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The Shapley NTU–Value via Surface Measures

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Abstract

We introduce the Maschler–Perles–Shapley value for NTU games composed by smooth bodies. This way we extend the M–P–S value established for games composed by Cephoids ("sums of deGua Simplices”).

The development is parallel to the one of the (generalized) Maschler–Perles bargaining solution. For Cephoidal bargaining problems this concept is treated in ([4], [11]). It is extended to smooth bargaining problems by the construction of surface measures. Such measures generalize the Maschler–Perles approach in two dimensions via a line integral – what the authors call their “donkey cart” ([6], [11]).

The Maschler–Perles–Shapley value for Cephoidal NTU Games extends the Cephoidal approach to Non Transferable Utility games with feasible sets consisting of Cephoids. The presentation is found in [10] and [11]. Using these results we formulate the Maschler–Perles–Shapley value for smooth NTU games.

We emphasize the intuitive justification of our concepts. The original Maschler–Perles approach is based on the axiom of superadditivity which we rate much more appealing than competing axioms like IIA etc. As a consequence, the construction of a surface measure (Maschler–Perles’ line integral) is instigated which renders concessions and gains of players during the bargaining process to be represented in a common space of “adjusted utility”.

Within this utility space side payments – transfer of utils – are feasible interpersonally as well as intrapersonally. Therefore, the barycenter/midpoint of the adjusted utility space is the natural base for the solution concept. This corresponds precisely to the Maschler–Perles “donkey card” reaching the solution by calling for equal concessions in terms of their line integral.

In addition, the adjusted utility space carries an obvious linear structure – thus admitting expectations in the sense of the Shapley value or “von Neumann–Morgenstern utility”. Consequently, we obtain a generally acceptable concept for bargaining problems as well as NTU games in the Cephoidal and in the smooth domain.

We collect the details of this reasoning along the development of our theory in Remarks 1.7., 2.6., and 3.2.. These remarks constitute a comprehensive view on M–P–S concepts for Cephoidal and smooth NTU games.

1 Introduction: Cephoids and the Maschler–Perles Envinronment

The notation within this paper equals the one used in [10],[12] see also [3], [4]. We denote $\boldsymbol{I} := \{1, \ldots, n\}$ to be the set of coordinates of $\mathbb{R}^n$, the positive orthant is $\mathbb{R}^n_+ := \{\boldsymbol{x} = (x_1, \ldots, x_n) \mid x_i \geq 0 \ , (i \in \boldsymbol{I})\}$. $\boldsymbol{e}^i$ denotes the i^{th} unit vector of $\mathbb{R}^n$ and $\boldsymbol{e} := (1, \ldots, 1) = \sum_{i=1}^n \boldsymbol{e}^i \in \mathbb{R}^n$ the "diagonal" vector.

The notation ***CovH*** A is used to denote the ***convex hull*** of a subset A of $\mathbb{R}^n_+$. Also ***CmpH*** A is the ***comprehensive hull*** of a set $A \subseteq \mathbb{R}^n_+$.

For $\boldsymbol{a} = (a_1, \ldots, a_n) > \boldsymbol{0} \in \mathbb{R}^n_+$, we define the multiples $\boldsymbol{a}^i := a_i \boldsymbol{e}^i \ \ (i \in \boldsymbol{I})$ of the unit vectors. Then

$$(1.1) \qquad \Delta^{\boldsymbol{a}} := \boldsymbol{CovH} \left\{ \boldsymbol{a}^1, \ldots, \boldsymbol{a}^n \right\}$$

is the ***Simplex*** resulting from $\boldsymbol{a}$ (we use capitals in this context). Slightly different,

$$(1.2) \qquad \Pi^{\boldsymbol{a}} := \boldsymbol{CovH} \left\{ \boldsymbol{0}, \boldsymbol{a}^1, \ldots, \boldsymbol{a}^n \right\} = \boldsymbol{CmpH}\, \Delta^{\boldsymbol{a}} \ .$$

is the ***deGua Simplex*** associated to $\boldsymbol{a}$.

For $\boldsymbol{J} \subseteq \boldsymbol{I}$ we write $\mathbb{R}^n_{\boldsymbol{J}} := \{\boldsymbol{x} \in \mathbb{R}^n \mid x_i = 0 \ (i \notin \boldsymbol{J})\}$. Accordingly, we obtain the ***Subsimplex*** $\Delta^{\boldsymbol{a}}_{\boldsymbol{J}} := \Delta^{\boldsymbol{a}} \cap \mathbb{R}^n_{\boldsymbol{J}}$ of $\Delta^{\boldsymbol{a}}$ and the ***deGua Subsimplex*** $\Pi^{\boldsymbol{a}}_{\boldsymbol{J}} = \Pi^{\boldsymbol{a}} \cap \mathbb{R}^n_{\boldsymbol{J}}$ of $\Pi^{\boldsymbol{a}}$

Figure 1.1 indicates the deGua Simplex $\Pi^{\boldsymbol{a}}$ generated by $\boldsymbol{a}$.

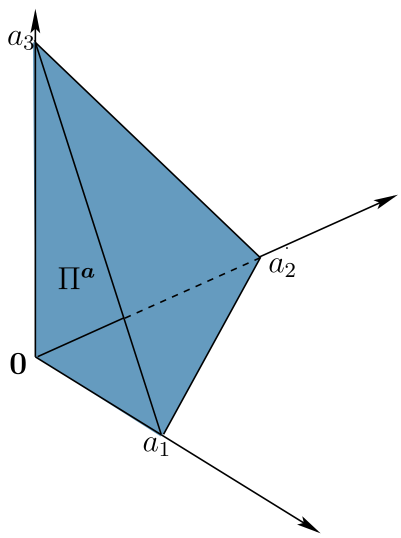

Figure 1.1: The deGua Simplex $\Pi^{\boldsymbol{a}}$; $\boldsymbol{a} = (a_1, a_2, a_3)$

The Simplex $\Delta^{\boldsymbol{a}}$ is the ***maximal (outward) face*** of $\Pi^{\boldsymbol{a}}$ In the terminology of Convex Analysis. Here we call $\Delta^{\boldsymbol{a}}$ the ***Pareto face*** of $\Pi^{\boldsymbol{a}}$.

A ***Cephoid*** is a *Minkowski sum of deGua Simplices*, precisely:

Definition 1.1. Let $\boldsymbol{K} = \{1, \ldots, K\}$ and let $\left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ denote a family of positive vectors. The Minkowski (or algebraic) sum

$$\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}} \tag{1.3}$$

is called a ***Cephoid***.

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In Figure 1.2 (the Cephoid "Odot" : ⊙) we depict the Pareto surface of a sum of 4 deGua Simplices in 3 dimensions. One of the deGua Simplices involved (brown) is sketched positioned in the origin. The copy on the Pareto surface is located in central position. This copy is the sum of three vertices and the brown deGua Simplex. Also, there appear 6 "rhombi" which are sums of subsimplices and vertices of the four deGua Simplices involved.

The relative position of the Pareto faces is what matters in most structural features of a Cephoid. Pareto faces form a lattice $\mathcal{V}$ representing the structure of the Pareto surface.

In order to describe this lattice we construct a *representation* which is an image $\mathcal{V}^0$ located in a K-fold multiple of the unit Simplex $K\Delta^{\boldsymbol{e}}$. Figure 1.3 is a typical example; it represents the lattice structure of the Cephoid "Odot".

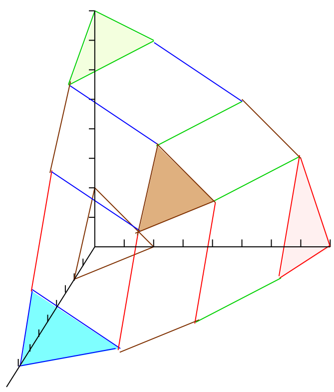

Figure 1.2: "Odot": a sum of 4 deGua Simplices

The inverse mapping of a representation constitutes a *parametrization* of the Pareto surface $\partial\Pi$ of a Cephoid Π (see [10] and [12]) This viewpoint is appropriate whenever we want to discuss measure and integration on the Pareto surface of a Cephoid.

For more detail, we recall the construction of the canonical parametrization $\boldsymbol{x}(\bullet)$ which is the inverse of the canonical representation ([10] CHAPTER I, DEFINITION 2.1, CHAPTER II, and [12]).

For short, any vertex $\boldsymbol{u} \in \partial\Pi$ can be represented uniquely as a sum of vertices of the Simplices involved, i.e.,

$$\boldsymbol{u} = \boldsymbol{a}^{\mathbf{i}_\bullet} := \sum_{k\in\boldsymbol{K}} \boldsymbol{a}^{(k)\mathbf{i}_k}. \tag{1.4}$$

This way it corresponds uniquely to a vector

$$\boldsymbol{u}^0 \;:=\; \sum_{k\in\boldsymbol{K}} \boldsymbol{a}^{0(k)\mathbf{i}_k} \in K\Delta^{\boldsymbol{e}} \;. \tag{1.5}$$

The parametrization maps bijectively

$$\boldsymbol{x}(\bullet) \;:=\; \boldsymbol{u}^0 \to \boldsymbol{u} \;. \tag{1.6}$$

The mapping is extended to Pareto faces via their extremals (vertices) and, accordingly, to the complete lattice of Pareto faces constituting $\partial\Pi$. This way one obtains an isomorphism between lattices $\mathcal{V}$ of Pareto faces of $\partial\Pi$ and the corresponding lattice $\mathcal{V}^0$ in $K\Delta^{\boldsymbol{e}}$.

Definition 1.2. Let $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ be a Cephoid. Let $\boldsymbol{a}^{0,(k)} = \boldsymbol{e}$ $(k \in \boldsymbol{K})$ be a family of copies of the diagonal vector $\boldsymbol{e}$ and let

$$K\Delta^{\boldsymbol{e}} \;=\; \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{0,(k)}}. \tag{1.7}$$

Let

$$\boldsymbol{x}(\bullet) \;:\; K\Delta^{\boldsymbol{e}} \mapsto \partial\Pi \tag{1.8}$$

be the bijection that preserves the lattice structure of Pareto faces of $\partial\Pi$ and of the corresponding polyhedra in $K\Delta^{\boldsymbol{e}}$. Then $(K\Delta^{\boldsymbol{e}}, \boldsymbol{x}(\bullet))$ is called the ***canonical parametrization***.

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Example 1.3. Figure 1.3 represents the lattice structure of the Cephoid "Odot" in Figure 1.2 (a sum of $K = 4$ deGua Simplices in $n = 3$ dimensions) within the Simplex $4\Delta^{\boldsymbol{e}}$. The lattice $\mathcal{V}^0$ is visualized demonstrating the isomorphism of the Pareto surfaces of Figure 1.2 and Figure 1.3.

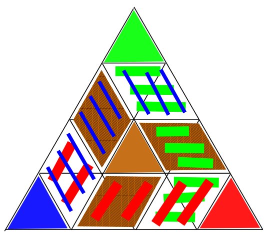

Figure 1.3: The lattice $\mathcal{V}^0$ on $4\Delta^{\boldsymbol{e}}$

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In our present context – bargaining solutions and the extended Shapley value – Cephoids are interpreted within the framework of cooperative NTU games as *bargaining problems*. Then, $\boldsymbol{I}$ is interpreted as the set of *players* and $\underline{\underline{\mathbf{P}}} = \{S \mid S \subseteq \boldsymbol{I}\} = \mathcal{P}(\boldsymbol{I})$ denotes the set of *coalitions*. Seen as a bargaining problem, a Cephoid Π allows players to distribute/receive utility according to a vector of $\partial\Pi$ upon agreement. If no agreement can be reached, then they will be reduced to the utility vector $\mathbf{0}$.

In the context of the Maschler–Perles solution and its extensions the notion of surface measures for Cephoids (and smooth bodies respectively) plays a central role. The original version of Maschler and Perles contains such idea, rudimentary expressed via the "speed" of their "donkey card" travelling on the Pareto curve of a (two dimensional) bargaining problem. More generally, surface measures are defined on the Pareto surface of a convex body. We focus our interest on two versions of a surface measure, the *Maschler–Perles surface measure* and the *deGua surface measure*.

We start out with a short introduction to the Maschler–Perles surface measure, its interpretation and applicability. Based on this concept we will present a more refined version of a parametrization, called "measure preserving".

Definition 1.4. 1. For $\mathbf{0} < \boldsymbol{a} \in \mathbb{R}^n_+$ the adjustment factor is

$$\boldsymbol{\tau}_{\Pi^{\boldsymbol{a}}} = \boldsymbol{\tau}_{\boldsymbol{a}} := \sqrt[n]{\prod_{i \in \boldsymbol{J}} a_i}. \tag{1.9}$$

For a family $\boldsymbol{a}^{\bullet} = \left\{\boldsymbol{a}^{(k)}\right\}_{k \in \boldsymbol{K}}$ of positive vectors and the corresponding Cephoid $\Pi^{\boldsymbol{a}^{\bullet}} = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ this notion is extended via

$$\boldsymbol{\tau}_{\Pi^{\boldsymbol{a}^{\bullet}}} = \boldsymbol{\tau}_{\boldsymbol{a}^{\bullet}} := \sum_{k \in \boldsymbol{K}} \boldsymbol{\tau}_{\boldsymbol{a}^{k}} , \tag{1.10}$$

see SECTION 2, CH XII of [10].

2. For positive $\boldsymbol{a} \in \mathbb{R}^n_+$ the ***Maschler–Perles measure*** assigned to $\Delta^{\boldsymbol{a}}$ is

$$\boldsymbol{\iota}_{\Delta}(\Delta^{\boldsymbol{a}}) := \boldsymbol{\tau}_{\boldsymbol{a}}^{n-1} . \tag{1.11}$$

In particular

$$\boldsymbol{\iota}_{\Delta}(\Delta^{\boldsymbol{e}}) = 1 . \tag{1.12}$$

Thus, the measure is normalized on the unit Simplex.

3. Let $\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ be Cephoid and let

$$\boldsymbol{F} = \sum_{k \in \boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \tag{1.13}$$

be a Pareto face with reference system $\mathfrak{J} = \left\{ \boldsymbol{J}^{(k)} \right\}_{k \in \boldsymbol{K}}$. Then the ***M–P measure*** of $\boldsymbol{F}$ is given by

$$\boldsymbol{\iota}_{\Delta}(\boldsymbol{F}) = c_{\mathfrak{J}} \sqrt[n]{\left[\prod_{i \in \boldsymbol{J}^{(1)}} a_i^1 \right]^{j_1 - 1} \cdot \ldots \cdot \left[\prod_{i \in \boldsymbol{J}^{(K)}} a_i^K \right]^{j_K - 1}} \tag{1.14}$$

with certain "normalizing coefficients" $c_{\mathfrak{J}}$.

For details and motivation see [10].

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We observe, that the relation (1.11) extends to the full Pareto surface of a Cephoid. More precisely, we have

Lemma 1.5. Let $\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ be a Cephoid, then

$$\boldsymbol{\tau}_{\Pi}^{n-1} \;=\; \boldsymbol{\iota}_{\Delta}(\Pi) \tag{1.15}$$

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Proof: This follows from the exposition concerning the Maschler–Perles measure for the Pareto faces of a Cephoid as presented in SECTION 2, CH XII of [10]. Accordingly, to any Pareto face

$$\boldsymbol{F} \;=\; \sum_{k \in \boldsymbol{K}} \Delta_{\boldsymbol{J}^{(k)}}^{(k)} \tag{1.16}$$

with reference system $\mathfrak{J} = \left\{ \boldsymbol{J}^{(k)} \right\}_{k \in \boldsymbol{K}}$ there is a suitable coefficient $c_{\mathfrak{J}}$ such that the Maschler–Perles surface measure of a Pareto face is given by

$$\boldsymbol{\iota}_{\Delta}(\boldsymbol{F}) = c_{\mathfrak{J}} \sqrt[n]{\left[\mathbf{P}_{\boldsymbol{I}}^{(1)} \right]^{j_1 - 1} \cdot \ldots \cdot \left[\mathbf{P}_{\boldsymbol{I}}^{(K)} \right]^{j_K - 1}} \tag{1.17}$$

with

$$\mathbf{P}_{\boldsymbol{I}}^{(k)} \;:=\; \mathbf{P}_{\boldsymbol{I}}^{\boldsymbol{a}^{(k)}} \;:=\; \prod_{i \in \boldsymbol{I}} a_i^{(k)} \quad (k \in \boldsymbol{K}) \; .$$

Denoting the collection of Pareto faces of Π by $\mathfrak{F}$, we obtain

$$\begin{aligned} \boldsymbol{\iota}_{\Delta}(\Pi) \;&=\; \sum_{\boldsymbol{F} \in \mathfrak{F}} \boldsymbol{\iota}_{\Delta}(\boldsymbol{F}) \\ &=\; \sum_{\boldsymbol{F} \in \mathfrak{F}} c_{\mathfrak{J}} \sqrt[n]{\left[\mathbf{P}_{\boldsymbol{I}}^{(1)} \right]^{j_1 - 1} \cdot \ldots \cdot \left[\mathbf{P}_{\boldsymbol{I}}^{(K)} \right]^{j_K - 1}} \\ &=\; \sum_{\boldsymbol{F} \in \mathfrak{F}} c_{\mathfrak{J}} \sqrt[n]{\boldsymbol{\tau}_{\boldsymbol{a}^{(1)}}^{j_1 - 1} \cdots \boldsymbol{\tau}_{\boldsymbol{a}^{(k)}}^{j_K - 1}} \\ &=\; \left(\boldsymbol{\tau}_{\boldsymbol{a}^{(1)}} + \ldots + \boldsymbol{\tau}_{\boldsymbol{a}^{(k)}} \right)^{n-1} \;=\; \boldsymbol{\tau}_{\Pi}^{n-1} \end{aligned} \tag{1.18}$$

q.e.d.

The *measure preserving parametrization* $\widehat{\boldsymbol{x}}(\bullet)$ is a variant of the canonical parametrization taking into account in addition the Maschler–Perles measure of the various Pareto faces. We proceed similar as in Definition 1.2.

The mapping $\widehat{\boldsymbol{x}}(\bullet)$ is arranged analogously to the construction of the canonical version $\boldsymbol{x}(\bullet)$. However, we choose $\widehat{\boldsymbol{a}}^{(k)} := \boldsymbol{\tau}_{\boldsymbol{a}^{(k)}}\boldsymbol{e}$ $(k \in \boldsymbol{K})$ to be a family of multiples of the diagonal vector $\boldsymbol{e}$.

Then, as previously, any vertex $\boldsymbol{u} \in \partial\Pi$ can be represented uniquely as a sum of vertices of the Simplices involved, i.e.,

$$\boldsymbol{u} \;=\; \sum_{k\in\boldsymbol{K}} \boldsymbol{a}^{(k)\mathbf{i}_k}. \tag{1.19}$$

This corresponds uniquely to a vector

$$\widehat{\boldsymbol{u}} \;:=\; \sum_{k\in\boldsymbol{K}} \widehat{\boldsymbol{a}}^{(k)\mathbf{i}_k} \in \boldsymbol{\tau}_{\Pi}\Delta^{\boldsymbol{e}}\;. \tag{1.20}$$

Now the (measure preserving) representation $\widehat{\boldsymbol{\kappa}}$ maps

$$\widehat{\boldsymbol{\kappa}} \;:=\; \boldsymbol{u} \;\to \widehat{\boldsymbol{u}} \tag{1.21}$$

and the (mesaure preserving) parametrization maps bijectively

$$\widehat{\boldsymbol{x}}(\bullet) \;:=\; \widehat{\boldsymbol{u}} \to \boldsymbol{u}\;. \tag{1.22}$$

The mappings are extended to Pareto faces via their extremals (vertices) and, accordingly, to the complete lattice of Pareto faces constituting $\partial\Pi$. This way one obtains an isomorphism between lattices $\mathcal{V}$ of Pareto faces of $\partial\Pi$ and the corresponding lattice $\widehat{\mathcal{V}}$ in $\boldsymbol{\tau}_{\Pi}\Delta^{\boldsymbol{e}}$. We repeat the formal definition:

Definition 1.6. Let $\left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ be a family of of positive vectors and let $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ be the Cephoid generated. For $k \in \boldsymbol{K}$ let

$$\widehat{\Pi}^{(k)} = \boldsymbol{\tau}_{\boldsymbol{a}^{(k)}}\Pi^{\boldsymbol{e}} \quad , \quad \widehat{\Delta}^{(k)} := \boldsymbol{\tau}_{\boldsymbol{a}^{(k)}}\Delta^{\boldsymbol{e}} \quad , \quad (k \in \boldsymbol{K}) \tag{1.23}$$

such that

$$\boldsymbol{\lambda}(\widehat{\Delta}^{(k)}) = \boldsymbol{\tau}_{\boldsymbol{a}^{(k)}}\boldsymbol{\lambda}(\Delta^{\boldsymbol{e}}) \;\text{ as well as }\; \boldsymbol{\iota}_{\Delta}(\widehat{\Delta}^{(k)}) \;=\; \boldsymbol{\iota}_{\Delta}(\Delta^{(k)}) \;\; (k \in \boldsymbol{K})\,, \tag{1.24}$$

holds true. Let

$$\widehat{\Delta} \;:=\; \sum_{k=1}^{K} \widehat{\Delta}^{(k)} \;=\; \boldsymbol{\tau}_{\Pi}\Delta^{\boldsymbol{e}} \quad , \qquad \widehat{\Pi} \;:=\; \sum_{k=1}^{K} \widehat{\Pi}^{(k)} \;=\; \boldsymbol{\tau}_{\Pi}\Pi^{\boldsymbol{e}} \tag{1.25}$$

so that

$$\boldsymbol{\lambda}(\widehat{\Delta}) = \boldsymbol{\tau}_{\Pi}\boldsymbol{\lambda}(\Delta^{\boldsymbol{e}})\;, \quad \boldsymbol{\iota}_{\Delta}(\widehat{\Delta}) \;=\; \boldsymbol{\iota}_{\Delta}(\Delta) \tag{1.26}$$

follows. Let

$$\widehat{\boldsymbol{x}}(\bullet) \;:\; \boldsymbol{\tau}_{\Pi}\Delta^{\boldsymbol{e}} \to \partial\Pi \tag{1.27}$$

be the bijection that preserves the lattice structure of Pareto faces of $\partial\Pi$ and of the corresponding polyhedra in $K\Delta^{\boldsymbol{e}}$. Then $(\boldsymbol{\tau}_{\Pi}\Delta^{\boldsymbol{e}}, \boldsymbol{x}(\bullet))$ is called the ***measure preserving parametrization*** of $\partial\Pi$.

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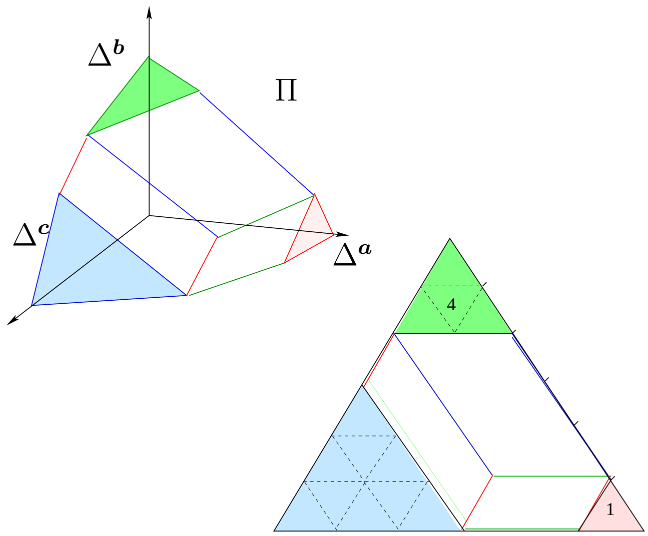

Figure 1.4: The measure preserving mappings of a Cephoid

Figure 1.4 suggests the action of $\widehat{\boldsymbol{\kappa}}$ and $\widehat{\boldsymbol{x}}(\bullet)$.

The (generalized) Maschler–Perles solution for a Cephoid Π is defined via the barycenter (i.e, the "midpoint") of $\boldsymbol{\tau}_{\Pi}\Delta^{\boldsymbol{e}}$. This barycenter is

$$\widehat{\boldsymbol{\mu}} \;:=\; \frac{\boldsymbol{\tau}_{\Pi}}{n}\boldsymbol{e} \;=:\; \boldsymbol{\mu}(\Pi^{\boldsymbol{\tau}_{\Pi}\boldsymbol{e}}) \tag{1.28}$$

and generally the Maschler–Perles solution of a Cephoid Π (regarded as a bargaining problem) is

$$\boldsymbol{\mu}(\Pi) \;:=\; \widehat{\boldsymbol{x}}(\widehat{\boldsymbol{\mu}}(\boldsymbol{\tau}_{\Pi}\Delta^{\boldsymbol{e}})) \;=\; \widehat{\boldsymbol{x}}(\frac{\boldsymbol{\tau}_{\Pi}}{n}\boldsymbol{e}) \tag{1.29}$$

Remark 1.7. We wish to again present the extensive interpretation given in [10] regarding the measure preserving mapping and its role in the definition of the Maschler–Perles solution.

To this end, let us first consider a simple bargaining situation such that n players can allocate a unit of a commodity ("money") by agreement about the distribution. Each player i has a linear utility function, say $u^i(t) = a_i t \;\; (t \in [0,1], a_i > 0)$. The feasible allocations of the commodity are represented by

$\Delta^{\boldsymbol{e}} = \{\widehat{\boldsymbol{x}} \in \mathbb{R}^n_+ \mid \sum_{i\in \boldsymbol{I}} \widehat{x}_i = 1\}$. The Pareto surface of the resulting bargaining problem ("in utility space") is

$$\begin{aligned} \left\{(u^1(\widehat{x}_1), \dots, u^n(\widehat{x}_n)) \,\middle|\, \widehat{\boldsymbol{x}} \in \Delta^{\boldsymbol{e}}\right\} = & \ \{(a_1\widehat{x}_1, \dots, a_n\widehat{x}_n) \mid \widehat{\boldsymbol{x}} \in \Delta^{\boldsymbol{e}}\} \\ = \boldsymbol{CovH}(\{\boldsymbol{a}^1, \dots, \boldsymbol{a}^n\}) = & \ \Delta^{\boldsymbol{a}} \ . \end{aligned} \tag{1.30}$$

That is, the bargaining problem is actually given by the deGua Simplex $\Pi^{\boldsymbol{a}}$ resulting from $\boldsymbol{a} = (a_1, \dots, a_n)$ with Pareto surface $\Delta^{\boldsymbol{a}}$.

Now we extend our framework assuming that our n players are involved in K bargaining situations of this type available in various locations ("countries") and with varying infrastructure. Players bargain about the distribution of one unit of money in each country $k \in \boldsymbol{K}$. Within every country each player i has a linear utility function $u^{(k)i}(t) = a^{(k)}_i t \ \ (t \in [0,1])$ referring to his utility of obtaining money in country $k \in \boldsymbol{K}$.

Thus, $\boldsymbol{a}^{(k)} = (a^{(k)}_1, \dots, a^{(k)}_n)$ represents the utility functions of the players regarding assignments in country $k \in \boldsymbol{K}$.

Now the Cephoid $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ reflects the assumption that players $i \in \boldsymbol{I}$ add their utilities in various countries accordingly. This can be seen as a version of *intra*–personal comparison of utility among the players w.r.t. different countries. Player $i \in \boldsymbol{I}$ will assess his holdings in different countries k, l according to his personal transfer rates a^i_k / a^i_l.

There is also a version of *inter*–personal transfer of utility. Exchanging a unit of money/commodity between two players has different effects when performed within the various locations/countries depending on the transfer rates a^i_k / a^j_k of two players i, j in country k.

We choose the space of commodity/money allocations for the joint bargaining problem to be $K\Delta^{\boldsymbol{e}}$. The canonical parametrization (Definition 1.2)

$$\boldsymbol{x}(\bullet) \ : \ K\Delta^{\boldsymbol{e}} \to \partial\Pi$$

provides a bijective mapping from commodity/money allocations into the utility space. (see also Figures 1.2 and 1.3). This way we obtain a representation of distributions of the total amount of K units of money K according to the Pareto face of $\partial\Pi$ being represented.

Now we construct a version of "adjusted utility" such that players exchange utility units on a universal scale and, simultaneously, have a universal intrapersonal transfer rate over all countries. This way we will obtain a consistent measurement of utility over the various countries and players resulting in a side payment situation.

To this end, we introduce a ("side payment") bargaining problem $\widehat{\Pi} = \boldsymbol{\tau}_{\Pi}\Delta^{\boldsymbol{e}}$. The utility set $\Delta^{\boldsymbol{a}^{(k)}}$ of each country k is mapped into a copy $\widehat{\Delta}^{(k)} \subseteq \widehat{\Delta}$. Within this universal utility space, concessions of players are measured by length measurements or, more generally by the Lebesgue measure. Also, intrapersonal comparison of utility takes place at a universal rate.

The construction of the adjusted utility space is obtained via the measure preserving parametrization

$$\widehat{\boldsymbol{x}}(\bullet) \; : \; \boldsymbol{\tau}_{\Pi}\Delta^{\boldsymbol{e}} \rightarrow \partial\Pi \; .$$

This mapping is based on a decomposition of the "adjusted utility space", i.e., the Simplex $\widehat{\Delta} = \boldsymbol{\tau}_{\Pi}\Delta^{\boldsymbol{e}}$ corresponding to the Pareto faces of $\partial\Pi$. The canonical identification of the Pareto faces of Π with their images is constructed via the isomorphism of the PO-sets of Pareto faces and their images.

Combining we obtain a mapping

$$\widehat{\boldsymbol{u}} \; := \; \boldsymbol{x}(\bullet) \circ \widehat{\boldsymbol{x}}^{-1}(\bullet) \; : \; K\Delta^{\boldsymbol{e}} \rightarrow \boldsymbol{\tau}_{\Pi}\Delta^{\boldsymbol{e}}$$

which maps allocations onto adjusted utilities. This set–up reflects a side-payment situation with total amount $\boldsymbol{\tau}_{\Pi}$ of utility to be distributed.

Figure 1.6 illustrates a typical shape of the adjusted utility side payment situation $\widehat{\Delta}$ corresponding to the three player bargaining problem given by the Cephoid Π ("Odot") of Figure 1.2.

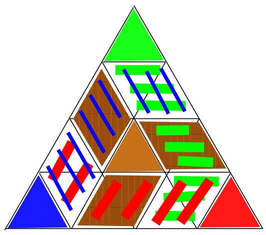

Figure 1.5: The canonical representation of "Odot"

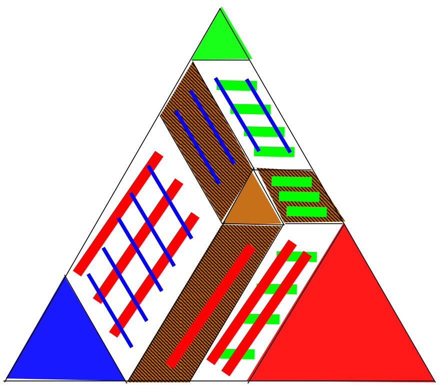

Figure 1.6: Measure Preserving Representation of Odot

The mapping $\widehat{\boldsymbol{u}} = \boldsymbol{x}(\bullet) \circ \boldsymbol{x}^{-1}(\bullet)$ maps commodity/money allocations bijectively from Figure 1.5 onto corresponding adjusted utilities of Figure 1.6.

We argue, that a consistent utility comparison of this type is dictated by the rationale of the Maschler–Perles solution.

For a bargaining problem in two dimensions, concessions of players during the bargaining process are measured in terms of the surface measure.

The Maschler–Perles solution requests equal concessions of players to be defined by equal volume (area) of line segments $\Delta^{\boldsymbol{a}} = \Delta^{(a_1,a_2)}$, i.e., by equal Maschler–Perles measures $\sqrt{a_1 a_2}$ along the Pareto surface. A concession of a player along a line segment $\Delta^{\boldsymbol{a}^{(1)}}$ is considered to be equal to the concession of another player along $\Delta^{\boldsymbol{a}^{(2)}}$ if and only if $\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2$ or $\boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}^1}) = \boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}^2})$ holds true.

The Maschler–Perles solution is the point in utility space at which players have made overall equal concessions along the Pareto surface (curve). This way, the size of a segment (Pareto face) $\widehat{\Delta}^{\boldsymbol{a}(k)}$ when transferred to the universal utility space is $\boldsymbol{\iota}_\Delta(\widehat{\Delta}^{(k)})$.

The Maschler–Perles solution is axiomatically based on the concept of *superadditivity*. Due to this axiom, the solution evaluates concessions of the players along line segments according to the corresponding *area* of the triangles (deGua Simplices). Superadditivity is the unique and outstanding axiom for this solution.

In three dimensions (for three players) the measurement of utility along boundary lines of a Simplex (a country) induces a measure for just this Simplex $\Delta^{\boldsymbol{a}^{(K)}}$. The measurement of utility should be consistent – i.e., the length of the boundary segments of a rhombus in Figure 1.6 should consistently be determined by the length measurement in the Simplices (triangles). As a rhombus has two linear boundary segments (determined by two triangles), this implies that the area should be consistently defined by the area in the generating Simplices.

This area, after some normalization (according to Definition 1.4) is obtained to be

$$\{\sqrt{a_1 a_2}\ \sqrt{a_1 a_3}\ \sqrt{a_2 a_3}\}^{\frac{2}{3}} = \sqrt[3]{(a_1 a_2 a_3)^2} = \boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}}) .$$

This way the construction of our universal ("adjusted") utiltiy space is determined by the axiomatic of the Maschler–Perles solution, i.e., by superadditivity.

Finally, as the measure preserving representation $\widehat{\Delta}$ (the adjusted utility space) allows for a universal comparison of utility, we hold that the barycenter or midpoint of this utility space establishes the bargaining solution. Consistently, we choose the barycenter/midpoint $\widehat{\boldsymbol{\mu}}$ of the side payment situation $\widehat{\Delta}$ as in (1.28). This results in the Maschler–Perles solution for polyhedral bargaining problems $\boldsymbol{\mu} = \widehat{\boldsymbol{\kappa}}_\Pi^{-1}(\widehat{\boldsymbol{\mu}}) = \widehat{\boldsymbol{x}}(\widehat{\boldsymbol{\kappa}})$ given by (1.29).

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Next we recall the notion of the deGua surface measure on the Pareto surface of a smooth body Γ.

To this end we consider a parametrization $(\boldsymbol{T}, (\boldsymbol{x}(\bullet))$ of $\partial\Gamma$. The functional determinants

(1.31) $$D_i(\bar{\boldsymbol{t}}) = (D_i\boldsymbol{x})(\bar{\boldsymbol{t}}) = \left|\frac{\partial x_k}{\partial t_j}\right|_{k\in\boldsymbol{I}\setminus\{i\},j\in\boldsymbol{I}\setminus\{n\}} \qquad (i \in \boldsymbol{I})$$

determine the normal at $\partial\Gamma$ in $\boldsymbol{x}(\boldsymbol{t})$:

(1.32) $$\bar{\mathfrak{n}} \;=\; \mathfrak{n}^{\boldsymbol{x}(\bar{\boldsymbol{t}})} \;=\; (D_1(\bar{\boldsymbol{t}}),\ldots,D_n(\bar{\boldsymbol{t}}))\ .$$

The ***deGua measure*** on $\partial\Gamma$ is

(1.33)
$$\begin{aligned}\boldsymbol{\vartheta}(\partial\Gamma) \;&=\; \int\limits_{\partial\Gamma} \sqrt[n]{d\mathfrak{n}_1\cdots d\mathfrak{n}_n} \;=\; \int\limits_{\boldsymbol{T}} \sqrt[n]{(D_1\cdots D_n)\circ\boldsymbol{x}}\; d\boldsymbol{\lambda} \\ &=\; \int\limits_{\boldsymbol{T}} \sqrt[n]{D_1(\boldsymbol{x}(t_1\cdots,t_{n-1}))\cdots D_n(\boldsymbol{x}(t_1,\cdots t_{n-1}))}\; dt_1\cdots dt_{n-1}.\end{aligned}$$

More generally for a measurable function $\boldsymbol{F}$ on $\partial\Gamma$ the integral w.r.t the deGua measure is

(1.34)
$$\begin{aligned}\int\limits_{\partial\Gamma} F d\boldsymbol{\vartheta} \;&=\; \int\limits_{\partial\Gamma} F\,\sqrt[n]{d\mathfrak{n}_1\cdots d\mathfrak{n}_n} \\ &=\; \int\limits_{\boldsymbol{T}} F(\boldsymbol{x}(\boldsymbol{t}))\,\sqrt[n]{D_1(\boldsymbol{x}(\boldsymbol{t}))\cdots D_n(\boldsymbol{x}(\boldsymbol{t}))} dt_1\cdots dt_n\ .\end{aligned}$$

The deGua measure, when defined analogously on Cephoids, differs from the Maschler–Perles measure. However, the main result of [12] is that for any filter of Cephoids converging towards Γ, the accompanying filter of Maschler–Perles measures converges towards the deGua measure on $\partial\Gamma$.

2 Coalitional Functions

The underlying concept of Game Theory within our present framework is the notion of a (cooperative) Non Transferable Utility Game. We recall

Definition 2.1. A (cooperative) ***NTU game*** is a triple $(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{V})$. Here, $\boldsymbol{I} = \{1, \ldots, n\}$ is the ***set of players***, $\underline{\underline{\mathbf{P}}} = \{S | S \subseteq \boldsymbol{I}\} = \mathcal{P}(\boldsymbol{I})$ is the ***system of coalitions***, and $\boldsymbol{V} : \underline{\underline{\mathbf{P}}} \to \mathcal{P}(\mathbb{R}^n)$ is the ***coalitional function***. $\boldsymbol{V}$ assigns to any coalition S a nonempty, compact, comprehensive, and convex set of "utility vectors" $\boldsymbol{V}(S) \subseteq \mathbb{R}^n_{S+}$.

We assume

$$\boldsymbol{V}(\{i\}) = \{\mathbf{0}\} \quad (i \in \boldsymbol{I}) . \tag{2.1}$$

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The standard interpretation has it that $\boldsymbol{V}(S)$ is the set of utility vectors that can be ensured to the members of coalition S by cooperation. The (utility) vector $\mathbf{0}$ reflects the fall back position ("status quo") for all players should cooperation fail in every coalition. The assumption that utilities are nonnegative and the status quo is 0 simultaneously for all players is not severe.

For simplicity we also refer to $\boldsymbol{V}$ as to "the game".

The aim of our present context is to extent the Maschler–Perles–Shapley value as developed in [10] to "smooth" case. Thus, we will admit that $\boldsymbol{V}(S)$ is either a Cephoid or a smooth body. Precisely:

Definition 2.2. An NTU game $\boldsymbol{V}$ is ***Cephoidal–smooth*** if, for $S \in \underline{\underline{\mathbf{P}}}$,

1. either

   $$\boldsymbol{V}(S) \quad \text{is a smooth body,} \tag{2.2}$$

2. or else $\boldsymbol{V}(S)$ is a Cephoid in $\mathbb{R}^n_{S+}$, i.e., there exists a (n.d.) family of positive vectors

   $$\left\{\boldsymbol{a}^{S,(k)}\right\}_{k \in \boldsymbol{K}_S} \subseteq \mathbb{R}^n_{S+} \tag{2.3}$$

   such that

   $$\boldsymbol{V}(S) = \sum_{k \in \boldsymbol{K}_S} \Pi^{\boldsymbol{a}^{S,(k)}} =: \sum_{k \in \boldsymbol{K}_S} \Pi^{S,(k)} . \tag{2.4}$$

$\boldsymbol{V}$ is called ***Cephoidal*** if *item* 2 is satisfied for all $S \in \underline{\underline{\mathbf{P}}}$.

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The TU game derived from an NTU game in the context of Cephoidal structures is derived from the adjustment factor. We recall this idea for Cephoids Π and then extend it to smooth bodies Γ (in the sense of [12], Section 3).

The adjustment factor (Definition 1.4) is defined by its values on deGua Simplices:

$$\boldsymbol{\tau}_{\Pi^{\boldsymbol{a}}} \; := \; \boldsymbol{\tau}_{\boldsymbol{a}} \; := \; \sqrt[n]{\prod_{i \in \boldsymbol{I}} a_i} \; . \tag{2.5}$$

This constitutes an additive function $\boldsymbol{\tau}_{\bullet}$ (Theorem 1.2. CH XIII of [10]) defined on Cephoids via Definition 1.4, formula (1.10). The connection to the Machler Perles measure $\boldsymbol{\iota}_{\Delta}$ (Definition 1.4) is given by Lemma 1.5, for any Cephoid $\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ we have

$$\boldsymbol{\tau}_{\Pi}^{n-1} \; = \; \boldsymbol{\iota}_{\Delta}(\Pi) \; . \tag{2.6}$$

By Corollary 4.12 of [12] we know that the Maschler Perles measure $\boldsymbol{\iota}_{\Delta}$ converges towards the deGua measure $\boldsymbol{\vartheta}$ along the filter of Cephoids approaching a smooth body.

$$\boldsymbol{\iota}_{\Delta}{}^{\boldsymbol{Q}} \underset{\mathfrak{Q}}{\longrightarrow} \boldsymbol{\iota}_{\Delta}^{\Gamma} \; = \; \boldsymbol{o}(n) \boldsymbol{\vartheta}^{\Gamma} \; . \tag{2.7}$$

Here, $\boldsymbol{o}(n)$ is a factor depending on the dimension only, it can be seen as a density of the Maschler Perles measure w.r.t. the deGua measure. Lemma 1.5 suggests that we use this fact in order to *define* the adjustment factor on smooth bodies.

Definition 2.3. The ***adjustment factor*** for a smooth body Γ is defined by

$$\boldsymbol{\tau}_{\Gamma} = \sqrt[n-1]{\boldsymbol{o}(n) \boldsymbol{\vartheta}(\partial \Gamma)} \; . \tag{2.8}$$

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Combining we obtain

Corollary 2.4.
1. Let Γ be a smooth body. Then there exists an approximating filter $\{\Pi^{\boldsymbol{Q}}\}_{\boldsymbol{Q} \in \mathfrak{Q}}$ such that

   $$\Pi^{\boldsymbol{Q}} \underset{\mathfrak{Q}}{\longrightarrow} \partial \Gamma \; . \tag{2.9}$$

   holds true uniformly. Here the topology maybe chosen to be the Haussdorff topology or equivalently a uniform topology suggested by the canonical parametrization of a convex comprehensive set.

2. The corresponding sequence of Maschler Perles measures $\{\boldsymbol{\iota}_{\Delta}^{\boldsymbol{Q}}\}_{\boldsymbol{Q} \in \mathfrak{Q}}$ has a weak limit $\boldsymbol{\iota}_{\Delta}^{\Gamma}$ satisfying

   $$\boldsymbol{\iota}_{\Delta}{}^{\boldsymbol{Q}} \underset{\mathfrak{Q}}{\longrightarrow} \boldsymbol{\iota}_{\Delta}^{\Gamma} \; = \; \boldsymbol{o}(n) \boldsymbol{\vartheta}^{\Gamma} \; . \tag{2.10}$$

   holds true. In particular we have

   $$\boldsymbol{\iota}_{\Delta}^{\boldsymbol{Q}}(\Pi^{\boldsymbol{Q}}) \; = \; \boldsymbol{\iota}_{\Delta}{}^{\partial \Gamma}(\Pi^{\boldsymbol{Q}}) \underset{\mathfrak{Q}}{\longrightarrow} \boldsymbol{o}(n) \boldsymbol{\vartheta}^{\Gamma}(\partial \Gamma) \tag{2.11}$$

3. The corresponding sequence of adjustment factors $\{\boldsymbol{\tau}_{\boldsymbol{Q}}\}_{\boldsymbol{Q}\in\mathfrak{Q}}$ has a weak limit which is the adjustment factor of Γ,

$$(2.12)\qquad \boldsymbol{\tau}_{\boldsymbol{Q}} \underset{\mathfrak{Q}}{\longrightarrow} \boldsymbol{\tau}_{\Gamma} \;=\; \sqrt[n-1]{\boldsymbol{o}(n)\boldsymbol{\vartheta}^{\Gamma}}\ .$$

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Proof: Follows from the results in [12] (see Lemma 4.3) and [11].

q.e.d.

Definition 2.5. Let $\boldsymbol{V}$ be a Cephoidal–smooth NTU game. The ***TU game induced by*** $\boldsymbol{V}$ is the coalitional function

$$(2.13)\qquad \widehat{\boldsymbol{v}} \;=\; \widehat{\boldsymbol{v}}^{\boldsymbol{V}} \;:\; \underline{\underline{\mathbf{P}}} \to \mathbb{R}$$

$$(2.14)\qquad \widehat{\boldsymbol{v}}(S) \;=\; \widehat{\boldsymbol{v}}^{\boldsymbol{V}}(S) \;=\; \boldsymbol{\tau}_{\boldsymbol{V}(S)} \quad (S \in \underline{\underline{\mathbf{P}}}).$$

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Remark 2.6. We pause for a moment in order to recall the motivation for choosing the TU–game as above. As we have seen, the environment of the Maschler–Perles solution induces players to evaluate concessions and gains in accordance with the coordinate product or rather the surface measure. In particular, the Maschler–Perles solution in two dimensions is the midpoint of the measure preserving image of the Pareto surface of the bargaining set. In two dimensions the evaluation of concessions and gains takes place according to the Maschler–Perles measure $\boldsymbol{\iota}_{\Delta}$ which is dictated by $\sqrt{-dx_1dx_2}$ – we are back at the origin with the Maschler–Perles donkey cart.

As a consequence, the "worth" of coalition S which comands utility vectors according to $\boldsymbol{V}(S)$ should be measured according to the total Maschler–Perles measure scaled appropriately. This is just the term provided by (2.8), i.e., $\boldsymbol{\tau}_{\boldsymbol{V}(S)}$.

Thus, other than the traditional approach of "admitting side payments" (initiated by SHAPLEY [14]) we do not just take the Lebesgue measure of a util for its worth but the corresponding Maschler–Perles measure. This strongly points to using Definition 2.5 for the "TU game" induced by an NTU game.

A glance at "side payment" or TU games is also helpful. To this end let

$$(2.15)\qquad \boldsymbol{v} : \underline{\underline{\mathbf{P}}} \to \mathbb{R}_{+}\ , \quad \boldsymbol{v}(\emptyset) = 0\ .$$

be a (nonnegative) ***TU game***. Consider a ***hyperplane NTU game*** $\boldsymbol{V}$ constructed as follows. For $S \in \underline{\underline{\mathbf{P}}}$, let $\boldsymbol{a}^S \in \mathbb{R}^n_{S+}$ be a positive vector and let

$$\boldsymbol{V}(S) = \boldsymbol{v}(S)\Pi^{\boldsymbol{a}^S} \quad (S \in \underline{\underline{\mathbf{P}}}).$$

That is, $\boldsymbol{V}(S)$ is a deGua Simplex determined by a hyperplane reflecting the rate of exchanging utility in coalition S. Then we have

$$\begin{aligned}\boldsymbol{\tau}_{\boldsymbol{V}(S)} &= \sqrt[|S|]{\prod_{i\in S}(\boldsymbol{v}(S)a_i^S)} = \boldsymbol{v}(S)\sqrt[|S|]{\prod_{i\in S}a_i^S} \\ &= \boldsymbol{v}(S)\boldsymbol{\tau}_{\prod \boldsymbol{a}^S} = \boldsymbol{v}(S)\boldsymbol{\tau}_{\boldsymbol{a}^S} \ .\end{aligned}$$

Hence, the TU game induced by $\boldsymbol{V}$ is given by

$$\widehat{\boldsymbol{v}}(S) = \boldsymbol{v}(S)\boldsymbol{\tau}_{\boldsymbol{a}^S} \quad (S \in \underline{\underline{\mathbf{P}}}) \tag{2.16}$$

We observe that the worth of coalition $S \in \underline{\underline{\mathbf{P}}}$ is adjusted or rescaled utilizing the adjustment factor. Clearly, $\widehat{\boldsymbol{v}}$ and $\boldsymbol{v}$ coincide whenever $\boldsymbol{a}^S = \boldsymbol{e}_S$ is the restriction of the unit vector $\boldsymbol{e}$ and hence $\boldsymbol{V}$ is the standard "embedding" of the side payment game $\boldsymbol{v}$ into the NTU framework.

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Now we use the results of [12] in order to approximate smooth games by Cephoidal Games.

Theorem 2.7. *Let $\boldsymbol{V}$ be a Cephoidal–smooth game. Then for all $S \in \underline{\underline{\mathbf{P}}}$ there exists an approximating filter $\left\{\boldsymbol{V}^{\boldsymbol{Q}}(S)\right\}_{\boldsymbol{Q}\in\mathfrak{Q}_S}$ of Cephoids such that*

$$\boldsymbol{V}^{\boldsymbol{Q}}(S) \underset{\mathfrak{Q}_S}{\longrightarrow} \boldsymbol{V}(S) \ . \tag{2.17}$$

holds true uniformly. Here the topology is the Haussdorff topology or a uniform topology suggested by the canonical parametrization of convex comprehensive sets.

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Proof: By *item* 2.9 of Corollary 2.4. Whenever $\boldsymbol{V}(S)$ is Cephoidal (i.e., obeys *item* 1), then one choose $\boldsymbol{V}^{\boldsymbol{Q}}(S) = \boldsymbol{V}(S)$ for the approximation. Whenever $\boldsymbol{V}(S)$ is a smooth body, one chooses a filter $\boldsymbol{V}^{\boldsymbol{Q}}(S)$ by means of Corollary 2.4.

q.e.d.

The filter $\mathfrak{Q}_S$ depends on the choice of S. For any system $\mathbf{Q} = \{Q_S\}_{S\in\underline{\underline{\mathbf{P}}}}$ satisfying $Q_S \in \mathfrak{Q}_S \quad (S \in \underline{\underline{\mathbf{P}}})$ we obtain a game

$$\boldsymbol{V}^{\mathbf{Q}} := \left\{\boldsymbol{V}^{\boldsymbol{Q}_S}(S)\right\}_{S\in\underline{\underline{\mathbf{P}}}} \tag{2.18}$$

such that

$$\boldsymbol{V}^{\mathbf{Q}} \underset{\mathfrak{Q}}{\longrightarrow} \boldsymbol{V} \ . \tag{2.19}$$

holds true in the sense of (2.17). We denote by $\underline{\underline{\mathfrak{Q}}}$ the collection of all these systems. Then, with reference to this convention, we state

Corollary 2.8. Let $\boldsymbol{V}$ be a Cephoidal–smooth game. Then there exists a filter

$$(2.20) \qquad \boldsymbol{V}^{\underline{\underline{\mathbf{Q}}}} := \left\{\boldsymbol{V}^{\mathbf{Q}}\right\}_{\mathbf{Q}\in\underline{\underline{\mathbf{Q}}}}$$

such that

$$(2.21) \qquad \boldsymbol{V}^{\mathbf{Q}} \underset{\underline{\underline{\mathbf{Q}}}}{\longrightarrow} \boldsymbol{V} \ .$$

Moreover, for the corresponding filter of TU games $\widehat{\boldsymbol{v}}^{\mathbf{Q}}$ it follows that, for $S \in \underline{\underline{\mathbf{P}}}$, we obtain

$$(2.22) \qquad \widehat{\boldsymbol{v}}^{\mathbf{Q}}(S) = \widehat{\boldsymbol{v}}^{\boldsymbol{V}^{\mathbf{Q}}}(S) = \boldsymbol{\tau}_{\boldsymbol{V}^{\mathbf{Q}}(S)} \underset{\underline{\underline{\mathbf{Q}}}}{\longrightarrow} \widehat{\boldsymbol{v}}^{\boldsymbol{V}}(S) = \widehat{\boldsymbol{v}}(S) \ ,$$

which we write

$$(2.23) \qquad \widehat{\boldsymbol{v}}^{\boldsymbol{Q}} \underset{\underline{\underline{\mathbf{Q}}}}{\longrightarrow} \widehat{\boldsymbol{v}} \ .$$

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Proof: By Theorem 2.7 and Corollary 2.4, see in particular (2.12) for the second part.

q.e.d.

For the remaining sections we focus on smooth games omitting the clumsy reference to the mixed “Cephoidal–smooth” version – the generalizations are obvious.

3 The Shapley Value

For the Shapley value we refer to Shapley's original work [14], see also [6] for a textbook treatment. A short summary follows.

Given a TU game $\boldsymbol{v}$ the Shapley value assigns a worth

$$(3.1)\qquad \Phi_i \;=\; \Phi_i(\boldsymbol{v}) \;:=\; \sum_{S\in\underline{\underline{\mathbf{P}}}} \frac{(n-|S|)!(|S|-1)!}{n!}(v(S)-v(S\setminus\{i\}))$$

to player $i \in \boldsymbol{I}$. The vector $\boldsymbol{\Phi} \;=\; (\Phi_1,\ldots,\Phi_n)$ constitutes an additive set function via

$$\boldsymbol{\Phi}(S) \;=\; \sum_{i\in S}\Phi_i \quad (S\in\underline{\underline{\mathbf{P}}}) \;.$$

The concept is characterized by an axiomatic foundation relying on anonymity, Pareto efficiency, additivity, and a dummy or null–player property.

The Shapley value of a game is a linear combination of the Shapley values of the unanimous games $\boldsymbol{e}^S$. If

$$(3.2)\qquad \boldsymbol{v}(\bullet) \;=\; \sum_{T\in\underline{\underline{\mathbf{P}}}} c_T\boldsymbol{e}^T(\bullet)$$

is the representation of a TU game $\boldsymbol{v}$ the basis $\{\boldsymbol{e}^S\}_{\Sigma\in\underline{\underline{\mathbf{P}}}}$ via the (Moebius) coefficients

$$(3.3)\qquad c_T \;=\; \sum_{S\subseteq T} -1^{|T\setminus S|}v(S) \quad (T\in\underline{\underline{\mathbf{P}}}),$$

then the Shapley value of $\boldsymbol{v}$ writes

$$(3.4)\qquad \boldsymbol{\Phi}(\boldsymbol{v}) \;=\; \sum_{T\in\underline{\underline{\mathbf{P}}}} c_T\boldsymbol{\mu}^T \;=\; \sum_{T\in\underline{\underline{\mathbf{P}}}} c_T\boldsymbol{\Phi}(\boldsymbol{e}^T) \;.$$

The additive function $\boldsymbol{\mu}^T$ is given by

$$(3.5)\qquad \boldsymbol{\mu}^T(S) \;=\; |S\cap T| \quad (S\in\underline{\underline{\mathbf{P}}})$$

and it is the Shapley value of the unanimous game $\boldsymbol{e}^T$.

Extending the Shapley value to NTU games does also have a tradition based on Shapley's conference paper [13]. This approach as most others hinges on a fixed point theorem.

In our present context, the Shapley value for a Cephoidal NTU game is treated in [12], Chapter XIV, Section 3 . We shortly recall the definition.

First of all we adapt the measure preserving parametrization (Section 1, Definition 1.6, Remark 1.7) to the framework of an NTU game $\boldsymbol{V}$. The representation of the Pareto surface $\partial\boldsymbol{V}(I)$ is

$$(3.6)\qquad \widehat{\boldsymbol{\kappa}}_{\boldsymbol{V}(I)} \;:\; \partial\boldsymbol{V}(I)\rightarrow \boldsymbol{\tau}_{\boldsymbol{V}(I)}\Delta^{\boldsymbol{e}} \;=\; \widehat{\boldsymbol{v}}^{\boldsymbol{V}}(\boldsymbol{I})\Delta^{\boldsymbol{e}} \;,$$

and the inverse mapping is the measure preserving parametrization

$$\widehat{\boldsymbol{x}}_{\boldsymbol{V}(I)}(\bullet) \;=\; \widehat{\boldsymbol{\kappa}}^{-1}_{\boldsymbol{V}(I)} \;:\; \boldsymbol{v}^{\boldsymbol{V}}(I)\Delta^{\boldsymbol{e}} \to \partial \boldsymbol{V}(I)\;. \tag{3.7}$$

As $\Phi(\widehat{\boldsymbol{v}})(\boldsymbol{I}) \;=\; \widehat{\boldsymbol{v}}(\boldsymbol{I})$, we find that $\boldsymbol{\Phi}(\boldsymbol{v})$ is located in the range of $\widehat{\boldsymbol{\kappa}}_{(\boldsymbol{V}(\boldsymbol{I})}$ which is the domain of $\boldsymbol{x}_{\boldsymbol{V}(I)}(\bullet)$, i.e., we have

$$\boldsymbol{\Phi}(\boldsymbol{v}^{\boldsymbol{V}}) \in \boldsymbol{\tau}_{\boldsymbol{V}(\boldsymbol{I})}\Delta^{\boldsymbol{e}} \;=\; \widehat{\boldsymbol{v}}^{\boldsymbol{V}}(\boldsymbol{I})\Delta^{\boldsymbol{e}}\;. \tag{3.8}$$

Therefore, we are in the position to define the Shapley value of the Cephoidal NTU game $\boldsymbol{V}$.

Definition 3.1. The Maschler–Perles–Shapley value of a Cephoidal NTU game $\boldsymbol{V}$ (for short the M–P–S value) is

$$\chi(\boldsymbol{V}) \;:=\; \widehat{\boldsymbol{\kappa}}^{-1}_{\boldsymbol{V}(I)}(\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{V}})) \;=\; \widehat{\boldsymbol{x}}_{\boldsymbol{V}(I)}(\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{V}}))\;. \tag{3.9}$$

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Remark 3.2. Let us pause for an interpretation along the guidelines of Remarks 1.7 and 2.6. The intuition behind formula (3.9) follows the arguments presented within those remarks.

Accordingly, players compare gains and concessions via the M–P measure $\boldsymbol{\iota}_{\Delta}$ (“adjusted utilities”) and this leads to focus on the side payment game $\widehat{\boldsymbol{v}} = \widehat{\boldsymbol{v}}^{\boldsymbol{V}}$. The feasible side payments for the grand coalition in terms of adjusted utilities are reflected by the multiple $\widehat{\boldsymbol{v}}(\boldsymbol{I})\Delta^{\boldsymbol{e}}$.

The corresponding side payment vectors for the various coalitions are then represented in $\widehat{\boldsymbol{v}}(S)\Delta^{\boldsymbol{e}_S}$. The Shapley value concept involves a linear structure: the (unanimous) games $\boldsymbol{e}^S$ constitute a basis of the linear space of side payment games. (see (3.2), (3.4)) This linear structure is referred to as “von Neumann–Morgenstern utility” as it involves lotteries and expectations of lotteries.

As the Shapley value essentially distributes the wealth of the grand coalition – respecting the power of smaller coalitions – we are led to consider the utility vectors of $\widehat{\boldsymbol{v}}(\boldsymbol{I})\Delta^{\varepsilon}$ as the relevant set for the determination “power based” utility agreement among the players. One is computing the Shapley value within the framework of adjusted utilities of the grand coalition.

This operation is performed consistently in the sidepayment environment of $\widehat{\boldsymbol{v}}(\boldsymbol{I})\Delta^{\varepsilon}$ which reflects “adjusted utility”. Then players agree upon the solution vector within $\partial\boldsymbol{V}(\boldsymbol{I})$ dictated by $\boldsymbol{\Phi}(\widehat{\boldsymbol{v}})$, that is, the result is lifted back to $\partial\boldsymbol{V}(\boldsymbol{I})$ via $\widehat{\boldsymbol{x}}(\bullet)$. This is the essence of the Maschler–Perles–Shapley value, as established by formula (3.9).

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Now we turn to the territory of smooth bodies. First of all, we extend the notion of the measure preserving parametrization $\widehat{\boldsymbol{x}}(\bullet)$. This procedure is only slightly more involved as the one for the Simplex parametrization used in [12]. The following is the analog to *Lemma 4.4.* in [12], the adjustment based on Corollary 2.4.

Lemma 3.3. Let Γ be smooth and let $\{\Pi^{\boldsymbol{Q}}\}_{\boldsymbol{Q}\subseteq\mathbb{N}}$ be an approximating filter, that is,

$$\Pi^{\boldsymbol{Q}} \underset{\mathfrak{Q}}{\rightarrow} \partial\Gamma \tag{3.10}$$

uniformly. Let the measure preserving parametrizations be given by

$$\widehat{\boldsymbol{x}}^{\boldsymbol{Q}}(\bullet)\ (\boldsymbol{Q} \in \mathfrak{Q})\ ,$$

such that

$$\widehat{\boldsymbol{x}}^{\boldsymbol{Q}}(\bullet) : \boldsymbol{\tau}_{\Pi^{\boldsymbol{Q}}}\Delta^{\boldsymbol{e}} \rightarrow \partial\Pi^{\boldsymbol{Q}} \quad (\boldsymbol{Q} \in \mathfrak{Q}) \tag{3.11}$$

holds true. Then these parametrizations converge uniformly to a mapping

$$\widehat{\boldsymbol{x}}^{\Gamma}(\bullet)\ : \boldsymbol{\tau}_{\Gamma}\Delta^{\boldsymbol{e}} \rightarrow \partial\Gamma \quad (\boldsymbol{Q} \in \mathfrak{Q}) \tag{3.12}$$

which is continuous and bijective, hence constitutes a parametrization of $\partial\Gamma$.

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Proof:

The proof is a slightly modified version of the one for Lemma 4.4. in [12] – a standard procedure which we omit. As in [12], bijectivity follows as $\partial\Gamma$ is smooth, i.e. in our terminology, there is a bijection between points $\boldsymbol{x} \in \partial\Gamma$ and normals $\mathfrak{n}^{\boldsymbol{x}}$.

q.e.d.

Definition 3.4. The mapping $\widehat{\boldsymbol{x}}^{\Gamma}(\bullet)$ constitutes the measure preserving parametrization $(\boldsymbol{\tau}_{\Gamma}\Delta^{\boldsymbol{e}}, \widehat{\boldsymbol{x}}^{\Gamma})$ of $\partial\Gamma$.

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We proceed as in the Cephoidal case. Let $\boldsymbol{V}$ be a smooth NTU game. The TU game derived is $\widehat{\boldsymbol{v}} = \widehat{\boldsymbol{v}}^{\boldsymbol{V}}$. In view of Pareto efficiency the Shapley value $\Phi(\widehat{\boldsymbol{v}})$ satisfies

$$(\Phi(\widehat{\boldsymbol{v}}))(\boldsymbol{I}) \;=\; \sum_{i\in\boldsymbol{I}} \Phi_i(\widehat{\boldsymbol{v}}) \;=\; \widehat{\boldsymbol{v}}(\boldsymbol{I}) \tag{3.13}$$

that is

$$\Phi(\widehat{\boldsymbol{v}}) \in \Delta^{\widehat{\boldsymbol{v}}(\boldsymbol{I})\boldsymbol{e}} \;=\; \widehat{\boldsymbol{v}}(\boldsymbol{I})\Delta^{\boldsymbol{e}}\ . \tag{3.14}$$

By Lemma 3.3 and Definition 3.4 the measure preserving parametrization

$$\widehat{\boldsymbol{x}}^{\boldsymbol{V}(\boldsymbol{I})} \;:\; \boldsymbol{\tau}_{\boldsymbol{V}(\boldsymbol{I})}\Delta^{\boldsymbol{e}} \;=\; \widehat{\boldsymbol{v}}^{\boldsymbol{V}}(\boldsymbol{I})\Delta^{\boldsymbol{e}} \to \partial\boldsymbol{V}(\boldsymbol{I})\;. \tag{3.15}$$

is defined such that $\Phi(\widehat{\boldsymbol{v}}) \in \boldsymbol{v}(\boldsymbol{I})\Delta^{\boldsymbol{e}}$ is located in its domain. This permits us to formulate the following definition.

Definition 3.5. Let $\boldsymbol{V}$ be a smooth NTU game and let $\widehat{\boldsymbol{v}}$ be the TU game derived. The ***Maschler–Perles–Shapley value*** (the M–P–S value) of $\boldsymbol{V}$ is

$$\chi(\boldsymbol{V}) \;:=\; \widehat{\boldsymbol{x}}^{\boldsymbol{V}(\boldsymbol{I})}(\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{V}})) \tag{3.16}$$

• ~~~~~~~ •

As this is the analog to Definition 3.1, we will be able to transfer the axioms and properties of the Maschler–Perles–Shapley NTU value as established for the Cephoidal case to the smooth case.

4 The Value Axioms

The Maschler–Perles–Shapley value $\boldsymbol{\chi}$ for Cephoidal NTU games $\boldsymbol{V}$ admits of an axiomatic justification, see [12], CHAPTER XIV. In what follows we will extend the axioms and properties as established for the Cephoidal case to the smooth case.

Theorem 4.1. *The M–P–S value on smooth NTU games*

1. *is Pareto efficient,*
2. *is symmetric,*
3. *respects affine transformations of utility.*
4. *is conditionally additive.*

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Proof: Conditional additivity (the last concept) is the only property that causes a problem for smooth NTU games. We postpone the proof to the next section.

Pareto efficiency is obvious by definition. Symmetry ("anonymity") and invariance under affine transformations of utility are straightforward by continuity. For completeness we treat the first concept.

To formulate symmetry we consider the actions of a permutation π of $\boldsymbol{I}$. For $\boldsymbol{x} \in \mathbb{R}^n$ we have

$$(\pi \boldsymbol{x})_i \;:=\; x_{\pi^{-1}(i)} \;\; (i \in \boldsymbol{I}) \; .$$

For a TU game $\boldsymbol{v}$ we have

$$\pi \boldsymbol{v}(S) \;:=\; \boldsymbol{v} \circ \pi^{-1}(S) \;\; (S \in \underline{\underline{\mathbf{P}}}) \; ,$$

while for NTU games, the appropriate definition is

$$\pi \boldsymbol{V}(S) \;:=\; \pi \circ \boldsymbol{V} \circ \pi^{-1}(S) \;\; (S \in \underline{\underline{\mathbf{P}}}) \; .$$

For a Cephoidal game $\boldsymbol{V}$ we know by Theorem 3.2 CHAPTER XIV of [10] that $\boldsymbol{\chi}$ is symmetric or "anonymous", that is, for any permutation π of $\boldsymbol{I}$ we have

$$\boldsymbol{\chi}(\pi \boldsymbol{V}) \;=\; \pi(\boldsymbol{\chi}(\boldsymbol{V})) \; . \tag{4.1}$$

Now let $\boldsymbol{V}$ be a smooth NTU game. Then, by Corollary 2.8 we find a filter of Cephoidal NTU games

$$\boldsymbol{V}^{\underline{\underline{\mathbf{Q}}}} \;:=\; \left\{ \boldsymbol{V}^{\mathbf{Q}} \right\}_{\mathbf{Q} \in \underline{\underline{\mathbf{Q}}}} \tag{4.2}$$

such that

$$V^{\mathbf{Q}} \underset{\underline{\underline{Q}}}{\longrightarrow} V \quad \text{and} \quad \widehat{v}^{Q} \underset{\underline{\underline{Q}}}{\longrightarrow} \widehat{v} \; . \tag{4.3}$$

holds true. It is not hard to see that for the permuted versions there is an appropriate ("permuted") filter converging to $\pi \boldsymbol{V}$ and $\pi \boldsymbol{v}$ as well. Therefore, it remains to establish continuity of $\boldsymbol{\chi}$ as a function on games along a filter of games as above. We refer to Definition 3.16. The TU Shapley value $\boldsymbol{\Phi}$ is clearly continuous (perceived as a function on $\mathbb{R}^{\underline{\underline{\mathbf{P}}}}$) and the parametrizations $\widehat{\boldsymbol{x}}^{\boldsymbol{Q}^{\boldsymbol{V}}}(\boldsymbol{I})$ will behave continuously by Lemma 3.3. This proves (4.1) for smooth games as well.

q.e.d.

5 Conditional Additivity

AUMANN'S [1] concept of conditional additivity refers to correspondences, i.e., a "value" is a set valued function defined on smooth surfaces (bargaining problems or NTU games).

The Maschler–Perles solution $\boldsymbol{\mu}$ and the Maschler–Perles–Shapley value $\boldsymbol{\chi}$ are functions defined on Cephoids. These functions are conditionally additive on Cephoidal bargaining problems or NTU games respectively. (CHAPTER XIV, Theorem 3.2., [10]). We recall the definition.

Definition 5.1. Let $\boldsymbol{\chi}$ be a mapping from a class of convex valued NTU games into $\mathbb{R}^n_+$ such that $\boldsymbol{\chi}(\boldsymbol{V}) \in \partial \boldsymbol{V}(\boldsymbol{I})$ holds true for all $\boldsymbol{V}$. $\boldsymbol{\chi}$ is ***conditionally additive*** if, for any two games $\boldsymbol{V}$ and $\boldsymbol{W}$ such that $\boldsymbol{\chi}(\boldsymbol{V})+\boldsymbol{\chi}(\boldsymbol{W})$ is Pareto efficient in $(\boldsymbol{V}+\boldsymbol{W})(\boldsymbol{I}) = \boldsymbol{V}(\boldsymbol{I}) + \boldsymbol{W}(\boldsymbol{I})$, it follows that

$$\boldsymbol{\chi}(\boldsymbol{V}+\boldsymbol{W}) \;=\; \boldsymbol{\chi}(\boldsymbol{V})+\boldsymbol{\chi}(\boldsymbol{W}) \tag{5.1}$$

holds true.

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Within this section we establish conditional additivity of the Shapley value for smooth NTU games. The result is based on the one for Cephoids. A problem arises as conditional additivity is not an l.h.c. property. That is, limiting Cephoids approaching a smooth body may lack limiting correct normals. But conditional superadditivity always hinges on the fact that the sum of two extremals is an extremal in the sum of two convex bodies if and only if they admit of the same normal. In what follows, we repair this deficiency.

Lemma 5.2. Let $\boldsymbol{V}$ be a smooth game and let $\boldsymbol{\chi}(\boldsymbol{V})$ be the Maschler–Perles–Shapley value of $\boldsymbol{V}$. Denote the normal at $\boldsymbol{V}(\boldsymbol{I})$ in $\boldsymbol{\chi}(\boldsymbol{V})$ by $\mathfrak{n}^{\overline{\boldsymbol{a}}} = (\frac{1}{\overline{a}_1}, \ldots, \frac{1}{\overline{a}_n})$. Then, for any $\varepsilon > 0$, there exists a Cephoidal Game $\boldsymbol{V}^{\boldsymbol{Q}}$ with Hausdorff distance $|\boldsymbol{V}, \boldsymbol{V}^{\boldsymbol{Q}}| < \varepsilon$ such that the normal at $\boldsymbol{V}^{\boldsymbol{Q}}(\boldsymbol{I})$ in $\boldsymbol{\chi}(\boldsymbol{V}^{\boldsymbol{Q}}(\boldsymbol{I}))$ is $\mathfrak{n}^{\overline{\boldsymbol{a}}}$.

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Proof:

Let $\varepsilon > 0$ and let $\eta < \frac{\varepsilon}{n}$. Define a ("unanimous") game $\boldsymbol{V}^{\overline{\boldsymbol{a}}}$ as follows:

$$\begin{array}{rcl} \boldsymbol{V}(\boldsymbol{I}) & = & \Delta^{\overline{\boldsymbol{a}}} \\ \boldsymbol{V}(S) & = & \{\boldsymbol{0}\} \quad (S \neq \boldsymbol{I}) \end{array} \tag{5.2}$$

such that the derived TU–game is the ("unanimous") game

$$\widehat{\boldsymbol{v}}^{\boldsymbol{V}^{\overline{\boldsymbol{a}}}} \;=\; \boldsymbol{\iota}_{\Delta}(\Delta^{\overline{\boldsymbol{a}}})\boldsymbol{e}^{\boldsymbol{I}} \;=\; \overline{\alpha}\boldsymbol{e}^{\boldsymbol{I}}\ . \tag{5.3}$$

with $\overline{\alpha} := \boldsymbol{\iota}_{\Delta}(\Delta^{\bar{\boldsymbol{a}}})$. We write $\widehat{\boldsymbol{v}}^{\overline{\boldsymbol{a}}} := \widehat{\boldsymbol{v}}^{\boldsymbol{V}^{\overline{\boldsymbol{a}}}}$ and $\widehat{\boldsymbol{v}}^{\boldsymbol{Q}} := \widehat{\boldsymbol{v}}^{\boldsymbol{V}^{\boldsymbol{Q}}}$.

Now choose $\boldsymbol{V}^{\boldsymbol{Q}}$ such that

$$\bar{\boldsymbol{x}} := \boldsymbol{\chi}(\boldsymbol{V}) = \widehat{\boldsymbol{x}}(\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{V}})) \in \partial \boldsymbol{V}(\boldsymbol{I}) \cap \partial \boldsymbol{V}^{\boldsymbol{Q}}(\boldsymbol{I}) \tag{5.4}$$

and $\mathfrak{n}^{\bar{\boldsymbol{a}}}$ is normal in $\bar{\boldsymbol{x}}$ to $\partial \boldsymbol{V}(\boldsymbol{I})$ as well as to $\partial \boldsymbol{V}^{\boldsymbol{Q}}(\boldsymbol{I})$; this is feasible by Theorem 1.5 in [11]. In addition, choose $\boldsymbol{Q}$ such that

$$|\Phi(\widehat{\boldsymbol{v}}^{\Gamma}) - \Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{Q}})| < \eta \ . \tag{5.5}$$

Define a game

$$\boldsymbol{V}^{\boldsymbol{Q},\varepsilon} := (1-\varepsilon)V^{\boldsymbol{Q}} + \varepsilon \boldsymbol{V}^{\bar{\boldsymbol{a}}} \tag{5.6}$$

As $\widehat{\boldsymbol{v}}^{\bullet}$ (i.e. $\boldsymbol{\tau}_{\bullet}$) is linear in $\boldsymbol{V}$ ([10], CHAPTER XVIII, Theorem 1.2.) we obtain

$$\widehat{\boldsymbol{v}}^{\boldsymbol{Q},\varepsilon} = \widehat{\boldsymbol{v}}^{\boldsymbol{V}^{\boldsymbol{Q},\varepsilon}} = \widehat{\boldsymbol{v}}^{(1-\varepsilon)V^{\boldsymbol{Q}}+\varepsilon \boldsymbol{V}^{\bar{\boldsymbol{a}}}} = (1-\varepsilon)\widehat{\boldsymbol{v}}^{\boldsymbol{Q}} + \varepsilon \widehat{\boldsymbol{v}}^{\bar{\boldsymbol{a}}} \tag{5.7}$$

The Shapley value for TU games is linear as well, consequently

$$\begin{aligned} \Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{Q},\varepsilon}) &= \Phi((1-\varepsilon)\widehat{\boldsymbol{v}}^{\boldsymbol{Q}} + \varepsilon \widehat{\boldsymbol{v}}^{\overline{\boldsymbol{a}}}) \\ &= (1-\varepsilon)\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{Q}}) + \varepsilon \Phi(\widehat{\boldsymbol{v}}^{\overline{\boldsymbol{a}}}) \end{aligned} \tag{5.8}$$

The Shapley value for the unanimous game $\widehat{\boldsymbol{v}}^{\bar{\boldsymbol{a}}}$ is

$$\Phi(\widehat{\boldsymbol{v}}^{\overline{\boldsymbol{a}}}) = \boldsymbol{\iota}_{\Delta}(\Delta^{\bar{\boldsymbol{a}}})\frac{\boldsymbol{e}}{n} =: \overline{\alpha}\frac{\boldsymbol{e}}{n} \tag{5.9}$$

using $\overline{\alpha} := \boldsymbol{\iota}_{\Delta}(\Delta^{\bar{\boldsymbol{a}}})$ as above.

Thus we continue (5.8) by

$$\begin{aligned} \Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{Q},\varepsilon}) &= (1-\varepsilon)\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{Q}}) + \varepsilon\overline{\alpha}\frac{\boldsymbol{e}}{n} \\ &= (1-\varepsilon)\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{V}}) + \varepsilon\overline{\alpha}\frac{\boldsymbol{e}}{n} + \boldsymbol{o}(\eta) \ . \end{aligned} \tag{5.10}$$

Here $\boldsymbol{o}(\eta)$ is a vector of the order of η, i.e., $|\boldsymbol{o}(\eta)| < \eta$ by (5.5).

As $\overline{\alpha}\frac{\bar{\boldsymbol{a}}}{n}$ is the barycenter of $\overline{\alpha}\Delta^{\overline{\boldsymbol{a}}}$ and $\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{Q},\varepsilon}) \in \widehat{\boldsymbol{v}}^{\boldsymbol{Q},\varepsilon}(\boldsymbol{I})\Delta^{\boldsymbol{e}}$ we conclude from (5.10) that for $\eta < \frac{\varepsilon}{n}$:

$$\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{Q},\varepsilon}) \in (1-\varepsilon)\Phi(\widehat{\boldsymbol{v}}^{\Gamma}) + \varepsilon\overline{\alpha}\Delta^{\boldsymbol{e}} \ . \tag{5.11}$$

Now $\mathfrak{n}^{\bar{\boldsymbol{a}}}$ is normal in $\bar{\boldsymbol{x}} = \boldsymbol{\chi}(\boldsymbol{V}(\boldsymbol{I}))$ to $\partial \boldsymbol{V}(\boldsymbol{I})$ as well as to $\partial \boldsymbol{V}^{\boldsymbol{Q}}(\boldsymbol{I})$. $\Delta^{\overline{\boldsymbol{a}}}$ has the same normal. Therefore,

$$\boldsymbol{F}^{\bar{\boldsymbol{x}},\varepsilon} := (1-\varepsilon)\boldsymbol{\chi}(\boldsymbol{V}^{\Gamma}) + \varepsilon\Delta^{\overline{\boldsymbol{a}}} \tag{5.12}$$

is a Pareto face of $\partial \boldsymbol{V}^{\boldsymbol{Q},\varepsilon}(\boldsymbol{I})$ with normal $\bar{\mathfrak{n}}$.

The parametrization $\widehat{\boldsymbol{x}}(\bullet) = \widehat{\boldsymbol{x}}^{\boldsymbol{V}^{\boldsymbol{Q},\varepsilon}}(\bullet)$ is the inverse of measure preserving representation $\widehat{\boldsymbol{\kappa}}^{\boldsymbol{V}^{\boldsymbol{Q},\varepsilon}}$. The latter one maps the Pareto face $\boldsymbol{F}^{\bar{\boldsymbol{x}},\varepsilon}$ via the isomorphism of the lattices of Pareto faces bijectively onto the face

$$\text{(5.13)} \qquad \boldsymbol{F}^{\boldsymbol{0},\varepsilon} := (1-\varepsilon)\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{V}}) + \varepsilon\overline{\alpha}\Delta^{\boldsymbol{e}}$$

of $\left((1-\varepsilon)\widehat{\boldsymbol{v}}^{\boldsymbol{V}}(\boldsymbol{I}) + \varepsilon\overline{\alpha}\right)\Delta^{\boldsymbol{e}}$ That is, we have

$$\text{(5.14)} \qquad \widehat{\boldsymbol{x}}(\bullet) = \widehat{\boldsymbol{x}}^{\boldsymbol{V}^{\boldsymbol{Q},\varepsilon}}(\bullet) : \boldsymbol{F}^{\boldsymbol{0},\varepsilon} \to \boldsymbol{F}^{\bar{\boldsymbol{x}},\varepsilon} .$$

Combining (5.11) and (5.14) we obtain

$$\text{(5.15)} \qquad \chi(\boldsymbol{V}^{\boldsymbol{Q},\varepsilon}) = \widehat{\boldsymbol{x}}(\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{Q},\varepsilon})) \in \boldsymbol{F}^{\bar{\boldsymbol{x}},\varepsilon} = (1-\varepsilon)\chi(\boldsymbol{V}^{\boldsymbol{Q}}) + \varepsilon\overline{\alpha}\Delta^{\boldsymbol{e}} .$$

Hence, $\chi(\boldsymbol{V}^{\boldsymbol{Q},\varepsilon})$ has the same normal to $\boldsymbol{V}^{\boldsymbol{Q},\varepsilon}(\boldsymbol{I})$ as $\chi(\boldsymbol{V})$ namely $\mathfrak{n}^{\overline{\boldsymbol{a}}}$.

q.e.d.

Theorem 5.3. *The Maschler–Perles–Shapley value χ is conditionally additive on smooth NTU games.*

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Proof:

1$^{\text{st}}$STEP : Let $\boldsymbol{V}$ and $\boldsymbol{W}$ be smooth games and let $\chi(\boldsymbol{V})+\chi(\boldsymbol{W})$ be Pareto efficient in $(\boldsymbol{V}+\boldsymbol{W})(\boldsymbol{I}) = \boldsymbol{V}(\boldsymbol{I})+\boldsymbol{W}(\boldsymbol{I})$. Then necessarily $\boldsymbol{V}(\boldsymbol{I})$, $\boldsymbol{W}(\boldsymbol{I})$ and $(\boldsymbol{V}+\boldsymbol{W})(\boldsymbol{I})$ admit of the same normal $\bar{\mathfrak{n}}$ in $\chi(\boldsymbol{W})$, $\chi(\boldsymbol{V})$ and $\chi(\boldsymbol{V})+\chi(\boldsymbol{W})$ respectively.

According to Corollary 2.8 we can approximate $\boldsymbol{V}$ and $\boldsymbol{W}$ by Cephoidal games $\boldsymbol{V}^{\boldsymbol{Q}}$ and $\boldsymbol{W}^{\boldsymbol{Q}}$ such that all Shapley values have the same normal $\bar{\mathfrak{n}}$ at $\boldsymbol{V}^{\boldsymbol{Q}}(\boldsymbol{I})$ and $\boldsymbol{W}^{\boldsymbol{Q}}(\boldsymbol{I})$ respectively. The approximating filter $\boldsymbol{V}^{\underline{\underline{\mathrm{Q}}}}$ can be chosen simultaneously for both games.

Then the values $\chi(\boldsymbol{V}^{\boldsymbol{Q}})$ and $\chi(\boldsymbol{W}^{\boldsymbol{Q}})$ approximate the corresponding values of $\chi(\boldsymbol{V}^{\boldsymbol{Q}})$ and $\chi(\boldsymbol{W}^{\boldsymbol{Q}})$ accordingly.

Now, it follows again from the common normal property that $\chi(\boldsymbol{V}^{\boldsymbol{Q}}) + \chi(\boldsymbol{W}^{\boldsymbol{Q}})$ is Pareto efficient in $(\boldsymbol{V}^{\boldsymbol{Q}} + \boldsymbol{W}^{\boldsymbol{Q}})(\boldsymbol{I})$. As the Maschler–Perles–Shapley value is conditionally additive on Cephoidal Games (Theorem 3.2, Chapter XIV of [10]), we conclude that for $\boldsymbol{Q} \in \underline{\underline{\mathrm{Q}}}$

$$\text{(5.16)} \qquad \chi(\boldsymbol{V}^{\boldsymbol{Q}} + \boldsymbol{W}^{\boldsymbol{Q}}) = \chi(\boldsymbol{V}^{\boldsymbol{Q}})) + \chi(\boldsymbol{W}^{\boldsymbol{Q}}) .$$

Consequently,

$$\text{(5.17)} \qquad \begin{aligned} \chi(\boldsymbol{V}+\boldsymbol{W}) &= \lim_{\underline{\underline{\mathrm{Q}}}} \chi(\boldsymbol{V}^{\boldsymbol{Q}} + \boldsymbol{W}^{\boldsymbol{Q}}) \\ &= \lim_{\underline{\underline{\mathrm{Q}}}} \chi(\boldsymbol{V}^{\boldsymbol{Q}}) + \lim_{\underline{\underline{\mathrm{Q}}}} \chi(\boldsymbol{W}^{\boldsymbol{Q}}) \\ &= \chi(\boldsymbol{V})) + \chi(\boldsymbol{W}) \end{aligned}$$

q.e.d.

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