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The Maschler–Perles–Shapley value for Taxation Games

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Abstract

We continue the discussion of the taxation game following our presentation in [12]. Our concept describes a cooperative game played between a set of jurisdictions (" countries"). These players admit the operation of a multinational enterprise (MNE, the "firm") within their jurisdiction. The original version of this game is due to W. F. Richter [3],[4]. We suggest an extension of the model by introducing the dual game of the firm's profits and the tax function game. The latter is the NTU game generated by introducing tax functions (the term "tariffs" will be avoided henceforth).

In [12] we treated the bargaining situation obtained when all countries decide to cooperate – otherwise everyone will fall back on their status quo point. However, in his basic paper, Richter argues that the share of the tax basis allotted to a country should be determined by the Shapley value of the taxation game. This idea establishes an interesting new field of applications.

The Shapley value "as a tool in theoretical economics" [13], [14] has widely been applied in Game Theory, Equilibrium Theory, applications to Cost Sharing problems, Airport Landing Fee games, and many others.

Based on these ideas, we continue our presentation by formulating the tax function game for the countries involved and introducing the Maschler–Perles–Shapley value as developed in [11].

To this end, we introduce the adjusted TU game, which reflects a rescaling of utility measurement as suggested by the superadditivity axiom of the Maschler–Perles solution. Then the Maschler–Perles–Shapley value of the tax function game is the image of the Shapley value of the adjusted TU game on the Pareto surface of the grand coalition.

We demonstrate that the Maschler-Perles–Shapley value for the tax function game is Pareto efficient, covariant with affine transformations of utility, and anonymous.

1 The Tax Function Game

The notation within this article is the one of [12]. We assume that the reader is familiar with concepts of a cooperative game ("with side payments") $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$ and Non–Transferable–Utility ("NTU") games $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{V})$.

We focus on a Multinational Enterprise ("the firm") operating in certain jurisdictions ("countries") , i.e., the ***players*** described by $\boldsymbol{I} = \{1, \ldots, n\}$. Countries consider an agreement about how to allot the firms profit for taxation purposes.

Subgroups of countries are ***coalitions***. If a coalition $S \subseteq \boldsymbol{I}$ agrees to cooperate, then the firm may carry out business simultaneously in each cooperating country. The profit $\boldsymbol{v}(S)$ earned is taxed according to a jointly agreed system of rules. Following W.F. Richter [3], [4] in a very simplified way, we adopt a profit concept called "pure return earned on know–how". Somewhat more precisely, we assume that an agency responsible for "unitary taxation" is established within the (joint) jurisdiction of all members of S.

Based on this idea, a firm operating in various groups of countries generates a ***coalitional function*** $\boldsymbol{v}$. We obtain a cooperative TU game $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$ which is called the ***profit game***.

In [12] we argue that the ***dual game*** to the profit game should be focused upon. For, otherwise the firm has incentives to ("after taxation rules") deviate from the planned production generating the profit $\boldsymbol{v}(\bullet)$. In the context of the dual game the firm has no such incentives and, hence, the basic model exhibits a higher degree of stability.

In order to obtain a most general theory, we agree that a profit game $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$ may be thought of as either referring to the primal or dual version of $\boldsymbol{v}$. In a particular application, this ambiguity should always be taken care of in advance.

Next we turn to the NTU context by introducing a ***Tax Function*** (for short a ***TF***). This is a function applied by the various countries to actually generate revenue by taxing the firm. A TF admits of a dual version as well and again we leave the details open for clarification in a specific context. A dual version is exhibited in Figure 1.1.

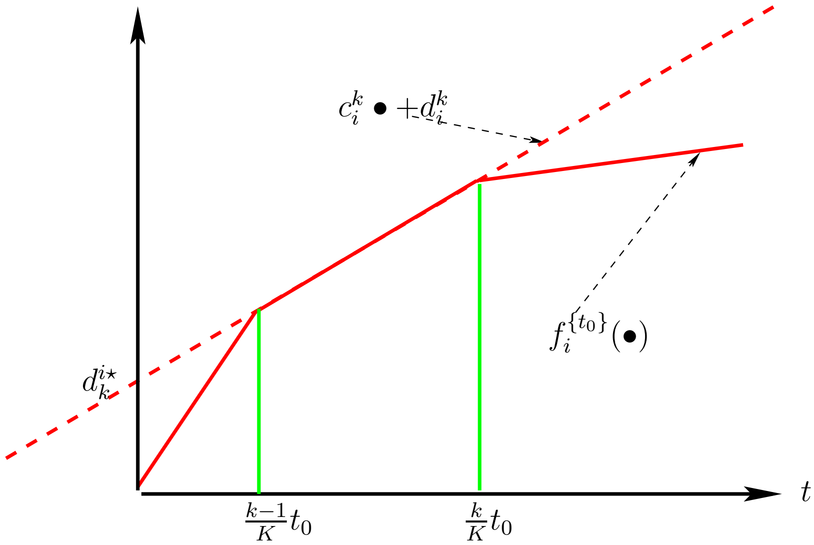

Figure 1.1: A dual TF of country i

The concave function in Figure 1.1 suggests a dual TF but we disregard the reference to primal and dual and just consider a piecewise linear and concave function as above. In order to provide description of such functions we introduce a "universal grid" as follows: We set

$$\boldsymbol{K} \;=\; \{1, \ldots, K\}\ . \tag{1.1}$$

Then for some $t_0 \in \mathbb{R}+$, the grid points are given by

$$\left\{0, \frac{t_0}{K}, \frac{2t_0}{K}, \ldots, \frac{(K-1)t_0}{K}\right\}\ . \tag{1.2}$$

A function $f^{\{t_0\}}$ as in Figure 1.1 is linear within intervals

$$\boldsymbol{J}_k^{\{t_0\}} \;:=\; \left[\frac{(k-1)t_0}{K}, \frac{kt_0}{K}\right], \quad (k \in \boldsymbol{K}) \tag{1.3}$$

Thus, a TF equals a linear function within each interval of the grid. Within this context we obtain

Definition 1.1. Let $t_0 \in \mathbb{R}_+, t_0 \leq K$.

1. A ***Tax Function (TF)*** (for taxation in $[0, t_0]$, adapted to $\boldsymbol{K}$) is a continuous, monotone, concave, and piecewise linear function

$$f^{\{t_0\}} \;:\; [0, t_0] \to \mathbb{R}_+, \quad f(0) = 0\ . \tag{1.4}$$

also written

$$f(t) \;=\; \min_{k \in \boldsymbol{K}} c^k t + d^k \quad (t \in \mathbb{R}_+ k, \quad k \in \boldsymbol{K}) \tag{1.5}$$

with suitable coefficients $c_k, d_k \quad (k = 1, \ldots, K)$.

2. Given the appropriate ***boundary conditions*** the coefficients $c^k, d^k \quad (k \in \boldsymbol{K})$ corresponding to f yield

$$f^{\{t_0\}}(t) \;=\; c^k t + d^k \quad (t \in \boldsymbol{J}_k^{\{t_0\}}, \quad k \in \boldsymbol{K}) \; . \tag{1.6}$$

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As to the boundary coefficients, the obvious definition – tedious and not further enlightening – is given in Definition 1.7 of [12]. Clearly, a piecewise linear function adapted to the grid and the corresponding coefficients representing the linear function within each interval of the grid are tantamount quantities.

Assume that a quantity t_0 is available for taxation. Then, as we assign a TF to each country $i \in \boldsymbol{I}$, we obtain

Definition 1.2. 1. Next, a ***TF system*** (for taxation in $[0, t_0]$, adapted to $\boldsymbol{K}$) is a family of TFs

$$\boldsymbol{f}^{\{t_0\}} \;=\; (f_1^{\{t_0\}}, \ldots, f_n^{\{t_0\}}) \tag{1.7}$$

such that for $i \in \boldsymbol{I}$

$$f_i^{\{t_0\}}(t) \;:\; [0, t_0] \to \mathbb{R}_+ \tag{1.8}$$

satisfies

$$f_i^{\{t_0\}}(t) \;=\; c_i^k t + d_i^k \quad (t \in \boldsymbol{J}_k^{\{t_0\}}, \quad k \in \boldsymbol{K}) \; . \tag{1.9}$$

with coefficients

$$c_i^k, d_i^k \quad (i \in \boldsymbol{I}, k \in \boldsymbol{K}) \; . \tag{1.10}$$

satisfying the boundary conditions.

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A distribution of monetary values is represented by a vector $\mathbf{t} \;=\; (t_1, \ldots, t_n) \in \mathbb{R}^n_+$. A TF system reflects the taxes obtained by such monetary distributions. The payoffs available to the countries are then obtained by the ***taxoid***

$$\Pi^{\{\boldsymbol{f}\}} = \Pi^{\{\boldsymbol{f}, t_0\}} \;:=\; \left\{ \boldsymbol{f}^{\{t_0\}}(\mathbf{t}) \,\middle|\, \mathbf{t} \in \mathbb{R}^n_+ \;, \; \sum_{i \in \boldsymbol{I}} t_i \leq t_0 \right\} \tag{1.11}$$

However, whenever we consider the influence of coalitions as well, we have to extend this notion in order to arrive at the NTU game generated. To this end we combine the TF system with the profit game $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$. That is, we determine, for each coalition $S \in \underline{\underline{\mathbf{P}}}$ the payoffs available via the TFs of the countries involved.

Let

$$\boldsymbol{f}^{\{\boldsymbol{v}(\boldsymbol{I})\}} = (f_1^{\{\boldsymbol{v}(\boldsymbol{I})\}}, \ldots, f_n^{\{\boldsymbol{v}(\boldsymbol{I})\}}) \tag{1.12}$$

be a TF system for the distribution of $t_0 = \boldsymbol{v}(\boldsymbol{I})$. Assume that, for $S \in \underline{\underline{\mathbf{P}}}$, the term $\boldsymbol{v}(S)$ is an element of the grid

$$\left\{0, \frac{\boldsymbol{v}(\boldsymbol{I})}{K}, \frac{2\boldsymbol{v}(\boldsymbol{I})}{K}, \ldots, \frac{(K-1)\boldsymbol{v}(\boldsymbol{I})}{K}, \boldsymbol{v}(\boldsymbol{I})\right\} , \tag{1.13}$$

say $\boldsymbol{v}(S) = \frac{K'\boldsymbol{v}(\boldsymbol{I})}{K}$ with some $K' \leq K$. Then, because of

$$\frac{\boldsymbol{v}(S)}{K'} = \frac{\boldsymbol{v}(\boldsymbol{I})}{K} , \tag{1.14}$$

the grid

$$\left\{0, \frac{\boldsymbol{v}(S)}{K'}, \frac{2\boldsymbol{v}(S)}{K'}, \ldots, \frac{(K'-1)\boldsymbol{v}(S)}{K'}, \boldsymbol{v}(S)\right\} \tag{1.15}$$

is a subgrid of the grid (1.13). Hence, the function

$$f_i^{\{\boldsymbol{v}(S)\}} := f_i^{\{\boldsymbol{v}(\boldsymbol{I})\}}\Big|_{[0,\boldsymbol{v}(S)]} \quad (i \in \boldsymbol{I}) \tag{1.16}$$

constitutes a TF for $i \in \boldsymbol{I}$ and

$$\boldsymbol{f}^{\{\boldsymbol{v}(S)\}} = (f_1^{\{\boldsymbol{v}(S)\}}, \ldots, f_n^{\{\boldsymbol{v}(S)\}}) \tag{1.17}$$

is a TF system for the distribution of $\boldsymbol{v}(S)$ with the same linear ingredients, i.e., we have

$$\begin{aligned} f_i^{\{\boldsymbol{v}(S)\}}(t) &= c_k^i t + d_k^i \quad t \in \left[\frac{(k-1)\boldsymbol{v}(S)}{K'}, \frac{k\boldsymbol{v}(S)}{K'}\right] = \boldsymbol{J}_k^{\{\boldsymbol{v}(S)\}} \\ &= c_k^i t + d_k^i \quad t \in \left[\frac{(k-1)\boldsymbol{v}(\boldsymbol{I})}{K}, \leq \frac{k\boldsymbol{v}(\boldsymbol{I})}{K}\right] = \boldsymbol{J}_k^{\{\boldsymbol{v}(\boldsymbol{I})\}} \\ &= f_i^{\{\boldsymbol{v}(\boldsymbol{I})\}}(t) \quad (i \in \boldsymbol{I}, k \in \boldsymbol{K}, k \leq K') . \end{aligned} \tag{1.18}$$

(The last term in the first row of (1.18) refers to the grid generated by K'). In other words, the function $\boldsymbol{f}^{\{\boldsymbol{v}(I)\}}$ serves simultaneously as a TF system for all coalitions via the restriction

$$\boldsymbol{f}^{\{\boldsymbol{v}(S)\}} := \boldsymbol{f}^{\{\boldsymbol{v}(\boldsymbol{I})\}}\Big|_{[0,\boldsymbol{v}(S)]} . \tag{1.19}$$

Definition 1.3. Let $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$ be the profit game. Also, let

$$\boldsymbol{f}^{\{\boldsymbol{v}(\boldsymbol{I})\}} = (f_1^{\{\boldsymbol{v}(\boldsymbol{I})\}}, \ldots, f_n^{\{\boldsymbol{v}(\boldsymbol{I})\}}) \tag{1.20}$$

be a TF system for the distribution of $\boldsymbol{v}(\boldsymbol{I})$. For $S \in \underline{\underline{\mathbf{P}}}$ let $\boldsymbol{f}^{\{\boldsymbol{v}(S)\}}$ be the derived tax function system according to (1.19).

The ***TF game*** (resulting from $\boldsymbol{v}$ and $\boldsymbol{f}^{\{\boldsymbol{v}(\boldsymbol{I})\}}$) is the NTU game $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{V}^{\{\boldsymbol{v}\}})$ defined via

$$\boldsymbol{V}^{\{\boldsymbol{v}\}}(S) := \left\{ \boldsymbol{f}^{\{\boldsymbol{v}(S)\}}(\mathbf{t}) \,\middle|\, \mathbf{t} \in \mathbb{R}_+^S , \sum_{i \in S} t_i \leq \boldsymbol{v}(S) \right\} \quad (S \in \underline{\underline{\mathbf{P}}}) . \tag{1.21}$$

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2 The Pareto Surface of a Taxoid

Given $t_0 \in \mathbb{R}_+$ and a TF system $\boldsymbol{f}^{\{t_0\}}$ we consider the taxoid $\Pi^{\{\boldsymbol{f}\}} = \Pi^{\{\boldsymbol{f},t_0\}}$ defined in (1.11). We introduce the unit simplex $\Delta^{\boldsymbol{e}} = \{\mathbf{t} \in \mathbb{R}^n_+ \mid \boldsymbol{e}\mathbf{t} = 1\}$ and its dilatation and $\Delta^{t_0\boldsymbol{e}} = t_0\Delta^{\boldsymbol{e}}$.

Then the Pareto surface of the taxoid (1.11) is

$$\text{(2.1)} \qquad \partial\Pi^{\{\boldsymbol{f},t_0\}} := \left\{ \boldsymbol{f}^{\{t_o\}}(\mathbf{t}) \,\middle|\, \mathbf{t} \in \mathbb{R}^n_+ \,,\; \sum_{i\in\boldsymbol{I}} t_i = t_0 \right\} = \boldsymbol{f}^{\{t_0\}}(\Delta^{t_0\boldsymbol{e}}) \,.$$

The Pareto surface constitutes the basic for the surface measure and a corresponding version of the Maschler–Perles–Shapley value regarding the TF game. Therefore, we cast a short view on this surface.

Consider a sequence

$$\mathbf{k} = (k_1, \ldots, k_n) \mid k_i \in \boldsymbol{K} \quad (i \in \boldsymbol{I})$$

of indices taken from $\boldsymbol{K}$. Because of

$$\text{(2.2)} \qquad f_i^{\{t_0\}}(s) = c^i_k s + d^i_k \quad (s \in \boldsymbol{J}_k^{\{t_0\}}, i \in \boldsymbol{I}, k \in \boldsymbol{K})$$

we obtain an n–dimensional cuboid

$$\text{(2.3)} \qquad \boldsymbol{Q}_{\mathbf{k}}^{\{t_0\}} = \boldsymbol{Q}_{k_1,\ldots,k_n}^{\{t_0\}} := \left\{ \mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n_+ \,\middle|\, \begin{array}{ll} t_1 & \in \boldsymbol{J}_{k_1}^{\{t_0\}} \\ & \cdot \\ & \cdot \\ & \cdot \\ t_n & \in \boldsymbol{J}_{k_n}^{\{t_0\}} \end{array} \right\}$$

such that $\boldsymbol{f}$ restricted to $\boldsymbol{Q}_{\mathbf{k}}^{\{t_0\}}$ coincides with the (affine) linear mapping induced by the local coefficients (Definition 1.2). According to (1.9), this mapping is given by

$$\text{(2.4)} \qquad \begin{aligned} \overset{\star}{\boldsymbol{x}}^{\mathbf{k}}(\bullet) &: \mathbb{R}^n_+ \to \mathbb{R}^n_+ \,, \\ \mathbf{t} &= (t_1, \ldots, t_n) \to (c^1_{k_1} t_1 + d^1_{k_1}, \ldots, c^n_{k_n} t_n + d^n_{k_n}) \,, \\ \overset{\star}{\boldsymbol{x}}^{\mathbf{k}}(\mathbf{t}) &= \mathbf{c}_{\mathbf{k}} \otimes \mathbf{t} + \mathbf{d}_{\mathbf{k}} \quad (\mathbf{t} \in \boldsymbol{Q}_{\mathbf{k}}^{\{t_0\}}) \,. \end{aligned}$$

Combining we obtain the following

Definition 2.1. 1. Let

$$\text{(2.5)} \qquad \mathbf{K} := \{\mathbf{k} = (k_1, \ldots, k_n) \mid k_i \in \boldsymbol{K} \quad (i \in \boldsymbol{I})\}$$

denote the system of all possible choices of indices taken from $\boldsymbol{K}$. For $t_0 > 0$ and $\mathbf{k} \in \mathbf{K}$ define

$$\text{(2.6)} \qquad \Delta_{\mathbf{k}}^{t_0\boldsymbol{e}} := \boldsymbol{Q}_{\mathbf{k}}^{\{t_0\}} \cap \Delta^{t_0\boldsymbol{e}} \,.$$

such that

$$\text{(2.7)} \qquad \boldsymbol{f}^{\{t_0\}}{}_{|\Delta_{\mathbf{k}}^{t_0\boldsymbol{e}}} = \overset{\star}{\boldsymbol{x}}^{\mathbf{k}}(\bullet)_{|\Delta_{\mathbf{k}}^{t_0\boldsymbol{e}}} \quad (\mathbf{k} \in \mathbf{K})$$

holds true.

2. We call a sequence $\mathbf{k} = (k_1, \ldots, k_n)$ such that $\Delta^{\boldsymbol{e}}_{\mathbf{k}} \neq \emptyset$ has *full dimension* relevant. Also, $\Delta^{\boldsymbol{e}}_{\mathbf{k}}$ is called a relevant polyhedron. The set of relevant sequences is denoted $\overset{\circ}{\mathbf{K}}$.

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Note that $\Delta^{t_0\boldsymbol{e}}_{\mathbf{k}}$ $(t_0 > 0)$ has full dimension if and only if $\Delta^{\boldsymbol{e}}_{\mathbf{k}}$ has full dimension, i.e., the relevance of some sequence $\mathbf{k} \in \overset{\circ}{\mathbf{K}}$ does not depend on the dilatation factor t_0. For a relevant polyhedron the polyhedron (2.7) has full dimension and constitutes a Pareto face of the taxoid $\Pi^{\{\boldsymbol{f},t_0\}}$. More precisely we have

Corollary 2.2. [**The parametrization of $\partial\Pi^{\{\boldsymbol{f},t_0\}}$**]

Let $t_0 \in \mathbb{R}$ and let $\Pi^{\{\boldsymbol{f},t_0\}}$ be the taxoid induced by a TF system $\boldsymbol{f}^{\{t_0\}}$. The Pareto faces of this taxoid are given by

$$\boldsymbol{F}^{\{\mathbf{k},t_0\}} := \overset{\star}{\boldsymbol{x}}{}^{\mathbf{k}}(\Delta^{t_0\boldsymbol{e}}_{\mathbf{k}}) \subseteq \partial\Pi^{\{\boldsymbol{f},t_0\}} \quad (\mathbf{k} \in \overset{\circ}{\mathbf{K}}) \ . \tag{2.8}$$

The bijection

$$\overset{\star}{\boldsymbol{x}}{}^{\mathbf{k}}(\bullet) \ : \ \Delta^{t_0\boldsymbol{e}}_{\mathbf{k}} \to \boldsymbol{F}^{\{\mathbf{k},t_0\}} \quad (\mathbf{k} \in \overset{\circ}{\mathbf{K}}) \ . \tag{2.9}$$

provides a parametrization of the Pareto face $\boldsymbol{F}^{\{\mathbf{k},t_0\}}$ (see [10] and [11]).

Therefore, the mapping

$$\overset{\star}{\boldsymbol{x}}(\bullet) : \Delta^{t_0\boldsymbol{e}} \to \partial\Pi^{\{\boldsymbol{f},t_0\}} \ , \quad \mathbf{t} \to \overset{\star}{\boldsymbol{x}}{}^{\mathbf{k}}(\mathbf{t}) \ \ (\mathbf{t} \in \Delta^{t_0\boldsymbol{e}}_{\mathbf{k}}, \mathbf{k} \in \overset{\circ}{\mathbf{K}}) \tag{2.10}$$

which is the composition of the local parametrizations

$$\overset{\star}{\boldsymbol{x}}(\bullet)_{|\Delta^{t_0\boldsymbol{e}}_{\mathbf{k}}} = \overset{\star}{\boldsymbol{x}}{}^{\mathbf{k}}(\bullet) \ \ (\mathbf{k} \in \overset{\circ}{\mathbf{K}}) \ . \tag{2.11}$$

constitutes a the canonical global parametrization of the Pareto surface $\Pi^{\{\boldsymbol{f},t_0\}}$.

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We emphasize that $\overset{\star}{\boldsymbol{x}}{}^{\mathbf{k}}(\bullet)$ is an (affine) linear mapping. The composed mapping $\overset{\star}{\boldsymbol{x}}(\bullet)$ is, in general, not linear. Indeed, $\overset{\star}{\boldsymbol{x}}(\bullet)$ coincides with $\boldsymbol{f}^{\{t_0\}}$ – the unifying notation of (2.10) emphasizes the locally linear character.

As it turned out ([12]), the Pareto faces $\boldsymbol{F}^{\mathbf{k}}$ show in general not a simplicial character. Rather they are unions of simplices located in a common hyperplane. This allows us to define a version of the Maschler-Perles surface measure. We describe our findings including the results of [12] within the following Theorem.

Theorem 2.3. *[The Pareto Surface of a Taxoid]*

Let $0 < t_0 \leq K$ *and let* $\boldsymbol{f}^{\{t_0\}}$ *be a TF system. Let* $\Pi^{\{\boldsymbol{f},t_0\}}$ *be the corresponding taxoid.*

1. *Let* $\mathbf{k} \in \overset{\circ}{\mathbf{K}}$ *be a relevant sequence. Then*

$$\boldsymbol{F}^{\{\mathbf{k},t_0\}} = \overset{\star}{\boldsymbol{x}}^{\mathbf{k}}(\Delta_{\mathbf{k}}^{t_0\boldsymbol{e}}) = \boldsymbol{c}_{\mathbf{k}} \otimes \Delta_{\mathbf{k}}^{t_0\boldsymbol{e}} + \boldsymbol{d}_{\mathbf{k}} \tag{2.12}$$

   is a Pareto face of $\Pi^{\{\boldsymbol{f},t_0\}}$. *The relevant sequences describe exactly all Pareto faces.*

2. *Because of*

$$\Delta^{t_0\boldsymbol{e}} = \bigcup_{\mathbf{k} \in \overset{\circ}{\mathbf{K}}} \Delta_{\mathbf{k}}^{t_0\boldsymbol{e}} \tag{2.13}$$

   the Pareto surface of $\Pi^{\{\boldsymbol{f},t_0\}}$ *is described by*

$$\partial\Pi^{\{\boldsymbol{f},t_0\}} = \bigcup_{\mathbf{k} \in \overset{\circ}{\mathbf{K}}} \boldsymbol{F}^{\{\mathbf{k},t_0\}} \ . \tag{2.14}$$

3. *Let* $\mathbf{k} \in \overset{\circ}{\mathbf{K}}$ *be a relevant sequence. Then there exists a family of simplices* $\left\{ \overset{\star}{\Delta}^l \,\middle|\, l \in \mathbf{L}^{\mathbf{k}} \right\}$ *such that*

$$\Delta_{\mathbf{k}}^{t_0\boldsymbol{e}} = \bigcup_{l \in \mathbf{L}^{\mathbf{k}}} \overset{\star}{\Delta}^l \tag{2.15}$$

4. *For* $l \in \mathbf{L}^{\mathbf{k}}$ *let*

$$\overset{\star}{\boldsymbol{F}}^l := \overset{\star}{\boldsymbol{x}}^{\mathbf{k}}(\overset{\star}{\Delta}^l) = \mathbf{c}_{\mathbf{k}} \otimes \overset{\star}{\Delta}^l + \mathbf{d}_{\mathbf{k}} \ . \tag{2.16}$$

   Then $\overset{\star}{\boldsymbol{F}}^l$ *is a simplex in* $\partial\Pi^{\{\boldsymbol{f},t_0\}}$ *such that*

$$\boldsymbol{F}^{\{\mathbf{k},t_0\}} = \bigcup_{l \in \mathbf{L}^{\mathbf{k}}} \overset{\star}{\boldsymbol{F}}^l \tag{2.17}$$

   holds true.

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Proof: No proof is offered. **q.e.d.**

3 Surface Measures

Within this section we extend the Maschler-Perles-Shapley value to the present context of TF games provided by Definition 1.3. The prerequisites can be found in [12], the basic concepts are due to Maschler–Perles [2], the extension has been developed in [8]. The rationale underlying this version of the Shapley value is (a weak form of) superadditivity.

We recall the definition of the surface measures involved.

Remark 3.1. The Maschler–Perles measure $\boldsymbol{\iota}_\Delta$ is defined for Cephoids and – by a limiting procedure – for smooth bodies, see [8], [9]. It is normalized such that $\boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{e}}) = 1$ holds true for the unit simplex and $\boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}}) = \sqrt[n]{\left(\prod_{i\in \boldsymbol{I}} a_i\right)^{n-1}}$ for a deGua simplex $\Pi^{\boldsymbol{a}}$ and its Pareto surface $\Delta^{\boldsymbol{a}}$. The Maschler–Perles Measure $\boldsymbol{\iota}_\Delta$ is then defined for Cephoids by additive extension, i.e., via the particular structure of such polyhedra. The deGua measure $\boldsymbol{\vartheta}$ is obtained by a standard calculus of measure and integral defined on certain surfaces. We refer to [10], [11].

For a deGua Simplex $\Delta^{\boldsymbol{a}}$, these "surface measures" satisfy $\boldsymbol{\iota}_\Delta(\bullet) = \boldsymbol{o}(n)\boldsymbol{\vartheta}(\bullet)$ with some constant $\boldsymbol{o}(n)$ depending on the dimension. As in [12] we unify our notation by working with the measure

$$\boldsymbol{\theta} := \boldsymbol{o}(n)\boldsymbol{\vartheta} \ . \tag{3.1}$$

somewhat sloppily we call $\boldsymbol{\theta}$ the Maschler–Perles measure.

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We recall our results of [12] as follows.

Theorem 3.2. ***[The Maschler–Perles Measure on a Taxoid]***

Let $0 < t_0 < K$ and let $\boldsymbol{f}^{\{t_0\}}$ be a TF system. Let $\Pi^{\{\boldsymbol{f},t_0\}}$ be the taxoid induced. Recall the structure of the Pareto surface presented in Theorem 2.3.

1. *For $n = 2$ and $\mathbf{k} \in \overset{\circ}{\mathbf{K}}$, $\Delta^{K\boldsymbol{e}}_{\mathbf{k}}$ is a unit simplex such that $\boldsymbol{\theta}(\Delta^{K\boldsymbol{e}}_{\mathbf{k}}) = 1$ holds true. Moreover, for some $\mathbf{k} = (k, K-(k-1))$,*

$$\boldsymbol{\theta}(\boldsymbol{F}^{\{\mathbf{k},t_0\}}) = \sqrt[2]{a_1^{(k)}a_2^{(k)}} = \frac{t_0}{K}\sqrt[2]{c^1_{k_1}c^2_{k_2}} = \frac{t_0}{K}\sqrt[2]{c^1_{k}c^2_{K-(k-1)}} \ . \tag{3.2}$$

2. *For $n = 3$, Lemma 2.5 of [12] characterizes the relevant sequences. The Maschler-Perles measure for $\mathbf{k} \in \overset{\circ}{\mathbf{K}}$ is*

$$\boldsymbol{\theta}(\Delta^{K\boldsymbol{e}}_{\mathbf{k}}) = 1, \quad \boldsymbol{\theta}(\Delta^{t_0\boldsymbol{e}}_{\mathbf{k}}) = \left(\frac{t_0}{K}\right)^2, \quad \boldsymbol{\theta}(\boldsymbol{F}^{\{\mathbf{k},t_0\}}) = \left(\frac{t_0}{K}\right)^2 \sqrt[3]{\left(\prod_{i\in\boldsymbol{I}} c^i_{k_i}\right)^2} \tag{3.3}$$

3. *In general, by (2.15) and (2.16), a Pareto face of a taxoid* $\Pi^{\{\boldsymbol{f},t_0\}}$ *is a union of simplices and the measure* $\boldsymbol{\theta}$ *for some* $\boldsymbol{F}^{\{\mathbf{k},t_0\}}$ *is given by additivity. For a relevant sequence* $\mathbf{k} \in \overset{\circ}{\mathbf{K}}$ *and the corresponding Pareto face* $\boldsymbol{F}^{\{\mathbf{k},t_0\}} := \overset{\star}{\boldsymbol{x}}^{\mathbf{k}}(\Delta_{\mathbf{k}}^{t_0\boldsymbol{e}})$ *of* $\Pi^{\{\boldsymbol{f},t_0\}}$*, let* $\left\{ \overset{\star}{\Delta}^l \,\middle|\, l \in \mathbf{L}^{\mathbf{k}} \right\}$ *be the family provided by Theorem 2.3,* item 3.

   The simplex $\overset{\star}{\Delta}^{\{l\}}$ *has Maschler–Perles measure*

$$\boldsymbol{\theta}(\overset{\star}{\Delta}^{\{l\}}) = \frac{t_0}{K} . \tag{3.4}$$

   Accordingly,it follows that

$$\overset{\star}{\boldsymbol{F}}^{l} := \overset{\star}{\boldsymbol{x}}^{\mathbf{k}}(\overset{\star}{\Delta}^{\{l\}}) = \mathbf{c}_{\mathbf{k}} \otimes (\overset{\star}{\Delta}^{l}) + \mathbf{d}_{\mathbf{k}} \tag{3.5}$$

   has Maschler–Perles measure

$$\begin{aligned} \boldsymbol{\theta}(\overset{\star}{\boldsymbol{F}}^{l}) &= \boldsymbol{\theta}(\mathbf{c}_{\mathbf{k}} \otimes (\overset{\star}{\Delta}^{\{l\}})) = \boldsymbol{\theta}(\mathbf{c}_{\mathbf{k}} \otimes \frac{t_0}{K}\Delta^{\{l\}}) \\ &= \left(\frac{t_0}{K}\right)^{n-1} \boldsymbol{\theta}(\mathbf{c}_{\mathbf{k}} \otimes \Delta^{\{l\}}) = \left(\frac{t_0}{K}\right)^{n-1} \sqrt[n]{\left(\prod_{i \in \boldsymbol{I}} (c_{k_i}^i)\right)^{n-1}} . \end{aligned} \tag{3.6}$$

4. *Hence, the Maschler–Perles measure of* $\boldsymbol{F}^{\{\mathbf{k},t_0\}}$ *is*

$$\boldsymbol{\theta}(\boldsymbol{F}^{\{\mathbf{k},t_0\}}) = \boldsymbol{\theta}(\bigcup_{l \in \mathbf{L}^{\mathbf{k}}} \boldsymbol{F}^{l\star}) = \left(\frac{t_0}{K}\right)^{n-1} |\mathbf{L}^{\mathbf{k}}| \sqrt[n]{\left(\prod_{i \in \boldsymbol{I}} (c_{k_i}^i)\right)^{n-1}} . \tag{3.7}$$

5. *Therefore, the Maschler–Perles measure of the Pareto surface of the taxoid* $\Pi^{\{\boldsymbol{f},t_0\}}$ *is*

$$\boldsymbol{\theta}(\partial\Pi^{\{\boldsymbol{f},t_0\}}) = \sum_{\mathbf{k} \in \overset{\circ}{\mathbf{K}}} \boldsymbol{\theta}(\boldsymbol{F}^{\{\mathbf{k},t_0\}}) = \left(\frac{t_0}{K}\right)^{n-1} \sum_{\mathbf{k} \in \overset{\circ}{\mathbf{K}}} |\mathbf{L}^{\mathbf{k}}| \sqrt[n]{\left(\prod_{i \in \boldsymbol{I}} (c_{k_i}^i)\right)^{n-1}} . . \tag{3.8}$$

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Proof:

No proof is offered.

q.e.d.

4 The Maschler–Perles–Shapley Value

The Shapley value for NTU games has a long–standing history beginning with Shapley's presentation on "La Decision" ([15])and branching for applications in various directions. Here we want to discuss the Maschler–Perles–Shapley value for the TF game of Definition 1.3. This topic is a continuation of the Maschler–Perles solution as presented in [12] as well as of the Maschler–Perles–Shapley value developed in [11].

First (Remark 4.2) we recall the motivation of the Maschler–Perles solution ([2], see also [10] for a textbook version). This demonstrates the connection of the original Maschler–Perles solution (based on their "donkey cart") and the Maschler–Perles surface measure which allows seeing the solution as the image of a weighted barycenter. That is, the solution is viewed as a centerpoint with respect to a simplex constructed according to adjusted utils.

The set–up for the Maschler–Perles–Shapley value will proceed in a directly analogous way (Remark 4.6). However, the application of an (induced) utility space has to be extended as the side payment Shapley value is involved. This context requires the introduction of the adjusted TU game which we have to provide in advance (Definition 4.4).

The basic axiom of the Maschler–Perles solution is superadditivity. The immediate consequence of this axiom is that players are induced to evaluate concessions and gains in accordance to an adjusted version of utility. In two dimensions (for two players in a bargaining problem) the "donkey card" of Maschler–Perles reflects this idea.

Given a bargaining problem $\boldsymbol{V} = (\boldsymbol{0}, \boldsymbol{U})$, imagine a point traveling along the Pareto surface $\partial\boldsymbol{U}$ with a certain speed. This speed is determined by two forces pulling in the direction of the axes (the donkeys pulling in different directions). The speed at each particular point is dictated by the superadditivity axiom to be $\sqrt{dx_1 dx_2}$. The resulting movement reflects the evaluation of concessions and gains. Accordingly, let both players continuously travel from their bliss point towards the solution point.

Both players are yielding to the demands of the opponent continuously until a point of stability is reached, at which both travelling points meet. This final point defines the Maschler–Perles solution: Each player has conceded the same amount of "utils" measured in the speed of concessions of the "donkey cards".

To be more intuitive we discuss the following traditional example.

Example 4.1. Consider two players involved in a a bargaining problem with piecewise linear Pareto curve. This curve consists of 5 line segments, hence the feasible set is an algebraic sum of 5 simplices as indicated by Figure 4.1.

As a consequence of the superadditivity axiom, players make equal concessions whenever they travel along a Pareto segment such that both terms $\sqrt{a_1 a_2}$ are equal. The solution is obtained when this term has been equal continuously on both sides during the movement.

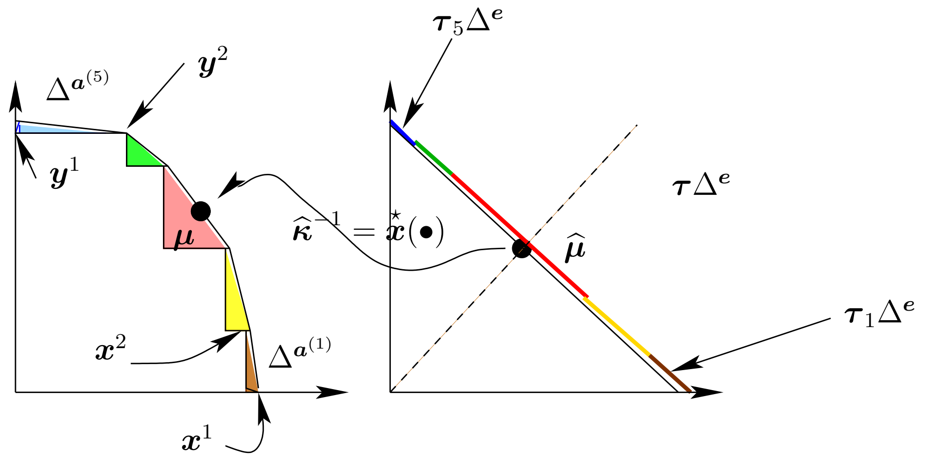

Figure 4.1: The M–P solution as the inverse image of the center point

We introduce the terms $\boldsymbol{\tau}_i := \sqrt{a_1^{(i)} a_2^{(i)}}$. The concession of player 1, when moving from the bliss point $\boldsymbol{x}^1$ to $\boldsymbol{x}^2$ along $\Delta^{\boldsymbol{a}^{(1)}}$ is considered to be equal to the concession of player 2 moving from $\boldsymbol{y}^1$ to $\boldsymbol{y}^2$ along $\Delta^{\boldsymbol{a}^{(5)}}$ whenever $\boldsymbol{\tau}_1 = \boldsymbol{\tau}_5$ holds true. Continuing this procedure the motions end at the Maschler–Perles solution $\boldsymbol{\mu}$ at which both players have made equal overall concessions (in terms of $\sqrt{dx_1 dx_2}$).

Now we change our intuition: we consider a *surface measure* defined on the Pareto curve with a density corresponding to the traveling speed specified above. Then the evaluation of concessions is performed via a density (replacing the "speed" concept). This evaluation of utils defines the Maschler–Perles measure. The density of this measure is $\sqrt{-dx_1 dx_2}$. Along some simplex/triangle in Figure 4.1 this amounts to $\boldsymbol{\tau}_i := \sqrt{a_1^{(i)} a_2^{(i)}}$.

In order to construct the solution, the total sum $\boldsymbol{\tau} := \sum_{k=1}^{5} \boldsymbol{\tau}_k$ determines the size of a new Simplex $\boldsymbol{\tau}\Delta^{\boldsymbol{e}}$. Each line segment $\Delta^{\boldsymbol{a}^{(k)}}$ is bijectively mapped onto a copy $\boldsymbol{\tau}_k\Delta^{\boldsymbol{e}} \subseteq \boldsymbol{\tau}\Delta^{\boldsymbol{e}}$, the size of this copy is the surface measure $\boldsymbol{\tau}_k$ of the line segment. This constitutes a bijective mapping $\widehat{\boldsymbol{\kappa}}$ of the Pareto curve $\partial\boldsymbol{U}$ onto a multiple $\boldsymbol{\tau}\Delta^{\boldsymbol{e}}$ of the unit simplex. We regard the simplex $\boldsymbol{\tau}\Delta^{\boldsymbol{e}}$ as an *adjusted utility space*. Measurement within this simplex reflects measuring utils of $\partial\boldsymbol{U}$ in terms of the MaschlerPerles density $\sqrt{-dx_1 dx_2}$.

In Figure 4.1 the right hand side reflects the adjusted utility space. The intervals in length correspond to the Maschler–Perles measure of the Pareto faces of the left side. The centerpoint $\widehat{\boldsymbol{\mu}}$ of $\boldsymbol{\tau}\Delta^{\boldsymbol{e}}$ is the barycenter of gravity of the transported measure, hence corresponds to the weighted centerpoint of the left hand side Pareto curve.

Next, the barycenter $\widehat{\boldsymbol{\mu}}$ is mapped onto the weighted center point $\boldsymbol{\mu}$ of $\partial\boldsymbol{U}$ by the inverse mapping $\widehat{\boldsymbol{\kappa}}^{-1}$, i.e., we obtain the solution $\boldsymbol{\mu}$ of the bargaining problem left hand by

$$\boldsymbol{\mu} = \widehat{\boldsymbol{\kappa}}^{-1}(\widehat{\boldsymbol{\mu}}). \tag{4.1}$$

This procedure defines the Maschler–Perles solution. A condensed verbal rephrasing runs as follows.

The adjusted utility space is the bijective image of the Pareto curve onto a multiple of the unit simplex endowed with the Maschler–Perles measure. The Maschler–Perles solution in turn is the inverse image of the barycenter of the adjusted utility space. It reflects the fair solution when utils are measured in accordance with the superadditivity axiom.

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Now we turn to our present context: tax games and TF games. Then the Maschler Perles measure of a Pareto surface $\partial \boldsymbol{U} = \partial \Pi^{\{\boldsymbol{f},t_0\}}$ is given by $\boldsymbol{\theta}(\partial \Pi^{\{\boldsymbol{f},t_0\}})$ as in (3.8). We adjust the traditional set up to our present framework. First we elaborate on a taxatioin bargaining problem. The mapping $\widehat{\boldsymbol{\kappa}}^{-1}(\bullet)$ is now regarded as a *parametrization* and replaced by $\overset{\star}{\boldsymbol{x}}(\bullet)$.

Remark 4.2. Consider a bargaining problem $(\mathbf{0}, \boldsymbol{U}^{\{t_0\}})$ resulting from a taxation game and a TF system $\boldsymbol{f}$. Within this context, the feasible set is a taxoid, i.e., $\boldsymbol{U}^{\{t_0\}} = \Pi^{\{\boldsymbol{f},t_0\}}$. Then we know the Pareto surface to be

$$\partial \boldsymbol{U}^{\{\boldsymbol{f},t_0\}} = \bigcup_{\mathbf{k} \in \overset{\circ}{\mathbf{K}}} \boldsymbol{F}^{\{\mathbf{k},t_0\}}. \tag{4.2}$$

1. The simplex $\Delta^{t_0 \boldsymbol{e}}$ is regarded as the utility space with respect to the distribution of t_0 units of currency available for taxation. According to Corollary 2.2, see (2.10), we obtain the parametrization

$$\overset{\star}{\boldsymbol{x}}(\bullet) : \Delta^{t_0 \boldsymbol{e}} \to \partial \boldsymbol{U}^{\{t_0\}} \ , \tag{4.3}$$

   which is the composition of the local parametrizations

$$\overset{\star}{\boldsymbol{x}}^{\mathbf{k}}(\bullet) \ := \ \Delta_{\mathbf{k}}^{t_0 \boldsymbol{e}} \to \boldsymbol{F}^{\{\mathbf{k},t_0\}} \tag{4.4}$$

2. The measure $\boldsymbol{\theta}$ is defined via (3.8) and transported from $\partial \boldsymbol{U}^{\{t_0\}}$ to $\Delta^{t_0 \boldsymbol{e}}$ via $\left(\overset{\star}{\boldsymbol{x}}(\bullet)\right)^{-1}$; the image is called $\overset{\star}{\boldsymbol{\theta}}$:

$$\overset{\star}{\boldsymbol{\theta}}(\bullet) := \boldsymbol{\theta} \circ \left(\overset{\star}{\boldsymbol{x}}(\bullet)\right)^{-1} = \boldsymbol{\theta}(\overset{\star}{\boldsymbol{x}}(\bullet)) \tag{4.5}$$

   (the transportation of measures occcurs contravariant) . We regard $\boldsymbol{\theta}$ to supply the measurements of utils according to the rationale of the Maschler–Perles solution as above.

3. The barycenter of $\Delta_{\mathbf{k}}^{t_0 \boldsymbol{e}}$ with respect to the measure $\overset{\star}{\boldsymbol{\theta}}$ is

$$\overset{\star}{\boldsymbol{\beta}}^{\mathbf{k}} = \frac{1}{\overset{\star}{\boldsymbol{\theta}}(\Delta_{\mathbf{k}}^{t_0 \boldsymbol{e}})} \int_{\Delta_{\mathbf{k}}^{t_0 \boldsymbol{e}}} \mathbf{t} \ \overset{\star}{\boldsymbol{\theta}}(d\mathbf{t}) \ . \tag{4.6}$$

Therefore, the barycenter of $\Delta^{t_0 \boldsymbol{e}}$ is

$$
\begin{aligned}
\stackrel{\star}{\boldsymbol{\beta}} \;&:=\; \sum_{\mathsf{k}\in\overset{\circ}{\mathsf{K}}} \stackrel{\star}{\boldsymbol{\theta}}(\Delta_{\mathsf{k}}^{t_0\boldsymbol{e}}) \frac{\stackrel{\star}{\boldsymbol{\beta}}^{\mathsf{k}}}{\sum_{\mathsf{l}\in\overset{\circ}{\mathsf{K}}} \stackrel{\star}{\boldsymbol{\theta}}(\Delta_{\mathsf{l}}^{t_0\boldsymbol{e}})} \\
&= \frac{1}{\sum_{\mathsf{l}\in\overset{\circ}{\mathsf{K}}} \stackrel{\star}{\boldsymbol{\theta}}(\Delta_{\mathsf{l}}^{t_0\boldsymbol{e}})} \sum_{\mathsf{k}\in\overset{\circ}{\mathsf{K}}} \int_{\Delta_{\mathsf{k}}^{t_0\boldsymbol{e}}} \mathbf{t}\, \stackrel{\star}{\boldsymbol{\theta}}(d\mathbf{t}) \qquad (4.7)\\
&= \frac{1}{\stackrel{\star}{\boldsymbol{\theta}}(\Delta^{t_0\boldsymbol{e}})} \int_{\Delta^{t_0\boldsymbol{e}}} \mathbf{t}\, \stackrel{\star}{\boldsymbol{\theta}}(d\mathbf{t}) \in \Delta^{t_0\boldsymbol{e}} \;.
\end{aligned}
$$

As in the traditional approach, the barycenter is the weighted centerpoint of the adjusted utility space. Te weights are supplied by the Maschler–Perles surface measure $\boldsymbol{\theta}$. Thus, concessions and gains are weighted accordingly, based on the superadditivity axiom.

4. Finally, the Maschler–Perles solution to the bargaining problem $\boldsymbol{V}^{t_0} = (\mathbf{0}, \boldsymbol{U}^{t_0})$ is obtained by inverse transporting of $\stackrel{\star}{\boldsymbol{\beta}}$ to $\partial \boldsymbol{U}^{\{t_0\}}$, i.e.,

$$
\boldsymbol{\mu}(\boldsymbol{V}^{t_0}) \;:=\; \stackrel{\star}{\boldsymbol{x}}(\stackrel{\star}{\boldsymbol{\beta}}) \;=\; \stackrel{\star}{\boldsymbol{x}}\left(\frac{1}{\stackrel{\star}{\boldsymbol{\theta}}(\Delta^{t_0\boldsymbol{e}})} \int_{\Delta^{t_0\boldsymbol{e}}} \mathbf{t}\, \stackrel{\star}{\boldsymbol{\theta}}(d\mathbf{t}) \right) \in \partial \boldsymbol{U}^{\{t_0\}} \;. \qquad (4.8)
$$

We refer to the last sentence in Example 4.1 as to a short verbal rephrasing.

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This development now leads to the final step: we extend the procedure in order to present the Maschler–Perles–Shapley value of a TF game. In this case, the simple notion of an adjusted utility space (defined by a multiple of the unit simplex) does not suffice. The generalization is provided by the *adjusted TU game* $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \widehat{\boldsymbol{v}})$. The values $\{\widehat{\boldsymbol{v}}(S) \,|\, (S \in \underline{\underline{\mathbf{P}}})\}$ each constitute an adjusted utility space for the coalitions.

We formulate this version within the framework of Remarks 4.3 and 4.6.

Remark 4.3. Consider the profit game $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$ and a TF system

$$
\boldsymbol{f}^{\{\boldsymbol{v}(\boldsymbol{I})\}} = (f_1^{\{\boldsymbol{v}(\boldsymbol{I})\}}, \ldots, f_n^{\{\boldsymbol{v}(\boldsymbol{I})\}}) \qquad (4.9)
$$

for the distribution of $\boldsymbol{v}(\boldsymbol{I})$. As in Definition 1.3, we assume that the function

$$
\boldsymbol{f}^{\{\boldsymbol{v}(S)\}} \;:=\; \boldsymbol{f}^{\{\boldsymbol{v}(\boldsymbol{I})\}}\Big|_{[0,\boldsymbol{v}(S)]} \;. \qquad (4.10)
$$

serves as a TF system for coalition $S \in \underline{\underline{\mathbf{P}}}$. This constitutes the TF game $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{V}^{\{\boldsymbol{v}\}})$ via (1.21), Definition 1.3. That is, for some $S \in \underline{\underline{\mathbf{P}}}$ we have

$$
\partial \boldsymbol{V}^{\{\boldsymbol{v}(S)\}} \;=\; \boldsymbol{f}^{\{\boldsymbol{v}(S)\}}\left(\Delta^{\{\boldsymbol{v}(S)\boldsymbol{e}\}}\right) \quad (S \in \underline{\underline{\mathbf{P}}}) \;. \qquad (4.11)
$$

as in (2.1).

In the context of Definition 1.3 the local linear parametrizations are defined by (2.4),(2.7) and hence do not change in dependence of the coalitions. Thus we have (cf. Definition 2.1)

$$\boldsymbol{f}_{\big|\Delta_{\mathsf{k}}^{\boldsymbol{v}(S)\boldsymbol{e}}} = \overset{\star}{\boldsymbol{x}}(\bullet)_{\big|\Delta_{\mathsf{k}}^{\boldsymbol{v}(S)\boldsymbol{e}}} = \overset{\star}{\boldsymbol{x}}^{\mathsf{k}}(\bullet) \quad (\mathsf{k} \in \overset{\circ}{\mathsf{K}}^{\{\boldsymbol{v}(S)\}}) \ . \tag{4.12}$$

The composition of these mappings is the universal linearization

$$\overset{\star}{\boldsymbol{x}}(\bullet) \ : \ \Delta^{\boldsymbol{v}(\boldsymbol{I})\boldsymbol{e}} \to \partial \boldsymbol{V}^{\{\boldsymbol{v}\}}(\boldsymbol{I}) \ . \tag{4.13}$$

We use this notation uniformly for the representation, i.e.,

$$\begin{gathered} \overset{\star}{\boldsymbol{x}}(\bullet) \ : \ \Delta^{\boldsymbol{v}(S)\boldsymbol{e}} \ \to \partial \boldsymbol{V}^{\{\boldsymbol{v}\}}(S) \ \ (S \in \underline{\underline{\mathbf{P}}}) \\ \text{implying} \\ \overset{\star}{\boldsymbol{x}}(\bullet)_{\big|\Delta^{\boldsymbol{v}(S)\boldsymbol{e}}} = \boldsymbol{f}^{\{\boldsymbol{v}(S)\}} \end{gathered} \tag{4.14}$$

Observe that all values $\{\boldsymbol{v}(S)\}_{S\in\underline{\underline{\mathbf{P}}}}$ are adapted to the grid $\mathsf{K}^{\{\boldsymbol{v}(I)\}}$ and in turn generate (sub–)grids $\mathsf{K}^{\{\boldsymbol{v}(S)\}}$. For $S \in \underline{\underline{\mathbf{P}}}$, we can then speak of relevant sequences $\mathsf{k} \in \overset{\circ}{\mathsf{K}}^{\{\boldsymbol{v}(S)\}}$ adapted to $\boldsymbol{v}(S)$. Recall that the grid size is referring to $K' = K\frac{\boldsymbol{v}(S)}{\boldsymbol{v}(\boldsymbol{I})}$ as in (1.14).

The Maschler–Perles measure on the Pareto surface of the taxoid $V^{\{\boldsymbol{v}(S)\}}$ is computed according to formula (3.8). Thus, we find

$$\boldsymbol{\theta}(\partial \boldsymbol{V}^{\{\boldsymbol{v}\}}(S)) = \left(\frac{\boldsymbol{v}(S)}{K'}\right)^{n-1} \sum_{\mathsf{k}\in\overset{\circ}{\mathsf{K}}^{\{\boldsymbol{v}\}}} (S)|\mathsf{L}^{\mathsf{k}}| \sqrt[n]{\left(\prod_{i\in\boldsymbol{I}}(c^{i}_{k_i})\right)^{n-1}} \ (S \in \underline{\underline{\mathbf{P}}}). \tag{4.15}$$

which in view of (4.11) and (4.12) reads

$$\boldsymbol{\theta}(\partial \boldsymbol{V}^{\{\boldsymbol{v}\}}(S)) = \boldsymbol{\theta}\left(\boldsymbol{f}^{\{\boldsymbol{v}(S)\}}(\Delta^{\{\boldsymbol{v}(S)\boldsymbol{e}\}})\right) = \boldsymbol{\theta}\left(\overset{\star}{\boldsymbol{x}}(\Delta^{\{\boldsymbol{v}(S)\boldsymbol{e}\}})\right) \tag{4.16}$$

using the locally linear representation $\overset{\star}{\boldsymbol{x}}(\bullet)$ of $\boldsymbol{f}$.

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The following definition incorporates the measurements of concessions and gains in terms of the Maschler–Perles measure to the context of the profit game. As such we view the adjusted game to be the appropriate generalization of the adjusted utility space exemplified in Remark 4.2.

Definition 4.4. [***The Adjusted Game***]

Let $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$ be the profit game and let

$$\boldsymbol{f}^{\{\boldsymbol{v}(\boldsymbol{I})\}} = (f_1^{\{\boldsymbol{v}(\boldsymbol{I})\}}, \ldots, f_n^{\{\boldsymbol{v}(\boldsymbol{I})\}}) \tag{4.17}$$

be a TF system for the distribution of $\boldsymbol{v}(\boldsymbol{I})$. For $S \in \underline{\underline{\mathbf{P}}}$ let $\boldsymbol{f}^{\{\boldsymbol{v}(S)\}}$ be the derived TF system according to (1.19), see also (4.10) and (4.11).

The ***adjusted TU game*** induced by $\boldsymbol{v}$ and $\boldsymbol{f}^{\{\boldsymbol{v}(\bullet)\}}$ is the coalitional function

$$\widehat{\boldsymbol{v}} = \widehat{\boldsymbol{v}}^{\boldsymbol{f}} = \widehat{\boldsymbol{v}}^{\boldsymbol{v},\boldsymbol{f}^{\{\boldsymbol{v}(\boldsymbol{I})\}}} : \underline{\underline{\mathbf{P}}} \to \mathbb{R} \tag{4.18}$$

given by

$$\begin{aligned}\widehat{\boldsymbol{v}}(S) &= \left(\frac{\boldsymbol{v}(S)}{K'}\right)^{|S|-1} \sum_{\mathbf{k}\in\mathring{\mathbf{K}}^{\{\boldsymbol{v}(S)\}}} |\mathbf{L}^{\mathbf{k}}| \sqrt[|S|]{\left(\prod_{i\in\boldsymbol{I}}(c^i_{k_i})\right)^{|S|-1}} \\ &= \boldsymbol{\theta}(\partial\boldsymbol{V}^{\{\boldsymbol{v}\}}(S)) = \boldsymbol{\theta}\left(\overset{\star}{\boldsymbol{x}}(\Delta^{\{\boldsymbol{v}(S)\boldsymbol{e}\}})\right) (S \in \underline{\underline{\mathbf{P}}}).\end{aligned} \tag{4.19}$$

In particular, we obtain for the grand coalition

$$\begin{aligned}\widehat{\boldsymbol{v}}(\boldsymbol{I}) &= \left(\frac{\boldsymbol{v}(\boldsymbol{I})}{K}\right)^{n-1} \sum_{\mathbf{k}\in\mathring{\mathbf{K}}^{\{\boldsymbol{v}(\boldsymbol{I})\}}} |\mathbf{L}^{\mathbf{k}}| \sqrt[n]{\left(\prod_{i\in\boldsymbol{I}}(c^i_{k_i})\right)^{n-1}} \\ &= \boldsymbol{\theta}(\partial\boldsymbol{V}^{\{\boldsymbol{v}\}}(\boldsymbol{I})) = \boldsymbol{\theta}\left(\overset{\star}{\boldsymbol{x}}(\Delta^{\{\boldsymbol{v}(\boldsymbol{I})\boldsymbol{e}\}})\right)\end{aligned} \tag{4.20}$$

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We illustrate the concept by an example of a *linear* TF system. The measurement in terms of adjusted utils (Remark 4.2) is particularly obvious within this context.

Example 4.5. [***The Linear TF System***]

Let $n \leq 3$ and let $\boldsymbol{v} : \underline{\underline{\mathbf{P}}} \to \mathbb{R}_+$ be the profit game. Then, for any relevant sequence $\mathbf{k}$ we know that $|L^{\mathbf{k}}| = 1$ ([12], SECTION 2).

First, let $|S| = 2$. Then $\boldsymbol{V}^{\boldsymbol{v}}(S)$ is a two–dimensional cephoid. The relevant sequences are given by $(k, (K' - (k-1))$ for $k,\ 1 \leq k \leq K'$.

That is,

$$\partial\boldsymbol{V}^{\{\boldsymbol{v}\}}(S) = \sum_{k=1}^{K'} = \Delta^{\frac{\boldsymbol{c}_{\mathbf{k}}}{K}} \tag{4.21}$$

is a concave Pareto curve consisting of line segments. We specify the example so that, for $\mathbf{k} \in \mathring{\mathbf{K}}$, all slopes are equal, say $\boldsymbol{c}_{\mathbf{k}} = \boldsymbol{c}_{\mathbf{o}}$ for all $\mathbf{k} \in \mathring{\mathbf{K}}$. Then Pareto curve degenerates into the straight line

$$\partial\boldsymbol{V}^{\{\boldsymbol{v}\}}(S) = K'\Delta^{\boldsymbol{c}_{\mathbf{o}}\frac{\boldsymbol{v}(\boldsymbol{I})}{K}} = \Delta^{\frac{K'\boldsymbol{v}(\boldsymbol{I})}{K}\boldsymbol{c}_{\mathbf{o}}} = \Delta^{\boldsymbol{v}(S)\boldsymbol{c}_{\mathbf{o}}}\ . \tag{4.22}$$

This indicates that $\partial\boldsymbol{V}^{\{\boldsymbol{v}(S)\}}$ reflects the NTU version of a side payment game with fixed rates of utility transfer specified by $\boldsymbol{c}_{\mathbf{o}}$. Now, if we turn to the adjusted TU game, we obtain by (4.19) for $|S| = 2$:

$$
\begin{aligned}
\widehat{\boldsymbol{v}}(S) &= \left(\frac{\boldsymbol{v}(S)}{K'}\right) \sum_{\mathbf{k}\in\overset{\circ}{\mathbf{K}}^{\{\boldsymbol{v}(S)\}}} \sqrt{\left(\prod_{i\in S}(c^i_{k_i})\right)} \\
&= \left(\frac{\boldsymbol{v}(S)}{K'}\right) \sum_{\mathbf{k}\in\overset{\circ}{\mathbf{K}}^{\{\boldsymbol{v}(S)\}}} \sqrt{(c^2_{\mathbf{o}}c^3_{\mathbf{o}})} \\
&= \left(\frac{\boldsymbol{v}(S)}{K'}\right) \sum_{k=1}^{K'} \sqrt{(c^2_o c^3_o)} \\
&= \left(\frac{\boldsymbol{v}(S)}{K'}\right) K'\sqrt{(c^2_o c^3_o)} \;=\; \boldsymbol{v}(S)\sqrt{(c^2_o c^3_o)}
\end{aligned}
\tag{4.23}
$$

Thus, the worth of coalition $S \in \underline{\underline{\mathbf{P}}}$ is adjusted by the density of the Maschler–Perles measure in two dimensions. In particular, $\widehat{\boldsymbol{v}}$ and $\boldsymbol{v}$ coincide whenever $\boldsymbol{c}_{\mathbf{o}} = \boldsymbol{e}$ is the unit vector and hence $\boldsymbol{V}^{\{\boldsymbol{v}(S)\}} = \Delta^{\boldsymbol{v}(S)\boldsymbol{e}}$.

Next, consider the case $n = 3$. As $|\mathbf{L}^{\mathbf{k}}| = 1$ holds true as well, we come up with an analogous result, i.e.,

$$
\widehat{\boldsymbol{v}}(\boldsymbol{I}) = \boldsymbol{v}(\boldsymbol{I})\sqrt[3]{(c^1_o c^2_o c^3_o)^2}
\tag{4.24}
$$

with the same interpretation: the adjustment is induced by the Maschler–Perles measure in 3 dimensions.

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Now we present the construction of the Maschler–Perles–Shapley value for a TF game, based on the notion of the adjusted TU game (Definition 4.4).

Formulae (4.19) and (4.20) are analogs to (3.8): the Maschler–Perles measure for the Pareto surface of a bargaining problem is replaced by the Pareto measures with respect to the TF game, that is by the adjusted TU game. Next the barycenter of the transported measure is replaced by the Shapley value and the solution is obtained by the inverse mapping $\boldsymbol{\kappa}^{-1}(\bullet)$ or rather $\overset{\star}{\boldsymbol{x}}(\bullet)$ onto the Pareto set of $\boldsymbol{V}^{\{\boldsymbol{v}\}}$.

Thus, other than the traditional approach of "admitting side payments" (initiated by SHAPLEY [13]) we do not just take the Lebesgue measure of a util for its worth but the corresponding Maschler–Perles measure. This explains the introduction of the adjusted TU game.

Remark 4.6. We proceed analogously to the development in Remark 4.2.

1. Given a profit game $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$ and a TF system $\boldsymbol{f}^{\{\boldsymbol{v}(\boldsymbol{I})\}}$, we construct the TF game $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{V}^{\{\boldsymbol{v}\}}$ accoeding to Definition 1.3.

2. The mapping $\overset{\star}{\boldsymbol{x}}(\bullet)$ is suitably modified to be the universal linearization suggested by (4.13). It satisfies

$$
\overset{\star}{\boldsymbol{x}}(\bullet) \;:\; \Delta^{\boldsymbol{v}(S)\boldsymbol{e}} \to \partial\boldsymbol{V}^{\boldsymbol{v}}(S) \quad (S \in \underline{\underline{\mathbf{P}}})
\tag{4.25}
$$

This mapping provides a parametrization. Essentially we obtain for the grand coalition:

$$(\Delta^{v(\boldsymbol{I})\boldsymbol{e}}, \overset{\star}{\boldsymbol{x}}(\bullet)) \quad \text{parametrizes} \quad \partial \boldsymbol{V}^{\{v\}}(\boldsymbol{I}). \tag{4.26}$$

The parametrization plays the role of the inverse mapping $\widehat{\boldsymbol{\kappa}}^{-1}$ in Example 4.1.

3. The Maschler–Perles measure $\boldsymbol{\theta}$ (see (3.8)) is transported from $\partial \boldsymbol{V}^{\{\boldsymbol{v}(\boldsymbol{I})\}}$ to $\Delta^{\boldsymbol{v}(\boldsymbol{I})\boldsymbol{e}}$ via $\left(\overset{\star}{\boldsymbol{x}}(\bullet)\right)^{-1}$. The image is called $\overset{\star}{\boldsymbol{\theta}}$.

$$\overset{\star}{\boldsymbol{\theta}}(\bullet) := \boldsymbol{\theta} \circ \left(\overset{\star}{\boldsymbol{x}}(\bullet)\right)^{-1} = \boldsymbol{\theta}(\overset{\star}{\boldsymbol{x}}(\bullet)) \tag{4.27}$$

We obtain

$$\begin{aligned} \overset{\star}{\boldsymbol{\theta}}(\Delta^{\boldsymbol{v}(\boldsymbol{I})\boldsymbol{e}}) &= \boldsymbol{\theta}(\overset{\star}{\boldsymbol{x}}(\Delta^{\boldsymbol{v}(\boldsymbol{I})\boldsymbol{e}})) = \boldsymbol{\theta}(\partial \boldsymbol{V}^{\{\boldsymbol{v}\}}(\boldsymbol{I})) \\ &= = \left(\frac{\boldsymbol{v}(\boldsymbol{I})}{K}\right)^{n-1} \sum_{\mathbf{k} \in \overset{\circ}{\mathbf{K}}^{\{\boldsymbol{v}(\boldsymbol{I})\}}} |\mathbf{L}^{\mathbf{k}}| \sqrt[n]{\left(\prod_{i \in \boldsymbol{I}} (c^i_{k_i})\right)^{n-1}} \\ &= \widehat{\boldsymbol{v}}^{\boldsymbol{f}}(\boldsymbol{I}) \;, \end{aligned} \tag{4.28}$$

by (4.20). Thus, the adjusted game enters the scene in analogy to the adjusted utility space. For $S \in \underline{\underline{\mathbf{P}}}$, there is a relation analogue to (4.28), namely

$$\overset{\star}{\boldsymbol{\theta}}(\Delta^{\boldsymbol{v}(S)\boldsymbol{e}}) = \widehat{\boldsymbol{v}}^{\boldsymbol{f}}(S) \quad (S \in \underline{\underline{\mathbf{P}}}) \;. \tag{4.29}$$

4. The barycenter is replaced by the (side payment) Shapley value Φ of the adjusted game. The correct normalization is implied by $\Phi(\widehat{\boldsymbol{v}})(\boldsymbol{I}) = \widehat{\boldsymbol{v}}(\boldsymbol{I})$ As a consequence we obtain

$$\Phi(\widehat{\boldsymbol{v}}) \in \widehat{\boldsymbol{v}}(\boldsymbol{I})\Delta^{\boldsymbol{e}} \quad \text{hence} \quad \frac{\boldsymbol{v}(\boldsymbol{I})}{\widehat{\boldsymbol{v}}(\boldsymbol{I})}\Phi(\widehat{\boldsymbol{v}}) \in \boldsymbol{v}(\boldsymbol{I})\Delta^{\boldsymbol{e}} \;. \tag{4.30}$$

5. By means of $\overset{\star}{\boldsymbol{x}}(\bullet)$ this quantity is now transported to $\partial \boldsymbol{V}^{\boldsymbol{v}}(\boldsymbol{I})$. The result obtained this way is MPS value.

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Combining, we obtain the the following definition.

Definition 4.7. Let $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$ be the profit game and let $\boldsymbol{f}^{\{\boldsymbol{v}(\boldsymbol{I})\}}$ be a TF system for the distribution of $\boldsymbol{v}(\boldsymbol{I})$. Let $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{V}^{\{\boldsymbol{v}\}})$ be the TF game resulting from $\boldsymbol{v}$ and $\boldsymbol{f}^{\{\boldsymbol{v}(\boldsymbol{I})\}}$ and let $(\Delta^{v(\boldsymbol{I})\boldsymbol{e}}, \overset{\star}{\boldsymbol{x}}(\bullet))$ be the global parametrization of $\partial \boldsymbol{V}^{\{\boldsymbol{v}\}}(\boldsymbol{I})$. Moreover, let $\widehat{\boldsymbol{v}}^{\boldsymbol{f}}$ be the corresponding adjusted TU game. The ***Maschler–Perles–Shapley value*** (the ***MPS value***) of $\boldsymbol{V}^{\{\boldsymbol{v}\}}$ is

$$\chi(\boldsymbol{V}^{\{\boldsymbol{v}\}}) \;:=\; \overset{\star}{\boldsymbol{x}}\left(\frac{\boldsymbol{v}(\boldsymbol{I})}{\widehat{\boldsymbol{v}}^{\boldsymbol{f}}(\boldsymbol{I})}\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{f}})\right) \;=\; \overset{\star}{\boldsymbol{x}}\left(\frac{\boldsymbol{v}(\boldsymbol{I})}{\widehat{\boldsymbol{v}}^{\boldsymbol{v},\boldsymbol{f}^{\{\boldsymbol{v}(\boldsymbol{I})\}}}(\boldsymbol{I})}\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{v},\boldsymbol{f}^{\{\boldsymbol{v}(\boldsymbol{I})\}}})\right). \tag{4.31}$$

We offer a short verbal rephrasing analogous to the one in Example 4.1.

> **The adjusted TU game reflects a bijective image of the Pareto curve for each coalition onto a multiple of the unit simplex endowed with the Maschler–Perles measure. The Maschler–Perles–Shapley value in turn is the inverse image of the Shapley value of the adjusted TU game. It constitutes the "value" or "power index" when utils are measured in accordance with the superadditivity axiom.**

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Within the remaining part we prove that the Maschler–Perles–Shapley value enjoys the traditional covarianvce properties. It is Pareto efficient, covariant with affine transformations of utility, and anonymous.

Lemma 4.8. The MPS–value provides an assignment feasible for the grand coalition of the NTU game $\boldsymbol{V}^{\{\boldsymbol{v}\}}$, i.e., χ is a Pareto efficient solution concept.

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Proof: Obvious because of (4.30) we have

$$\frac{\boldsymbol{v}(\boldsymbol{I})}{\widehat{\boldsymbol{v}}(\boldsymbol{I})}\Phi(\widehat{\boldsymbol{v}}) \in \boldsymbol{v}(\boldsymbol{I})\Delta^{\boldsymbol{e}} \tag{4.32}$$

and therefore by (4.13)

$$\chi(\boldsymbol{V}^{\{\boldsymbol{v}\}}) = \overset{\star}{\boldsymbol{x}}\left(\frac{\boldsymbol{v}(\boldsymbol{I})}{\widehat{\boldsymbol{v}}^{\boldsymbol{f}}(\boldsymbol{I})}\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{f}})\right) \in \overset{\star}{\boldsymbol{x}}(\boldsymbol{v}(\boldsymbol{I})\Delta^{\boldsymbol{e}}) = \partial\boldsymbol{V}^{\{\boldsymbol{v}\}}(\boldsymbol{I}) \;. \tag{4.33}$$

q.e.d.

Lemma 4.9. The MPS–value is covariant with linear transformations of utility (l.t.u.). That is, for a linear mapping

$$L \;:\; \mathbb{R}^n \to \mathbb{R}^n, \quad L(u_1,\ldots,u_n) = (\alpha_1 u_1,\ldots,\alpha_n u_n) \tag{4.34}$$

with positive coefficients $(\alpha_1,\ldots,\alpha_n)$ it follows that

$$\chi(L\boldsymbol{V}^{\{\boldsymbol{v}\}}) = L(\chi(\boldsymbol{V}^{\{\boldsymbol{v}\}})) \tag{4.35}$$

• ~~~◦~~~ •

Proof:

Given a profit game $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$ and a TF system $\boldsymbol{f}^{\{\boldsymbol{v}(\boldsymbol{I})\}}$, the proof comprises clarifying the action of an l.t.u. on the various objects involved.

1st STEP :

Let $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{V}^{\{\boldsymbol{v}\}})$ be the TF game generated. The action of an l.t.u. on $\boldsymbol{V}^{\{\boldsymbol{v}\}}$ is induced by the action on $\boldsymbol{f}$. For $t_0 \in \mathbb{R}_+$ we have

$$(4.36) \qquad L\boldsymbol{f}^{\{t_0\}}(\bullet) \;=\; L\left(f_1^{\{t_0\}}(\cdot), \ldots, f_n^{\{t_0\}}(\cdot)\right) \;=\; \left(\alpha_1 f_1^{\{t_0\}}(\cdot), \ldots, \alpha_n f_n^{\{t_0\}}(\cdot)\right).$$

The action on the NTU game is suitably taken to be

$$(4.37) \qquad \begin{aligned} L_S(\boldsymbol{V}^{\{\boldsymbol{v}\}}(S)) \;&=\; \left(\alpha_i f^{\{\boldsymbol{v}(S)\}}\right)_{i\in S} \quad (S \in \underline{\underline{\mathbf{P}}}) \;, \\ \left(L\boldsymbol{V}^{\{\boldsymbol{v}\}}\right)(S) \;&:=\; \frac{\prod_{i\in \boldsymbol{I}} \alpha_i}{\prod_{i\in S} \alpha_i} L_S(\boldsymbol{V}^{\{\boldsymbol{v}\}}(S)) \quad (S \in \underline{\underline{\mathbf{P}}}) \;, \end{aligned}$$

and in particular

$$(4.38) \qquad \left(L\boldsymbol{V}^{\{\boldsymbol{v}\}}\right)(\boldsymbol{I}) \;:=\; L\left(\boldsymbol{V}^{\{\boldsymbol{v}\}}(\boldsymbol{I})\right) \;=\; L(\boldsymbol{f}^{\boldsymbol{v}(\boldsymbol{I})}(\mathbf{t}) \Big| \mathbf{t} \in \Delta^{\boldsymbol{v}(\boldsymbol{I})\boldsymbol{e}}) \;=\; (L\boldsymbol{f}^{\boldsymbol{v}(\boldsymbol{I})})(\mathbf{t}) \Big| \mathbf{t} \in \Delta^{\boldsymbol{v}(\boldsymbol{I})\boldsymbol{e}})$$

2nd STEP :

The action of L on the adjusted game $\widehat{\boldsymbol{v}} \;=\; \widehat{\boldsymbol{v}}^{\{\boldsymbol{v},\boldsymbol{f}^{\{v(\boldsymbol{I})\}}\}}$ follows from an inspection of (4.20). By (4.37) it follows that the "local coefficients" c_k^i as specified to represent $\boldsymbol{f}^{\{\boldsymbol{v}\}}$ in (4.20) are just multiplied by the factor

$$(4.39) \qquad \prod_{i\in\boldsymbol{I}} \alpha_i \;=:\; \boldsymbol{P}^{\boldsymbol{\alpha}} \;.$$

We obtain

$$(4.40) \qquad \widehat{\boldsymbol{v}}^{\{\boldsymbol{v},L\boldsymbol{f}^{\{\boldsymbol{v}(\boldsymbol{I})\}}\}} = \boldsymbol{\theta}\left(\partial L\boldsymbol{V}^{\{\boldsymbol{v}\}}(\boldsymbol{I})\right) \;=\; \boldsymbol{P}^{\boldsymbol{\alpha}}\boldsymbol{\theta}\left(\partial \boldsymbol{V}^{\{\boldsymbol{v}(\boldsymbol{I})\}}\right) \;=\; \boldsymbol{P}^{\boldsymbol{\alpha}}\widehat{\boldsymbol{v}}^{\{\boldsymbol{v},\boldsymbol{f}^{\{v(\boldsymbol{I})\}}\}} \;;$$

for short

$$(4.41) \qquad \widehat{\boldsymbol{v}}^{L\boldsymbol{f}} \;=\; \boldsymbol{P}^{\boldsymbol{\alpha}}\widehat{\boldsymbol{v}}^{\boldsymbol{f}} \;.$$

3rd STEP :

Finally, we discuss the parametrizing mappings. We have

$$(4.42) \qquad \begin{aligned} \overset{\star}{\boldsymbol{x}}(\bullet) \;&=\; \Delta^{\boldsymbol{v}(\boldsymbol{I})\boldsymbol{e}} \to \partial\boldsymbol{V}^{\boldsymbol{v}}(\boldsymbol{I}) \\ &\text{with inverse} \\ \overset{\star}{\kappa}(\bullet) \;&=\; \partial\boldsymbol{V}^{\boldsymbol{v}}(\boldsymbol{I}) \to \Delta^{\boldsymbol{v}(\boldsymbol{I})\boldsymbol{e}} \;. \end{aligned}$$

Analogously, the inverse for the transformed NTU game is

$$(4.43) \qquad \begin{aligned} &\overset{\wedge}{\kappa}(\bullet) \;:\; \partial L\boldsymbol{V}^{\{\boldsymbol{v}\}}(\boldsymbol{I}) \to \boldsymbol{P}^{\boldsymbol{\alpha}}\Delta^{\boldsymbol{v}(\boldsymbol{I})\boldsymbol{e}} \\ &\text{which provides the parametrization} \\ &\widehat{\boldsymbol{x}}(\bullet) \;:\; \boldsymbol{P}^{\boldsymbol{\alpha}}\Delta^{\boldsymbol{v}(\boldsymbol{I})\boldsymbol{e}} \to \partial L\boldsymbol{V}^{\{\boldsymbol{v}\}}(\boldsymbol{I}) = \partial L(\boldsymbol{V}^{\{\boldsymbol{v}\}}(\boldsymbol{I})) \;. \end{aligned}$$

The first line in (4.43) writes

$$\overset{\wedge}{\kappa}(L(\bullet)) = \boldsymbol{P}^{\boldsymbol{\alpha}}\overset{\star}{\kappa}(\bullet)\ . \tag{4.44}$$

This is now reformulated

$$\overset{\wedge}{\kappa}(L(\overset{\star}{\boldsymbol{x}}(\circ))) = \boldsymbol{P}^{\boldsymbol{\alpha}}\circ \tag{4.45}$$

or

$$\overset{\wedge}{\boldsymbol{x}}(\boldsymbol{P}^{\boldsymbol{\alpha}}\circ) = L(\overset{\star}{\boldsymbol{x}}(\circ))\ . \tag{4.46}$$

4^th STEP :

Combining these results, we obtain :

$$\begin{aligned}
\chi(L\boldsymbol{V}^{\{\boldsymbol{v}\}}) &= \widehat{\boldsymbol{x}}\left(\tfrac{\boldsymbol{P}^{\boldsymbol{\alpha}}\boldsymbol{v}(\boldsymbol{I})}{\widehat{\boldsymbol{v}}^{Lf}(\boldsymbol{I})}\Phi(\widehat{\boldsymbol{v}}^{Lf})\right) && - \text{ applying (4.43) ,}\\
&= \widehat{\boldsymbol{x}}\left(\tfrac{\boldsymbol{P}^{\boldsymbol{\alpha}}\boldsymbol{v}(\boldsymbol{I})}{\boldsymbol{P}^{\boldsymbol{\alpha}}\widehat{\boldsymbol{v}}^{f}(\boldsymbol{I})}\Phi(\boldsymbol{P}^{\boldsymbol{\alpha}}\widehat{\boldsymbol{v}}^{f})\right) && - \text{ by (4.41) ,}\\
&= \widehat{\boldsymbol{x}}\left(\boldsymbol{P}^{\boldsymbol{\alpha}}\tfrac{\boldsymbol{v}(\boldsymbol{I})}{\widehat{\boldsymbol{v}}^{f}(\boldsymbol{I})}\Phi(\widehat{\boldsymbol{v}}^{f})\right) && - \text{ by linearity of } \Phi\ ,\\
&= L\left(\overset{\star}{x}\left(\tfrac{\boldsymbol{v}(\boldsymbol{I})}{\widehat{\boldsymbol{v}}^{f}(\boldsymbol{I})}\Phi(\widehat{\boldsymbol{v}}^{f})\right)\right) && - \text{ by (4.46) ,}\\
&= L(\chi(\boldsymbol{V}^{\{\boldsymbol{v}\}}))\ .
\end{aligned} \tag{4.47}$$

q.e.d.

Lemma 4.10. The MPS–value is anonymous. That is, for a permutation

$$\pi\ :\ \boldsymbol{I} \to \boldsymbol{I}, \tag{4.48}$$

it follows that

$$\chi(\pi\boldsymbol{V}^{\{\boldsymbol{v}\}}) = \pi(\chi(\boldsymbol{V}^{\{\boldsymbol{v}\}})) \tag{4.49}$$

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Proof:

Again we first clear up the action of a permutation on vectors, games, etc. Recall that, for $\boldsymbol{u} \in \mathbb{R}^n$ and a coalitional function $\boldsymbol{v}$, the action of π is given by

$$(\pi(\boldsymbol{u}))_i\ :=\ \boldsymbol{u}_{\pi^{-1}(i)} \quad (i \in \boldsymbol{I}) \tag{4.50}$$

and

$$(\pi\boldsymbol{v})(S)\ :=\ \boldsymbol{v}(\pi^{-1}(S)) \quad (S \in \underline{\underline{\mathbf{P}}})\ . \tag{4.51}$$

1stSTEP : Given a profit game $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$ and a TF system $\boldsymbol{f}^{\{\boldsymbol{v}(\boldsymbol{I})\}}$ we have

$$(4.52) \qquad (\pi \boldsymbol{f}^{\{t_0\}})(\bullet) \;=\; \pi\left(f_1^{\{t_0\}}(.),\; \dots\;, f_n^{\{t_0\}}(.)\right) \;=\; \left(f_{\pi^{-1}(i)}^{\{t_0\}}\right)_{i\in\boldsymbol{I}}$$

The action on the NTU game (listed for $\boldsymbol{I}$ only) is, therefore,

$$(4.53) \qquad \begin{aligned}\left(\pi \boldsymbol{V}^{\{\boldsymbol{v}\}}\right)(\boldsymbol{I}) \;:=\; \pi(\boldsymbol{V}^{\{\boldsymbol{v}\}}(\boldsymbol{I})) \;&=\; \left\{(\pi \boldsymbol{f}^{\{\boldsymbol{v}(\boldsymbol{I})\}})(\mathbf{t}) \,\middle|\, \mathbf{t}\in\Delta^{\boldsymbol{v}(\boldsymbol{I})\boldsymbol{e}}\right\} \\ &=\; \left\{\pi^{-1}\left(f_1^{\{\boldsymbol{v}(\boldsymbol{I})\}}(\mathbf{t}),\; \dots\;, f_n^{\{\boldsymbol{v}(\boldsymbol{I})\}}(\mathbf{t})\right) \,\middle|\, \mathbf{t}\in\Delta^{\boldsymbol{v}(\boldsymbol{I})\boldsymbol{e}}\right\}\end{aligned}$$

2ndSTEP :

The action of π on the adjusted game $\widehat{\boldsymbol{v}} \;=\; \widehat{\boldsymbol{v}}^{\{\boldsymbol{v},\boldsymbol{f}^{\{v(\boldsymbol{I})\}}\}}$ follows from the way permutations act on measures. This is

$$(4.54) \qquad \pi\boldsymbol{\theta}(\bullet) := \boldsymbol{\theta}(\pi^{-1}(\bullet))\;.$$

We obtain

$$(4.55) \qquad \widehat{\boldsymbol{v}}^{\{\boldsymbol{v},\pi\boldsymbol{f}^{\{v(\boldsymbol{I})\}}\}} = \boldsymbol{\theta}\left(\partial(\pi\boldsymbol{V}^{\{\boldsymbol{v}\}})(\boldsymbol{I})\right) \;=\; \pi\boldsymbol{\theta}\left(\partial\boldsymbol{V}^{\{\boldsymbol{v}(\boldsymbol{I})\}}\right) \;=\; \pi\widehat{\boldsymbol{v}}^{\{\boldsymbol{v},\boldsymbol{f}^{\{v(\boldsymbol{I})\}}\}}\;.$$

3rdSTEP :

Finally, we discuss the parametrizing mappings. We have

$$(4.56) \qquad \begin{aligned}\overset{\star}{\boldsymbol{x}}(\bullet) \;&=\; \Delta^{\boldsymbol{v}(\boldsymbol{I})\boldsymbol{e}} \to \partial\boldsymbol{V}^{\boldsymbol{v}}(\boldsymbol{I}) \\ &\text{with inverse} \\ \overset{\star}{\kappa}(\bullet) \;&=\; \partial\boldsymbol{V}^{\boldsymbol{v}}(\boldsymbol{I}) \to \Delta^{\boldsymbol{v}(\boldsymbol{I})\boldsymbol{e}}\;.\end{aligned}$$

Analogously, the inverse for the permuted NTU game is (as $\pi\widehat{\boldsymbol{v}}(\boldsymbol{I}) = \widehat{\boldsymbol{v}}(\boldsymbol{I})$)

$$(4.57) \qquad \begin{aligned}&\overset{\wedge}{\kappa}(\bullet) \;:\; \partial(\pi\boldsymbol{V})^{\{\boldsymbol{v}\}}(\boldsymbol{I}) \to \Delta^{\boldsymbol{v}(\boldsymbol{I})\boldsymbol{e}} \;=\; \Delta^{\pi\boldsymbol{v}(\boldsymbol{I})\boldsymbol{e}} \\ &\text{which provides the parametrization} \\ &\widehat{\boldsymbol{x}}(\bullet) \;:\; \Delta^{\pi\boldsymbol{v}(\boldsymbol{I})\boldsymbol{e}} \to \partial(\pi\boldsymbol{V})^{\{\boldsymbol{v}\}}(\boldsymbol{I}) = \partial\pi(\boldsymbol{V}^{\{\boldsymbol{v}\}}(\boldsymbol{I}))\;.\end{aligned}$$

The first line in (4.57) writes

$$(4.58) \qquad \overset{\wedge}{\kappa}(\pi(\bullet)) = \pi(\overset{\star}{\kappa}(\bullet))\;.$$

This is now reformulated

$$(4.59) \qquad \overset{\wedge}{\kappa}(\pi(\overset{\star}{\boldsymbol{x}}(\circ))) = \pi(\circ)$$

or

$$(4.60) \qquad \overset{\wedge}{\boldsymbol{x}}(\pi(\circ)) = \pi(\overset{\star}{\boldsymbol{x}}(\circ))\;.$$

4thSTEP :

Combining these results we obtain :

$$
\begin{aligned}
\chi(\pi \boldsymbol{V}^{\{v\}}) &= \widehat{\boldsymbol{x}}\left(\frac{\boldsymbol{v}(\boldsymbol{I})}{\widehat{\boldsymbol{v}}^{\pi \boldsymbol{f}}(\boldsymbol{I})}\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{v},\pi \boldsymbol{f}})\right) && - \quad \text{applying (4.57) ,}\\
&= \widehat{\boldsymbol{x}}\left(\frac{\boldsymbol{v}(\boldsymbol{I})}{\pi\widehat{\boldsymbol{v}}^{\boldsymbol{f}}(\boldsymbol{I})}\Phi(\pi\widehat{\boldsymbol{v}}^{\boldsymbol{f}})\right) && - \quad \text{by (4.55) ,}\\
&= \widehat{\boldsymbol{x}}\left(\frac{\boldsymbol{v}(\boldsymbol{I})}{\pi\widehat{\boldsymbol{v}}^{\boldsymbol{f}}(\boldsymbol{I})}\pi\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{f}})\right) && - \quad \text{by anonymity of } \Phi \text{ ,}\\
&= \widehat{\boldsymbol{x}}\left(\frac{\boldsymbol{v}(\boldsymbol{I})}{\widehat{\boldsymbol{v}}^{\boldsymbol{f}}(\boldsymbol{I})}\pi\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{f}})\right) && - \quad \text{as } \pi\widehat{\boldsymbol{v}}(\boldsymbol{I}) = \widehat{\boldsymbol{v}}(\boldsymbol{I})\\
&= \widehat{\boldsymbol{x}}\left(\pi\left(\frac{\boldsymbol{v}(\boldsymbol{I})}{\widehat{\boldsymbol{v}}^{\boldsymbol{f}}(\boldsymbol{I})}\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{f}})\right)\right) && - \quad \text{as } \pi \text{ is a linear mapping}\\
&= \pi\left(\overset{\star}{x}\left(\frac{\boldsymbol{v}(\boldsymbol{I})}{\widehat{\boldsymbol{v}}^{\boldsymbol{f}}(\boldsymbol{I})}\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{f}})\right)\right) && - \quad \text{by (4.60) ,}\\
&= \pi(\chi(\boldsymbol{V}^{\{v\}})) \ .
\end{aligned}
\tag{4.61}
$$

q.e.d.

We conclude this presentation by collecting all results.

Theorem 4.11. *The Maschler-Perles–Shapley value for TF games is Pareto efficient, covariant with affine transformations of utility, and anonymous.*

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