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NTU–Solutions for the Taxation Game

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Abstract

The Taxation Game is a cooperative game played between a set of countries $\boldsymbol{I} = \{1, \ldots, n\}$ admitting the operation of a multinational enterprise (MNE, "the firm") within their jurisdiction. The firm, when operating within the territory of a subgroup $S \subseteq \boldsymbol{I}$ of countries (a coalition) will generate a profit. Cooperating countries can agree about a share of the profit to be available for imposing taxes according to their rules and specifications, i.e., their tax rate or *tariff*.

If the taxation basis, i.e., the profit obtained by the firm, is taken as the (monetary) value of the game, then we obtain a side payment or TU game, represented by a coalitional function $\overset{\circ}{\boldsymbol{v}}$ defined on coalitions. This version is discussed by W. F. Richter [5],[6]. This author suggests that countries should agree on an allocation of the tax basis, i.e., the total profit $v(\boldsymbol{I})$ generated when the firm is operating in all participating countries. However, the profit obtained by the firm when (hypothetically) operating in in a subgroup (a coalition) $S \subseteq \boldsymbol{I}$ of all countries should be taken into account. Consequently, the share of the tax basis alloted to a country should be determined by the Shapley value of the taxation game.

The Shapley value "as a tool in theoretical economics" [14], [15] has widely been applied in Game Theory and Equilibrium Theory, but also in applications to Cost Sharing problems. We recall the Tenessee valley project ([18]), the determination of airport fees ([1]), and many others . By his approach Richter creates an interesting new field of applications.

Within this paper we present a modification of this model by introducing the game $\overset{\star}{\boldsymbol{v}}$ dual to $\overset{\circ}{\boldsymbol{v}}$. Moreover, we introduce the tariffs of countries, as the incentives of all parties involved (and actually of the firm) are based on their actual tax income depending on the tariffs.

Then the resulting game is of NTU character and involves the tariffs. To this NTU game we apply a version of the Shapley NTU value. This way we characterize an agreement of the countries involved regarding the share of profit and the taxes resulting.

A particular version of an NTU–game is given by a "bargaining problem" for n countries. For simplicity of the argument we start out with this version and dicusse the bargaining solution which are taylored versions of a Shapley version concept. We particularly deal with the Maschler–Perles solution ([4]) based on superadditivity. Here, we focus on a suitable generalization of the Maschler–Perles solution as presented in [12].

1 Profit and Tariffs

The coalitional function as a concept in Game Theory appears in the TU and NTU version, i.e., Transferable Utility games and Non Transferable Utility games. We recall some essential definitions.

Definition 1.1. A cooperative ("side payment") ***TU game*** is a triple

$$(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{v}) \ . \tag{1.1}$$

$\boldsymbol{I} = \{1, \ldots, n\}$ is the set of ***players***, $\underline{\underline{\mathbf{P}}} = \{S \,|\, S \subseteq \boldsymbol{I}\}$ is the system of ***coalitions***, and

$$\boldsymbol{v} \ : \ \underline{\underline{\mathbf{P}}} \rightarrow \mathbb{R}_+ \ , \quad \boldsymbol{v}(\emptyset) = 0 \ , \quad \boldsymbol{v}(\{i\}) = 0 \ \ (i \in \boldsymbol{I})$$

is the ***coalitional function***.

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Intuitively, for any coalition $S \in \underline{\underline{\mathbf{P}}}$ of players, $\boldsymbol{v}(S)$ denotes the worth coalition S can achieve whenever the members cooperate.

We consider a Multinational Enterprise ("MNE", the firm) operating in certain n countries ("jurisdictions") that consider an agreement about how to allot the firms profit for taxation purposes. Thus, the firm induces a cooperative game between the countries, the coalitional function of which is specified by its profit generated within various groups of countries available to the firm for operation. This model is formulated by W.F. Richter [5], [6]. The author suggests that a firm residing in one of the cooperating countries is allowed to carry out business in each of those country and that the profit earned on such business is *taxed according to a jointly agreed system of rules*. Within this model *only the tax basis* is considered for allocation, the tariffs of the various countries are not mentioned.

The model involves a simplified version to the profit concept: it is "pure return earned on know how". We accept this approach for further discussion.

Based on this idea, a firm allowed to be operating in various groups of countries determines a coalitional function. We obtain a cooperative TU game $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \mathring{\boldsymbol{v}})$ as follows.

The ***players*** in $\boldsymbol{I} = \{1, \ldots, n\}$ are the countries to engage in bargaining about an agreement regarding the allocation of the firm's profit for taxation. $\underline{\underline{\mathbf{P}}} = \{S \,|\, S \subseteq \boldsymbol{I}\}$ describes subgroups ("***coalitions***") of countries that will agree about the taxation of the total revenue of the firm. The ***coalitional function*** $\mathring{\boldsymbol{v}}$ denotes the taxable profit ("earned on know how") obtained by the firm whenever it operates freely within the territory/jurisdiction of all members of a coalition S. Then following Richter's definition ([5],[6]) we obtain

Definition 1.2. $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \mathring{\boldsymbol{v}})$ is the ***profit game***.

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For a TU game $\boldsymbol{v}$, an ***imputation*** is a vector $\boldsymbol{x} \in \mathbb{R}^n_+$ satisfying $\sum_{i \in \boldsymbol{I}} x_i = v(\boldsymbol{I})$. Equivalently, this amounts to an additive function

$$\boldsymbol{x} \; : \; \underline{\underline{\mathbf{P}}} \to \mathbb{R}_+, \quad \boldsymbol{x}(\boldsymbol{I}) = v(\boldsymbol{I}) \quad , \boldsymbol{x}(S) = \sum_{i \in S} x_i \;\; (S \in \underline{\underline{\mathbf{P}}}) \; .$$

An imputation is a possible distribution of the worth obtainable by the grand coalition. In the present context, an imputation for $\overset{\circ}{\boldsymbol{v}}$ can be seen as a "profit allocation", meaning that the share $x_i = \boldsymbol{x}(\{i\})$ is allotted for taxation to country $i \in \boldsymbol{I}$ if all countries $i \in \boldsymbol{I}$ (the grand coalition) agree on $\boldsymbol{x}$.

The question arises which imputation $\boldsymbol{x}$ will be supported, justified, or agreed upon. This discussion does not take place on the scene but its incorporated in a weighing of power derived from the game. That is, a player/country $i \in \boldsymbol{I}$ derives influence or power from their abilities reflected by their membership in various coalitions $S \in \underline{\underline{\mathbf{P}}}$. From the viewpoint of Game Theory, this points to an imputation identified as a "power index" based on an axiomatization, that is, to the ***Shapley value*** $\Phi(\boldsymbol{v})$, a concept widely used in Game Theory. The Shapey value provides an imputation which is based on axioms of Pareto efficiency (i.e., it yields a imputation), symmetry (i.e., it is independent on the players names or appearance in an ordering), a dummy axiom (i.e., a player who never contributes will not get a share), and additivity.

This additivity axiom is crucial and far reaching and, as always in Game Theoretical models, stimulates debate. It is frequently based on von Neumann–Morgenstern utility (which requires the introduction of lotteries) and thus emphasizes the side payment character of the set–up. However, there occurs a serious problem inasmuch as "adding two firms" or considering the "expected firm" under the rule of lotteries can be justified in the present context. Richter does not argue additivity directly but considers the "extension" of one and the same firm when its market is opened up to another country $i \in \boldsymbol{I}$. This amounts to exploiting marginal properties of the Shapley value which *result* from additivity.

This approach (according to the rationale of the Shapley value) implies, that eventually all countries agree to cooperate within the grand coalition $\boldsymbol{I}$, that is, the firm is permitted to operate in the full market of all countries involved.

Presently we entertain some doubts regarding the profit game to constitute the appropriate framework. The actual payoff for a country depends essentially on its tariff. Also, the firm reporting tax returns may find itself in a different situation after its payoffs are computed according to the tariff of countries involved.

Taxable revenue obtained via the firm causes various incentives in the countries involved because of the varying tariffs. In addition, the variation of tariffs causes friction between countries. This holds typically true for the members of the EU. Tariffs are used strategically to provide incentives to a firm for moving operations to a more benevolent country. Critically, this could provide incentives to the firm *after* countries have agreed upon a share of its profit for taxation.

The firm is not a player in the context of the profit game, but it may well influence the profit by a decision no to enter a "common market" after its tax burden has been agreed upon.

Thus, one could argue that – contrary to the generally accepted character of the coalitional function – a coalition S of countries is *not* capable of ensuring itself the profit $\overset{\circ}{\boldsymbol{v}}(S)$. In particular, an agreement according to the Shapley value that allots certain shares of the firms profit to countries for taxation completely disregards the reaction of the firm to this situation – which seems to constitute a self contradictory element in the model. If countries - after an agreement is known – set up high tariffs in order to increase their income, this may result in a reaction of the firm not to operate in such countries – rendering the model obsolete.

A firm forced to operate in a high taxation country (Germany) via an agreed "profit allocation" may well decide to shift operations to a low taxation country (Ireland) – which is actually the present situation and which the attempt to share taxes should counterbalance. The firm could thus decrease or stop activities in the high taxation country for the benefit of competing countries. This basic instability reflects the nature of the profit game – it shows "increasing returns to scale" for the countries involved. By contrast, the firm faces increasing incentives to leave a country when its share to be taxed in this country is growing.

At this stage we suggest, therefore, that the game under consideration should be given by the *dual* profit game. We recall the general definition.

Definition 1.3. Let $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$ be a a cooperative side payment game. The dual game $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \overset{\star}{\boldsymbol{v}})$ is provided by the coalitional function

$$\overset{\star}{\boldsymbol{v}} \; : \; \underline{\underline{\mathbf{P}}} \rightarrow \mathbb{R}_+ \; , \tag{1.2}$$

defined via

$$\overset{\star}{\boldsymbol{v}}(S) \; := \; \boldsymbol{v}(\boldsymbol{I}) - \boldsymbol{v}(S^c) \;\; (S \in \underline{\underline{\mathbf{P}}}) \tag{1.3}$$

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Intuitively, for any coalition $S \in \underline{\underline{\mathbf{P}}}$, $\overset{\star}{\boldsymbol{v}}(S)$ denotes the worth coalition S^c can *prevent coalition S to achieve.*

We interpret this in our context of the profit game: $\overset{\circ}{\boldsymbol{v}}(S^c)$ denotes the profit the firm can achieve by operating completely outside of S. Then $\overset{\circ}{\boldsymbol{v}}(\boldsymbol{I}) - \overset{\circ}{\boldsymbol{v}}(S^c)$ is the possible profit coalition S can be prevented to make use of (eventually in $\boldsymbol{I}$) by the firm completely blocking all operations within S. That is, as the discussion is centered around the distribution of the profit $\overset{\circ}{\boldsymbol{v}}(\boldsymbol{I}) = \overset{\star}{\boldsymbol{v}}(\boldsymbol{I})$, the quantity $\overset{\star}{\boldsymbol{v}}(S)$ reflects the bargaining power of S with respect of what S could definitely be blocked off if the firm decides to act most unfriendly towards S. If these quantities are being regarded as to influence the final outcome for $\boldsymbol{I}$, the firm would be considered to have no incentive to deviate from the distribution agreed upon.

Now luckily, all of Richters arguments are being preserved when switching from $\overset{\circ}{\boldsymbol{v}}$ to $\overset{\star}{\boldsymbol{v}}$. This follows from the invariance of the Shapley value under the duality operation, i.e., from

$$\Phi(\overset{\star}{\boldsymbol{v}}) = \Phi(\boldsymbol{v}) \; . \tag{1.4}$$

for any coalitional function $\boldsymbol{v}$. Thus, his model can be upheld and is even strengthened by considering the dual game. This justifies our emphasis on the dual game within what follows.

Obviously, the tariffs of the countries are essential. We, therefore, aim to incorporate tariffs thus generating a more elaborate model of a game without side payments.

We recall

Definition 1.4. A (cooperative) ***NTU game*** is a triple $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{V})$. Here, $\boldsymbol{I} = \{1, \ldots, n\}$ is the ***set of players***, $\underline{\underline{\mathbf{P}}} = \{S | S \subseteq \boldsymbol{I}\}$ is the ***system of coalitions***, and $\boldsymbol{V} : \underline{\underline{\mathbf{P}}} \to \mathcal{P}(\mathbb{R}^n_+)$ is the ***coalitional function***. $\boldsymbol{V}$ assigns to any coalition S a compact and comprehensive set of "utility vectors" $\boldsymbol{V}(S) \subseteq \mathbb{R}^n_{S+}$.

We assume

$$\boldsymbol{V}(\{i\}) = \{\mathbf{0}\} \quad (i \in \boldsymbol{I}) \; . \tag{1.5}$$

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The standard interpretation has it that $\boldsymbol{V}(S)$ is the set of utility vectors that can be ensured to the members of coalition S by cooperation. The (utility) vector $\mathbf{0}$ reflects the fall back position ("status quo") for all players should cooperation fail in every coalition. The assumption that utilities are nonnegative and the status quo is $\mathbf{0}$ is not severe. Casually, we refer to $\boldsymbol{V}$ as to "the game" as well.

A particular version of an NTU game is obtained when only the grand coalition $\boldsymbol{V}(\boldsymbol{I})$ is effective. Formally, all coalition S except $\boldsymbol{I}$ satisfy $\boldsymbol{V}(S) = \{\emptyset\}$. This situation is emphasized by

Definition 1.5. A ***bargaining problem*** is a pair $\boldsymbol{V}^0 = (\mathbf{0}, \boldsymbol{U})$ such that $\boldsymbol{U} \subseteq \mathbb{R}^n_+$ is a nonempty and comprehensive set.

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Within NTU territory, the concept of an imputation is generalized to the one of a Pareto efficient vector.

Definition 1.6. The ***Pareto surface*** of $\boldsymbol{U}$ is denoted $\partial \boldsymbol{U}$. For $S \in \underline{\underline{\mathbf{P}}}$, the ***Pareto surface*** of $\boldsymbol{V}(S)$ is denoted $\partial \boldsymbol{V}(\boldsymbol{S})$. A vector $\boldsymbol{x} \in \partial \boldsymbol{U}$ $(\boldsymbol{x} \in \partial \boldsymbol{V}(S))$ is called ***Pareto efficient*** for $\boldsymbol{U}$ or $\boldsymbol{V}(S)$ respectively.

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Now we describe situations derived from the distribution of a financial asset when players apply different revenue generating tax systems. In most countries tariffs are convex functions, at least in a certain domain. This property reflects increasing returns to scale of revenue for the state – allegedly by arguments of "fairness" and "just distribution of burdens". The concept frequently flounders when the marginal taxation is set to be constant, e.g., once the 50% mark is reached. Yet we will consider a ***tariff*** to be a continuous, monotone, convex, and piecewise linear function. Such a function can be written as the convex hull of a family of linear functions, say

$$f(t) \;=\; \min_{k\in\boldsymbol{K}} c_k t + d_k \quad (t \in \mathbb{R}_+ k, \quad k \in \boldsymbol{K}) \tag{1.6}$$

with suitable coefficients $c_k, d_k \quad (k = 1, \ldots, K)$. We introduce the "grid" suggested by this notation via

$$\boldsymbol{K} \;=\; \{1, \ldots, K\}\ . \tag{1.7}$$

Then a tariff equals a linear function within each interval of the grid. When we assign a tariff to each player/country $i \in \boldsymbol{I}$, we end up with

Definition 1.7. Let $t_0 \in \mathbb{R}_+, t_0 \leq K$.

1. A ***tariff*** (for taxation in $[0, t_0]$, adapted to $\boldsymbol{K}$) is a continuous, monotone, convex, and piecewise linear function

$$\begin{aligned} \overset{\circ}{f}{}^{\{t_0\}} &: [0, t_0] \to \mathbb{R}_+, \quad \overset{\circ}{f}(0) = 0\ . \\ \overset{\circ}{f}{}^{\{t_0\}}(t) &= \overset{\circ}{c}_k t + \overset{\circ}{d}_k \quad \left(\frac{(k-1)t_0}{K} \leq t \leq \frac{kt_0}{K}\right), \quad (k \in \boldsymbol{K}) \end{aligned} \tag{1.8}$$

with suitable coefficients $\overset{\circ}{c}_k, \overset{\circ}{d}_k \quad (k \in \boldsymbol{K})$ satisfying the necessary boundary conditions.

2. A set of positive coefficients (i.e., two matrices)

$$\overset{\circ}{\boldsymbol{C}} \;=\; \left(\overset{\circ}{c}{}^i_k\right)^{i\in\boldsymbol{I}}_{k\in\boldsymbol{K}}, \overset{\circ}{\boldsymbol{D}} \;=\; \left(\overset{\circ}{d}{}^i_k\right)^{i\in\boldsymbol{I}}_{k\in\boldsymbol{K}} \tag{1.9}$$

obeys the ***boundary conditions*** (for the construction of a tariff system), if

$$\begin{aligned} &\text{For } i \in \boldsymbol{I},\ c^i_k \quad \text{is monotone in } k \in \boldsymbol{K}\ , \\ &\text{for } i \in \boldsymbol{I},\ d^i_k \quad \text{is antitone in } k \in \boldsymbol{K}\ , \\ &c^i_k \frac{(k-1)t_0}{K} + d^i_k = c^i_{k-1} \frac{(k-1)t_0}{K} + d^i_k \quad \text{for} \quad (i \in \boldsymbol{I}, k \in \boldsymbol{K}) \end{aligned} \tag{1.10}$$

holds true.

3. A ***tariff system*** (for taxation in $[0, t_0]$, adapted to $\boldsymbol{K}$) is a family of tariffs

$$\overset{\circ}{\boldsymbol{f}}{}^{\{t_0\}} \;=\; (\overset{\circ}{f}{}^{\{t_0\}}_1, \ldots, \overset{\circ}{f}{}^{\{t_0\}}_n) \tag{1.11}$$

such that for $i \in \boldsymbol{I}$

$$\overset{\circ}{f}{}^{\{t_0\}}_i(t) \;:\; [0, t_0] \to \mathbb{R}_+ \tag{1.12}$$

satisfies

$$(1.13) \qquad \overset{\circ}{f}{}_i^{\{t_0\}}(t) \;=\; \overset{\circ}{c}{}_k^i t + \overset{\circ}{d}{}_k^i \quad \left(\frac{(k-1)t_0}{K} \leq t \leq \frac{kt_0}{K}\right), \quad (k \in \boldsymbol{K})$$

with coefficients

$$(1.14) \qquad \overset{\circ}{c}{}_k^i, \overset{\circ}{d}{}_k^i \quad (i \in \boldsymbol{I}, k \in \boldsymbol{K}) \ .$$

satisfying the conditions (1.10).

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Figure 1.1 depicts a typical tariff.

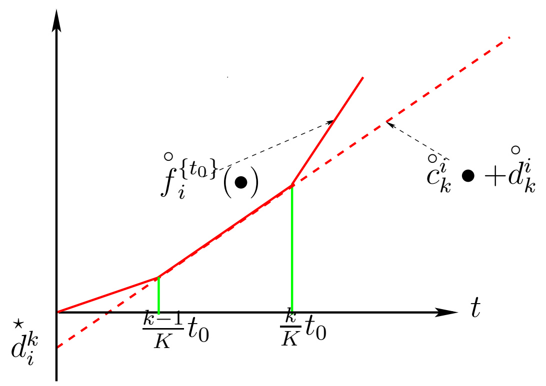

Figure 1.1: A tariff of country i

The above formulation does not constitute a severe restriction, one can partition intervals in several subintervals and keep the coefficients $c_{\bullet}^{\bullet}$ and $d_{\bullet}^{\bullet}$ unchanged so as to imitate any more general version. And by choosing K sufficiently large, one can adapt a tariff system to every requirement needed. Indeed, in the context of legal formulation tariffs are always represented as piecewise linear functions.

Now, in view of of our propensity regarding the dual version of a game, we introduce the *dual version* of tariffs.

Definition 1.8. 1. Let $\overset{\circ}{f} : [0, t_0] \to \mathbb{R}$ be a tariff. The ***dual tariff*** is the continuous, monotone, concave, and piecewise linear function

$$(1.15) \qquad \begin{gathered} \overset{\star}{f}{}^{\{t_0\}} \;:\; [0,t_0] \to \mathbb{R}_+, \quad f(0) = 0 \ , \\ \text{given by} \\ \overset{\star}{f}{}^{\{t_0\}}(s) \;:=\; \overset{\circ}{f}{}^{\{t_0\}}(t_0) - \overset{\circ}{f}{}^{\{t_0\}}(t_0 - s) \quad (s \in [0, t_0]) \ . \end{gathered}$$

2. Let $\boldsymbol{K} = \{1, \ldots, K\}$ and let $\boldsymbol{f}^{\{t_0\}}$ be a tariff system (adapted to $\boldsymbol{K}$). The ***dual tariff system*** is the family of dual tariffs

$$(1.16) \qquad \overset{\star}{\boldsymbol{f}}{}^{\{t_0\}} \;=\; (\overset{\star}{f}{}_1^{\{t_0\}}, \ldots, \overset{\star}{f}{}_n^{\{t_0\}}) \ ,$$

such that for $i \in \boldsymbol{I}$

$$\overset{\star}{f}{}_i^{\{t_0\}}(t) \;=\; \overset{\star}{c}{}_k^i t + \overset{\star}{d}{}_k^i \;\left(\frac{(k-1)t_0}{K} \le t \le \frac{kt_0}{K}\right), \quad (k \in \boldsymbol{K})\ . \tag{1.17}$$

holds true. The coefficients

$$\overset{\star}{c}{}_k^i, \overset{\star}{d}{}_k^i \quad (i \in \boldsymbol{I}, k \in \boldsymbol{K}) \tag{1.18}$$

result from the coefficients (1.14). E.g. we have

$$\overset{\star}{c}{}_k^i = \overset{\circ}{c}{}_{(K-k+1)}^i \quad (i \in \boldsymbol{I}, k \in \boldsymbol{K})\ . \tag{1.19}$$

and the boundary conditions (1.10) are satisfied with due modifications.

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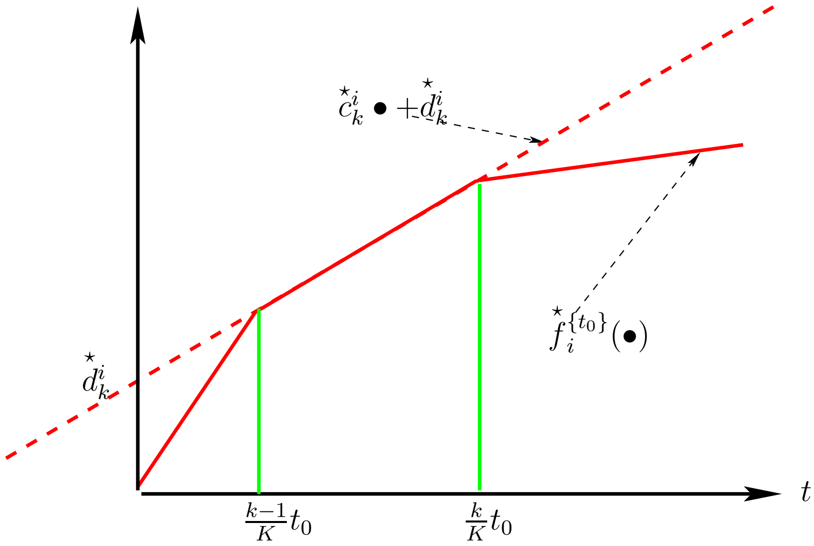

Figure 1.2: A dual tariff of country i

Figure 1.2 indicates the dual tariff to the one depicted in Figure 1.1.

A distribution of monetary values is represented by a vector $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{R}^n_+$. A tariff system reflects the taxes obtained by such monetary distributions. Simultaneously we obtain the bargaining problem presented by such data.

Definition 1.9. Let $\boldsymbol{f}^{\{t_0\}}$ be a tariff system (primal or dual) and let $t_0 \ge 0$. Then

$$\Pi^{\{\boldsymbol{f}\}} = \Pi^{\{\boldsymbol{f},t_0\}} \;:=\; \left\{ \boldsymbol{f}^{\{t_0\}}(\mathbf{t}) \;\middle|\; \mathbf{t} \in \mathbb{R}^n_+\ ,\ \sum_{i \in \boldsymbol{I}} t_i \le t_0 \right\} \tag{1.20}$$

is called a ***taxoid***. The resulting bargaining problem (for the distribution of t_0) is the pair $\boldsymbol{V}^{\{t_0\}} = (\mathbf{0}, \boldsymbol{U}^{\{t_0\}})$ with $\boldsymbol{U}^{\{t_0\}} = \Pi^{\{\boldsymbol{f},t_0\}}$.

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More generally, consider a TU game. We assume that each of the quantities $\boldsymbol{v}(S)$ ($S \in \underline{\underline{\mathbf{P}}}$) is multiple of $\frac{\boldsymbol{v}(\boldsymbol{I})}{K}$. Within the following definition, $\boldsymbol{v}$ can represent a profit game and $\boldsymbol{f}$ may be primal or dual.

Definition 1.10. Let $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{v})$ be a TU game. Also, for $S \in \underline{\underline{\mathbf{P}}}$. let

$$\boldsymbol{f}^{\{\boldsymbol{v}(S)\}} = (f_1^{\{\boldsymbol{v}(S)\}}, \ldots, f_n^{\{\boldsymbol{v}(S)\}}) \tag{1.21}$$

be a tariff system for the distribution of $\boldsymbol{v}(S)$ (primal or dual). The ***tariff game*** (resulting from $\boldsymbol{v}$ and $\boldsymbol{f}^{\{\boldsymbol{v}(\bullet)\}}$) is the NTU game $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{V}^{\{\boldsymbol{v}\}})$ defined via

$$\boldsymbol{V}^{\{\boldsymbol{v}\}}(S) := \left\{ \boldsymbol{f}^{\{\boldsymbol{v}(S)\}}(\mathbf{t}) \,\middle|\, \mathbf{t} \in \mathbb{R}_+^S \,, \ \sum_{i \in S} t_i \leq \boldsymbol{v}(S) \right\} \quad (S \in \underline{\underline{\mathbf{P}}}) \,. \tag{1.22}$$

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The dilemma regarding the definition of cooperative games in the taxation context appears even more obvious for the NTU version. If $\boldsymbol{f}$ is primal, then the feasible sets $\boldsymbol{U}$ or $\boldsymbol{V}^{\{\boldsymbol{v}\}}(S)$ are not convex as the Pareto boundary is a convex curve. This is in eclatant contradiction to generally accepted versions of NTU Theory; with "increasing returns to scale" one obtaines a situation intractable for solution concepts. Obviously this is a very real obstacle for any agreement within governments: if shares of taxable income are firmly allotted to tax–happy governments, they will have a tendency to raise taxes in the name of social benefits.

This we feel motivates our emphasis on dual versions of the game.

2 Taxoids: The Pareto Surface

Given $t_0 \in \mathbb{R}_+$ and a tariff system $\boldsymbol{f}^{\{t_0\}}$ we consider the Pareto surface of the taxoid $\Pi^{\{\boldsymbol{f}\}} = \Pi^{\{\boldsymbol{f},t_0\}}$, i.e.,

$$(2.1) \qquad \partial\Pi^{\{\boldsymbol{f},t_0\}} := \left\{ \boldsymbol{f}^{\{t_0\}}(\mathbf{t}) \,\middle|\, \mathbf{t} \in \mathbb{R}^n_+ \,,\ \sum_{i\in \boldsymbol{I}} t_i = t_0 \right\} = \boldsymbol{f}^{\{t_0\}}(\Delta^{t_0\boldsymbol{e}}) \;.$$

For $i \in \boldsymbol{I}$

$$(2.2) \qquad f_i^{\{t_0\}}(s) = c_k^i s + d_k^i \quad \left(\frac{(k-1)t_0}{K} \leq s \leq \frac{kt_0}{K}\right) \quad (i \in \boldsymbol{I}, k \in \boldsymbol{K})$$

holds true. Let

$$\mathbf{k} = (k_1,\ldots,k_n) \,|\, k_i \in \boldsymbol{K} \quad (i \in \boldsymbol{I})$$

be a sequence of indices taken from $\boldsymbol{K}$. Then, $\boldsymbol{f}$ restricted to

$$(2.3) \quad \boldsymbol{Q}^{\{t_0\}}_{\mathbf{k}} = \boldsymbol{Q}^{\{t_0\}}_{k_1,\ldots,k_n} := \left\{ \mathbf{t} = (t_1,\ldots,t_n) \in \mathbb{R}^n_+ \,\middle|\, \begin{array}{ccc} \frac{(k_1-1)t_0}{K} & \leq t_1 & \leq \frac{k_1 t_0}{K} \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ \frac{(k_n-1)t_0}{K} & \leq t_n & \leq \frac{k_n t_0}{K} \end{array} \right\}$$

coincides with the (affine) linear mapping

$$(2.4) \qquad \begin{aligned} \overset{\star}{\boldsymbol{x}}{}^{\mathbf{k}}(\bullet) &: \mathbb{R}^n_+ \to \mathbb{R}^n_+ \,, \\ \mathbf{t} &= (t_1,\ldots,t_n) \to (c^1_{k_1} t_1 + d^1_{k_1}, \ldots, c^n_{k_n} t_n + d^n_{k_n}) \\ \overset{\star}{\boldsymbol{x}}{}^{\mathbf{k}}(\mathbf{t}) &= \mathbf{c}_{\mathbf{k}} \otimes \boldsymbol{t} + \mathbf{d}_{\mathbf{k}} \quad (\boldsymbol{t} \in \boldsymbol{Q}^{\{t_0\}}_{\mathbf{k}}) \end{aligned}$$

Definition 2.1. 1. Let

$$(2.5) \qquad \mathbf{K} := \{\mathbf{k} = (k_1,\ldots,k_n) \,|\, k_i \in \boldsymbol{K} \quad (i \in \boldsymbol{I})\}$$

denote the system of all possible choices of indices taken from $\boldsymbol{K}$. For $\mathbf{k} \in \mathbf{K}$ define $\boldsymbol{Q}^{\{t_0\}}_{\mathbf{k}}$ by (2.3) and

$$(2.6) \qquad \Delta^{t_0\boldsymbol{e}}_{\mathbf{k}} := \boldsymbol{Q}^{\{t_0\}}_{\mathbf{k}} \cap \Delta^{t_0\boldsymbol{e}} \;.$$

such that

$$(2.7) \qquad \boldsymbol{f}_{\,|\Delta^{t_0\boldsymbol{e}}_{\mathbf{k}}} = \overset{\star}{\boldsymbol{x}}{}^{\mathbf{k}}(\bullet)_{|\Delta^{t_0\boldsymbol{e}}_{\mathbf{k}}} \quad (\mathbf{k} \in \mathbf{K})$$

holds true.

2. We call a sequence $\mathbf{k} = (k_1,\ldots,k_n)$ such that $\Delta^{t_0\boldsymbol{e}}_{\mathbf{k}} \neq \emptyset$ has full dimension relevant. Also, $\Delta^{t_0\boldsymbol{e}}_{\mathbf{k}}$ is called a relevant polyhedron. The set of relevant sequences is denoted $\overset{\circ}{\mathbf{K}}$.

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Note that $\Delta^{t_0 e}_{\mathbf{k}}$ ($t_0 > 0$) has full dimension if and only if $\Delta^{e}_{\mathbf{k}}$ has full dimension, i.e., the relevance of some sequence $\mathbf{k} \in \overset{\circ}{\mathbf{K}}$ does not depend on the dilatation factor t_0. For a relevant polyhedron the image suggested by (2.7) has full dimension and constitutes a Pareto face of the taxoid under consideration. More precisely we have

Corollary 2.2. Let $t_0 \in \mathbb{R}$ and let $\Pi^{\{\boldsymbol{f}\}} = \Pi^{\{\boldsymbol{f},t_0\}}$ be the taxoid induced by a tariff system $\boldsymbol{f}^{\{t_0\}}$. The Pareto faces of this taxoid are

$$(2.8)\qquad \boldsymbol{F}^{\mathbf{k}} = \boldsymbol{F}^{\{\mathbf{k},t_0\}} := \overset{\star}{\boldsymbol{x}}{}^{\mathbf{k}}(\Delta^{t_0 e}_{\mathbf{k}}) \subseteq \partial\Pi^{\{\boldsymbol{f}\}} \quad (\mathbf{k} \in \overset{\circ}{\mathbf{K}}) \ .$$

The bijection

$$(2.9)\qquad \overset{\star}{\boldsymbol{x}}{}^{\mathbf{k}}(\bullet) \ : \ \Delta^{t_0 e}_{\mathbf{k}} \to \boldsymbol{F}^{\mathbf{k}} \quad (\mathbf{k} \in \overset{\circ}{\mathbf{K}}) \ .$$

provides a parametrization of the Pareto face $\boldsymbol{F}^{\{\mathbf{k}\}}$ in the sense of [12] and [13]. Therefore, there is a canonical global parametrization of the Pareto surface of the taxoid

$$(2.10)\qquad \overset{\star}{\boldsymbol{x}}(\bullet) : \Delta^{t_0 e} \to \partial\Pi^{\{\boldsymbol{f}\}} \ , \quad \mathbf{t} \to \overset{\star}{\boldsymbol{x}}{}^{\mathbf{k}}(\mathbf{t}) \ \ (\mathbf{t} \in \Delta^{t_0 e}_{\mathbf{k}}, \mathbf{k} \in \overset{\circ}{\mathbf{K}})$$

which is just the composition of the local parametrizations, i.e.,

$$(2.11)\qquad \overset{\star}{\boldsymbol{x}}(\bullet)_{|\Delta^{t_0 e}_{\mathbf{k}}} \ = \ \overset{\star}{\boldsymbol{x}}{}^{\mathbf{k}}(\bullet) \ \ (\mathbf{k} \in \overset{\circ}{\mathbf{K}}) \ .$$

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We emphasize that $\overset{\star}{\boldsymbol{x}}{}^{\mathbf{k}}(\bullet)$ is an (affine) linear mapping, the composed mapping $\overset{\star}{\boldsymbol{x}}(\bullet)$, is in general not linear. Indeed, $\overset{\star}{\boldsymbol{x}}(\bullet)$ coincides with $\boldsymbol{f}^{\{t_0\}}$, however the unifying notation of (2.9) and (2.10) emphasizes the locally linear character represented by coefficient matrices $\overset{\circ}{\boldsymbol{C}}$ and $\overset{\circ}{\boldsymbol{D}}$ (or rather their dual versions) of the boundary conditions (1.9) or (1.19) .

We approach the concept of a taxoid by discussing a set of examples. We start out for $n = 2$. Here the situation is well accessible. For, in two dimensions every convex comprehensive polyhedron is a Cephoid, so we combine both aspects.

Example 2.3. Let $n = 2$ and let $\boldsymbol{f}$ be a dual taxation system. Then the $\partial\Pi^{\boldsymbol{f}}$ is described by a piecewise linear concave curve. As all such polyhedra are Cephoids (MASCHLER–PERLES [4], see also [10] for a textbook treatment) we can describe the taxoid $\Pi^{\boldsymbol{f}}$ in two ways.

On one hand we have to consider a Cephoid $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$. The Pareto surface consists of line segments which are the translates of the simplices $\Delta^{(k)} = \Delta^{\boldsymbol{a}^{(k)}}$, typically we have a Pareto face

$$(2.12)\qquad \boldsymbol{F}^{(k)} \ = \ \sum_{\kappa\in\boldsymbol{K},\kappa<k} \boldsymbol{a}^{(\kappa)1} + \Delta^{\boldsymbol{a}^{(k)}} + \sum_{\kappa\in\boldsymbol{K},\kappa>k} \boldsymbol{a}^{(\kappa)2} = \Delta^{\boldsymbol{a}^{(k)}} + \boldsymbol{a}^{(-k)} \ ,$$

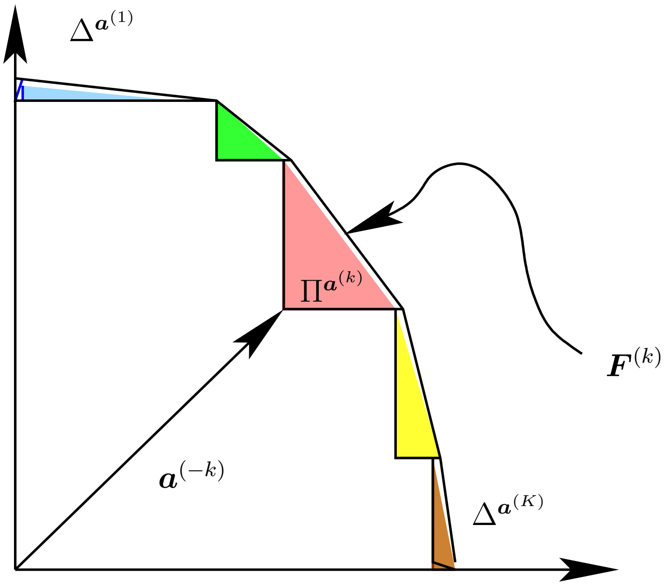

Figure 2.1: A Cephoid Π in two dimensions

assuming the enumeration follows the slope of the triangles (see Figure 2.1).

On the other hand,the taxoid $\Pi^{\boldsymbol{f}} = \Pi^{\boldsymbol{f},t_0}$ has Pareto faces

$$\begin{aligned}\partial\Pi^{\boldsymbol{f},t_0}_{k_1,k_2} &= \Big\{ \boldsymbol{f}^{t_0}(\mathbf{t}) \,\Big|\, t_1 + t_2 = t_0, \\ &= \frac{k_1-1}{K}t_0 \le t_1 \le \frac{k_1}{K}t_0, \quad \frac{k_2-1}{K}t_0 \le t_2 \le \frac{k_2}{K}t_0 \Big\} \,. \end{aligned} \tag{2.13}$$

We obtain a relevant sequence $\mathbf{k} = (k_1, k_2)$ if and only if we choose $k_1 = k$ and $k_2 = K - (k-1)$, i.e., $\mathbf{k} = (k, K-(k-1))$. The corresponding relevant polyhedron (i.e., triangle) is

$$\begin{aligned}\partial\Pi^{\boldsymbol{f},t_0}_{\mathbf{k}} &= \Big\{ \boldsymbol{f}^{t_0}(\mathbf{t}) \,\Big|\, t_1 + t_2 = t_0, \\ &= \frac{k-1}{K}t_0 \le t_1 \le \frac{k}{K}t_0, \quad \frac{K-k}{K}t_0 \le t_2 \le \frac{K-(k-1)}{K}t_0 \Big\} \\ &= \Big\{ \boldsymbol{f}^{t_0}(\mathbf{t}) \,\Big|\, \mathbf{t} \in \Delta^{t_0\boldsymbol{e}}_{\mathbf{k}} \Big\} \\ &= \overset{\star}{\boldsymbol{x}}^{\mathbf{k}}(\Delta^{t_0\boldsymbol{e}}_{\mathbf{k}}) = \big\{ \boldsymbol{c}_{\mathbf{k}} \otimes \mathbf{t} \,\big|\, \mathbf{t} \in \Delta^{t_0\boldsymbol{e}}_{\mathbf{k}} \big\} + \boldsymbol{d}_{\mathbf{k}} \\ &= \boldsymbol{F}^{\mathbf{k}} \,. \end{aligned} \tag{2.14}$$

Hence, as a consequence to enumerating the simplices (triangles) involved according to slope, we find the system of relevant sequences to be

$$\mathbf{K} = \{(1, K), (2, K-1), \ldots, (K-1, 2), (K, 1)\} \,. \tag{2.15}$$

The Pareto faces of $\partial\Pi^{\{\boldsymbol{f}\}}$ are listed according to the slope of the boundary simplices $\Delta^{\{\boldsymbol{a}^{\mathbf{k}}\}}$.

For completeness we note the data when comparing the representations (2.14) and

(2.12). After some computation it turns out that

$$
(2.16)\qquad \boldsymbol{c}_{\mathbf{k}} = \frac{K}{t_0}\boldsymbol{a}^{(k)}\ , \quad \boldsymbol{d}_{\mathbf{k}} = \boldsymbol{a}^{(-k)} - \left((k-1)a_1^{(k)},\ (K-k)a_2^{(k)}\right) = \boldsymbol{a}^{(-k)} - \mathbf{k}\boldsymbol{a}^{(k)} + \boldsymbol{a}^{(k)} = \sum_{k\in\boldsymbol{K}} \boldsymbol{a}^{(k)} - \mathbf{k}\boldsymbol{a}^{(k)}\ .
$$

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Next we turn to 3 dimensions. Here it is no longer true that taxoids and Cephoids coincide. Hence the theory of Cephoids is not applicable. Rather we have to develop an autonomous theory of taxoids. To initiate this, we continue with a set of examples in 3 and 4 dimensions.

The graphical description of relevant polyhedra is simplified by tentatively choosing $t_0 = K$. We can then study the decomposition of $\Delta^{K\boldsymbol{e}}$ into relevant subsimplices $\Delta^{K\boldsymbol{e}}_{\mathbf{k}}$. This version is easily translated into the version for $\Delta^{t_0\boldsymbol{e}}$ and $\Delta^{t_0\boldsymbol{e}}_{\mathbf{k}}$ via

$$
(2.17)\qquad \Delta^{t_0\boldsymbol{e}} = \frac{t_0}{K}\Delta^{K\boldsymbol{e}}\ , \quad \Delta^{t_0\boldsymbol{e}}_{\mathbf{k}} = \frac{t_0}{K}\Delta^{K\boldsymbol{e}}_{\mathbf{k}}\ .
$$

The structure of relevant sequences and polyhedra is not affected by the transition suggested by (2.17).

Example 2.4. Figure 2.2 reflects the case $n = 3, K = 4$. First we exhibit the situation for $t_0 = K = 4$, i.e., we study $\Delta^{4\boldsymbol{e}}$. There are two types of $n-1 = 2$ dimensional simplices. One type is just a (translated) unit simplex, the other one is a rotated version.

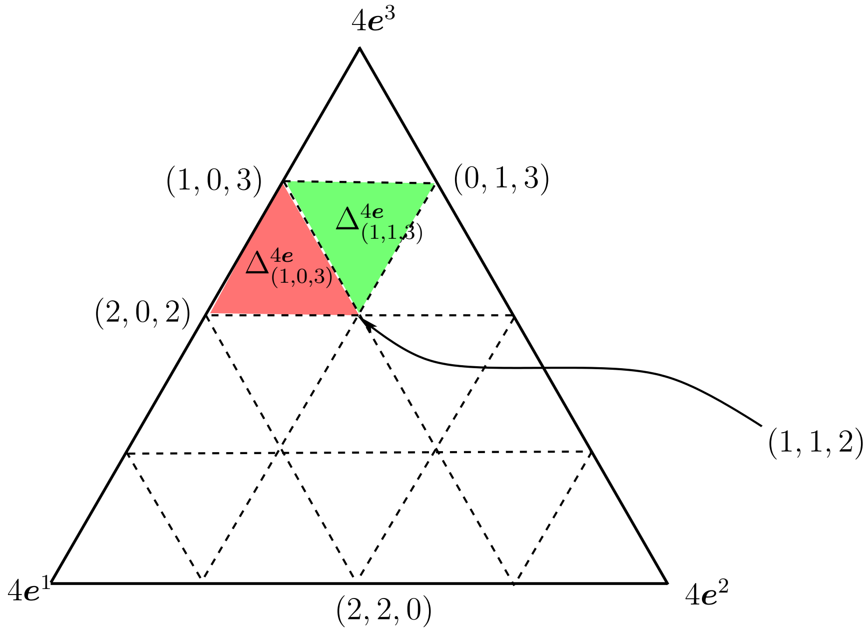

Figure 2.2: Relevant polyhedra in $\Delta^{4\boldsymbol{e}}$ for $n = 3, K = 4$

Each of these constitutes a relevant polyhedron. To see this, consider the subsimplex $\Delta^{4\boldsymbol{e}}_{\mathbf{k}}$ of $\Delta^{4\boldsymbol{e}}$ given by $\mathbf{k} = (2,1,3)$, i.e.,

$$
(2.18)\qquad \Delta^{4\boldsymbol{e}}_{(2,1,3)} = \left(\boldsymbol{t} \in \Delta^{4\boldsymbol{e}} \,\middle|\, 1 \le t_1 \le 2,\ 0 \le t_2 \le 1,\ 2 \le t_3 \le 3\right)
$$

with vertices

$$(2.19)\qquad (2,0,2),\ (1,1,2),\ (2,1,3)$$

which is a translated copy of the unit simplex $\Delta^{\boldsymbol{e}}$. The parametrization of $\Delta^{4\boldsymbol{e}}_{(2,1,3)}$ can be directly taken from the sketch of Figure 2.2, we obtain

$$(2.20)\qquad \Delta^{4\boldsymbol{e}}_{(2,1,3)} = \{(\mathbf{t}\,|\,\boldsymbol{e}\mathbf{t} = 4, 1 \le t_1 \le 2, 0 \le t_2 \le 1, 2 \le t_3 \le 3)\}\ .$$

For general t_0 this leads to

(2.21)

$$\Delta^{t_0\boldsymbol{e}}_{(2,1,3)} = \frac{t_0}{4}\Delta^{4\boldsymbol{e}}_{(2,1,3)} = \left\{\mathbf{t}\,\middle|\,\boldsymbol{e}\mathbf{t} = t_0, \frac{t_0}{4} \le t_1 \le \frac{2t_0}{4}, 0 \le t_2 \le \frac{t_0}{4}, \frac{2t_0}{4} \le t_3 \le \frac{3t_0}{4}\right\}\ .$$

Given some taxation system $\boldsymbol{f}$ and its linear representation $\overset{\star}{\boldsymbol{x}}^{\mathsf{k}}$, the corresponding Pareto face of the taxoid is

$$(2.22)\qquad \begin{aligned}\boldsymbol{F}^{\mathsf{k}} = \boldsymbol{F}^{(2,1,3)} &= \overset{\star}{\boldsymbol{x}}^{(2,1,3)}(\Delta^{t_0\boldsymbol{e}}_{(2,1,3)})\\ &= \{\boldsymbol{c}_{(2,1,3)} \otimes \mathbf{t}\,\big|\,\mathbf{t} \in \Delta^{t_0\boldsymbol{e}}_{(2,1,3)}\} + \boldsymbol{d}_{(2,1,3)}\\ &= \left\{(c^1_2t_1, c^2_1t_2, c^3_3t_3)\,\middle|\,\boldsymbol{e}\mathbf{t} = t_0, \frac{t_0}{4} \le t_1 \le \frac{2t_0}{4}, 0 \le t_2 \le \frac{t_0}{4}, \frac{2t_0}{4} \le t_3 \le \frac{3t_0}{4}\right\} + \boldsymbol{d}_{(2,1,3)}\end{aligned}$$

On the other hand consider the subsimplex of $\Delta^{4\boldsymbol{e}}$ given by $\mathsf{k} = (1,1,3)$, i.e.,

$$(2.23)\qquad \Delta^{4\boldsymbol{e}}_{(1,1,3)} = \big(\boldsymbol{t} \in \Delta^{4\boldsymbol{e}}\,\big|\,0 \le t_1 \le 1,\ 0 \le t_2 \le 1,\ 2 \le t_3 \le 3\big)$$

with vertices

$$(2.24)\qquad (0,1,3)\,,\ (1,0,3)\,,\ (1,1,2)\,.$$

which is an rotated copy of the unit simplex. The corresponding taxoidal face is

$$(2.25)\qquad \begin{aligned}\boldsymbol{F}^{(1,1,3)} &= \overset{\star}{\boldsymbol{x}}^{(1,1,3)}(\Delta^{t_0\boldsymbol{e}}_{(1,1,3)})\\ &= \{\boldsymbol{c}_{(1,1,3)} \otimes \mathbf{t}\,\big|\,\mathbf{t} \in \Delta^{t_0\boldsymbol{e}}_{(1,1,3)}\}\ + \boldsymbol{d}_{(113)}\\ &= \{\boldsymbol{c}_{(1,1,3)} \otimes \mathbf{t}\,\big|\,\boldsymbol{e}\mathbf{t} = t_0, 0 \le t_1 \le \frac{t_0}{4}, 0 \le t_2 \le \frac{t_0}{4}, \frac{2t_0}{4} \le t_3 \le \frac{3t_0}{4}\}\\ &\quad +\ \boldsymbol{d}_{(113)}\end{aligned}$$

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These results are generalized for 3 dimensions:

Lemma 2.5. Let $n = 3$ and let $\boldsymbol{f}^{\{t_0\}}$ be a tariff system. Then the relevant sequences are given by

$$(2.26)\qquad \overset{\circ}{\mathsf{K}} = \{\mathsf{k} \in \mathsf{K}\,\big|\,\boldsymbol{e}\mathsf{k} = K+1\} \cup \{\mathsf{k} \in \mathsf{K}\,\big|\,\boldsymbol{e}\mathsf{k} = K+2\}\ . =: \overset{\circ}{\mathsf{K}}^{K+1} \cup \overset{\circ}{\mathsf{K}}^{K+2}$$

Accordingly, $\Pi^{\boldsymbol{f}}$ has Pareto faces

$$(2.27)\qquad \boldsymbol{F}^{\mathsf{k}} = \overset{\star}{\boldsymbol{x}}^{\mathsf{k}}(\Delta^{t_0\boldsymbol{e}}_{\mathsf{k}}) = \{\boldsymbol{c}_{\mathsf{k}} \otimes \mathbf{t}\,\big|\,\mathbf{t} \in \Delta^{t_0\boldsymbol{e}}_{\mathsf{k}}\}\ + \boldsymbol{d}_{\mathsf{k}}$$

such that

1. $\Delta^{K\boldsymbol{e}}_{\mathbf{k}}$, $\Delta^{t_0\boldsymbol{e}}_{\mathbf{k}}$, and $\boldsymbol{F}^{\mathbf{k}} = \overset{\star}{\boldsymbol{x}}^{\mathbf{k}}(\Delta^{t_0\boldsymbol{e}}_{\mathbf{k}})$ are (deGua) simplices if and only if

   $$\mathbf{k} \in \mathring{\mathbf{K}}^{K+2}, \tag{2.28}$$

2. $\Delta^{K\boldsymbol{e}}_{\mathbf{k}}$, $\Delta^{t_0\boldsymbol{e}}_{\mathbf{k}}$, and $\boldsymbol{F}^{\mathbf{k}} = \overset{\star}{\boldsymbol{x}}^{\mathbf{k}}(\Delta^{t_0\boldsymbol{e}}_{\mathbf{k}})$ are rotated simplices if and only if

   $$\mathbf{k} \in \mathring{\mathbf{K}}^{K+1} . \tag{2.29}$$

3.

In particular, for $K = 4$ we have

$$\mathring{\mathbf{K}} = \{\mathbf{k} \in \mathbf{K} \,\big|\, \boldsymbol{e}\mathbf{k} = 3\} \cup \{\mathbf{k} \in \mathbf{K} \,\big|\, \boldsymbol{e}\mathbf{k} = 4\} =: \mathring{\mathbf{K}}^3 + \mathring{\mathbf{K}}^4 \tag{2.30}$$

Hence $\Pi^{\boldsymbol{f}}$ has Pareto faces

$$\boldsymbol{F}^{\mathbf{k}} = \overset{\star}{\boldsymbol{x}}^{\mathbf{k}}(\Delta^{t_0\boldsymbol{e}}_{\mathbf{k}}) = \{\boldsymbol{c}_{\mathbf{k}} \otimes \mathbf{t} \,\big|\, \mathbf{t} \in \Delta^{t_0\boldsymbol{e}}_{\mathbf{k}}\} \; + \boldsymbol{d}_{\mathbf{k}} \tag{2.31}$$

such that

1. $\Delta^{4\boldsymbol{e}}_{\mathbf{k}}$, $\Delta^{t_0\boldsymbol{e}}_{\mathbf{k}}$, and $\boldsymbol{F}^{\mathbf{k}} = \overset{\star}{\boldsymbol{x}}^{\mathbf{k}}(\Delta^{t_0\boldsymbol{e}}_{\mathbf{k}})$ are (deGua) simplices if and only if

   $$\mathbf{k} \in \mathring{\mathbf{K}}^6 = \{(4,1,1), (3,2,1), (2,2,2) \text{ (and permutations) } \} \tag{2.32}$$

2. $\Delta^{4\boldsymbol{e}}_{\mathbf{k}}$, $\Delta^{t_0\boldsymbol{e}}_{\mathbf{k}}$, and $\boldsymbol{F}^{\mathbf{k}} = \overset{\star}{\boldsymbol{x}}^{\mathbf{k}}(\Delta^{t_0\boldsymbol{e}}_{\mathbf{k}})$ are rotated simplices if and only if

   $$\mathbf{k} \in \mathring{\mathbf{K}}^5 = \{(3,1,1), (2,2,1) \text{ (and permutations) } \} . \tag{2.33}$$

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For dimensions $n \geq 4$ it turns out that the Pareto faces of a taxoid are not necessarily simplices.

Example 2.6. Figure 2.3 represents the relevant polyhedra of $\Delta^{2\boldsymbol{e}}$ for $n = 4, K = 2$. A simplex is located in each vertex. E.g., the simplex

$$\Delta^{2\boldsymbol{e}}_{(2,1,1,1)} \;=\; \{\mathbf{t} \in \Delta^{2\boldsymbol{e}} \,\big|\, 1 \leq t_1 \leq 2, 0 \leq t_i \leq 1 (i = 2,3,4)\} \tag{2.34}$$

is located at the vertex $2\boldsymbol{e}^1$. This is a unit simplex and the Maschler–Perles measure is

$$\boldsymbol{\theta}(\Delta^{2\boldsymbol{e}}_{(2,1,1,1)}) \;=\; 1 . \tag{2.35}$$

However, the polyhedron

$$\Delta^{2\boldsymbol{e}}_{(1,1,1,1)} \;=\; \{\mathbf{t} \in \Delta^{2\boldsymbol{e}} \,\big|\, 0 \leq t_i \leq 1 (i = 1,2,3,4)\} \tag{2.36}$$

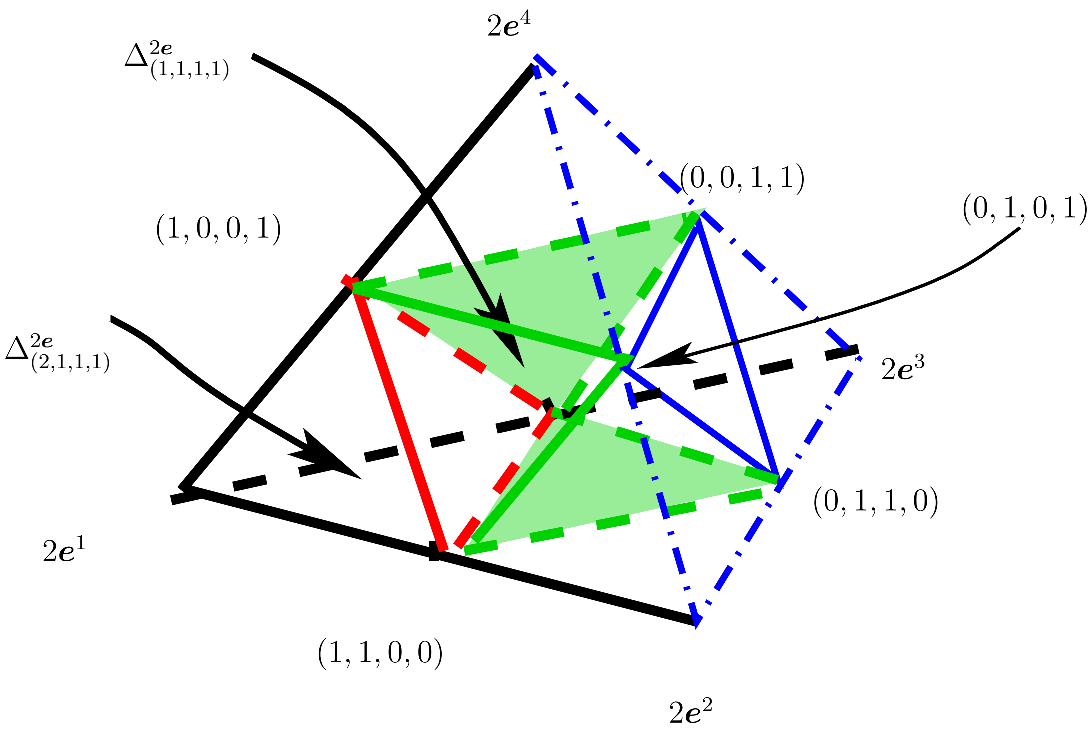

Figure 2.3: The octahedron Δ^2_{1111}

is not a simplex but an octahedron. The Maschler Perles volume of this octahedron is

$$\boldsymbol{\theta}(\Delta^{2e}_{(1,1,1,1)}) = 4 \ , \tag{2.37}$$

as the total measure of D^{2e} equals 8 and each of the simplices in the corner has measure 1.

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Example 2.7. Consider the case $n = 4$ and $K = 6$. We demonstrate the construction of simplices with unit Maschler–Perles measure filling the simplex Δ^{6e}.

1^st STEP : The simplex

$$\Delta^{6e}_{(6,1,1,1)} \tag{2.38}$$

has vertices

$$\begin{aligned} \boldsymbol{a}^1 &:= (6,0,0,0), & \boldsymbol{a}^2 &:= (5,1,0,0), \\ \boldsymbol{a}^3 &:= (5,0,1,0), & \boldsymbol{a}^4 &:= (5,0,0,1), \end{aligned} \tag{2.39}$$

this is the unit simplex located at the vertex $6\boldsymbol{e}^1$ of Δ^{6e} see Figure 2.4.

we have $\boldsymbol{e}\mathbf{k} = 9$ and $\boldsymbol{\theta}(\Delta^{6e}_{(6,1,1,1)}) = 1$.

Next,

$$\Delta^{6e}_{(5,2,1,1)} \tag{2.40}$$

has $\boldsymbol{e}\mathbf{k} = 9$ and vertices

$$\begin{array}{cc} (5,1,0,0) & (4,2,0,0) \\ (4,1,1,0) & (4,1,0,1) \end{array} \tag{2.41}$$

This is a unit simplex as well with $\boldsymbol{\theta}(\Delta^{6\boldsymbol{e}}_{(5,2,1,1)}) = 1$.

2^nd STEP : Next consider

$$\Delta^{6\boldsymbol{e}}_{(4,1,1,1)} \tag{2.42}$$

This simplex is a rotated version of the unit simplex, it can be identified in Figure 2.4. We obtain $\boldsymbol{e}\mathbf{k} = 7$ and vertices

$$\begin{array}{rclrcl} \boldsymbol{a}^{234} & = & (3,1,1,1), & \boldsymbol{a}^{134} & = & (4,0,1,1), \\ \boldsymbol{a}^{124} & = & (4,1,0,1), & \boldsymbol{a}^{123} & = & (4,1,1,0). \end{array} \tag{2.43}$$

Up to rotation, this is a unit simplex and the Maschler–Perles measure, therefore, equals

$$\boldsymbol{\theta}(\Delta^{6\boldsymbol{e}}_{(4,1,1,1)}) \ = \ 1 \ . \tag{2.44}$$

4^th STEP : Next consider

$$\Delta^{6\boldsymbol{e}}_{(3,2,1,1)} \tag{2.45}$$

which is rotated with $\boldsymbol{e}\mathbf{k} = 7$ and $\boldsymbol{\theta}(\Delta^{6\boldsymbol{e}}_{(3,2,1,1)}) = 1$. Vertices are

$$\begin{array}{cc} (3,2,1,0) & (3,2,0,1) \\ (3,1,1,1) & (2,2,1,1) \ . \end{array} \tag{2.46}$$

Similarly

$$\Delta^{6\boldsymbol{e}}_{(2,2,2,1)} \tag{2.47}$$

has $\boldsymbol{e}\mathbf{k} = 7$ and is rotated, thus $\boldsymbol{\theta}(\Delta^{6\boldsymbol{e}}_{(2,2,2,1)}) = 1$. Vertices are

$$\begin{array}{cc} (2,2,2,0) & (2,2,1,1) \\ (2,1,2,1) & (1,2,2,1) \end{array} \tag{2.48}$$

3^rd STEP :

Finally, we focus on

$$\Delta^{6\boldsymbol{e}}_{(5,1,1,1)} \tag{2.49}$$

whic has $\boldsymbol{e}\mathbf{k} = 8$ and vertices

$$\begin{array}{rclrcl} \boldsymbol{a}^{12} & := & (5,1,0,0), & \boldsymbol{a}^{13} & := & (5,0,1,0), \\ \boldsymbol{a}^{14} & := & (5,0,0,1), & \boldsymbol{a}^{23} & := & (4,1,1,0), \\ \boldsymbol{a}^{24} & := & (4,1,0,1), & \boldsymbol{a}^{34} & := & (4,0,1,1). \end{array} \tag{2.50}$$

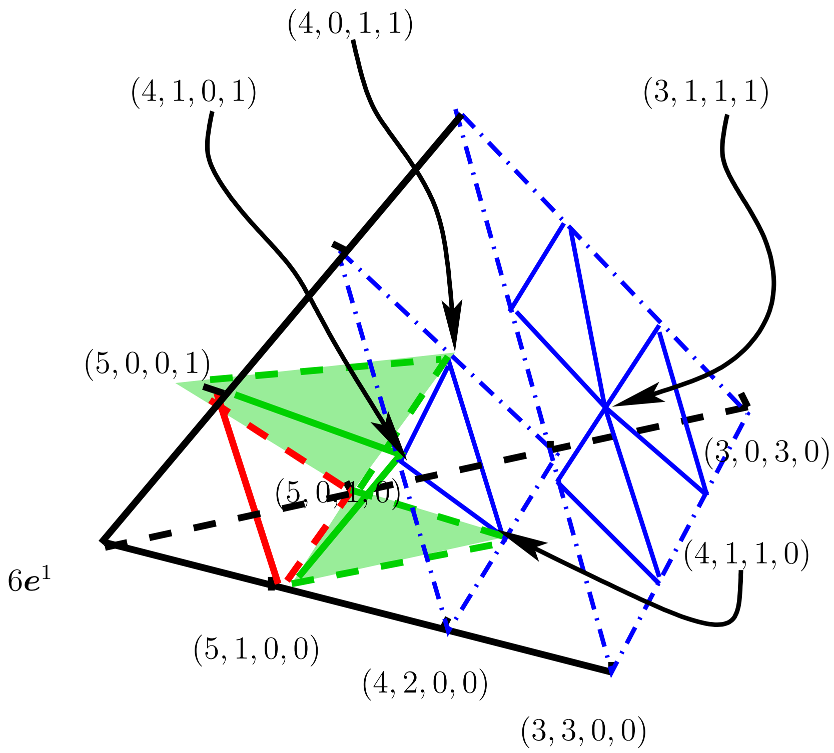

Figure 2.4: Focus on the octahedron

This is not a simplex. The vertices are depicted in Figure 2.4. It is seen that $\Delta^{6\boldsymbol{e}}_{(5,1,1,1)}$ is a regular octahedron . The Maschler–Perles measure of this polyhedron is

$$\boldsymbol{\theta}(\Delta^{6\boldsymbol{e}}_{(5,1,1,1)}) = 4 \ , \tag{2.51}$$

which we obtain as in Example 2.6.

We conclude; for $n = 4, K = 6$ we find that the relevant sequences are

$$\mathring{\mathbf{K}} = \mathbf{K}^7 \cup \mathbf{K}^8 \cup \mathbf{K}^9$$

such that

$$\mathbf{K}^7 = \{\mathbf{k} \,|\, \boldsymbol{e}\mathbf{k} = 7\} \tag{2.52}$$

contains rotated simplices,

$$\mathbf{K}^9 = \{\mathbf{k} \,|\, \boldsymbol{e}\mathbf{k} = 9\} \tag{2.53}$$

contains unit simplices, and

$$\mathbf{K}^8 = \{\mathbf{k} \,|\, \boldsymbol{e}\mathbf{k} = 8\} \tag{2.54}$$

contains octahedra.

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Based on these examples we now describe the Pareto surface of a taxoid.

Theorem 2.8. *[**The Pareto Surface of a Taxoid**]*

Let $\boldsymbol{f}$ be a tariff system and let $t_0 > K$. Let $\Pi^{\{\boldsymbol{f}\}} = \Pi^{\{\boldsymbol{f},t_0\}}$ be the taxoid generated.

1. *Let $\mathbf{k} \in \overset{\circ}{\mathbf{K}}$ be a relevant sequence. Then*

$$\boldsymbol{F}^{\mathbf{k}} = \overset{\star}{\boldsymbol{x}}^{\mathbf{k}}(\Delta_{\mathbf{k}}^{t_0\boldsymbol{e}}) = \boldsymbol{c}_{\mathbf{k}} \otimes \Delta_{\mathbf{k}}^{t_0\boldsymbol{e}} + \boldsymbol{d}_{\mathbf{k}} \tag{2.55}$$

   is a Pareto face of $\Pi^{\{\boldsymbol{f}\}}$. The relevant sequences describe exactly all Pareto faces.

2. *Because of*

$$\Delta^{t_0\boldsymbol{e}} = \bigcup_{\mathbf{k}\in\overset{\circ}{\mathbf{K}}} \Delta_{\mathbf{k}}^{t_0\boldsymbol{e}} \tag{2.56}$$

   the Pareto surface of $\Pi^{\{\boldsymbol{f}\}}$ is described by

$$\partial\Pi^{\{\boldsymbol{f}\}} = \bigcup_{\mathbf{k}\in\overset{\circ}{\mathbf{K}}} \boldsymbol{F}^{\mathbf{k}} . \tag{2.57}$$

3. *Let $\mathbf{k} \in \overset{\circ}{\mathbf{K}}$ be a relevant sequence. Then there exists a family of simplices $\left\{ \overset{\star}{\Delta}^{l} \,\middle|\, l \in \mathbf{L}^{\mathbf{k}} \right\}$ such that*

$$\Delta_{\mathbf{k}}^{t_0\boldsymbol{e}} = \bigcup_{l\in\mathbf{L}^{\mathbf{k}}} \overset{\star}{\Delta}^{l} \tag{2.58}$$

4. *For $l \in \mathbf{L}^{\mathbf{k}}$ let*

$$\overset{\star}{\boldsymbol{F}}^{l} := \overset{\star}{\boldsymbol{x}}^{\mathbf{k}}(\overset{\star}{\Delta}^{l}) = \mathbf{c}_{\mathbf{k}} \otimes \overset{\star}{\Delta}^{l} + \mathbf{d}_{\mathbf{k}} . \tag{2.59}$$

   Then $\overset{\star}{\boldsymbol{F}}^{l}$ is a simplex on $\partial\Pi^{\{\boldsymbol{f}\}}$ and

$$\boldsymbol{F}^{\mathbf{k}} = \bigcup_{l\in\mathbf{L}^{\mathbf{k}}} \overset{\star}{\boldsymbol{F}}^{l} \tag{2.60}$$

   holds true.

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Proof: No proof is offered. **q.e.d.**

3 The Maschler–Perles Solution

We wish to extend the solution concept of Maschler–Perles [4] as developed in [10] to tariff bargaining solutions. In a later version we will further extend this concept to tariff games. Within this section we start out with the (generalized) Maschler–Perles bargaining solution. Thus we focus on the bargaining problem $\boldsymbol{V}^{t_0} = (\boldsymbol{0}, \boldsymbol{U}^{\{t_0\}})$ with $\boldsymbol{U}^{\{t_0\}} = \Pi^{\{\boldsymbol{f},t_0\}}$ (Definition 1.9).

This version may be relevant whenever the countries involved in the bargaining process need to come up with a unanimous agreement and no data regarding the profit of the firm within the jurisdiction of subcoalitions are available. The solution will assign a distribution of payoffs $\boldsymbol{\mu}(\boldsymbol{V}^{\{t_0\}})$ to the players/countries involved. This amounts also to a share of the total revenue t_0.

Let $\bar{\boldsymbol{f}}$ and $\widehat{\boldsymbol{f}}$ be tariff systems and let $\bar{\Pi} = \Pi^{\{\bar{\boldsymbol{f}}\}}$ and $\widehat{\Pi} = \Pi^{\{\widehat{\boldsymbol{f}}\}}$ be the corresponding taxoids. For $n = 2$, both taxoids are cephoids (Section Example 2.3). The (algebraic) sum $\overset{\star}{\Pi} = \Pi^{\{\bar{\boldsymbol{f}}\}} + \Pi^{\{\widehat{\boldsymbol{f}}\}}$ is a cephoid as well and represented by some tariff system, say $\overset{\star}{\boldsymbol{f}}$. In general, the Pareto curve of $\overset{\star}{\Pi}$ will consist of $2K$ line segments, so $\overset{\star}{\boldsymbol{f}}$ is a tariff system constructed by means of the grid $\overset{\star}{\boldsymbol{K}} = \{1, \ldots, 2K\}$.

For $n = 2$ we know that the Maschler–Perles solution $\boldsymbol{\mu}$ is superadditive, that is, it satisfies

$$\boldsymbol{\mu}(\bar{\boldsymbol{V}}^{\boldsymbol{t_0}}) + \boldsymbol{\mu}(\widehat{\boldsymbol{V}}^{\boldsymbol{t_0}}) \leq \boldsymbol{\mu}(\bar{\boldsymbol{V}}^{\boldsymbol{t_0}} + \widehat{\boldsymbol{V}}^{\boldsymbol{t_0}}) = \boldsymbol{\mu}(\overset{\star}{\boldsymbol{V}}{}^{t_0}) \tag{3.1}$$

with the bargaining problem $\overset{\star}{\boldsymbol{V}}{}^{t_0}$ generated by the tariff system $\overset{\star}{\boldsymbol{f}}$.

Superadditivity is rationalized via the concept of von Neumann Morgenstern utility in Game Theoretical context. However, we provide an argument without referring to vNM-utility as provided in [10].

Assume that two groups of countries – let us call them **US** and **EU** – are engaged in two "remote" tariff bargaining problems $\boldsymbol{V}^1$ and $\boldsymbol{V}^2$ (one in Brussels and one in Washington) simultaneously. Initially, these are separate affairs of two sets of administrations. Each one regards liberalizing the markets and introducing tariffs for the mutual unions of countries. This way, joint standards are provided for firms operating in the complete area of both jurisdictions. Given a solution concept, in our case the Maschler–Perles solution $\boldsymbol{\mu}$, the representing officials settle for $\boldsymbol{\mu}(\boldsymbol{V}^1)$ and $\boldsymbol{\mu}(\boldsymbol{V}^2)$ respectively and separately.

Later on negotiations are initiated between the upper political echelons of the US and the EU concerning a joint tariff system and taxation of firms active in both unions. It is consent that the result should be advantageous for both parties, otherwise there would be no possible agreement. Both administrations consider concessions in $\boldsymbol{V}^1$ versus gains in $\boldsymbol{V}^2$ and *vice versa*. The utilities and the status quo point available are now $\{\boldsymbol{x}^1 + \boldsymbol{x}^2 | \boldsymbol{x}^1 \in \boldsymbol{U}^1, \boldsymbol{x}^2 \in \boldsymbol{U}^2\} =: \boldsymbol{U}^1 + \boldsymbol{U}^2$ and $\boldsymbol{0} + \boldsymbol{0} = \boldsymbol{0}$. That is, the countries involved face the bargaining problem $\boldsymbol{V}^1 + \boldsymbol{V}^2$. The solution is then $\boldsymbol{\mu}(\boldsymbol{V}^1 + \boldsymbol{V}^2)$.

Now, as the solution concept is superadditive, i.e., $\boldsymbol{\mu}(\boldsymbol{V}^1 + \boldsymbol{V}^2) \geq \boldsymbol{\mu}(\boldsymbol{V}^1) + \boldsymbol{\mu}(\boldsymbol{V}^2)$, then *all countries of both communities profit* from a *quid quo pro*.

For general n we cannot claim a version of superadditivity as the structure of taxoids does not allow for algebraic summation. However, the axiomatic justification of the Maschler–Perles solution in case of 2 dimensions is applied to justify generalization to higher dimensions, see the development for cephoids in [10].

Now we summarize shortly a version of the Maschler–Perles solution adapted to tariff bargaining solutions (and tariff games). To this end, let us first recall the definition of the surface measures involved.

Remark 3.1. The Maschler–Perles measure $\boldsymbol{\iota}_\Delta$ is defined for Cephoids and – by a limiting procedure – for smooth bodies, see [10], [11]. It is normalized such that $\boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{e}}) = 1$ holds true for the unit simplex and $\boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}}) = \sqrt[n]{\left(\prod_{i\in \boldsymbol{I}} a_i\right)^{n-1}}$ for a deGua simplex $\Delta^{\boldsymbol{a}}$. The Maschler Perles Measure $\boldsymbol{\iota}_\Delta$ is then defined for Cephoids by additive extension, i.e., via the particular structure of such polyhedra. By contrast, the deGua measure $\boldsymbol{\vartheta}$ is obtained by a standard calculus of measure and integral defined on certain surfaces. Formally, for some integrable function F defined on a surface Γ, a parametrization $\boldsymbol{x}(\bullet) : \boldsymbol{T} \to \Gamma$ with functional determinants D_i, we have

$$
\begin{aligned}
\int_{\partial\Gamma} F d\boldsymbol{\vartheta} &= \int_{\partial\Gamma} F \sqrt[n]{d\mathfrak{n}_1 \cdots d\mathfrak{n}_n} \\
&= \int_{\boldsymbol{T}} F(\boldsymbol{x}(\boldsymbol{t})) \sqrt[n]{D_1(\boldsymbol{x}(\boldsymbol{t})) \cdots D_n(\boldsymbol{x}(\boldsymbol{t}))} dt_1 \cdots dt_n \ .
\end{aligned}
\tag{3.2}
$$

Here, $\mathfrak{n}_1, \ldots, \mathfrak{n}_n$ describes a linear system of normals in direction of the axes. As the continuous version is not applied in our context, we do not elaborate on the subject. We refer to [12], in particular Lemma 3.8., for the details.

The relation between both "surface measures" for a deGua Simplex $\Delta^{\boldsymbol{a}}$ is given by

$$
\boldsymbol{\iota}_\Delta(\bullet) = \boldsymbol{o}(n)\boldsymbol{\vartheta}(\bullet) \quad \text{on} \quad \Delta^{\boldsymbol{a}} \ , \tag{3.3}
$$

with some constant $\boldsymbol{o}(n)$ depending on the dimension only. In order to avoid permanent reference to the factor $\boldsymbol{o}(n)$ we introduce a measure

$$
\boldsymbol{\theta} := \boldsymbol{o}(n)\boldsymbol{\vartheta} \ . \tag{3.4}
$$

Then we have $\boldsymbol{\theta} = \boldsymbol{\iota}_\Delta$ on cephoids and in particular for two dimensions on line segments. Hence, we continue calling $\boldsymbol{\theta}$ the Maschler–Perles measure. The extension to taxoids is then naturally given by (3.2) and does not require a reference to $\boldsymbol{\iota}_\Delta$.

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Based on our results in SECTION 2 we collect the data for taxoids.

Theorem 3.2. *[**The Maschler–Perles Measure**]*

Let $\boldsymbol{f}$ be a tariff system and let $t_0 > K$. Let $\Pi^{\{\boldsymbol{f}\}} = \Pi^{\{\boldsymbol{f},t_0\}}$ be the taxoid induced.

1. *For* $n = 2$ *and* $\mathsf{k} \in \overset{\circ}{\mathsf{K}}$, $\Delta_{\mathsf{k}}^{K\boldsymbol{e}}$ *is a unit simplex, consequently* $\boldsymbol{\theta}(\Delta_{\mathsf{k}}^{K\boldsymbol{e}}) = 1$. *Moreover, for some* $\mathsf{k} = (k, K-(k-1))$,

$$\boldsymbol{\theta}(\boldsymbol{F}^{\mathsf{k}}) = \sqrt[2]{a_1^{(k)} a_2^{(k)}} = \frac{t_0}{K}\sqrt[2]{c_{k_1}^1 c_{k_2}^2} = \frac{t_0}{K}\sqrt[2]{c_k^1 c_{K-(k-1)}^2}\,, \tag{3.5}$$

by (2.12) *and* (2.16).

2. *For* $n = 3$, *Lemma 2.5 characterizes the relevant sequences. The Maschler-Perles measure for* $\mathsf{k} \in \overset{\circ}{\mathsf{K}}$ *is*

$$\boldsymbol{\theta}(\Delta_{\mathsf{k}}^{K\boldsymbol{e}}) = 1,\quad \boldsymbol{\theta}(\Delta_{\mathsf{k}}^{t_0\boldsymbol{e}}) = \left(\frac{t_0}{K}\right)^2,\quad \boldsymbol{\theta}(\boldsymbol{F}^{\mathsf{k}}) = \left(\frac{t_0}{K}\right)^2 \sqrt[3]{\left(\prod_{i\in\boldsymbol{I}} c_{k_i}^i\right)^2} \tag{3.6}$$

3. *In general , by* (2.58) *and* (2.59), *a Pareto face* $\boldsymbol{F}^{\mathsf{k}}$ *of a taxoid* $\Pi^{\{\boldsymbol{f}\}}$ *is a union of simplices and the measure* $\boldsymbol{\theta}$ *for some* $\boldsymbol{F}^{\mathsf{k}}$ *is given by additivity. For amy relevant sequence* $\mathsf{k} \in \overset{\circ}{\mathsf{K}}$ *and the corresponding Pareto face* $\boldsymbol{F}^{\mathsf{k}} := \overset{\star}{\boldsymbol{x}}^{\mathsf{k}}(\Delta_{\mathsf{k}}^{t_0\boldsymbol{e}})$ *of* $\Pi^{\{\boldsymbol{f}\}}$, *let* $\left\{\overset{\star}{\Delta}^l \,\middle|\, l \in \mathsf{L}^{\mathsf{k}}\right\}$ *be the family provided by Theorem 2.8,* item 3.

The simplex $\overset{\star}{\Delta}^{\{l\}}$ *has Maschler–Perles measure*

$$\boldsymbol{\theta}(\overset{\star}{\Delta}^{\{l\}}) = \frac{t_0}{K}\,. \tag{3.7}$$

Accordingly,it follows that

$$\overset{\star}{\boldsymbol{F}}^l := \overset{\star}{\boldsymbol{x}}^{\mathsf{k}}(\overset{\star}{\Delta}^{\{l\}}) = \mathsf{c}_{\mathsf{k}} \otimes (\overset{\star}{\Delta}^l) + \mathsf{d}_{\mathsf{k}} \tag{3.8}$$

has Maschler–Perles measure

$$\begin{aligned}\boldsymbol{\theta}(\overset{\star}{\boldsymbol{F}}^l) &= \boldsymbol{\theta}(\mathsf{c}_{\mathsf{k}} \otimes (\overset{\star}{\Delta}^{\{l\}})) = \boldsymbol{\theta}(\mathsf{c}_{\mathsf{k}} \otimes \frac{t_0}{K}\Delta^{\{l\}}) \\ &= \left(\frac{t_0}{K}\right)^{n-1} \boldsymbol{\theta}(\mathsf{c}_{\mathsf{k}} \otimes \Delta^{\{l\}}) = \left(\frac{t_0}{K}\right)^{n-1} \sqrt[n]{\left(\prod_{i\in\boldsymbol{I}} (c_{k_i}^i)\right)^{n-1}}\,.\end{aligned} \tag{3.9}$$

4. *Hence, the Maschler–Perles measure of* $\boldsymbol{F}^{\mathsf{k}}$ *is*

$$\boldsymbol{\theta}(\boldsymbol{F}^{\mathsf{k}}) = \boldsymbol{\theta}(\bigcup_{l\in\mathsf{L}^{\mathsf{k}}} \boldsymbol{F}^{l\star}) = \left(\frac{t_0}{K}\right)^{n-1} |\mathsf{L}^{\mathsf{k}}| \sqrt[n]{\left(\prod_{i\in\boldsymbol{I}} (c_{k_i}^i)\right)^{n-1}}\,. \tag{3.10}$$

5. *Therefore, the Maschler–Perles measure of the Pareto surface of the taxoid is*

$$\boldsymbol{\theta}(\partial\Pi^{\{\boldsymbol{f}\}}) = \sum_{\mathsf{k}\in\overset{\circ}{\mathsf{K}}} \boldsymbol{\theta}(\boldsymbol{F}^{\{\mathsf{k}\}}) = \left(\frac{t_0}{K}\right)^{n-1} \sum_{\mathsf{k}\in\overset{\circ}{\mathsf{K}}} |\mathsf{L}^{\mathsf{k}}| \sqrt[n]{\left(\prod_{i\in\boldsymbol{I}} (c_{k_i}^i)\right)^{n-1}}\,.\,. \tag{3.11}$$

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Proof:

Obvious.

q.e.d.

We shorten our notation as follows.

Corollary 3.3. Let $t_0 \in \mathbb{R}_+$ and let $\boldsymbol{f}^{\{t_0\}}$ be a tariff system. For $\mathbf{k} \in \overset{\circ}{\mathbf{K}}$ define

$$\boldsymbol{\sigma}_{\mathbf{k}} := \left(\frac{t_0}{K}\right)^{n-1} |\mathbf{L}^{\mathbf{k}}| \sqrt[n]{\left(\prod_{i \in \boldsymbol{I}} (c^i_{k_i})\right)^{n-1}} = \left(\frac{t_0}{K}\right) \sqrt[n]{\left(\prod_{i \in \boldsymbol{I}} (c^i_{k_i})\right)} \; . \tag{3.12}$$

Then, for relevant $\mathbf{k} \in \overset{\circ}{\mathbf{K}}$ and $l \in \mathbf{L}$ we have

$$\boldsymbol{\theta}(\overset{\star}{\boldsymbol{F}}{}^{l}) = \boldsymbol{\sigma}_{\mathbf{k}}^{n-1} \; , \; \boldsymbol{\theta}(\boldsymbol{F}^{\mathbf{k}}) = |\mathbf{L}^{\mathbf{k}}| \boldsymbol{\sigma}_{\mathbf{k}}^{n-1} \; , \tag{3.13}$$

and hence

$$\boldsymbol{\theta}(\partial \Pi^{\{\boldsymbol{f}, t_0\}}) = \sum_{\mathbf{k} \in \overset{\circ}{\mathbf{K}}} |\mathbf{L}^{\mathbf{k}}| \boldsymbol{\sigma}_{\mathbf{k}}^{n-1} \tag{3.14}$$

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We follow the procedure for the Maschler Perles solution as in [12].

1. For $\mathbf{k} \in \overset{\circ}{\mathbf{K}}$ the measure $\boldsymbol{\theta}$ is transported from $\boldsymbol{F}^{\mathbf{k}} = \boldsymbol{F}^{\{\mathbf{k}, t_0\}}$ to $\Delta_{\mathbf{k}}^{t_0 \boldsymbol{e}}$ via $\left(\overset{\star}{\boldsymbol{x}}{}^{\mathbf{k}}(\bullet)\right)^{-1}$. As $\overset{\star}{\boldsymbol{x}}{}^{\mathbf{k}}(\bullet)$ acts contravariant on measures we write

$$\overset{\star}{\boldsymbol{\theta}}{}^{\mathbf{k}} := \boldsymbol{\theta} \circ \overset{\star}{\boldsymbol{x}}{}^{\mathbf{k}}(\bullet) := \left(\overset{\star}{\boldsymbol{x}}{}^{\mathbf{k}}(\bullet)\right)^{-1} \boldsymbol{\theta} \tag{3.15}$$

   The transported measure $\overset{\star}{\boldsymbol{\theta}}{}^{\mathbf{k}}$ on $\Delta_{\mathbf{k}}^{t_0 \boldsymbol{e}}$ has a constant density w.r.t the measure $\boldsymbol{\theta}$ – which, on $\Delta_{\mathbf{k}}^{t_0 \boldsymbol{e}}$ – is a multiple of the Lebesgue measure. Thus,

$$\overset{\star}{\boldsymbol{\theta}}(\Delta_{\mathbf{k}}^{t_0 \boldsymbol{e}}) = \boldsymbol{\theta}(\boldsymbol{F}^{\{\mathbf{k}, t_0\}}) = |\mathbf{L}^{\mathbf{k}}| \boldsymbol{\sigma}_{\mathbf{k}}^{n-1} \; , \tag{3.16}$$

   as seen in (3.13).

2. The barycenter of $\Delta_{\mathbf{k}}^{t_0 \boldsymbol{e}}$ w.r.t. $\boldsymbol{\theta}$ is

$$\overset{\star}{\beta}{}^{\mathbf{k}} := \frac{1}{|\mathbf{L}^{\mathbf{k}}|} \int_{\Delta_{\mathbf{k}}^{t_0 \boldsymbol{e}}} \mathbf{t} \; \boldsymbol{\theta}(d\mathbf{t}) \; , \tag{3.17}$$

3. This quantity is also the barycenter of $\Delta_{\mathsf{k}}^{t_0 e}$ w.r.t to the transported measure $\overset{\star}{\boldsymbol{\theta}}{}^{\mathsf{k}}$. Indeed, for $\mathsf{k} \in \overset{\circ}{\mathsf{K}}$, we have

$$
\begin{aligned}
\overset{\star}{\boldsymbol{\beta}}{}^{\mathsf{k}} &= \frac{1}{|\mathsf{L}^{\mathsf{k}}|} \int_{\Delta_{\mathsf{k}}^{t_0 e}} \mathbf{t}\ \boldsymbol{\theta}(d\mathbf{t}) = \frac{1}{\boldsymbol{\theta}(\Delta_{\mathsf{k}}^{t_0 e})} \int_{\Delta_{\mathsf{k}}^{t_0 e}} \mathbf{t}\ \boldsymbol{\theta}(d\mathbf{t})\ , \\
&= \frac{1}{\overset{\star}{\boldsymbol{\theta}}{}^{\mathsf{k}}(\Delta_{\mathsf{k}}^{t_0 e})} \int_{\Delta_{\mathsf{k}}^{t_0 e}} \mathbf{t}\ \overset{\star}{\boldsymbol{\theta}}{}^{\mathsf{k}}(d\mathbf{t})\ ,
\end{aligned}
\tag{3.18}
$$

as the density factor cancels out.

4. Therefore, the barycenter of the total transported mass on $\Delta^{t_0 e}$ is

$$
\begin{aligned}
\overset{\star}{\boldsymbol{\beta}} &:= \sum_{\mathsf{k} \in \overset{\circ}{\mathsf{K}}} \overset{\star}{\boldsymbol{\theta}}{}^{\mathsf{k}}(\Delta_{\mathsf{k}}^{t_0 e}) \frac{\overset{\star}{\boldsymbol{\beta}}{}^{\mathsf{k}}}{\sum_{\mathsf{k} \in \overset{\circ}{\mathsf{K}}} \overset{\star}{\boldsymbol{\theta}}{}^{\mathsf{l}}(\Delta_{\mathsf{l}}^{t_0 e})} \\
&= \frac{1}{\sum_{\mathsf{l} \in \mathsf{K}} \boldsymbol{\sigma}_{\mathsf{l}}} \sum_{\mathsf{k} \in \overset{\circ}{\mathsf{K}}} \boldsymbol{\sigma}_{\mathsf{k}} \int_{\Delta_{\mathsf{k}}^{t_0 e}} \mathbf{t}\ \boldsymbol{\theta}(d\mathbf{t})
\end{aligned}
\tag{3.19}
$$

5. Finally, the Maschler–Perles solution to the bargaining problem $\boldsymbol{V}^{t_0} = (\mathbf{0}, \boldsymbol{U}^{t_0})$ is obtained by a transporting backwards to $\partial \boldsymbol{U}^{\{t_0\}}$. More precisely, it is the image of the barycenter $\overset{\star}{\boldsymbol{\beta}}$ under the mapping $\overset{\star}{\boldsymbol{x}}(\bullet)$, i.e.,

$$
\begin{aligned}
\boldsymbol{\mu}(\boldsymbol{V}^{t_0}) &:= \overset{\star}{\boldsymbol{x}}(\overset{\star}{\boldsymbol{\beta}}) = \overset{\star}{\boldsymbol{x}} \left(\frac{1}{\sum_{\mathsf{l} \in \overset{\circ}{\mathsf{K}}} \boldsymbol{\sigma}_{\mathsf{l}}} \sum_{\mathsf{k} \in \overset{\circ}{\mathsf{K}}} \boldsymbol{\sigma}_{\mathsf{k}} \int_{\Delta_{\mathsf{k}}^{t_0 e}} \mathbf{t}\ \boldsymbol{\theta}(d\mathbf{t}) \right) \\
&= \overset{\star}{\boldsymbol{x}} \left(\frac{1}{\boldsymbol{\theta}(\partial \boldsymbol{U}^{\{t_0\}})} \int_{\partial \boldsymbol{U}^{\{t_0\}}} \left(\overset{\star}{\boldsymbol{x}}(\bullet) \right)^{-1} d\boldsymbol{\theta}(\bullet) \right) .
\end{aligned}
\tag{3.20}
$$

We mention that the above version is exactly the one presented and elaborated in [12]. The factor $\boldsymbol{o}(n)$ that appeared previously cancels in the above formulae, so we may use the deGua measure $\boldsymbol{\vartheta}$ as well as the Maschler–Perles $\boldsymbol{\iota}_{\Delta}$ measure within the above context. Using $\boldsymbol{\theta}$ unifies our notation.

The subsequent example is generated by a linear tariff system. The result obtained is then analogous to side payment bargaining solution with constant rates of utility transfer. This result is not exactly an innovation compared to the profit game as is represents just the embedding of a TU game into the NTU framework. However it serves as a convenient test to scrutiny the consistency of our concepts.

Example 3.4. A simple example is provided by a ***linear tariff system*** which results in a ***hyperplane NTU game***. The situation is quite analogous to the one in the context of Cephoid Theory, see [9],[10] (in particular Example 2.2 in

CHAPTER XIV). We obtain an NTU game $\boldsymbol{V}^{\{t_0\}}$ which reflects a side payment game with constant rates of utility transfer.

The result obtained is analogous to side payment bargaining solution with constant rates of utility transfer. This result is not exactly progress compared to the profit game as is represents just the embedding of a TU game into the NTU framework. However it serves as a convenient test to scrutiny the consistency of our concepts. More elaborate examples will be left for separate treatment.

We start out with $n \leq 3$. Then $|L^{\mathbf{k}}| = 1$ holds true for any relevant sequence $\mathbf{k}$ (see SEC 2, Examples 2.3 and 2.4).

Recall (2.57) and (2.55). With regard to

$$\partial\Pi^{\{\boldsymbol{f}\}} = \bigcup_{\mathbf{k}\in\overset{\circ}{\mathbf{K}}} \boldsymbol{F}^{\mathbf{k}} . \tag{3.21}$$

with

$$\boldsymbol{F}^{\mathbf{k}} = \overset{\star}{\boldsymbol{x}}^{\mathbf{k}}(\Delta_{\mathbf{k}}^{t_0\boldsymbol{e}}) = \boldsymbol{c}_{\mathbf{k}} \otimes \Delta_{\mathbf{k}}^{t_0\boldsymbol{e}} + \boldsymbol{d}_{\mathbf{k}} \quad (\mathbf{k} \in \overset{\circ}{\mathbf{K}}) , \tag{3.22}$$

we observe that every $\Delta_{\mathbf{k}}^{t_0\boldsymbol{e}}$ and hence every $\boldsymbol{F}^{\mathbf{k}}$ is a simplex as the decomposition suggested by Theorem 2.8 *item 3* and formula (2.58) degenerates to a simpleton.

Assume now for the "linear case" that all coefficients $\boldsymbol{c}_{\bullet}^{i}$ are equal, say $\boldsymbol{c}_{k}^{\bullet} = \boldsymbol{c}_{o}^{\bullet}$ $(k \in \boldsymbol{K})$ so that each $\boldsymbol{f}_{i}^{\{t_0\}}$ is indeed a linear function with slope c_{o}^{i}. then for every relevant sequence $\mathbf{k} = (k_1, \ldots, k_n)$ we write

$$\mathbf{c}_{\mathbf{k}} = (c_{k_1}^{1}, \ldots, c_{k_n}^{n}) = (c_{o}^{1}, \ldots, c_{o}^{n}) = \mathbf{c}_{\mathbf{o}} . \tag{3.23}$$

Next, by (2.56) and (2.57) and (2.55) we derive for the Pareto surface

$$\begin{aligned} \partial\Pi^{\{\boldsymbol{f}\}} &= \bigcup_{\mathbf{k}\in\overset{\circ}{\mathbf{K}}} \boldsymbol{F}^{\mathbf{k}} = \bigcup_{\mathbf{k}\in\overset{\circ}{\mathbf{K}}} \overset{\star}{\boldsymbol{x}}^{\mathbf{k}}(\Delta_{\mathbf{k}}^{t_0\boldsymbol{e}}) = \bigcup_{\mathbf{k}\in\overset{\circ}{\mathbf{K}}} \boldsymbol{c}_{\mathbf{k}} \otimes \Delta_{\mathbf{k}}^{t_0\boldsymbol{e}} + \boldsymbol{d}_{\mathbf{k}} \\ &= \mathbf{c}_{\mathbf{o}} \otimes \bigcup_{\mathbf{k}\in\overset{\circ}{\mathbf{K}}} \Delta_{\mathbf{k}}^{t_0\boldsymbol{e}} + \boldsymbol{d}_{\mathbf{o}} = \mathbf{c}_{\mathbf{o}} \otimes \Delta^{t_0\boldsymbol{e}} + \boldsymbol{d}_{\mathbf{o}} = \Delta^{t_0\mathbf{c}_{\mathbf{o}}} + \boldsymbol{d}_{\mathbf{o}} , \end{aligned} \tag{3.24}$$

with a sight abuse of notation regarding $\boldsymbol{d}_{\mathbf{o}}$. In other words, the Pareto surface (up to a shift with $\boldsymbol{d}_{\mathbf{o}}$ degenerates into a simplex representing constant transfer of utilities dictated by $\mathbf{c}_{\mathbf{o}}$.

It is clear that the Maschler–Perles solution is the midpoint of this simplex, i.e., for $\boldsymbol{V}^{\{t_0\}} = (\mathbf{0}, \boldsymbol{U}^{\{t_0\}})$ we have

$$\boldsymbol{\mu}(\boldsymbol{V}^{\{t_0\}}) = t_0 \left(\frac{1}{\boldsymbol{c}_{o}^{1}}, \ldots, \frac{1}{\boldsymbol{c}_{o}^{n}} \right) + \boldsymbol{d}_{\mathbf{o}} ; \tag{3.25}$$

some formal computations are e.g.listed in Example 5.3 of [12].

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Essentially, the example demonstrates that, in a unanimous taxation bargaining problem, all players/countries will get an equal share - adjusted to transfer rates if need be. This is not a surprising result, we will have to provide stronger examples to support our concept. This we leave to further work in progress.

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