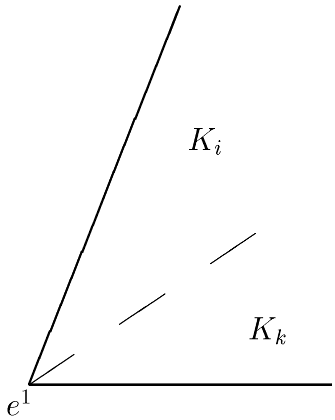



Figure 1.5: Nondegeneracy prohibits a unit vector in $K_i \cap K_k$

A further edge cannot be of the form $\mathbf{H}_{T',U'}$ with $| T' | \geq 2$. Because of Remark 1.7 we would have

$$\bar{\boldsymbol{y}} \in \mathbf{H}_{\{i\},\{2,\ldots,n\}} \cap \mathbf{H}_{T',U'} = \mathbf{H}_{T' \cup \{i\},\{2,\ldots,n\} \cup U'}$$

which again is a contradiction to nondegeneracy. That is, the "obvious edges" are all the edges which possibly contain our unit vector.

**2$^{\text{nd}}$STEP** : Now let us argue with respect to the alternative. Suppose $\bar{\boldsymbol{y}}$ is not a basis vector and assume that

$$\{\bar{\boldsymbol{y}}\} = \mathbf{H}_{T,U}$$

with $| T | \geq 2$ and $| T | + | U | = n$ holds true.

Once again we have $n$ "obvious edges" available which contain $\bar{\boldsymbol{y}}$. These are

$$\mathbf{H}_{T-i,U} \quad (i \in T)$$

and

$$\mathbf{H}_{T,U-j} \quad (j \in U)$$

Now if $\mathbf{H}_{T',U'}$ is a further edge, then again we would have $\bar{\boldsymbol{y}} \in \mathbf{H}_{T \cup T',U \cup U'}$ and because of $| T | + | U | = n$ it would follow that $| T \cup T' | = | T |$ and $| U \cup U' |$, hence $T' \subseteq T, U' \subseteq U$. This means of course that again $\mathbf{H}_{T',U'}$ is one of the "obvious edges", which finishes our proof. **q.e.d.**

In the future we frequently claim that in view of non degeneracy it is sufficient to consider the "obvious edges". The proofs will be omitted as they are in principle most of the time a complete repetition of the argument we have just given above.

**Example 1.11.** Consider again the situation reflected by Figure 1.3. Here $n = 4$. The basis vector $\boldsymbol{e}^1$ is located exactly in $\mathbf{K}_1$, in fact we have $\boldsymbol{e}^1 \in \mathbf{H}_{\{1\},\{2,\ldots n\}} = \mathbf{K}_1 \cap \mathbf{Y}_{\{2,\ldots n\}}$.

The obvious $n-1 = 3$ edges adjcent to $\boldsymbol{e}^1$ are obtained by removing just one index $i \neq 1$ from $\{2, \ldots n\}$.

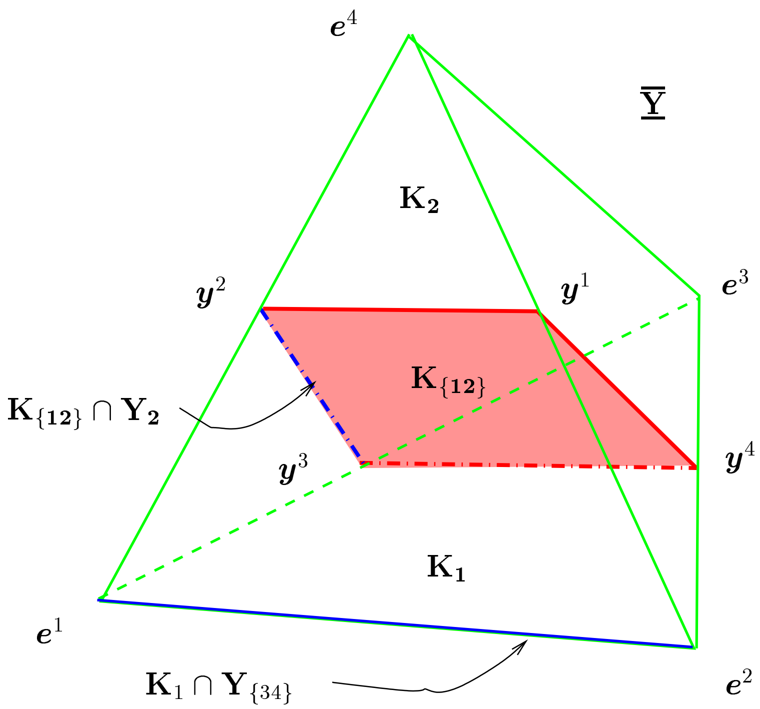



Figure 1.6: Example 1.4 revisited

E.g., the edge $H^{(2)} := \mathbf{H}_{\{1\},\{3,\ldots n\}} = \mathbf{K}_1 \cap \mathbf{Y}_{\{3,\ldots n\}}$ is the one obtained by admitting the $2nd$ coordinate to become positive. Or, more colloquially, by dropping the equation $x_2 = 0$ we start moving into the direction of the $2nd$ unit vector.

Consider, on the other hand, the vertex $\boldsymbol{y}^2 \in \mathbf{K}_{\{12\}} \cap \mathbf{Y}_{\{23\}}$. Now, there are 4 edges adjacent. E.g., we can admit positive $3rd$ coordinates (i.e., drop the index 3 with respect to $\mathbf{Y}_{\{23\}}$, drop the equation $x_3 = 0$), this way obtaining the edge $\mathbf{K_{\{12\}}} \cap \mathbf{Y_{\{2\}}}$ . Or else we can drop the index 1 with respect to $\mathbf{K}_{\{12\}}$ (leave the compartment $\mathbf{K}_1$ this way moving towards $\boldsymbol{e}^4$ along the edge $\mathbf{K_2} \cap \mathbf{Y_{\{23\}}}$.

## 2 Alternating Moves

Now we have to slightly change our viewpoint: we shall have to consider elements of a "net" in the product $\underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}}$. The reason is as follows: the Lemke–Howson algorithm to be presented formally works in the Cartesian product $\underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}}$. It moves from one vertex of the hypothetical net to the next on along an edge (all this taking place in $\underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}}$). The situation in one of the simplices induces the conditions for the movement working on the other one.

Later on it is realized that the single steps can be interpreted as alternating movements in $\underline{\overline{\mathbf{X}}}$ or $\underline{\overline{\mathbf{Y}}}$. That is, the present algorithm is a relative to the simplex algorithm but essentially proceeds alternatingly in both simplicies.

**Remark 2.1.** *In the following version* ***the index*** $n \in J$ ***plays an exceptional role*** *throughout all of the presentation. Eventually, it will be argued that this particular role can be played by any other index. It is just a convenience of a notation to take* $n$*.*

**Remark 2.2.** *Let* $\emptyset \neq T \subseteq I$ *and* $J \neq R \subseteq J$ *satisfy* $\mid T \mid = \mid R \mid +1, \ \ n \notin R$*. Pick* $i_0 \in T$ *and* $j_0 \neq n, \ j_0 \notin R$*. Then we have*

$$\begin{array}{rcll} \mid T \mid + \mid R^c - n \mid & = & \mid R \mid +1+ \mid R^c - n \mid & = n \\ \mid R \mid + \mid T^c + i_0 \mid & = & \mid R \mid +m- \mid T - i_0 \mid & = m \\ \mid R + j_0 \mid + \mid T^c \mid & = & \mid T \mid + \mid T^c \mid & = n \ . \end{array}$$

*The result is obvious by just counting the number of indices.*

**Definition 2.3.** *Let* $\emptyset \neq T \subseteq I$ *and* $J \neq R \subseteq J$ *satisfy* $\mid T \mid = \mid R \mid +1, \ \ n \notin R$*. A nonempty set*

$$(1) \qquad (\mathbf{L}_R \cap \mathbf{X}_{T^c+i_0}) \times (\mathbf{K}_T \cap \mathbf{Y}_{R^c-n}) \quad (i_0 \in T)$$

*is called an* ***admissible product vertex*** *(for short a* $\mathbf{\Pi}$***–vertex****), provided it is not empty.*

*Similarly, a nonempty set*

$$(2) \qquad (\mathbf{L}_{R+j_0} \cap \mathbf{X}_{T^c}) \times (\mathbf{K}_T \cap \mathbf{Y}_{R^c-n}) \quad (j_0 \notin R, j_0 \neq n)$$

*called an* ***admissible product vertex*** *(for short a* $\mathbf{\Pi}$***–vertex****).*

Because of nondegeneracy and in view of Remark 2.2, the sets involved in the cartesian products do have dimension 0, thus they just contain a single point. Therefore, a $\mathbf{\Pi}$–vertex consists of a pair of two vertices in $\underline{\overline{\mathbf{X}}}$ and $\underline{\overline{\mathbf{Y}}}$

respectively. The unique point within this set, say $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ will of course also be called "a $\boldsymbol{\Pi}$–vertex."

Consider for some $i_0 \in T$ the $\boldsymbol{\Pi}$–vertex

$$(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) = (\mathbf{L}_R \cap \mathbf{X}_{T^c + i_0}) \times (\mathbf{K}_T \cap \mathbf{Y}_{R^c - n})$$

Obviously $\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}$ satisfies:

If $j \in R$, then $\bar{y}_j > 0$ and $\bar{\boldsymbol{x}} \in \mathbf{L}_R$,
(this ist one part of the optimality criterion),
if $i \in T - i_0$, then $\bar{x}_i > 0$ and $\bar{\boldsymbol{y}} \in \mathbf{K}_T$
(and again, this is part of the optimality criterion).

However, the indices $n$ and $i_0$ play a particular role: $\bar{\boldsymbol{y}}_n > 0$ holds true - but $\bar{\boldsymbol{x}} \in \mathbf{L}_n$ is not satisfied. And on the other hand we have $\bar{\boldsymbol{y}} \in \mathbf{K}_{i_0}$ and $\bar{x}_{i_0} > 0$ is *not* satisfied.

That is, two conditions of the (*stronger*) optimality criterion (cf. Corollary 1.9, formula (13)), namely

$$(\boldsymbol{y}_n > 0 \Leftrightarrow \bar{\boldsymbol{x}} \in \mathbf{L}_n)$$

and

$$(\bar{x}_{i_0} > 0 \Leftrightarrow \bar{\boldsymbol{y}} \in \mathbf{K}_{i_0})$$

are violated. In this sense we could call $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ an *almost equilibrium point.*

Next we turn to a particular type of edge in the cartesian product $\underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}}$. In order to have a one–dimensional line segment, we expect an "edge" to be a pair which consists of a vertex and an edge in either one of the two simplices respectively.

**Definition 2.4.** *Let* $\mid R \mid \leq \mid T \mid \leq \mid R \mid +1$ *and* $n \notin R$*. The set*

$$(\mathbf{L}_R \cap \mathbf{X}_{T^c}) \times (\mathbf{K}_T \cap \mathbf{Y}_{R^c - n}) \tag{3}$$

*is called an* ***admissible product edge*** *(for short a* $\boldsymbol{\Pi}$*–edge). We shall say that* $\boldsymbol{\Pi}$*–edges satisfying*

$\mid R \mid = \mid T \mid$ *represent a "motion" in* $\underline{\overline{\mathbf{Y}}}$

*and* $\boldsymbol{\Pi}$*–edges satisfying*

$\mid R \mid = \mid T \mid -1$ *represent a "motion" in* $\underline{\overline{\mathbf{X}}}$*.*

We are indeed dealing with an edge in the cartesian product.

For, if $|R| = |T|$, then $|R| + |T^c| = n$, hence the left hand factor in (3) constitutes a vertex in $\underline{\overline{\mathbf{X}}}$. The right hand side constitues an edge in $\underline{\overline{\mathbf{Y}}}$ as $|T| + |R^c - n| = n - 1$ is true. To speak of a "motion in $\underline{\overline{\mathbf{Y}}}$" reflects the idea that "moving along the edge (3)" (which the algorithm shall perform in a typical step) actually means fixing a vertex in $\underline{\overline{\mathbf{X}}}$ and "moving along an edge in $\underline{\overline{\mathbf{Y}}}$ ".

If, on the other hand, $\mid R \mid = \mid T \mid -1$ holds true, the the situation is symmetric, but constitutes an edge in $\underline{\overline{\mathbf{X}}}$ and a vertex in $\underline{\overline{\mathbf{Y}}}$. We reflect the situation in Figure 2.1

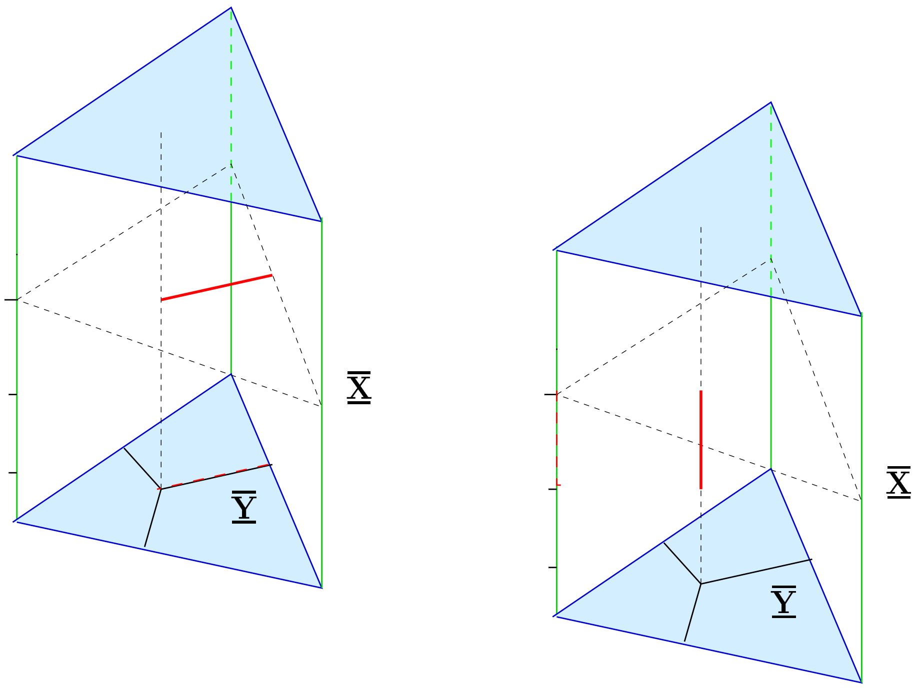



Figure 2.1: $\mathbf{\Pi}$–edges as motions in

The final definition concerns a particular vertex that will serve as the starting point or initial vertex for the algorithm. *Recall that $n$ plays an exeptional role within the presentation.*

**Definition 2.5.** *Let $i_0 \in I$ be such that $\boldsymbol{e}^n \in \mathbf{K}_{i_0}$ (that is, $a_{i_0 n} > a_{kn} (k \in I)$). Then $(\boldsymbol{e}^{i_0}\ ,\ \boldsymbol{e}^n)$ is called the **initial $\mathbf{\Pi}$–vertex**.*

The algorithm reflects a motion along certain vertices and edges that takes place in $\underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}}$. It starts at the $\mathbf{\Pi}$–vertex and is restricted to $\mathbf{\Pi}$ vertices and

**Π**–edges such that for each **Π**–vertex an edge is constructed leading to the "next" **Π**–vertex. The **Π**–edges in $\underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}}$ always consist of a pair (vertex and edge) in the $\underline{\overline{\mathbf{X}}}$–net and the $\underline{\overline{\mathbf{Y}}}$–net respectively. It will turn out that the "motion" alternatingly takes place in $\underline{\overline{\mathbf{X}}}$ and $\underline{\overline{\mathbf{Y}}}$.

**Definition 2.6.** *The **LH–net** is the set $\mathfrak{L}$ that consists of the initial* **Π**–*vertex, all* **Π**–*vertices, all* **Π**–*edges and all Nash equilibrium points.*

It will be our aim to, simultaneously with defining the algorithm, exhibit the structure of the LH–net.

**Example 2.7.** We return to Example 1.4. Here we have $m = 2$ and $n = 4$ and the decomposition of the two simplices is repeated in Figure 2.2

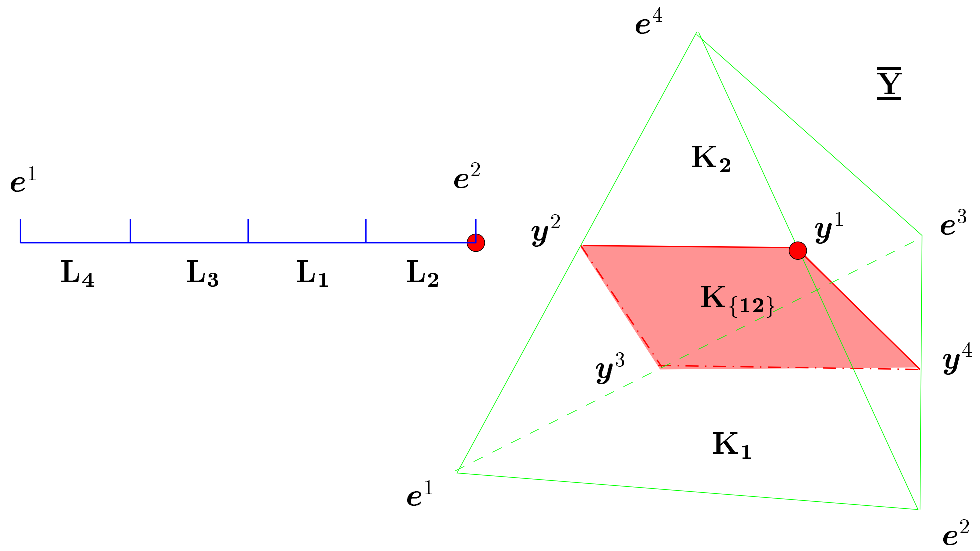



Figure 2.2: A **Π** Vertex in Example 2.7

shows that $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ with

$$\bar{\boldsymbol{x}} = \boldsymbol{e}^2 = (0, 1), \quad \bar{\boldsymbol{y}} = \boldsymbol{y}^1 = (0, \frac{1}{2}, 0, \frac{1}{2})$$

is an admissible **Π**–vertex: We have $(n = 4)$ and the following relations can be verified.

$$\begin{aligned} \bar{\boldsymbol{y}}_2 > 0 \Rightarrow &\quad \bar{\boldsymbol{x}} \in \mathbf{L}_2 \\ \bar{\boldsymbol{y}}_n > 0 \Rightarrow &\quad \cdots \\ \bar{\boldsymbol{x}}_2 > 0 \Rightarrow &\quad \bar{\boldsymbol{y}} \in \mathbf{K}_2 \\ \cdots &\quad \bar{\boldsymbol{y}} \in \mathbf{K}_1 \end{aligned}$$

This means that the case ($i_0 = 1$) prevails. A more formal way to represent this situation is as follows:

$$\{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})\} = (\mathbf{L}_2 \cap \mathbf{X}_{\{1\}}) \times (\mathbf{K}_{\{1,2\}} \cap \mathbf{Y}_{\{1,3\}})$$

Now come up with the first essential theorem which regulates the start of our procedure:

**Theorem 2.8.** *The initial* $\mathbf{\Pi}$*–vertex*

$$(\boldsymbol{e}^{i_0}, \boldsymbol{e}^n)$$

*is either admissible or an equilibrium and not both. If it is admissible, then it is located at exactly one admissible* $\mathbf{\Pi}$*–edge. If it is not admissible, then it is located at no* $\mathbf{\Pi}$*–edge.*

**Proof:** We have chosen $i_0$ in a way such that $\boldsymbol{e}^n \in \mathbf{K}_{i_0}$ is true. Assume first of all that $\boldsymbol{e}^{i_0} \notin \mathbf{L}_n$ holds true. Then $\boldsymbol{e}^{i_0} \in \mathbf{L}_{j_0}$ for a suitable $j_0 \neq n$. Hence, we have

$$\{(\boldsymbol{e}^{i_0}, \boldsymbol{e}^n)\} = (\mathbf{L}_{j_0} \cap \mathbf{X}_{I-i_0}) \times (\mathbf{K}_{i_0} \cap \mathbf{Y}_{J-n})$$

with $j_0 \neq n$. Obviously $(\boldsymbol{e}^{i_0}, \boldsymbol{e}^n)$ is a $\mathbf{\Pi}$–vertex, compare the second equation (2) in Definition 2.3. Now exactly the $\mathbf{\Pi}$–edge

$$\mathbf{L}_{j_0} \cap \mathbf{X}_{I-i_0}) \times (\mathbf{K}_{i_0} \cap \mathbf{Y}_{J-n-j_0}) \tag{4}$$

is admissible and contains the initial $\mathbf{\Pi}$–vertex $(\boldsymbol{e}^{i_0}, \boldsymbol{e}^n)$.

On the other hand, assume now that $\boldsymbol{e}^{i_0} \in \mathbf{L}_n$ is true. Then $(\boldsymbol{e}^{i_0}, \boldsymbol{e}^n)$ is an equilibrium. The only $\underline{\overline{\mathbf{X}}}$-edges at $\boldsymbol{e}^{i_0}$ are of the form

$$\mathbf{L}_{j_0} \cap \mathbf{X}_{I-i_0-i} \qquad (i \in I \ , \ i \neq i_0)$$

but since $\boldsymbol{e}^n \in \mathbf{K}_{i_0}$ and $\boldsymbol{e}^n \notin \mathbf{K}_i$, the corresponding $\mathbf{\Pi}$–edges are not of the correct form as required.

Analogously the only edges at $\boldsymbol{e}^n$ are of the form

$$\mathbf{K}_{i_0} \cap \mathbf{Y}_{J-n-j} \qquad (j \in J, j \neq n)$$

and again these together with $\boldsymbol{e}^{i_0}$ do not yield an admissible $\mathbf{\Pi}$–edge.

**q.e.d.**

Consider again the case that $\boldsymbol{e}^{i_0} \notin \mathbf{L}_n$ holds true in the above situation. Note that the $\mathbf{\Pi}$–edge constructed, i.e., the one indicated in (4), reflects a motion in $\underline{\overline{\mathbf{Y}}}$, that is if $e^{i_0} \in \mathbf{L}_{j_0}$ then we move in $\underline{\overline{\mathbf{Y}}}$ and we move in direction $e^{j_0}$. Verbally: there are two violations of the Stronger Optimality Criterion manifested by the conditions defining the initial $\mathbf{\Pi}$–vertex $(e^{i_0}, e^n)$. Exactly one of them can be removed, this way we "leave" this vertex on exactly one $\mathbf{\Pi}$–edge.

**Example 2.9.** Once again we return to Example 1.4. Here we have $n = 4$ and $e^4 \in \mathbf{K}_2$. Since $e^2 \notin \mathbf{L}_4$ we have not an equilibrium at hand. Rather it is true that $e^2 \in \mathbf{L}_2$ holds true - hence we move in $\underline{\overline{\mathbf{Y}}}$ in direction towards $e^2$. This eventually leads to the point $y^1 = (0, 1/2, 0, 1/2)$.

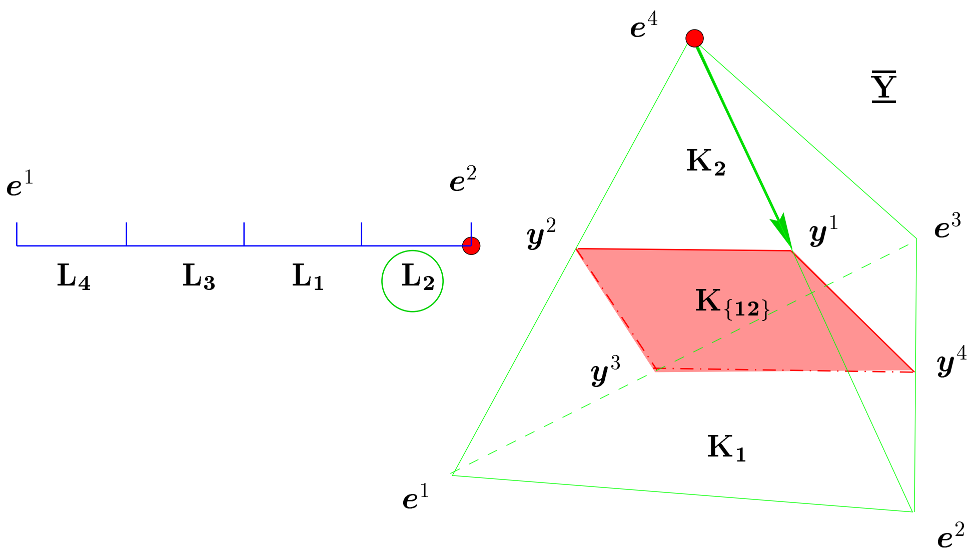



Figure 2.3: Starting the Algorithm in Example 2.9

**Remark 2.10.** *We recall the simple geometric procedure that governs the beginning of our algorithm:*

1. ***Choose*** $i_0$ ***such that (in*** $\underline{\overline{\mathbf{Y}}}$***)*** $(e^n \in \mathbf{K}_{i_0}$.
2. ***If this is not an equilibrium, move in the direction*** $j_0$ ***that is uniquely described by*** $e^{i_0} \in \mathbf{L}_{j_0}$.

**Theorem 2.11.** *Every Nash equilibrium which is not the initial* $\mathbf{\Pi}$*–vertex is located in exactly one admissible* $\mathbf{\Pi}$*–edge.*

**Proof:**
**1st STEP** : Let $(\bar{x}, \bar{y})$ be an equilibrium. In view of Remark 1.3 and Corollary

1.9 we may introduce the sets $R = \{j \mid \bar{y}_j > 0\}$ and $T = \{i \mid \boldsymbol{x}_i > 0\}$ and we obtain

$$\{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})\} = (\mathbf{L}_R \cap \mathbf{X}_{T^c}) \times (\mathbf{K}_T \cap \mathbf{Y}_{R^c})$$

with $\mid R \mid = \mid T \mid$.

We shall distinguish two cases.

**2**[nd]**STEP** : Assume $\bar{\boldsymbol{y}}_n > 0$.

Then $n \in R$. Now, it cannot happen that $\mid R \mid = 1$ for otherwise we would be dealing with the initial $\mathbf{\Pi}$–vertex $(\boldsymbol{e}^{i_0}, \boldsymbol{e}^n)$. Therefore, we have necessarily $\mid R \mid \geq 2$ and it is possible to remove $n$ from $R$. We observe that

$$(5) \qquad (\mathbf{L}_{R-n} \cap \mathbf{X}_{T^c}) \times (\mathbf{K}_T \cap \mathbf{Y}_{(R-n)^c-n})$$

is an admissible $\mathbf{\Pi}$–edge which contains $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$. There are no further edges containing $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ - just the obvious ones are those to be taken into consideration (compare the arguments given in Lemma 1.10).

The geometric meaning of this procedure is that we can leave the equilibrium point by moving in $\underline{\overline{\mathbf{X}}}$ and leaving $\mathbf{L}_n$. This means just that the condition of the Stronger Optimality Criterion which says

$$(6) \qquad \bar{\boldsymbol{y}}_n > 0 \Leftrightarrow \bar{\boldsymbol{x}} \in \mathbf{L}_n$$

is sacrificed. Or else we can say that we *reach* an equilibrium point moving on the edge described by the missing optimality condition (6). That is, we reach $\mathbf{L}_n$ in $\underline{\overline{\mathbf{X}}}$ and thereby supply the missing condition.

Indeed, this edge represents a "motion in $\underline{\overline{\mathbf{X}}}$" since $\mid R_n \mid = \mid R \mid -1 = \mid T \mid -1$ holds true.

**3**[rd]**STEP** : Within this step we supply the formal proof for the fact that only the "obvious" edges are adjacent at the equilibrium point. This maybe useful for the reader who wants to follow the formally exact proof. The statement that further edges are "obviously" not attached to $\bar{\boldsymbol{y}}$ appeals to an insight we have gained in the proof of Lemma 1.10. We learned that "too many edges" (that is too many equations) contradict the non–degeneracy condition. This Step appears to be redundant, but we want to discuss the detailed arguments again. The reader who wants to proceed can as well continue with the fourth step (which treats the case $\bar{\boldsymbol{y}}_n = 0$).

Let us assume, therefore, that apart from the $\mathbf{\Pi}$–edge which is defined by (5) there is another edge indicated, say by

$$(\mathbf{L}_{R'} \cap \mathbf{X}_{T'^c}) \times (\mathbf{K}_{T'} \cap \mathbf{Y}_{R'^c - n}). \tag{7}$$

We know that $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ is located within this edge. Hence, we argue as follows: because of $\bar{x}_i = 0 \quad (i \in T^c)$ we must have

$$T'^c \subseteq T^c$$
$$T' \supseteq T$$

Now, if it so happens that $T' \supsetneq T$ holds true, then we would have

$$y \in \mathbf{K}_{T'} \cap \mathbf{Y}_{R^c}.$$

However, this cannot happen since we have already $\mid T \mid + \mid R^c \mid = n$. Consequently, we must have $T = T'$ (again the argument in Lemma 1.10 should be compared).

Analogously, because of $\bar{y}_j = 0$ we obtain

$$R'^C - n \subseteq R^c \ .$$

Now if it would happen that $j \in R'$ and $j \notin R$, then we would obtain

$$\bar{\boldsymbol{x}} \in \mathbf{L}_{R+j} \cap \mathbf{X}_{T^c},$$

which by the same reasoning as above is impossible. Since $n \notin R$ holds true again we have $R = R' + n$. As a consequence the hypothetical edge (7) and the one we have originally considered (5) coincide - this finishes our consideration of the $3^{rd}$*step*.

**$4^{\text{th}}$STEP** : Now consider an equilibrium in which by $\bar{y}_n = 0$ is the case. Define $R$ and $T$ in the obvious way as in the $2^{nd}$*step*. We see immediately that exactly the $\mathbf{\Pi}$–edge

$$(\mathbf{L}_R \cap \mathbf{X}_{T^c}) \times (\mathbf{K}_T \cap \mathbf{Y}_{R^c - n})$$

contains our equilibrium point $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}.)$ Further edges are ("obviously") not available.

Note that this is a motion in $\underline{\overline{\mathbf{Y}}}$ (we reach a vertex reflecting the condition $y_n = 0$ in $\underline{\overline{\mathbf{Y}}}$ – or we leave this vertex by admitting $y_n > 0$ ).

The reader will realize that arrival or departure at this kind of equilibrium on an admissible track occurs in a way such that the optimality criterion (6) is eventually satisfied by the appearance of a coordinate $y_n = 0$; this happens by means of a "motion" in $\underline{\overline{\mathbf{Y}}}$.

**q.e.d.**

**Remark 2.12.** *Again it is useful to describe the simple Geometry of the algorithm verbally:*

***Whenever we reach a vertex such that the index*** $n$ ***is involved either by entering*** $\mathbf{L}_n$ ***or by a vanishing coordinate*** $y_n = 0$***, then we are stuck at some equilibrium (and there is no departure edge).***

**Example 2.13.** We repeat our standard example that occurred the last time in Example 2.9:

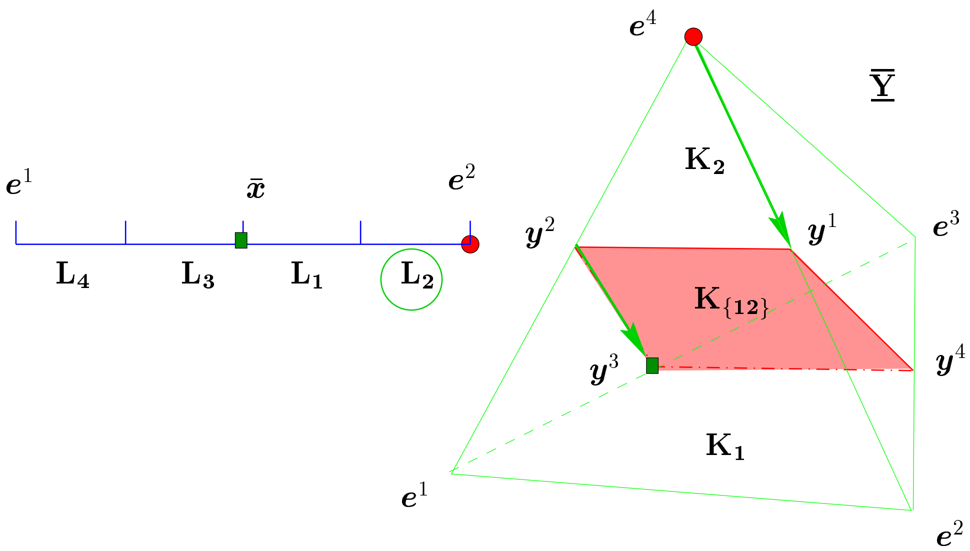



Figure 2.4: Stopping the Algorithm at some Nash equilibrium

The pair of mixed strategies $(\bar{\boldsymbol{x}}, \boldsymbol{y}^3)$ is a Nash equilibrium. For, one one hand, the positive $\boldsymbol{x}$–coordinates are 1 and 2 (i.e., all coordinates) – and $\boldsymbol{y}^3 \in \mathbf{K}_{12}$ is satisfied. On the other hand, the positive $\boldsymbol{y}$–coordinates are 1 and 3 and, indeed, we have $\bar{\boldsymbol{x}} \in \mathbf{L}_{13}$. As $n = 4$ and the fourth coordinate of $\boldsymbol{y}^3$ vanishes, it is the Fourth Step of the previous proof that applies: exactly by admitting $y_4 > 0$ we define the unique $\mathbf{\Pi}$–edge on which to leave or on which to arrive at $\boldsymbol{y}^3$.

Roughly speaking, we have characterized the departure from the initial $\mathbf{\Pi}$–vertex and the arrival at some equilibrium. What happens intermediately?

**Theorem 2.14.** *Every admissible* $\mathbf{\Pi}$*–vertex is located in exactly two admissible* $\mathbf{\Pi}$*–edges.*

**Proof:**

Consider a $\mathbf{\Pi}$–vertex $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ such that with a suitable $i_0 \in T$. We assume

$$\{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})\} = (\mathbf{L}_R \cap \mathbf{X}_{T^c+i_0}) \times (\mathbf{K}_T \cap \mathbf{Y}_{R^c-n}) \tag{8}$$

with $| T |=| R | +1$ , $n \in R^c$ and consequently

$$\begin{aligned} | R | + | T^c | +1 &= m \\ | T | + | R^c | -1 &= n \ . \end{aligned}$$

(The remaining case from Definition 2.3 can be treated quite analogously.) We present a shortened proof just indicating the $\mathbf{\Pi}$–edges on which one can leave or reach $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$. These edges are given by

$$(\mathbf{L}_R \cap \mathbf{X}_{T^c}) \times (\mathbf{K}_T \cap \mathbf{Y}_{R^c-n}) \tag{9}$$

("motion in $\overline{\underline{\mathbf{X}}}$" – "relieve $x_{i_0} = 0$") and

$$(\mathbf{L}_R \cap \mathbf{X}_{T^c+i_0}) \times (\mathbf{K}_{T-i_0} \cap \mathbf{Y}_{R^c-n}), \tag{10}$$

( "motion in $\overline{\underline{\mathbf{Y}}}$" – "relieve $\mathbf{K}_{i_0}$"), and there are ("obviously") no further edges. **q.e.d.**

Let us again offer a geometrical interpretation. In case of the $\mathbf{\Pi}$–vertex given by (8) there exists, apart from the specified index $n$, another particular index $i_0$ which corresponds to an "obsolete" equation $x_{i_0} > 0$. Accordingly, this vertex can be left reached on two tracks: either by admitting $x_{i_0} > 0$ - meaning that we move in the direction towards $\boldsymbol{e}^{i_0}$ - or by leaving $\mathbf{K}_{i_0}$. The first motion type takes place in $\overline{\underline{\mathbf{X}}}$ and the second takes place in $\overline{\underline{\mathbf{Y}}}$.

**Example 2.15.** Our standard example appears in Figure 2.5. The pair $(\boldsymbol{e}^2, \boldsymbol{y}^1)$ is of the type treated explicitely in the last theorem: the decisive index is $i_0 = 2$. We have an equation $x_2 = 0$ and yet $\boldsymbol{y} \in \mathbf{K}_2$ is satisfied. We can leave (or arrive) by either admitting $x_2 > 0$, i.e., moving in $\overline{\underline{\mathbf{X}}}$ from $\boldsymbol{e}^2$ to $\widehat{\boldsymbol{x}}$ or by leaving $\mathbf{K}_2$, i.e., moving in $\overline{\underline{\mathbf{Y}}}$ from $\boldsymbol{y}^1$ to $\boldsymbol{e}^4$.

On the other hand, consider the pair $(\widehat{\boldsymbol{x}}, \bar{\boldsymbol{y}}^2)$. The decisive coordinate is $j_0 = 2$. We can either leave by admitting $y_2 > 0$, i.e., by moving in $\overline{\underline{\mathbf{Y}}}$ from $\boldsymbol{y}^2$ to $\boldsymbol{y}^3$. Or else we can leave by leaving $\mathbf{L}_2$, i.e., by moving in $\overline{\underline{\mathbf{X}}}$ from $\widehat{\boldsymbol{x}}$ to $\bar{\boldsymbol{x}}$.

The direction indicated by the arrows is suggested by the fact that we started out from $(\boldsymbol{e}^2, \boldsymbol{e}^4)$ and, whenever we arrive at some $\mathbf{\Pi}$–vertex on some admissible track we want to leave on the other one.

**Remark 2.16.** *We offer the usual short geometrical description "for the browser:"*

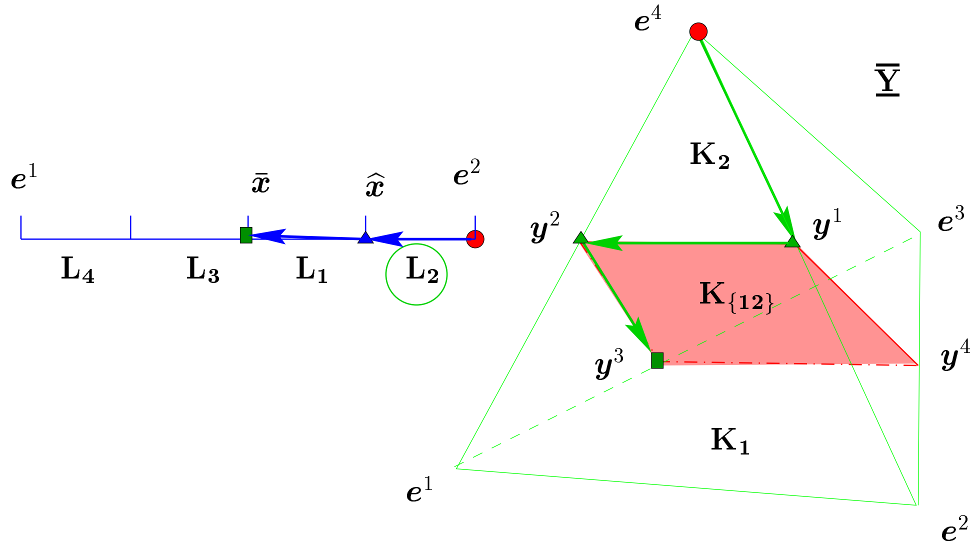



Figure 2.5: Motions of the Algorithm at $\mathbf{\Pi}$–vertices

1. ***If, at some $\mathbf{\Pi}$ vertex, we arrive via a motion in $\overline{\underline{\mathbf{Y}}}$ and enter some $\mathbf{K}_{i_0}$, then we leave via a motion in $\overline{\underline{\mathbf{X}}}$ by admitting $\boldsymbol{x}_{i_0} > 0$. (Or the other way around: if $\boldsymbol{x}_{i_0}$ becomes $0$, then we leave $\mathbf{K}_{i_0}$.)***

2. ***If, at some $\mathbf{\Pi}$ vertex, we arrive via a motion in $\overline{\underline{\mathbf{X}}}$ and newly reach some $\mathbf{L}_{j_0}$, then we leave via a motion in $\overline{\underline{\mathbf{Y}}}$ by admitting $\boldsymbol{y}_{j_0} > 0$. (Or the other way around: if $\boldsymbol{y}_{j_0}$ becomes $0$, then we leave $\mathbf{L}_{j_0}$.)***

**Theorem 2.17.** *Any $\mathbf{\Pi}$–edge contains exactly two $\mathbf{\Pi}$-vertices. These vertices are either admissible or equilibria.*

**Proof:** Recall that an admissible $\mathbf{\Pi}$–edge has the form

$$(\mathbf{L}_R \cap \mathbf{X}_{T^c}) \times (\mathbf{K}_T \cap \mathbf{Y}_{R^c - n}).$$

This describes a convex polyhedron of dimension 1, i.e., a line segment. This line segment has two endpoints which have dimension 1 and are contained in the line segment.

We consider the case $| R |=| T |$ which reflects a "motion in $\overline{\underline{\mathbf{Y}}}$" (the second alternative of Definition 2.4 is treated analogously). Then we have $dim(\mathbf{L}_R \cap \overline{\underline{\mathbf{X}}}_{T^c}) = 0$ and $dim(\mathbf{K}_T \cap \overline{\underline{\mathbf{Y}}}_{R^c - n}) = 1$; the first part is a vertex in $\overline{\underline{\mathbf{X}}}$ and the second a line segment in $\overline{\underline{\mathbf{Y}}}$.

Necessarily the endpoints of that line segment occur when an additional $\mathbf{K}_i$ or some boundary simplex $\overline{\underline{\mathbf{Y}}}_j$ is reached. That is, the endpoints will have one of the following forms.

$$\begin{array}{rcrl} \mathbf{K}_{T+i_0} & \cap & \mathbf{Y}_{R^c-n} & \mathbf{K}_{i_0} \text{ is reached} \\ \mathbf{K}_{T} & \cap & \mathbf{Y}_{R^c+j_0-n} & \mathbf{Y}_{j_0} \text{ is reached} \\ \mathbf{K}_{T} & \cap & \mathbf{Y}_{R^c} & \mathbf{Y}_{n} \text{ is reached} \end{array}$$

Exactly in the last case the procedure arrives at an equilibrium (as $y_n = 0$ holds true and all equilibrium conditionas are satisfied). The other cases mentioned above generate $\mathbf{\Pi}$-vertices - for example the $\mathbf{\Pi}$–vertex

$$(\mathbf{L}_R \cap \mathbf{X}_{T^c}) \times (\mathbf{K}_{T+i_0} \cap \mathbf{Y}_{R^c-n}) = (\mathbf{L}_R \cap \mathbf{X}_{(T+i_0)^c+i_0}) \times (\mathbf{K}_{T+i_0} \cap \mathbf{Y}_{R^c-n})$$

satisfies the conditions specified in Definition 2.3. **q.e.d.**

Again, it is seen that an equilibrium is located at an admissible edge "whenever index $n$ appears", that is, "whenever one moves into $\mathbf{Y}_n$", "one moves into $\mathbf{L}_n$', or "$y_n = 0$ occurs."

# 3 The Structure of Equilibria

We are now in the position to discuss in detail the structure of the LH–net $\mathfrak{L}$ that consists of all $\mathbf{\Pi}$–edges and –vertices, the equilibrium points and the initial vertex (see Definition 2.6). This in turn allows to give a rather precise account of the set of Nash equilibria of a non degenerate bimatrix game. In addition, we obtain a second existence proof for such equilibria which is, in a sense, constructive and does not hinge on the Brouwer Fixed Point Theorem.

As previously, we assume that all games we are dealing with are non degenerate.

**Theorem 3.1.** *1. The LH–net satisfies*

$$\begin{aligned}\mathfrak{L} \;=\; & \{(\boldsymbol{x},\boldsymbol{y}) \in \underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}} \mid x_i > 0 \Rightarrow \boldsymbol{y} \in \mathbf{K}_i \quad (i \in I),\\ & y_j > 0 \Rightarrow \boldsymbol{x} \in \mathbf{L}_j \quad (j \in J - n)\},\\ =\; & \{(\boldsymbol{x},\boldsymbol{y}) \in \underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}} \mid \textit{ there exists either } i_0 \in I \textit{ or } n \neq j_0 \in J\\ & \textit{such that } x_i > 0 \Rightarrow \boldsymbol{y} \in \mathbf{K}_i \quad (i \in I - i_0),\\ & \textit{or } y_j > 0 \Rightarrow \boldsymbol{x} \in \mathbf{L}_j \quad (j \in J - n - j_0) \textit{ respectively } \},\end{aligned}$$

*hence it can be seen as the set of equilibria and "almost equilibria".*

*2. The LH–net $\mathfrak{L}$ is nonempty.*

*3. The LH–net $\mathfrak{L}$ is a compact subset of $\underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}}$.*

**Proof:** The first statement is an obvious reformulation of the definition in view of nondegeneracy. The LH–net is nonempty as it contains the initial $\mathbf{\Pi}$–vertex and closed as it is a union of finitely many closed sets (points and line segments).

**q.e.d.**

The following is a collection of the results obtained in the previous sections.

**Theorem 3.2.** *1. The initial $\mathbf{\Pi}$–vertex is either an equilibrium or admits of a $\mathbf{\Pi}$–edge*

*2. Any equilibrium which is not the initial $\mathbf{\Pi}$–vertex admits of exactly one $\mathbf{\Pi}$–edge.*

*3. Any $\mathbf{\Pi}$–vertex which is not an equilibrium admits of exactly two $\mathbf{\Pi}$-edges.*

4. *There are only finitely many* $\mathbf{\Pi}$*-vertices and edges.*

As a consequence, we obtain the structure of $\mathfrak{L}$ as follows.

**Theorem 3.3.** *Let* $\Gamma = (\underline{\overline{\mathbf{X}}}, \underline{\overline{\mathbf{Y}}}; A, B)$ *be a non degenerate bimatrix game. Then the following holds true.*

1. $\Gamma$ *has at least one equilibrium.*

2. *The number of equilibria of* $\Gamma$ *is odd.*

3. *The LH–net* $\mathfrak{L}$ *consists of finitely many connected path without loops, each path consisting of finitely many edges and vertices. The path containing the initial* $\mathbf{\Pi}$*–vertex contains exactly* 1 *equilibrium located at its endpoint. Each other path contains two equilibria, one at each endpoint.*

**Proof:** We define the decomposition of $\mathfrak{L}$ into a set of disjoint path by specifying a procedure that reflects the motion along $\mathbf{\Pi}$-vertices and $\mathbf{\Pi}$-edges only. The procedure starts at the initial $\mathbf{\Pi}$–vertex If this is not an equilibrium, then there is exactly one $\mathbf{\Pi}$-edge on which to leave the starting point. There is a second endpoint (a $\mathbf{\Pi}$-vertex) to this edge. If this happens to be no equilibrium, then there is exactly one further $\mathbf{\Pi}$-edge on which to leave this vertex. The path can never return to meet itself as this would require three $\mathbf{\Pi}$-edges to join at one $\mathbf{\Pi}$-vertex. Since there are only finitely many such vertices and edges, the path has to terminate eventually. This final $\mathbf{\Pi}$-vertex admits of one joining $\mathbf{\Pi}$-edge only and, hence, is an equilibrium point.

Suppose there are further $\mathbf{\Pi}$-vertices that have not been visited so far. Unless it is an equilibrium, each such vertex admits of exactly two $\mathbf{\Pi}$-edges, one for arrival, one for departure. A path constructed as above necessarily has to yield two endpoints not admitting departure. That is, there are two further equilibria. Any further $\mathbf{\Pi}$-vertices that have not been visited so far can be grouped together so as to yield an even number of equilibria. Thus, there is one equilibrium resulting from the initial path and possibly an even number of additional equilibria resulting from the second type of path,

**q.e.d.**

**Remark 3.4.** *(The Lemke-Howson Algorithm)*
*The previous theorem constitutes not only an existence theorem but also an algorithm to find or compute equilibria. This algorithm is constructed as follows.*

1. *The procedure starts with the starting point* $(\boldsymbol{e}^{i_0}, \boldsymbol{e}^n)$ *as defined by 3.10. If this is an equilibrium, then the algorithm terminates. Otherwise, if* $(\boldsymbol{e}^{i_0}, \boldsymbol{e}^n)$ *is not an equilibrium, then there is a unique "departure* $\mathbf{\Pi}$*–edge" in* $\underline{\overline{\mathbf{Y}}}$*. This edge is described by a motion towards* $\boldsymbol{e}^{j_0}$ *or "*$y_{j_0} > 0$*"*

2. *By inspection of 3.11 the following facts can be observed: If, upon arrival, a motion in* $\underline{\overline{\mathbf{X}}}$ *occurs, then there appears a new index* $i_0$ *satisfying* $\boldsymbol{x} \in \mathbf{X}_{T^c+i_0}$*. This means that we obtain "*$x_{i_0} = 0$*". When departure takes place (in* $\underline{\overline{\mathbf{Y}}}$*) that is, when* $\mathbf{K}_{T-i_0} \cap \mathbf{Y}_{R^c-n}$ *is the departure edge in* $\underline{\overline{\mathbf{Y}}}$*, then we leave the polyhedron* $\mathbf{K}_{i_0}$*. Obviously we have the same specified index ruling departure as well as arrival and the motion changes from taking place in* $\underline{\overline{\mathbf{X}}}$ *to* $\underline{\overline{\mathbf{Y}}}$*.*

3. *Inspection of 3.12 shows the following. If the "specified index" is* $n$*, then the equilibrium has been reached.*

4. *For the geometric behavior the combinatorics of the polyhedra of best reply is decisive.*

**Example 3.5.** We return to our favorite example which is reproduced by Figure 3.1.

We obtain the following sequence representing the initial path of the Lemke-Howson procedure. Note that $m = 2$ and $n = 4$ holds true, thus $4 \in J$ is the critical index and we take $\boldsymbol{e}^4 \in \overline{\underline{\mathbf{Y}}}$ to be the second part of the starting point.

| Arrival | new vertex | *criterion* | critical index | type of motion | simplex |
|---|---|---|---|---|---|
| | | $e^4 \in \mathbf{K}_2$ | $2 \in I$ | choose $\boldsymbol{e}^2$ | in $\overline{\underline{\mathbf{X}}}$ |
| $\mathbf{e}^2$ | $(\boldsymbol{e}^2, \boldsymbol{e}^4)$ | $e^2 \in \mathbf{L}_2$ | $2 \in J$ | $y_2 > 0$ | in $\overline{\underline{\mathbf{Y}}}$ |
| $\boldsymbol{y}^1$ | $(\boldsymbol{e}^2, \boldsymbol{y}^1)$ | $\boldsymbol{y}^1 \in \mathbf{K}_1$ | $1 \in I$ | $x_1 > 0$ | in $\overline{\underline{\mathbf{X}}}$ |
| $\widehat{\boldsymbol{x}}$ | $(\widehat{\boldsymbol{x}}, \boldsymbol{y}^1)$ | $\widehat{\boldsymbol{x}} \in \mathbf{L}_1$ | $1 \in J$ | $y_1 > 0$ | in $\overline{\underline{\mathbf{Y}}}$ |
| $\boldsymbol{y}^2$ | $(\widehat{\boldsymbol{x}}, \boldsymbol{y}^2)$ | $y_2 = 0$ | $2 \in J$ | leave $\mathbf{L}_2$ | in $\overline{\underline{\mathbf{X}}}$ |
| $\bar{\boldsymbol{x}}$ | $(\bar{\boldsymbol{x}}, \boldsymbol{y}^2)$ | $\bar{\boldsymbol{x}} \in \mathbf{L}_3$ | $3 \in J$ | $y_3 > 0$ | in $\overline{\underline{\mathbf{Y}}}$ |
| $\boldsymbol{y}^3$ | $(\bar{\boldsymbol{x}}, \boldsymbol{y}^3)$ | $y_4 = 0$ | $4 \in J$ | specified index | equilibrium |

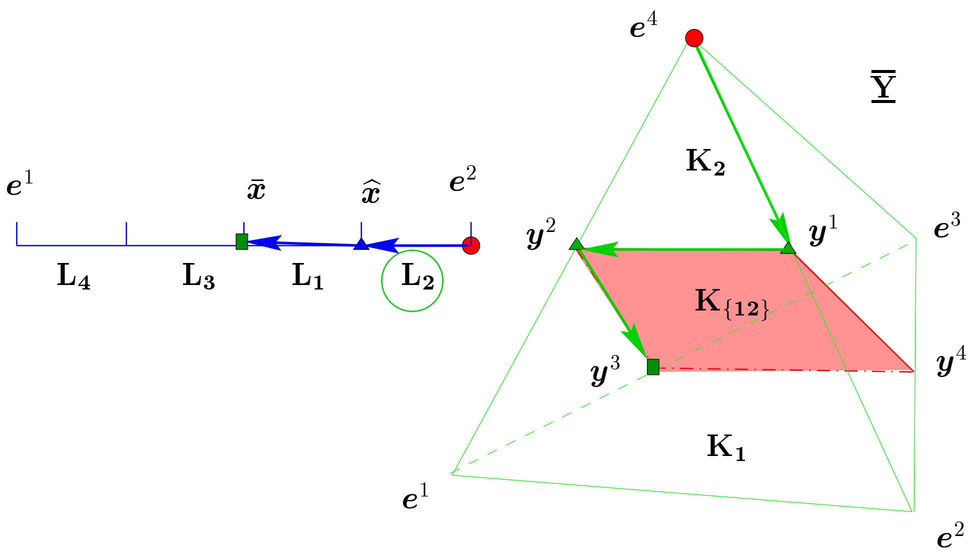



Figure 3.1: The LH–Algorithm in Example 2.15

**Example 3.6.** For $t > 0, s > 1$, consider the matrices

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ s & s & -2s \end{pmatrix} \quad , \quad \boldsymbol{B} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & 0 & 0 \\ t & 0 & 0 \end{pmatrix}$$

The corresponding decomposition of $\overline{\underline{\mathbf{X}}}$ and $\overline{\underline{\mathbf{Y}}}$ into the polyhedra of best reply is represented in Figure 3.2. The mixed strategies playing an exceptional role are given by

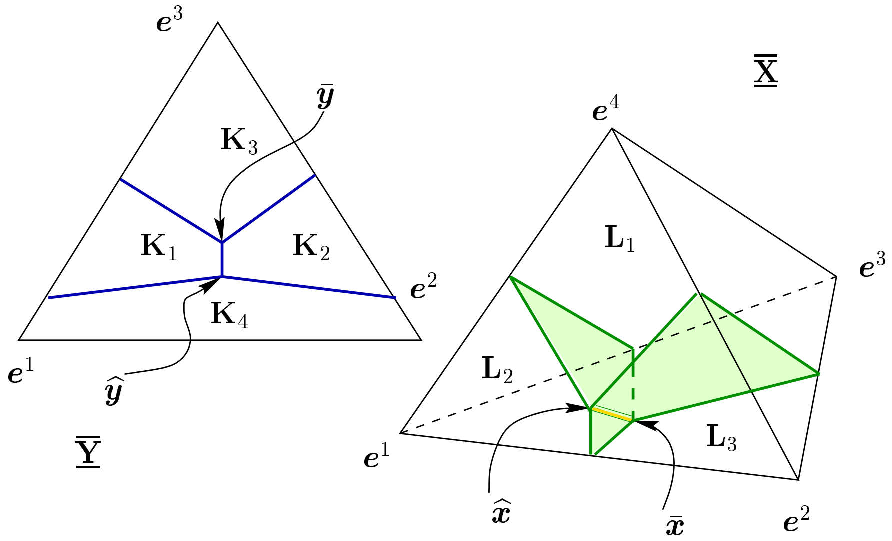



Figure 3.2: The Decomposition in Example 3.6

$$\bar{\boldsymbol{x}} := \frac{1}{2t+1}(t,t,1,0), \quad \bar{\boldsymbol{y}} := \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$$

$$\widehat{\boldsymbol{x}} := \frac{1}{2t+1}(t,t,0,1), \quad \widehat{\boldsymbol{y}} := \frac{1}{6s-1}(2s,2s,2s-1)$$

There are 3 equilibria given by

$$(\boldsymbol{e}^4, \boldsymbol{e}^1) \quad , \quad (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \quad , \quad (\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}})$$

which are indicated in Figure 3.3

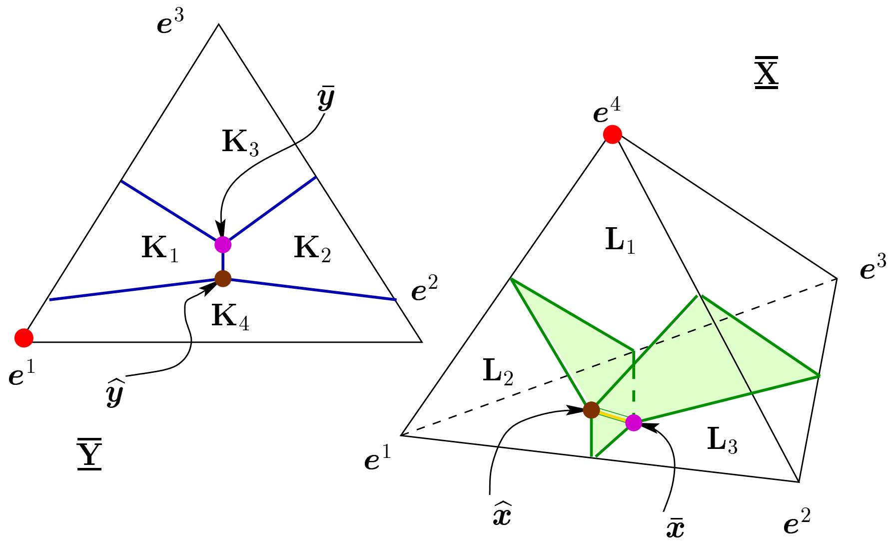



Figure 3.3: Equilibria in Example 3.6

Figure 3.4 reflects the LH–algorithm starting from $\boldsymbol{e}^3$ in $\underline{\overline{\mathbf{Y}}}$ and ending up at the Nash equilibrium $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$. The remaining two equilibria are connected by means of a second path in $\mathfrak{L}$ so the fugure suggests the full LH–net . Of course, $\mathfrak{L}$ is a 4–dimensional set, so we cannot actually depict it directly. One has to take into account the alterneating motion within both simplices in order to actually concieve the structure of $\mathfrak{L}$.

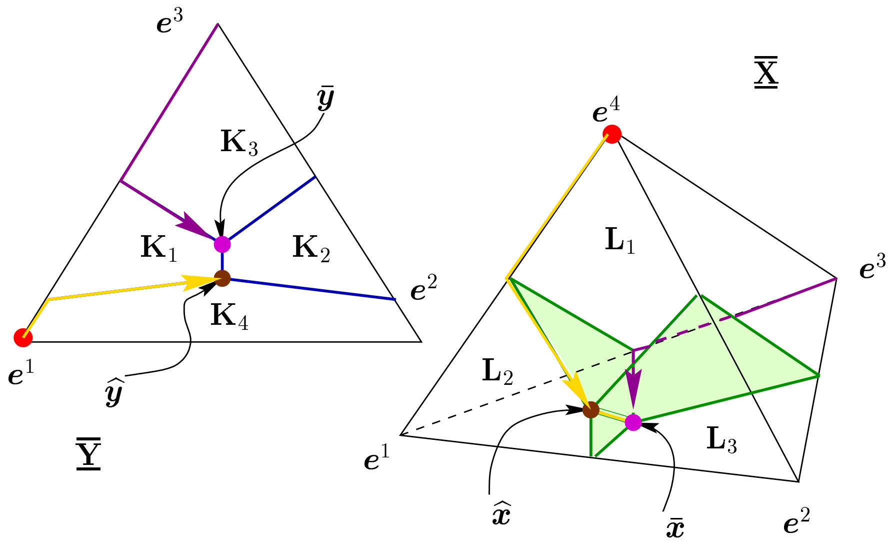



Figure 3.4: The LH–net in Example 3.6

Instead of choosing $n = 3$ as the critical index we might choose $2 \in J$. Then the algorithm ends up with the equilibrium $(\boldsymbol{e}^4, \boldsymbol{e}^1)$. Choosing $2 \in I$ yields the equilibrium $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ again. After some inspection one realizes that the pasth never reaches $(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}})$ whatever the critical index and the resulting initial $\boldsymbol{\pi}$–vertex.

Yet there is a way to find all equilibria because, when $n \in J$ is critical, then the pure equilibrium is connected to $(\widehat{\boldsymbol{x}}, \widehat{)}y$. So if we use within this setup the pure equilibrium as a starting point we may obtain the missing one. While this a change of the procedure it clearly leads to successfully establishing all equilibria.

Another nice feature of this example is that the equilibria $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ and $(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}})$ move arbitrarily close to each other with increasing parameters $t$ and $s$. This property may be employed for a test of precision or of numerical errors when implementing the algorithm by some computer program. A test of convergence of approximative procedures is also useful.

**Example 3.7.** A permutation of the rows of $\boldsymbol{A}$ changes the picture. E.g., exchanging the first and second row amounts to exchanging the first and second index of the polyhedra $\mathbf{K}_i$, hence the result looks almost identical up the fact that $\mathbf{K}_1$ and $\mathbf{K}_2$ have exchanged their roles.

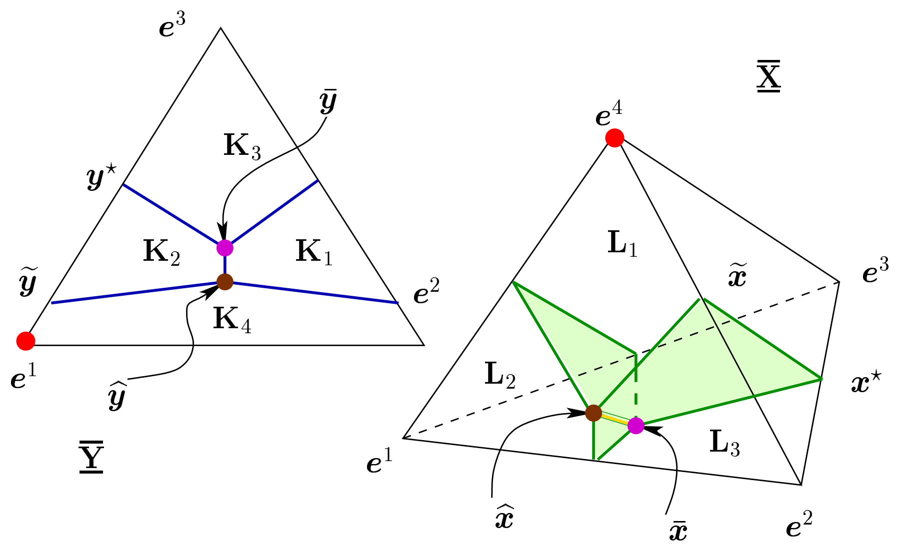



Figure 3.5: Permuting the Rows creates further Equilibria

Now there appear two nice additional equilibria described by $(\boldsymbol{x}^\star, \boldsymbol{y}^\star)$ and $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{y}})$. The exact coordinates are

$$\begin{aligned} \boldsymbol{x}^\star &= \frac{1}{t+1}(0, t, 1, 0) & \boldsymbol{y}^\star &= (\frac{1}{2}, 0, \frac{1}{2}) \\ \widetilde{\boldsymbol{x}} &= \frac{1}{t+1}(0, t, 0, 1) & \widetilde{\boldsymbol{y}} &= \frac{1}{3s-1}(2s, 0, s-1) \end{aligned}$$

Again $\boldsymbol{x}^\star$ and $\widetilde{\boldsymbol{x}}$ are arbitrarily close for large $t$ while $\boldsymbol{y}^\star$ and $\widetilde{\boldsymbol{y}}$ are not close for any choice of the parameters. The example may now serve for all possible versions of the LH and other algorithms.

**Example 3.8.** The previous examples may be generalized as follows. Take

$$(1) \qquad \boldsymbol{A} = \begin{pmatrix} 1 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & \cdots & 0 \\ . & \cdots & \cdots & \cdots & . \\ . & \cdots & \cdots & \cdots & . \\ 0 & 0 & \cdots & \cdots & 1 \\ s & s & \cdots & s & -(n-1)s \end{pmatrix}$$

and

$$(2) \qquad \boldsymbol{B} = \begin{pmatrix} 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ . & \cdots & \cdots & \cdots & . \\ 0 & 0 & \cdots & \cdots & 1 \\ t & 0 & \cdots & \cdots & 0 \\ t & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

Both matrices are supposed to be $(n+1) \times n$.

Again there are 3 equilibria

$$(\boldsymbol{e}^4, \boldsymbol{e}^1) \quad , \quad (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \quad , \quad (\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}})$$

which, in this case are given by

$$(3) \qquad \begin{aligned} \bar{\boldsymbol{x}} &= \frac{1}{(n-1)t+1}(t, \ldots, t, 1, 0) \\ \bar{\boldsymbol{y}} &= \left(\frac{1}{n}, \ldots, \frac{1}{n}\right) \\ \widehat{\boldsymbol{x}} &= \frac{1}{(n-1)t+1}(t, \ldots, t, 0, 1) \\ \widehat{\boldsymbol{y}} &= \frac{1}{sn(n-1)-1}(s(n-1), \ldots, s(n-1), s(n-1)-1) \end{aligned}$$

One can now consider these equilibria (which are all) and observe again that some of the equilibrium coordinates are "close" for large parameters. Also, permuting the first n rows of $\boldsymbol{A}$ and the last $n-1$ columns of $\boldsymbol{B}$ amounts to exchanging just the numbering of the polyhedra, not the geometric shape of our basic picture. Then further equilibria appear. These examples are, therefore, quite useful for testing procedures to compute equilibria.

# 4 Remarks on the Alternating Simplex Procedure

We shall shortly discuss the formal procedure resulting from from the Lemke–Howson algorithm. This procedure resembles the simplex method but, due to the nature of the algorithm, involves two tableaus. For short one can speak of an alternating simplex algorithm as indeed the motion induced by manipulating the tableaus alternatingly takes place in the two simplices $\underline{\overline{\mathbf{X}}}$ and $\underline{\overline{\mathbf{Y}}}$.

As a prerequisit we introduce the "canonical" parametrization appropriate to an edge in $\underline{\overline{\mathbf{Y}}}$. To this end, let $\emptyset \neq T \subseteq I$ and $J \neq U \subseteq J$ be such that

$$|T| + |U| = n$$

and

$$\mathbf{H}_{T,U} \;=\; \mathbf{K}_T \cap \mathbf{Y}_U \neq 0$$

holds true. By nondegeneracy it follows that the dimension of this set is 0, hence it containes just one element, say

$$\mathbf{H}_{T,U} \;=\; \{\bar{\boldsymbol{y}}\}\;. \tag{1}$$

Define

$$\bar{\lambda} \;:=\; \boldsymbol{A}_{i\bullet}\bar{\boldsymbol{y}} > \boldsymbol{A}_{l\bullet}\bar{\boldsymbol{y}} \quad (i \in T,\; l \notin T).$$

Then $(\bar{\boldsymbol{y}}, \bar{\lambda})$ is the unique solution of the non homogeneous linear system of equations

$$\begin{aligned} \boldsymbol{A}_{i\bullet}\boldsymbol{y} &= \;\lambda \quad (i \in T) \\ y_j &= \;0 \quad (j \in U) \\ \sum_{J \in U} y_j &= \;1\;\;. \end{aligned} \tag{2}$$

As we consider the variables to be $(\boldsymbol{y}, \lambda)$, the coefficient matrix is obtained by augmenting the index sets $I, J$. Accordingly, we write

$$I \cup \{\diamond\}\;,\quad J \cup \{\star\}$$

and introduce the matrix

$$\mathcal{A}_I^J = \begin{array}{c} I \left\{ \begin{array}{c} \\ \\ \\ \\ \end{array} \right. \\ \diamond \end{array} \left( \begin{array}{ccccc|c} & & \overbrace{\qquad\qquad\qquad}^{J} & & & \star \\ & & \vdots & & & -1 \\ & \dots & \mathbf{A} & \dots & & \vdots \\ & & \vdots & & & \vdots \\ & & & & & -1 \\ \hline 1 & \dots & 1 & \dots & 1 & 0 \end{array} \right)$$

Then, for $W \subseteq J + \{\star\}$, $\;Z \subseteq I + \{\diamond\}$ we obtain submatrices

$$\mathcal{A}_Z^W \;=\; (\mathcal{A}_{ij})_{i \in W, j \in Z}$$

of the matrix $\mathcal{A}$. We use symbols $\mathcal{A}_{T-i_0+\diamond}^{-U}$ etc. analogously, $-U$ refers to the complement of $U$.

This notation is useful in order to represent affine or linear subspaces that are obtained by systems of equations like (2).

For instance, suppose we want to parametrize an edge $\mathbf{K}_{T-i_0} \cap \mathbf{Y}_U$ adjacent to $\bar{\boldsymbol{y}}$. We imagine that the situation depicted in Figure 4.1 has to be treated.

Then , consider the *linear* subspace

$$\text{(3)} \qquad \begin{array}{rcl} \mathfrak{L}_{T-i_0}^{-U} & := & \left\{ \boldsymbol{\mu} = (\boldsymbol{\gamma}, \nu) \in \mathbb{R}^{J-U} \times \mathbb{R} \,\middle|\, \mathcal{A}_{T-i_0+\diamond}^{-U} \boldsymbol{\mu} = \mathbf{0} \right\} \\ & \subseteq & \mathbb{R}^{J-U} \times \mathbb{R} \;=\; \mathbb{R}^{J-U+\{\star\}} \;. \end{array}$$

If we introduce the zero vector $\mathbf{0}_U$ of $\mathbb{R}^U$, then the above linear subspace can be imbedded into $\mathbb{R}^{J+\star}$, this subspace is denoted by

$$\text{(4)} \qquad \mathfrak{L}_{T-i_0}^{-U} \oplus \mathbf{0}_U \;=\; \left\{ (\boldsymbol{y}, \lambda) \,\middle|\, \begin{array}{rcll} \boldsymbol{A}_{i\bullet} \boldsymbol{y} & = & \lambda & (i \in T - i_0), \\ y_j & = & 0 & (j \in U), \\ \sum_{j \in J} & = & 0 & \end{array} \right\}$$

Therefore, the *affine* subspace

$$\text{(5)} \qquad \{(\bar{\boldsymbol{y}}, \bar{\lambda})\} + \left(\mathfrak{L}_{T-i_0}^{-U} \oplus \mathbf{0}_U\right)$$

is the candidate for "canonical representation" as its projection on $\mathbb{R}^n$ contains an edge of the form $\mathbf{K}_{T-i_0} \cap \mathbf{Y}_U$ adjacent to $(\bar{\boldsymbol{y}}, \bar{\lambda})$.

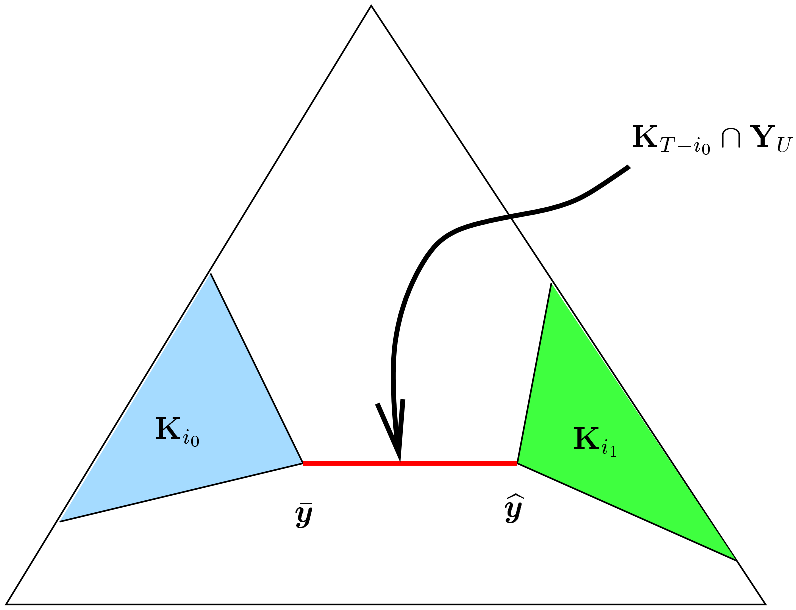



Figure 4.1: Parametrizing an edge

As $L_{T-i_0}^{-U}$ is a linear subspace, we can specify a vector

$$\bar{\boldsymbol{\mu}}^{i_0} \;:=\; (\bar{\boldsymbol{\gamma}}^{i_0}, \nu^{i_0}) \in L_{T-i_0}^{-U} \tag{6}$$

by the additional requirement

$$\mathcal{A}_{i_0\bullet}^{-U} \bar{\boldsymbol{\mu}}^{i_0} \;=\; 1. \tag{7}$$

Now we know that the pair $(\bar{\boldsymbol{y}}, \bar{\lambda})$ satisfies

$$\mathcal{A}_T(\bar{\boldsymbol{y}}, \bar{\lambda}) \;=\; (0, \ldots, 0, \ldots, 0, 1) \in \mathbb{R}^{T+\{\star\}} \tag{8}$$

and as a consequence of equation (4) we obtain

$$\mathcal{A}_T\left((\bar{\boldsymbol{y}}, \bar{\lambda}) + (\bar{\boldsymbol{\mu}}^{i_0} + \mathbf{0}_U)\right) \;=\; (0, \ldots, 1, 0, \ldots, 0, 1) \in \mathbb{R}^{T+\{\star\}} \tag{9}$$

Verbally: we sacrifice one degree of freedom of $L_{T-i_0}^{-U}$ by the requirement (7) and obtain a vector pointing in direction of the subspace. (Figure 4.2 ). This vector is now used for parametrizing the subspace. Figure 4.2 represents the situation projected onto the unit simplex $\underline{\overline{\mathbf{Y}}}$ and the subspace containing it.

We are now in the position to suggest the shape of the "canonical representstion".

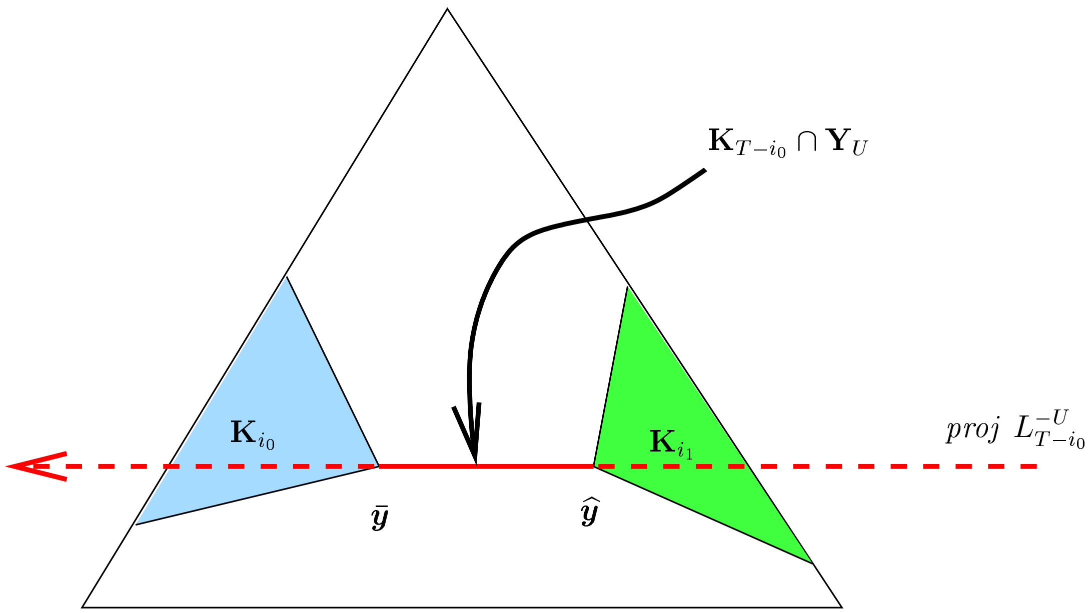



Figure 4.2: The projection of $L_{T-i_0}^{-U}$

**Definition 4.1.** *The **canonical representation** of the affine subspace* (5) *is the bijective mapping*

$$\theta \longrightarrow \left(\boldsymbol{y}^\theta, \lambda^\theta\right) \ := \ \left(\bar{\boldsymbol{y}}, \bar{\lambda}\right) - \theta\left(\bar{\boldsymbol{\mu}}^{i_0} \oplus \mathbf{0}_U\right) \tag{10}$$

It is not hard to see that there exists a parameter $\bar{\theta}^{i_0}$ such that the canonical parametrization does what it is supposed to do, that is, satisfies

$$\mathbf{H}_{T-i_0,U} \ = \ \left\{\boldsymbol{y}^\theta \,\middle|\, 0 \leq \theta \leq \overline{\theta}^{i_0}\right\}. \tag{11}$$

Assuming that some $\widehat{\boldsymbol{y}}$ is the second vertex of $\mathbf{H}_{T-i_0,U}$ and that, in addition, the decisive index $i_1$ occurs because

$$\boldsymbol{y}^{\theta^{i_0}} \in \mathbf{K}_{i_1}$$

holds true, we conclude that a set of equations

$$\boldsymbol{A}_{i_1\bullet}\boldsymbol{y}^{\bar{\theta}^{i_0}} \ = \ \boldsymbol{A}_{i\bullet}\boldsymbol{y}^{\bar{\theta}^{i_0}} \ = \ \lambda^{\bar{\theta}^{i_0}} \tag{12}$$

is satisfied. From these equations one can actually *compute* the parameter $\bar{\theta}_{i_0}$. The "Alternating Simplex Procedure" is a procedure that, analogously to the Simplex Procedure as discussed in CHAPTER 2, provides the formal transition from $(\bar{\boldsymbol{y}}, \overline{\lambda})$ to $(\widehat{\boldsymbol{y}}, \widehat{\lambda}) = (\boldsymbol{y}^{\bar{\theta}_{i_0}}, \lambda^{\bar{\theta}_{i_0}})$.

The procedure formalizes

1. The computation of $(\widehat{\boldsymbol{y}}, \widehat{\lambda})$ by means of $(\bar{\boldsymbol{y}}, \overline{\lambda})$ *and* a set of data carried along (the "tableau elements").

2. The computation of the tableau elements *at* $(\widehat{\boldsymbol{y}}, \widehat{\lambda})$ by by means of the ones at the previous vertex.

3. The alternating movement in the simplices $\overline{\underline{\mathbf{X}}}$ and $\overline{\underline{\mathbf{Y}}}$. Correspondingly, the procedure actually involves *two* tableaus and switches alternatingly between both of them.

4. The decision to terminate as the decisive index ($n$ in most of the previous discussion) appears.

5. The final generation of a Nash equilibrium.

The two tableaus involved are linked to the two payoff matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. The generic form of the $\boldsymbol{A}$–tableau involves tw o index sets $T \subset I$ and $U \subseteq J$ and lists the tableau–elements proper, the parametrizing parameters and the present strategy $\bar{\boldsymbol{y}}$ as follows. Note that, due to the dual nature of index sets in both simplices there appears further partition as indicated; accordingly, the indices from $I$ and $J$ are separated.

$$
\mathsf{T}^{\boldsymbol{A}} = \mathsf{T}^{\boldsymbol{A}}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{D} & \bar{\boldsymbol{\theta}}_{T^c} \\ \boldsymbol{\Gamma} & \boldsymbol{\Delta} & \bar{\boldsymbol{y}}_{U^c} \end{pmatrix}
$$

$$
= \begin{array}{ll} & \\ T^c \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. & i_1 \\ U^c \left\{ \begin{array}{l} \\ \\ \\ \end{array} \right. & j_1 \end{array}
\overset{\begin{array}{ccc} \overbrace{\qquad\qquad i_0 \qquad\qquad}^{T} & \overbrace{\qquad\qquad j_0 \qquad\qquad}^{U} & \star \end{array}}{\left( \begin{array}{ccc|ccc|c} & \vdots & & & \vdots & & \vdots \\ \dots & \bar{c}_{i_1}^{i_0} & \dots & \dots & \bar{d}_{i_1}^{j_0} & \dots & \bar{\theta}_{i_1} \\ & \vdots & & & \vdots & & \vdots \\ \hline & \vdots & & & \vdots & & \vdots \\ \dots & \bar{\gamma}_{j_1}^{i_0} & \dots & \dots & \bar{\delta}_{j_1}^{j_0} & \dots & \bar{y}_{j_1} \\ & \vdots & & & \vdots & & \vdots \end{array} \right)}
$$

These data are basically determined by $\boldsymbol{A}$ , $\bar{\boldsymbol{y}}$, and $\bar{\lambda}$ ; e.g., the $T^c$–vector $\bar{\theta}$

is given by

$$
\begin{aligned}
\bar{\theta} &= -\boldsymbol{A}_{-T}\bar{\boldsymbol{y}} + \bar{\lambda}\boldsymbol{e}_{-T} \\
&= \begin{pmatrix} & \vdots & & \vdots \\ - & \boldsymbol{A}_{i\bullet}\bar{\boldsymbol{y}} & + & \bar{\lambda} \\ & \vdots & & \vdots \end{pmatrix}_{i \in T^c}
\end{aligned}
\tag{13}
$$

while the generic element of the uppermost left corner is given by

$$
\begin{aligned}
\bar{c}^{i_0}_{i_1} &= \mathcal{A}^{-U-i_0}_{i_1\bullet}\bar{\mu}^{i_0} \\
&= \boldsymbol{A}^{-U-i_0}_{i_1\bullet}\left(\bar{\gamma}^{i_0}, \bar{\nu}^{i_0}\right) \ ,
\end{aligned}
\tag{14}
$$

etc. The computation during the process, however, shall *not* be performed by the solution of this kind of equation but by the *Rectangle Rule* !

To this end, the following alternating procedure is established:

1. A pivot determines the change of the Tableau according to the Rectangle Rule. Accordingly, a criterion is developed which yields either an index $i_1 \in T^c$ –
   (corresponding to a change to $\mathbf{H}_{T-i_0+i_1,U}$),
   or an index $j_1 \in U^c$ –
   (corresponding to a change to $H_{T-i_0,U+j_1}$). This choice respects the partition within the Tableau.

2. The alternating movement in the geometrical context is reflected by the alternating action in the $\boldsymbol{A}$– and $\boldsymbol{B}$–Tableau.

3. The generic step proceeds as follows: Let $i_0$ be given from an action in $\mathsf{T}^{\boldsymbol{A}}$ and let $j_0$ be known from an action in $\mathsf{T}^{\boldsymbol{B}}$. Assume $i_1$ to be the minimal element in the $\star$–columnn of $\mathsf{T}^{\boldsymbol{A}}$. Then convert the Tableau according to

   $$\widehat{\mathsf{T}}^{\boldsymbol{A}} = \boldsymbol{R}^{i_1}_{j_0}\bar{\mathsf{T}}. \tag{15}$$

   Now $i_1$ is known from $\widehat{\mathsf{T}}^{\boldsymbol{A}}$. this index is transferred to $\widehat{\mathsf{T}}^{\boldsymbol{B}}$ and pivoting takes place in this second tableau. This way an index is alternatingly shifted from one tableau to the next one simultaneously with pivotation.

It remains to describe the initial tableau. This is canonically produced by the inbital steps of the algorithm. We list the following steps.

1. Choose the specified index $n_0 \in J$ (or $m_0 \in I$, this case is treated analogously).

2. Determine $i_0$ by the requirement

$$a_{i_0 j_0} = \max_{i \in I} a_{i n_0} \; . \tag{16}$$

3. Determine $j_0 \in J$ by the requirement

$$b_{i_0 j_0} = \max_{j \in J} b_{i_0 j} \; . \tag{17}$$

4. If $j_0 = n_0$ happens to be the case, then terminate the algorithm: $(\boldsymbol{e}^{i_0}, \boldsymbol{e}^{j_0})$ is a Nash equilibrium.

5. Otherwise define, for $i \in I, i \neq i_0$

$$\begin{array}{rcl} \bar{\theta}_i & := & a_{i_0 n_0} - a_{i n_0} \\ \bar{d}_i^j & := & a_{ij} - a_{i n_0} + a_{i_0 n_0} - a_{i_0 j} \end{array} \tag{18}$$

and set up the tableaus

$$\mathbf{T}^{\boldsymbol{A}} = \mathbf{T}^{\boldsymbol{A}}(\boldsymbol{e}^{i_0}, \boldsymbol{e}^{n_0}) = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{D} & \bar{\boldsymbol{\theta}}_{T^c} \\ \boldsymbol{\Gamma} & \boldsymbol{\Delta} & \boldsymbol{e}^{n_0}_{U^c} \end{pmatrix}$$

$$= \begin{array}{rr} & \\ I - i_0 \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. & i_1 \\ \{n_0\} \; \{ & n_0 \end{array} \left( \begin{array}{c|ccc|c} \overbrace{\;\;\;}^{T} & & \overbrace{\qquad\qquad\qquad}^{U} & & \star \\ i_0 & & j_0 & & \\ & & \vdots & & \vdots \\ -1 & \dots & \bar{d}_{i_1}^{j_0} & \dots & \bar{\theta}_{i_1} \\ & & \vdots & & \vdots \\ \hline -1 & 1 \dots & 1 \dots & & 1 \end{array} \right)$$

and$\mathbf{T}^{\boldsymbol{B}}$ analogously.

Now by choosing the pivot one may change from one tableau to the other one alternatingly and stopp once the specified index $n_0$ appears.

In the end, the positive coordinates of the equilibrium appear in the $\star$–column: the ones for $\boldsymbol{y}$ in $\mathbf{T}^{\boldsymbol{A}}$. The miissing coordinates are zero and the same applies *mutatis mutandis* for the tableau $\mathbf{T}^{\boldsymbol{A}}$.

We have described the Alternating Simplex Procedure. No proofs have been provided and the reader looking for more details may consult the presentation in [18]. For comparison, the original version of the Lemke–Howson Algorithm is found in [20].

# Chapter 6

# Selection of Equilibria

Some Nash equlibria are less desirable that others and frequently the abundance of equilibria is disappointing. Selection of certain equilibria according to some basic principles is a major problem of Noncooperative Game Theory.

# 1 The Trembling Hand

Within this chapter we consider the first type of a selection mechanism for Nash equilibria.

Thus we define the notion of a *perfect equilibrium* in the context of bimatrix games.

We consider a bimatrix game in mixed strategies

$$\Gamma = (\underline{\overline{\mathbf{X}}}, \underline{\overline{\mathbf{Y}}}; \boldsymbol{A}, \boldsymbol{B}) \tag{1}$$

As usual, $\boldsymbol{A}$ and $\boldsymbol{B}$ are $m \times n$ matrices and $\underline{\overline{\mathbf{X}}}$ and $\underline{\overline{\mathbf{Y}}}$ are the unit simplices of $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively; the generic elements of these strategy spaces are denoted by $\boldsymbol{x}$ and $\boldsymbol{y}$.

A mixed strategy $\widetilde{\boldsymbol{x}} \in \underline{\overline{\mathbf{X}}}$ is said to be ***weakly dominated*** if the following holds true.

$$\begin{aligned} &\text{There exists } \widehat{\boldsymbol{x}} \in \underline{\overline{\mathbf{X}}} \text{ such that} \\ &\qquad \widehat{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y} \geq \widetilde{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y} \qquad (\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}) \\ &\text{and} \\ &\qquad \widehat{\boldsymbol{x}}\boldsymbol{A}\widehat{\boldsymbol{y}} > \widetilde{\boldsymbol{x}}\boldsymbol{A}\widehat{\boldsymbol{y}} \\ &\text{for at least one } \widehat{\boldsymbol{y}} \in \underline{\overline{\mathbf{Y}}}. \end{aligned} \tag{2}$$

Furthermore, $\widetilde{\boldsymbol{x}} \in \underline{\overline{\mathbf{X}}}$ is ***strictly dominated*** if

$$\begin{aligned} &\text{there exists } \widehat{\boldsymbol{x}} \in \underline{\overline{\mathbf{X}}} \text{ such that} \\ &\qquad \widehat{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y} > \tilde{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y} \qquad (\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}) \\ &\text{holds true.} \end{aligned} \tag{3}$$

**Lemma 1.1.** *Let $\widetilde{\boldsymbol{x}} \in \underline{\overline{\mathbf{X}}}$ be **not** weakly dominated. Then there exists $\tilde{\boldsymbol{y}} \in \underline{\overline{\mathbf{Y}}}$, $\tilde{\boldsymbol{y}} > 0$ such that $\widetilde{\boldsymbol{x}}$ is a best reply against $\tilde{\boldsymbol{y}}$ (i.e. $\widetilde{\boldsymbol{x}}\boldsymbol{A}\tilde{\boldsymbol{y}} \geq \boldsymbol{x}\boldsymbol{A}\tilde{\boldsymbol{y}}$ $(\boldsymbol{x} \in \underline{\overline{\mathbf{X}}})$).*

**Proof: (Sketch)**

**1$^{\text{st}}$STEP** :

We may assume $\widetilde{\boldsymbol{x}}\boldsymbol{A} = \mathbf{0}$ – otherwise replace $\boldsymbol{A}$ by

$$\tilde{\boldsymbol{A}} = \boldsymbol{A} - \frac{1}{\widetilde{\boldsymbol{x}}\boldsymbol{e}} \begin{pmatrix} \widetilde{\boldsymbol{x}}\boldsymbol{A} \\ \vdots \\ \widetilde{\boldsymbol{x}}\boldsymbol{A} \end{pmatrix} = \boldsymbol{A} - \begin{pmatrix} \widetilde{\boldsymbol{x}}\boldsymbol{A} \\ \vdots \\ \widetilde{\boldsymbol{x}}\boldsymbol{A} \end{pmatrix}$$

with $\boldsymbol{e} = (1, \ldots, 1)$.

Indeed, if we write

$$\begin{pmatrix} \widetilde{\boldsymbol{x}}\boldsymbol{A} \\ \vdots \\ \widetilde{\boldsymbol{x}}\boldsymbol{A} \end{pmatrix} = \begin{pmatrix} \widetilde{\boldsymbol{x}}\boldsymbol{A}_{\bullet 1}, \ldots, \widetilde{\boldsymbol{x}}\boldsymbol{A}_{\bullet n} \\ \vdots \\ \widetilde{\boldsymbol{x}}\boldsymbol{A}_{\bullet 1}, \ldots, \widetilde{\boldsymbol{x}}\boldsymbol{A}_{\bullet n} \end{pmatrix},$$

then we see that

$$\widetilde{\boldsymbol{x}} \begin{pmatrix} \widetilde{\boldsymbol{x}}\boldsymbol{A} \\ \vdots \\ \widetilde{\boldsymbol{x}}\boldsymbol{A} \end{pmatrix} = (\widetilde{\boldsymbol{x}}\boldsymbol{A}_{\bullet 1}, \ldots, \widetilde{\boldsymbol{x}}\boldsymbol{A}_{\bullet n})$$

and hence

$$\widetilde{\boldsymbol{x}}\widetilde{\boldsymbol{A}} \;=\; \widetilde{\boldsymbol{x}}\boldsymbol{A} - \widetilde{\boldsymbol{x}}\boldsymbol{A} \;=\; \mathbf{0}.$$

Moreover, for any $\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}$ and $\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}$ we obtain

$$\boldsymbol{x} \begin{pmatrix} \widetilde{\boldsymbol{x}}\boldsymbol{A} \\ \vdots \\ \widetilde{\boldsymbol{x}}\boldsymbol{A} \end{pmatrix} \boldsymbol{y} \;=\; \boldsymbol{x} \begin{pmatrix} \widetilde{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y} \\ \vdots \\ \widetilde{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y} \end{pmatrix} \;=\; \widetilde{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y}\;.$$

This implies

$$\boldsymbol{x}\widetilde{\boldsymbol{A}}\boldsymbol{y} \;=\; \boldsymbol{x}\boldsymbol{A}\boldsymbol{y} - \widetilde{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y}.$$

Now we observe that all dominance relations are not influenced by the presence of the second term as $\boldsymbol{x}$ does not appear.

**2^nd^STEP** :

$\widetilde{\boldsymbol{x}}$ is not weakly dominated, hence there is *no* $\boldsymbol{x}$ such that

$$\boldsymbol{x}\boldsymbol{A}_{\bullet j} \geq \widetilde{\boldsymbol{x}}\boldsymbol{A}_{\bullet j} \qquad (j \in \{1, \ldots, n\})$$

and

$$\boldsymbol{x}\boldsymbol{A}_{\bullet j_0} > \widetilde{\boldsymbol{x}}\boldsymbol{A}_{\bullet j_0}$$

for at least one $j_0$, $1 \leq j_0 \leq n$.

Now define

$$U = \{\boldsymbol{x}\boldsymbol{A} \mid \boldsymbol{x} \in \underline{\overline{\mathbf{X}}}\}.$$

Then $U$ is a convex, compact polyhedron. We know that $\widetilde{\boldsymbol{u}} := \widetilde{\boldsymbol{x}}\boldsymbol{A} = \mathbf{0} \in U$. And we know that there is no $\boldsymbol{u} \in U$, $\boldsymbol{u} \geq \widetilde{\boldsymbol{u}}$ and $u_{j_0} > \tilde{u}_{j_0}$ for at least on $j_0$, $1 \leq j_0 \leq n$. That is, $U$ does *not* contain an element of the closed nonnegative orthant *apart from the origin* $\mathbf{0}$.

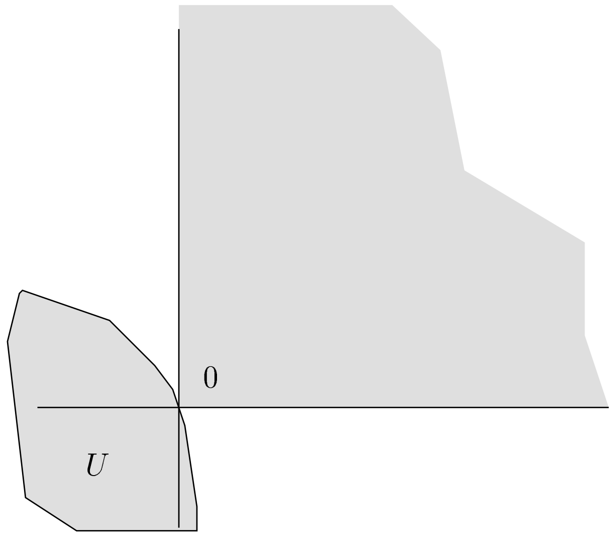



Figure 1.1: Separating $U$ and the non–negative orthant

**3$^{\text{rd}}$STEP** :

As it turns out, it suffices to construct a hyperplane with normal $\widetilde{\boldsymbol{p}} > \mathbf{0}$ which separates $U$ and $\mathbf{0}$ weakly, i.e. satisfies

$$\widetilde{\boldsymbol{p}}\widetilde{\boldsymbol{u}} \geq \widetilde{\boldsymbol{p}}\boldsymbol{u} \qquad (\boldsymbol{u} \in U). \tag{4}$$

Indeed, suppose we are given $\widetilde{\boldsymbol{p}} > \mathbf{0}$ satisfying (4). Then we put $\tilde{\boldsymbol{y}} := \frac{\widetilde{\boldsymbol{p}}}{\boldsymbol{e}\widetilde{\boldsymbol{p}}}$, and obtain $\tilde{\boldsymbol{y}} \in \underline{\overline{\mathbf{X}}}$, $\tilde{\boldsymbol{y}} > \mathbf{0}$ and

$$\widetilde{\boldsymbol{x}}\boldsymbol{A}\tilde{\boldsymbol{y}} = \widetilde{\boldsymbol{u}}\frac{\widetilde{\boldsymbol{p}}}{\boldsymbol{e}\widetilde{\boldsymbol{p}}} \geq \boldsymbol{u}\frac{\widetilde{\boldsymbol{p}}}{\boldsymbol{e}\widetilde{\boldsymbol{p}}} = \boldsymbol{x}\boldsymbol{A}\tilde{\boldsymbol{y}} \qquad (\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}) \ .$$

**4$^{\text{th}}$STEP** :

Hence we have to appeal to an appropriate separation theorem. Note that the polyhedral shape of $U$ is important. Nevertheless, we will not enter into a more detailed exposition (see Figure 1.2). **q.e.d.**

Now we change our viewpoint in order to specify a particular type of equilibrium. The following definition is due to R. SELTEN ([37])

**Definition 1.2.** *Let* $\Gamma = (\underline{\overline{\mathbf{X}}}, \underline{\overline{\mathbf{Y}}}; \boldsymbol{A}, \boldsymbol{B})$ *be a bimatrix game. A pair* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}}$ *is a* ***trembling hand perfect equilibrium*** *if there exists a sequence*

$$(\boldsymbol{x}^n, \boldsymbol{y}^n)_{n\in\mathbb{N}} \in \underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}}$$

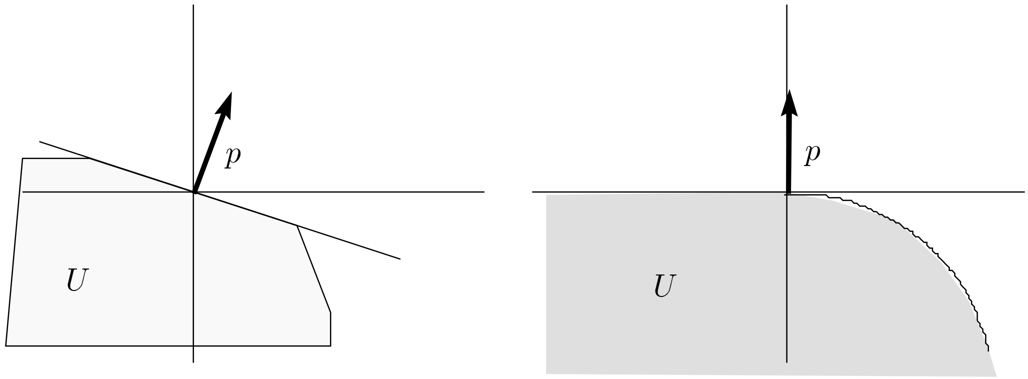



Figure 1.2: A polyhedral $U$ ensures a positive normal

*such that the following holds true*

1. $(\boldsymbol{x}^n, \boldsymbol{y}^n) \longrightarrow (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \quad (n \longrightarrow \infty)$
2. $(\boldsymbol{x}^n, \boldsymbol{y}^n) > 0 \qquad (n \in \mathbb{N})$
3. $\bar{\boldsymbol{x}}$ *is best reply against* $\boldsymbol{y}^n$ *(i.e.* $\bar{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y}^n \geq \boldsymbol{x}\boldsymbol{A}\boldsymbol{y}^n$*)* $(\boldsymbol{x} \in \underline{\overline{\mathbf{X}}},\ n \in \mathbb{N})$
4. $\bar{\boldsymbol{y}}$ *is best reply against* $\boldsymbol{x}^n$ *(i.e.* $\boldsymbol{x}^n\boldsymbol{B}\bar{\boldsymbol{y}} \geq \boldsymbol{x}^n\boldsymbol{B}\boldsymbol{y}$*)* $(\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}},\ n \in \mathbb{N})$

The elements $(\boldsymbol{x}^n, \boldsymbol{y}^n)$ of the sequence are interpreted as "small deviations" from equilibrium. E.g., if player 2 when choosing his (equilibrium) strategy $\bar{\boldsymbol{y}}$, slightly "trembles", thus choosing $\boldsymbol{y}^n$ instead, then nevertheless player 1 should have a best reply in his equilibrium strategy $\bar{\boldsymbol{x}}$. And vice versa.

Conceivably, a critique against this interpretation rests on the fact that there is just one sequence $(\boldsymbol{x}^n, \boldsymbol{y}^n)_{n\in\mathbb{N}}$ required in order to approximate $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$. This would (only) justify a statement that the pair $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ is robust against a certain type of trembling.

In our present context we shall use the term ***t.h.-perfect*** in order to refer to SELTEN's concept as indicated by Definition 1.2.

**Remark 1.3.** *Equivalently, given* $\Gamma = (\underline{\overline{\mathbf{X}}}, \underline{\overline{\mathbf{Y}}}; \boldsymbol{A}, \boldsymbol{B})$*, a pair of mixed strategies* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}}$ *is t.h.-perfect if and only if in every neighborhood of* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ *there exists a positive* $(\boldsymbol{x}, \boldsymbol{y}) \in \underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}}$ *such that the following statement holds true:*

1. $\bar{\boldsymbol{x}}$ *is best reply against* $\boldsymbol{y}$
2. $\bar{\boldsymbol{y}}$ *is best reply against* $\boldsymbol{x}$.

**Example 1.4.** Let $m = n = 3$ and consider the two matrices

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \qquad \boldsymbol{B} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix}$$

First of all we claim that $(\boldsymbol{e}^1, \boldsymbol{e}^1)$ is a Nash equilibrium but *not* t.h.-perfect.

Indeed, let $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{y}}) > 0$ be such that $\boldsymbol{e}^1$ is best reply against $\widetilde{\boldsymbol{y}}$ and $\boldsymbol{e}^1$ is best reply against $\widetilde{\boldsymbol{x}}$. Then, according to the optimality criterion for bimatrix games, we would infer that

$$e_1^1 > 0 \Longrightarrow \boldsymbol{A}_{1\bullet}\widetilde{y} = \max_i \boldsymbol{A}_{i\bullet}\widetilde{y} \tag{5}$$

holds true. However, we find

$$\boldsymbol{A}_{2\bullet}\widetilde{y} = \widetilde{y}_2 + 2\widetilde{y}_3 > \boldsymbol{A}_{3\bullet}\widetilde{y} = 2\widetilde{y}_3 > \boldsymbol{A}_{1\bullet}\widetilde{y} = 0 \tag{6}$$

holds true, contradicting (5) immediately. (Analogously for $\boldsymbol{e}^1$ and $\widetilde{\boldsymbol{x}}$). Thus, $(\boldsymbol{e}^1, \boldsymbol{e}^1)$ is not t.h.-perfect

The next observation concerns $(\boldsymbol{e}^3, \boldsymbol{e}^3)$. Again it is immediately seen that this is a Nash equilibrium.

However, $(\boldsymbol{e}^3, \boldsymbol{e}^3)$ is not t.h.-perfect. For, similarly to the above argument, if $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{y}}) > 0$ is such that $e^3$ is best reply against $\widetilde{\boldsymbol{y}}$, then the requirement

$$e_3^3 > 0 \Longrightarrow \boldsymbol{A}_{3\bullet}\widetilde{\boldsymbol{y}} = \max_i \boldsymbol{A}_{i\bullet}\widetilde{\boldsymbol{y}}$$

is immediately contradicted by

$$\boldsymbol{A}_{2\bullet}\widetilde{\boldsymbol{y}} = \widetilde{y}_2 + 2\widetilde{y}_3 > \boldsymbol{A}_{3\bullet}\widetilde{y} = 2\widetilde{y}_3.$$

Finally, $(\boldsymbol{e}^2, \boldsymbol{e}^2)$ turns out to be a Nash equilibrium which is as well t.h.-perfect. To verify this, consider for $\varepsilon > 0$ the pair of strategies defined by

$$\begin{aligned} \boldsymbol{x}^\varepsilon &:= (\varepsilon, 1 - 2\varepsilon, \varepsilon) \\ \boldsymbol{y}^\varepsilon &:= (\varepsilon, 1 - 2\varepsilon, \varepsilon) \end{aligned} \tag{7}$$

Now we obtain

$$\begin{aligned} \boldsymbol{A}_{2\bullet}\boldsymbol{y}^\varepsilon &= 1(1 - 2\varepsilon) + 2\varepsilon \\ &\geq \boldsymbol{A}_{3\bullet}\boldsymbol{y}^\varepsilon = 2\varepsilon \\ &\geq \boldsymbol{A}_{1\bullet}\boldsymbol{y}^\varepsilon = 0; \end{aligned}$$

thus, as $e^2_2 > 0$ and $\boldsymbol{A}_{3\bullet}\boldsymbol{y}^\varepsilon = \max_i \boldsymbol{A}_{i\bullet}\boldsymbol{y}^\varepsilon$, it is indeed true that $\boldsymbol{e}^2$ is best reply against $\widetilde{\boldsymbol{y}}$ (and similarly, $\boldsymbol{e}^2$ is best reply against $\widetilde{\boldsymbol{x}}$).

Note that with respect to the strategy sets of both players, $\boldsymbol{e}^1$ and $\boldsymbol{e}^3$ are weakly dominated while $\boldsymbol{e}^2$ is not.

**Theorem 1.5.** *Let $n = 2$. A pair of mixed strategies $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}}$ is t.h.-perfect if and only if the following conditions are satisfied.*

1. *$(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ is a Nash equilibrium.*
2. *$\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{y}}$ are not weakly dominated.*

**Proof:**

**1st STEP** : We assume that $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ is t.h.-perfect. Then we can find a sequence $(\boldsymbol{x}^n, \boldsymbol{y}^n)_{n\in\mathbb{N}}$ of positive elements of $\underline{\overline{\mathbf{X}}} \times \underline{\overline{\mathbf{Y}}}$ such that $(\boldsymbol{x}^n, \boldsymbol{y}^n) \to (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ $(n \to \infty)$ and

$$(8) \qquad \begin{array}{ll} \bar{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y}^n \geq \boldsymbol{x}\boldsymbol{A}\boldsymbol{y}^n & (n \in \mathbb{N}, \boldsymbol{x} \in \underline{\overline{\mathbf{X}}}) \\ \boldsymbol{x}^n\boldsymbol{B}\bar{\boldsymbol{y}} \geq \boldsymbol{x}^n\boldsymbol{B}\boldsymbol{y} & (n \in \mathbb{N}, \boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}) \end{array}$$

Passing through the limit yields

$$(9) \qquad \bar{\boldsymbol{x}}\boldsymbol{A}\bar{\boldsymbol{y}} \geq \boldsymbol{x}\boldsymbol{A}\bar{\boldsymbol{y}} \quad (\boldsymbol{x} \in \underline{\overline{\mathbf{X}}}) \quad , \quad \bar{\boldsymbol{x}}\boldsymbol{B}\bar{\boldsymbol{y}} \geq \boldsymbol{x}\boldsymbol{B}\boldsymbol{y} \quad (\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}),$$

i.e., the Nash equilibrium conditions.

Next, suppose that $\bar{\boldsymbol{x}}$ is weakly dominated, say by $\widehat{\boldsymbol{x}}$, i.e., we find $\widehat{\boldsymbol{x}}$ such that

$$(10) \qquad \begin{array}{lll} \widehat{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y} & \geq \bar{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y} & (\boldsymbol{y} \in \mathbb{N}) \\ \widehat{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y}^0 & > \bar{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y}^0 & \end{array}$$

for some $\boldsymbol{y}^0 \in \underline{\overline{\mathbf{Y}}}$.

From (10) we conclude that there is $j \in J$ such that

$$(11) \qquad \begin{array}{lll} \widehat{\boldsymbol{x}}\boldsymbol{A}_{\bullet l} & \geq \bar{\boldsymbol{x}}\boldsymbol{A}_{\bullet l} & (l \in J) \\ \widehat{\boldsymbol{x}}\boldsymbol{A}_{\bullet j} & > \bar{\boldsymbol{x}}\boldsymbol{A}_{\bullet j} & \end{array}$$

Now, pick any arbitrary $\boldsymbol{y} > 0$, $\boldsymbol{y} \in \underline{\overline{\mathbf{Y}}}$. Then

$$(12) \qquad \widehat{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y} = \sum_{l=1}^{n} \widehat{\boldsymbol{x}}\boldsymbol{A}_{\bullet l}y_l > \sum_{l=1}^{n} \bar{\boldsymbol{x}}\boldsymbol{A}_{\bullet l}y_l = \bar{\boldsymbol{x}}\boldsymbol{A}\boldsymbol{y}$$

in view of (11). That is, $\bar{\boldsymbol{x}}$ cannot be best reply against positive $\boldsymbol{y} \in \underline{\mathbf{Y}}$, contradicting then t.h.-perfectness (or (8) for that matter). Hence $\bar{\boldsymbol{x}}$ (and $\bar{\boldsymbol{y}}$) is *not* weakly dominated. This settles the first step of our proof.

**2[nd]STEP** :

Now consider $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \underline{\mathbf{X}} \times \underline{\mathbf{Y}}$ constituting a Nash equilibrium such that $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{y}}$ are *not* weakly dominated. According to Lemma 1.1 there is $\widetilde{\boldsymbol{y}} \in \underline{\mathbf{Y}}$, $\widetilde{\boldsymbol{y}} > 0$ against which $\bar{\boldsymbol{x}}$ is best reply. Also there is $\widetilde{\boldsymbol{x}} \in \underline{\mathbf{X}}$, $\widetilde{\boldsymbol{x}} > 0$ against which $\bar{\boldsymbol{y}}$ is best reply. Now the set

$$\mathbf{K}_{\bar{\boldsymbol{x}}} = \{\boldsymbol{y} \mid \bar{\boldsymbol{x}} \text{ is best reply against } \boldsymbol{y}\} = \bigcap_{\{i \mid \bar{x}_i > 0\}} \mathbf{K}_i$$

is a convex compact polyhedron; hence as $\widetilde{\boldsymbol{y}}$ and $\bar{\boldsymbol{y}}$ are contained in $K_{\bar{\boldsymbol{x}}}$, so is

$$\boldsymbol{y}^{\varepsilon} := (1-\varepsilon)\bar{\boldsymbol{y}} + \varepsilon\widetilde{\boldsymbol{y}}$$

for all $\varepsilon > 0$. Similarly,

$$\boldsymbol{x}^{\varepsilon} := (1-\varepsilon)\bar{\boldsymbol{x}} + \varepsilon\widetilde{\boldsymbol{x}}$$

is best replied against by $\bar{\boldsymbol{y}}$ $(\varepsilon < 0)$.

As $(\boldsymbol{x}^{\varepsilon}, \boldsymbol{y}^{\varepsilon}) > 0$ and $\varepsilon > 0$ can be arbitrarily small, this shows that $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ is t.h.-perfect, **q.e.d.**

**Theorem 1.6 (SELTEN).** *Every bimatrix game* $\Gamma = (\underline{\mathbf{X}}, \underline{\mathbf{Y}}; \boldsymbol{A}, \boldsymbol{B})$ *has t.h.-perfect Nash equilibria.*

**Proof:** Define for $\varepsilon > 0$

$$\begin{aligned} \underline{\mathbf{X}}^{\varepsilon} &:= \{\boldsymbol{x} \in \underline{\mathbf{X}} \mid x_i \geq \varepsilon \quad (i = 1, \ldots, m)\} \\ \underline{\mathbf{Y}}^{\varepsilon} &:= \{\boldsymbol{y} \in \underline{\mathbf{Y}} \mid y_j \geq \varepsilon \quad (j = 1, \ldots, n)\} \end{aligned} \tag{13}$$

and let

$$\Gamma^{\varepsilon} := (\underline{\mathbf{X}}^{\varepsilon}, \underline{\mathbf{Y}}^{\varepsilon}; \boldsymbol{A}, \boldsymbol{B}) \tag{14}$$

be the game obtained by restricting the strategy spaces. As $\underline{\mathbf{X}}^{\varepsilon}$ and $\underline{\mathbf{Y}}^{\varepsilon}$ are convex and compact and $\boldsymbol{A}$ and $\boldsymbol{B}$ are linear functions in each coordinate it follows that $\Gamma^{\varepsilon}$ has a Nash equilibrium, say $(\bar{\boldsymbol{x}}^{\varepsilon}, \bar{\boldsymbol{y}}^{\varepsilon})$ $(\varepsilon > 0)$.

Pick a subsequence $(\varepsilon_n)_{n\in\mathbb{N}}$, $\varepsilon_n \to 0$ $(n \to \infty)$ such that

$$(\bar{\boldsymbol{x}}^{\varepsilon_n}, \bar{\boldsymbol{y}}^{\varepsilon_n}) \longrightarrow (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \underline{\mathbf{X}} \times \underline{\mathbf{Y}} \qquad (n \to \infty)$$

holds true, clearly $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ is a Nash equilibrium for $\Gamma$. We would like to show that $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ is t.h.-perfect.

To this end it is sensible to attempt a proof of the fact that $\bar{\boldsymbol{x}}$ is best reply against $\bar{\boldsymbol{y}}^\varepsilon$ (and $\bar{\boldsymbol{y}}$ against $\bar{\boldsymbol{x}}^\varepsilon$).

In $\Gamma^\varepsilon$ the "optimality criterion" is formulated as follows:

$(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}})$ is a Nash equilibrium in $\Gamma^\varepsilon$ if and only if

$$\widehat{x}_l > \varepsilon \Longrightarrow \boldsymbol{A}_{l\bullet}\widehat{\boldsymbol{y}} = \max_r \boldsymbol{A}_{r\bullet}\widehat{\boldsymbol{y}}$$
$$\widehat{y}_k > \varepsilon \Longrightarrow \widehat{\boldsymbol{x}}\boldsymbol{B}_{\bullet k} = \max_s \widehat{\boldsymbol{x}}G_{\bullet s}$$

holds true.

Therefore, if $\bar{x}_i > 0$ for some $i \in \{1, \ldots, m\}$ then $\bar{x}_i^\varepsilon > \varepsilon$ for sufficiently small $\varepsilon > 0$ and hence

$$\boldsymbol{A}_{i\bullet}\bar{\boldsymbol{y}}^\varepsilon = \max_r \boldsymbol{A}_{r\bullet}\bar{\boldsymbol{y}}^\varepsilon. \tag{15}$$

Hence, by choosing $\varepsilon_0$ small such that $\bar{x}_i > 0$ implies $\bar{x}_i^\varepsilon$ for *all* $i = 1, \ldots, m$, and $\varepsilon < \varepsilon_0$, we conclude that

$$\bar{x}_i > 0 \Longrightarrow \boldsymbol{A}_{i\bullet}\bar{\boldsymbol{y}}^\varepsilon = \max_r \boldsymbol{A}_{r\bullet}\bar{\boldsymbol{y}}^\varepsilon \qquad (\varepsilon < \varepsilon_0).$$

Thus, $\bar{\boldsymbol{x}}$ is best reply against $\bar{\boldsymbol{y}}^\varepsilon > 0$ for all $\varepsilon > 0$, $\varepsilon < \varepsilon_0$, **q.e.d.**

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