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# Cephoids

## Minkowski Sums

## of DeGua Simplices

Theory

and Applications

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## Abstract

This volume is a monograph on the geometric structure of a certain class of ("comprehensive") compact polyhedra called *Cephoids*. A Cephoid is a Minkowski sum of finitely many standardized simplices. The emphasis rests on the Pareto surface of Cephoids which consists of certain translates of simplices, algebraic sums of subsimplices etc.

Cephoids appear in Operations Research (Optimization), in Mathematical Economics (Free Trade theory), and in Cooperative Game Theory.

In particular, in the context of Cooperative Game Theory the notions of a Cephoid serves to construct "solutions" or "values" for bargaining problems and non–side payment games.

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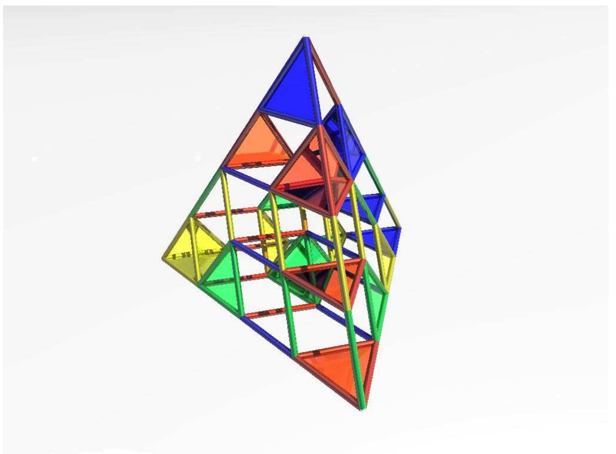

# Preface

Within this volume we describe the geometric structure of a certain class of ("comprehensive") compact polyhedra in $\mathbb{R}^n_+$, called *Cephoids*. A Cephoid is a (Minkowski, algebraic) sum of finitely many standardized simplices, called *DeGua Simplices* (referring to DeGua [4]). The *outward* or *Pareto surface* of a deGua Simplex is a simplex as well (of lower dimension), but the outward or Pareto surface of a Cephoid is generally much more involved and consists of certain translates of simplices, algebraic sums of subsimplices etc. To completely describe this structure is the main purpose of this book.

Cephoids appear in Applications of Mathematics like Operations Research (Optimization), in Mathematical Economics (Free Trade theory), and in Cooperative Game Theory.

A first and rather simple example is the "Rucksack"– or "Knapsack"–problem, which constitutes an elementary exercise in Linear Programming.

A further example discussed by Economists exhibits rudimentary concepts of Cephoids in the context of "Ricardian Production" which establishes the comparative advantages of free trade between countries or economies with different specialization abilities (David Ricardo [23]). We mention Graham, [8], Jones [11], McKenzie [14]. Amazingly, Jones computes by hand and without any underlying idea of the structure he is dealing with, an extremal of a cephoid in 10 dimensions which is a sum of 10 deGua simplices – quite a formidable achievement. Though there are indications that these authors have been aware of a need for treating the general structure, economists have never attempted to provide a full scenery of the realm. Naturally, they feel that this is a Mathematical objective and not an Economical one.

Finally, in the context of (Cooperative) Game Theory authors have been applying the notions of a Cephoid more or less explicitly in the context of constructing "solutions" or "values" for non–side payment games or bargaining problems. Most important we mention Maschler–Perles [22]; see also

Pallaschke–Rosenmüller [18]).

It would seem that authors in Convex Geometry or Convex Analysis have not been interested in studying Cephoids. This may be so as Cephoids are subsets of $\mathbb{R}^n_+$ and we assume them to be *comprehensive*, i.e., containing the full south-west orthant of any point located within. This assumption – dictated by some obvious restrictions of models in economical context – seems to be alien to the protagonists of Convex Geometry. The general sum of simplices in $\mathbb{R}^n$ is possibly much more complicated structure, yet the study of Cephoids has eluded the attention of researchers in that field.

The term "Cephoid" will be made more suggestive during the development of our theory. Originally the term referred to the "cephalopodic" structure discovered in the family of maximal faces. The present name is more manageable.

However, we want to remind the reader of an almost synchrone sounding topic in astronomy. Here, a "Cepheid" is a variable star with periodic changes in radiance being connected to absolute brightness. As the spectrum can be well identified, a comparison of the apparent brightness and the absolute brightness allows for a rather exact determination of the distance. Thus, Cepheids have been found to establish pegs in the universe to measure the distances to galaxies and nebulae. ("The Shapley–Curtis debate"). In this context, the name of Harlow Shapley surfaces – we use the opportunity to cite both Shapleys in the context of this volume.

As to the origin of the Theory of Cephoids, this volume is based on a series of papers by the author and Diethard Pallaschke, see [18],[17], [19], [16], [15], [29].

Diethard's activity and influence was decisive for the bouquet of final results, it cannot be overestimated. Within every level of the new concept, its development and extension, its applicability and smooth handling Diethard was essentially involved; full credit goes to his essential contribution.

# Contents

# Chapter 1

# Cephoids: Sums of DeGua Simplices

Within this chapter, we provide introductory definitions and motivations. Some notations are standard and nevertheless presented. Some others – especially the essential description of our basic structure, the Cephoid – are not and we strive to provide a uniform notation. There are also a few conventions used in Convex Geometry (and not necessarily familiar to the Mathematical Economist) or in Mathematical Economy (not necessarily familiar to the Geometrician) that have to be brought on board to satisfy parts of the community embarking.

We provide a few introductory examples. These are mainly taken from the geometric background and ranging in two, three, or four dimensions, thus providing ample opportunity to view the appropriate sketches. But, on the other hand, it seems worthwhile (as a motivation for those sailing with us that are not familiar with the waters of Convex Geometry) to provide the basic applications we will treat *in extenso* later on.

# 1 Notations and Definitions

We consider specific convex compact polyhedra located within the nonnegative orthant of $\mathbb{R}^n$. To this end, let $\boldsymbol{I} := \{1, \ldots, n\}$ denote the set of coordinates of $\mathbb{R}^n$, the positive orthant is $\mathbb{R}^n_+ := \{\boldsymbol{x} = (x_1, \ldots, x_n) \mid x_i \geq 0 \, , i \in \boldsymbol{I}\}$. Let $\boldsymbol{e}^i$ denote the $i^{th}$ unit vector of $\mathbb{R}^n$ and $\boldsymbol{e} := (1, \ldots, 1) = \sum_{i=1}^n \boldsymbol{e}^i \in \mathbb{R}^n$ the "diagonal" vector. The notation ***CovH*** $A$ is used to denote the ***convex hull*** of a subset $A$ of $\mathbb{R}^n_+$.

Given a vector $\boldsymbol{a} = (a_1, \ldots, a_n) > \mathbf{0} \in \mathbb{R}^n_+$ with positive coordinates, we consider the $n$ multiples $\boldsymbol{a}^i := a_i \boldsymbol{e}^i \;\; (i \in \boldsymbol{I})$ of the unit vectors. The the set

(1) $$\Delta^{\boldsymbol{a}} := \boldsymbol{CovH} \left\{ \boldsymbol{a}^1, \ldots, \boldsymbol{a}^n \right\}$$

is the ***Standard Simplex*** or for short, the ***Simplex*** resulting from $\boldsymbol{a}$ (we use capitals in this context). Figure 1.1 represents a Simplex in $\mathbb{R}^3_+$.

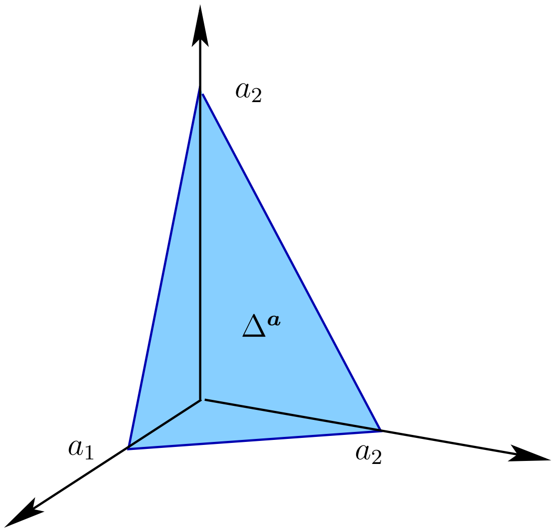


Figure 1.1: The Simplex in $\mathbb{R}^3_+$ generated by $\boldsymbol{a} = (a_1, a_2, a_3)$

Next, for $\boldsymbol{J} \subseteq \boldsymbol{I}$ we write $\mathbb{R}^n_{\boldsymbol{J}} := \{\boldsymbol{x} \in \mathbb{R}^n \,|\, x_i = 0 \; (i \notin \boldsymbol{J})\}$. Accordingly, we obtain the ***Standard Subsimplex*** or just ***Subsimplex***

(2)
$$\Delta^{\boldsymbol{a}}_{\boldsymbol{J}} := \{x \in \Delta^{\boldsymbol{a}} \,|\, x_i = 0 \; (i \notin \boldsymbol{J})\} = \Delta^{\boldsymbol{a}} \cap \mathbb{R}^n_{\boldsymbol{J}} = \boldsymbol{CovH} \left\{ \boldsymbol{a}^i \,\middle|\, i \in \boldsymbol{J} \right\} .$$

There is a second type of simplex we want to associate with a positive vector $\boldsymbol{a} \in \mathbb{R}^n_+$. This is the one spanned by the vectors $\boldsymbol{a}^i$ plus the vector $\mathbf{0} \in \mathbb{R}^n_+$, that is

$$\Pi^{\boldsymbol{a}} \;:=\; \boldsymbol{CovH}\left\{\mathbf{0}, \boldsymbol{a}^1, \ldots, \boldsymbol{a}^n\right\} \;. \tag{3}$$

In order to distinguish both types verbally we call $\Pi^{\boldsymbol{a}}$ the ***deGua Simplex*** associated to $\boldsymbol{a}$, paying homage to J.P. de Gua de Malves [4] who generalized the Pythagorean theorem for simplices of this type. Consistently we write, for any $\boldsymbol{J} \subseteq \boldsymbol{I}$ the corresponding ***deGua Subsimplex*** of $\Pi^{\boldsymbol{a}}$ as

$$\begin{aligned} \Pi^{\boldsymbol{a}}_{\boldsymbol{J}} \;&:=\; \{x \in \Pi^{\boldsymbol{a}} \,|\, x_i = 0 \; (i \notin \boldsymbol{J})\} \\ &=\; \Pi^{\boldsymbol{a}} \cap \mathbb{R}^n_{\boldsymbol{J}} \;=\; \boldsymbol{CovH}\left\{\{\mathbf{0}\}\{\boldsymbol{a}^i \,\middle|\, i \in \boldsymbol{J}\}\right\} \;. \end{aligned} \tag{4}$$

A set $A \subseteq \mathbb{R}^n_+$ is called ***comprehensive*** if, for any $\boldsymbol{x} \in A$ it contains all vectors $\boldsymbol{y} \in \mathbb{R}^n_+$ satisfying $\boldsymbol{y} \leq \boldsymbol{x}$ (inequalities between vectors to be interpreted coordinatewise). The ***comprehensive*** hull of a set $A \subseteq \mathbb{R}^n_+$ is given by

$$\boldsymbol{CmpH}\, A \;:=\; \left\{\boldsymbol{y} \in \mathbb{R}^n_+ \,\middle|\, \exists \boldsymbol{x} \in A \;:\; \boldsymbol{y} \leq \boldsymbol{x}\right\} \;.$$

clearly we have also

$$\Pi^{\boldsymbol{a}} \;=\; \boldsymbol{CmpH}\, \Delta^{\boldsymbol{a}} \quad , \quad \Pi^{\boldsymbol{a}}_{\boldsymbol{J}} \;=\; \boldsymbol{CmpH}\, \Delta^{\boldsymbol{a}}_{\boldsymbol{J}} \;,$$

and Figure 1.2 indicates the deGua Simplex $\Pi^{\boldsymbol{a}}$ generated by $\boldsymbol{a}$. All vectors below $\Delta^{\boldsymbol{a}}$ including the vector $\mathbf{0} \in \mathbb{R}^3_+$ are included.

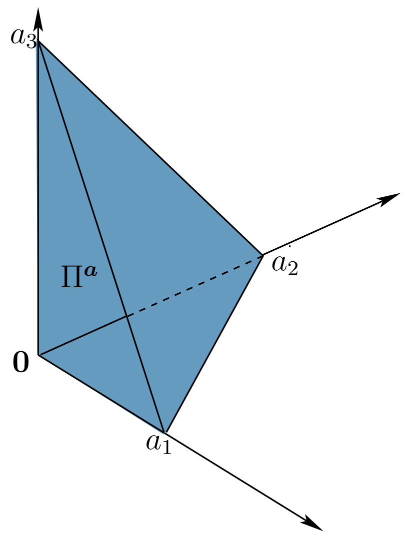


Figure 1.2: The deGua Simplex $\Pi^{\boldsymbol{a}}$; $\boldsymbol{a} = (a_1, a_2, a_3)$

In the terminology of Convex Analysis, $\Delta^{\boldsymbol{a}}$ is the ***maximal (outward) face*** of $\Pi^{\boldsymbol{a}}$. Here we prefer the MathEcon notation, calling $\Delta^{\boldsymbol{a}}$ the ***Pareto face*** of $\Pi^{\boldsymbol{a}}$.

A ***normal*** to some convex set $\boldsymbol{C}$ in some (boundary) point $\bar{\boldsymbol{x}} \in C$ is a vector that generates a separating hyperplane. A vector that is a normal to some face $\boldsymbol{F}$ of a convex set $\boldsymbol{C}$ in *all* points of $\boldsymbol{F}$is called normal to $\boldsymbol{F}$.

$\Delta^{\boldsymbol{a}}$ admits of a normal

$$\mathfrak{n}^{\boldsymbol{a}} := \left(\frac{1}{a_1}, \ldots, \frac{1}{a_n}\right) .$$

All other normals to $\Delta^{\boldsymbol{a}}$ are positive multiples of this one, i.e., the ***normal cone*** to $\Delta^{\boldsymbol{a}}$ is

$$\mathcal{N}^{\boldsymbol{a}} := \{t\mathfrak{n}^{\boldsymbol{a}} \,|\, t > 0\} .$$

We refer to this situation saying that the normal of $\Delta^{\boldsymbol{a}}$ is "unique up to a multiple" or "essentially unique" etc.

The projection of $\mathfrak{n}^{\boldsymbol{a}}$ to $\mathbb{R}^n_{\boldsymbol{J}+}$ is denoted by $\mathfrak{n}^{\boldsymbol{a}}_{\boldsymbol{J}} := \mathfrak{n}^{\boldsymbol{a}} \mid_{\mathbb{R}^n_{\boldsymbol{J}+}}$. The subface $\Delta^{\boldsymbol{a}}_{\boldsymbol{J}}$ of the Pareto face admits of a normal cone $\mathcal{N}^{\boldsymbol{a}}_{\boldsymbol{J}}$ generated by the normals

$$\{\mathfrak{n}^{\boldsymbol{a}}_{\boldsymbol{J}'} \,|\, \boldsymbol{J} \subseteq \boldsymbol{J}' \subseteq \boldsymbol{I}\} .$$

Certain operations on convex sets are a standard in Convex Geometry. For two subsets $A, B \subseteq \mathbb{R}^n_+$ the ***algebraic*** or ***Minkowski*** sum is

$$A + B := \{\boldsymbol{x} + \boldsymbol{y} \,|\, \boldsymbol{x} \in A,\ \boldsymbol{y} \in B\}$$

and for $\lambda \in \mathbb{R}_+$ the multiple of $A$ is defined via

$$\lambda A := \{\lambda \boldsymbol{x} \,|\, \boldsymbol{x} \in A\} .$$

If $A$ and $B$ are convex sets, then the sets $A + B$ and $\lambda A$ are also convex and if $A$ and $B$ are polytopes, so are $A + B$ and $\lambda A$.

Now we are in the position to define the subject of this treatise, a ***Cephoid*** which is a *Minkowski sum of deGua Simplices*. More precisely, we introduce for some integer $K$ the set

$$\boldsymbol{K} := \{1, \ldots, K\} .$$

**Definition 1.1.** *Let* $\left\{\boldsymbol{a}^{(k)}\right\}_{k \in \boldsymbol{K}}$ *denote a family of positive vectors and let*

$$\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}} \tag{5}$$

*be the Minkowski sum. Then* $\Pi$ *is called a* ***Cephoid****.*

There are basically two ways to describe the structure of a polyhedron or, more specifically, a Cephoid. One approach is provided by constructing the extremal points and the other one consists of a description of the faces or rather the maximal faces of such a polyhedron. Both methods are in a certain sense dual to each other and both sets of data provide easy access to the other one. E.g., if we have a description of the maximal faces, then it is not hard to also provide a list of the extremal points or ***vertices*** of the polyhedron. First of all we concerned with the maximal outward faces of a Cephoid which constitute the *Pareto subface.* We provide the following

**Definition 1.2.** *1. A face $\boldsymbol{F}$ of a Cephoid $\Pi$ is* ***maximal*** *if, for any face $\boldsymbol{F}^0$ of $P$ with $\boldsymbol{F} \subseteq \boldsymbol{F}^0$ it follows that $\boldsymbol{F} = \boldsymbol{F}^0$ is true.*

*2. The (**outward** or) **Pareto surface** of a compact convex set (specifically: of a Cephoid $\Pi$) is the set*

$$\partial\Pi \;:=\; \{\boldsymbol{x} \in \Pi \mid \; \nexists \boldsymbol{y} \in \Pi, \nexists i \in \boldsymbol{I} \;:\; \boldsymbol{y} \geq \boldsymbol{x},\; y_i > x_i \;\} \,. \tag{6}$$

*3. The points of the Pareto surface are called **Pareto efficient**.*

*4. Maximal faces in the Pareto surface are called **Pareto faces**.*

The vector $\mathbf{0}$ is always an extremal point of a Cephoid in $\mathbb{R}^n$ but it is not Pareto efficient. All other extremal points of a Cephoid are Pareto efficient.

**Definition 1.3.** *Let $\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ be a Cephoid and let $i \in \boldsymbol{I}$. Define*

$$\Pi^{(-)i} \;:=\; \Pi \cap \mathbb{R}_{\boldsymbol{I}\setminus\{i\}} \,. \tag{7}$$

*Then $\Pi^{(-i)}$ constitutes a maximal face of $\Pi$ but not a Pareto face. $\Pi^{(-i)}$ is called the **$\boldsymbol{i}$–face** of $\Pi$.*

Indeed, $\Pi^{(-i)}$ is clearly a maximal face but not located in the Pareto surface as not all points of $\Pi^{(-i)}$ are Pareto efficient (Definition 1.2). All maximal faces of a Cephoid $\Pi$ are either Pareto faces or intersections of $\Pi$ with some $\mathbb{R}_{\boldsymbol{I}\setminus\{i\}}$ as in (7). On the other hand, $\Pi^{(-i)} \subseteq \mathbb{R}^n_{\boldsymbol{I}\setminus\{i\}+}$ is a Cephoid in its own right, generated by the family of vectors

$$\left\{\boldsymbol{a}^{(k)}_{\boldsymbol{I}\setminus\{i\}}\right\}_{k \in \boldsymbol{K}} \,.$$

We also introduce a notation for the reduction of a Cephoid in members of the family as follows.

**Definition 1.4.** *Let* $\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ *be a Cephoid and let* $k \in \boldsymbol{K}$*. Define*

$$\Pi^{[-k]} = \sum_{k \in \boldsymbol{K} \setminus \{k\}} \Pi^{\boldsymbol{a}^{(k)}} \tag{8}$$

*Then* $\Pi^{[-k]}$ *is called the* ***k–missing*** *Cephoid to* $\Pi$*. This is a Cephoid in* $\mathbb{R}^n_+$*.*

Clearly, $\Delta^{\boldsymbol{a}}$ is the only Pareto face of $\Pi^{\boldsymbol{a}}$; similarly for $\Delta^{\boldsymbol{a}}_{\boldsymbol{J}}$ and $\Pi^{\boldsymbol{a}}_{\boldsymbol{J}}$. The Pareto surface of a general Cephoid will be the main topic of our discussion.

The following well known theorem (see e.g. Ewald [7] or Pallaschke–Urba´nski [20]) is basic tool for testing Pareto efficiency of a sum of polyhedra.

**Theorem 1.5.** *Let* $A$ *and* $B$ *be compact convex sets and let* $\boldsymbol{x} \in A$ *and* $\boldsymbol{y} \in B$ *be Pareto efficient vectors of* $A$ *and* $B$ *respectively. Then* $\boldsymbol{x} + \boldsymbol{y}$ *is a Pareto efficient vector in* $A + B$ *if and only if the normal cone of* $A$ *in* $\boldsymbol{x}$ *and the normal cone of* $B$ *in* $\boldsymbol{y}$ *have a nonempty intersection. That is, if and only if* $A$ *and* $B$ *admit of a joint normal in* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *respectively.*

*On the other hand, every extremal point* $\boldsymbol{z}$ *of* $A + B$ *is the sum* $\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y}$ *of two extremal points* $\boldsymbol{x} \in A$ *and* $\boldsymbol{y} \in B$*, such that the intersection of the normal cones of* $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ *has a nonempty intersection.*

Similarly, we have for faces or extremal sets of two convex and compact sets the following

**Theorem 1.6.** *Let* $A$ *and* $B$ *be compact convex sets and let* $\boldsymbol{F}^1 \in A$ *and* $\boldsymbol{F}^2 \in B$ *be faces of* $A$ *and* $B$ *respectively. Then* $\boldsymbol{F}^1 + \boldsymbol{F}^2$ *is a face of* $A + B$ *if and only if the normal cone of* $\boldsymbol{F}^1$ *with respect to* $A$ *and the normal cone of* $\boldsymbol{F}^2$ *with respect to* $B$ *have a nonempty intersection. That is, if and only if* $A$ *and* $B$ *admit of a joint normal in* $\boldsymbol{F}^1$ *and* $\boldsymbol{F}^2$ *respectively.*

*On the other hand, every face* $\boldsymbol{F}$ *of* $A + B$ *is the sum* $\boldsymbol{F} = \boldsymbol{F}^1 + \boldsymbol{F}^2$ *of two faces* $\boldsymbol{F}^1$ *of* $A$ *and* $\boldsymbol{F}^2$ *of* $B$*, such that the intersection of normal cones of* $\boldsymbol{F}, \boldsymbol{F}^1, \boldsymbol{F}^2$ *have a nonempty intersection.*

**Remark 1.7.** *Let* $\{\boldsymbol{a}^{(k)}\}_{k \in \boldsymbol{K}}$ *denote a family of positive vectors and let* $\Pi^{(k)} := \Pi^{\boldsymbol{a}^{(k)}}$ *denote the corresponding deGua Simplices. Then*

$$\overline{\Pi} := \boldsymbol{CovH}\left\{ \bigcup_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}} \right\} = \boldsymbol{CovH}\left\{ \bigcup_{k \in \boldsymbol{K}} \Pi^{(k)} \right\} \tag{9}$$

*is a deGua Simplex. We will use the term "maximum" for this convex hull referring to the partial ordering induced by inclusion on convex sets; hence we write*

$$\overline{\Pi} \;=\; \bigvee_{k \in \boldsymbol{K}} \Pi^{(k)}. \tag{10}$$

*Clearly, if*

$$a_i^\star \;:=\; \max_{k \in \boldsymbol{K}} \; a_i^{(k)} \quad (i \in \boldsymbol{I}) \;,$$

*then* $\boldsymbol{a}^\star \;:=\; (a_i^\star)_{i \in \boldsymbol{I}}$ *yields* $\overline{\Pi} \;=\; \Pi^{\boldsymbol{a}^\star}$*. The operation* $\bigvee$ *is well defined for any family of Subsimplices as well. Therefore, given a family* $\left\{c_{(k)}\right\}_{k \in \boldsymbol{K}}$ *of positive coefficients and a family* $\left\{\boldsymbol{J}^{(k)}\right\}_{k \in \boldsymbol{K}}$*, the deGua Simplex*

$$\bigvee_{k \in \boldsymbol{K}} c_k \Pi^{(k)}_{\boldsymbol{J}^{(k)}}. \tag{11}$$

*is well defined.*

# 2 First Examples and Non–Degeneracy

Examples in lower dimensions can be informally discussed as the geometrical aspects are obvious. Indeed, any comprehensive compact polyhedron $\Pi$ in two dimensions is a Cephoid. For, such a polyhedron is completely described by the line segments that constitute the Pareto surface and these line segments uniquely determine the deGua Simplices involved.

To make this somewhat more precise we discuss, in what follows, the nature of Cephoids in 2 dimensions.

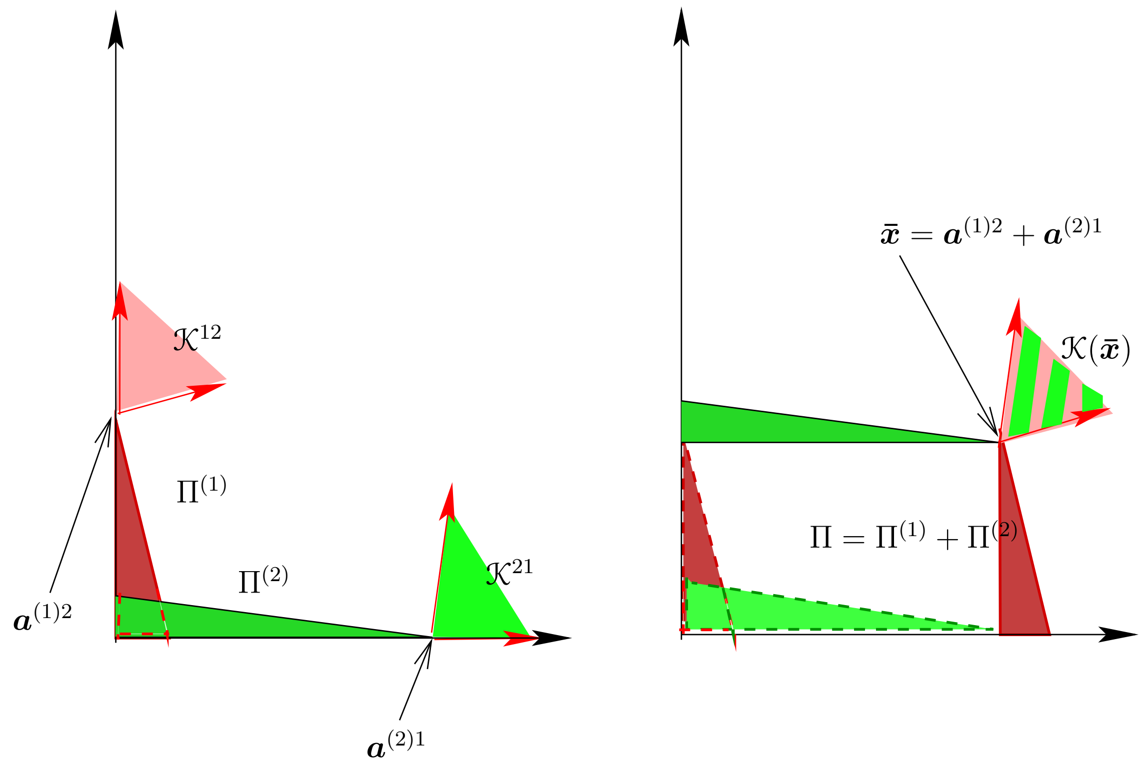


Figure 2.1: Cephoids and normal cones in $\mathbb{R}^2_+$

**Example 2.1.** *The first sketch (Figure 2.1) shows the situation for two deGua Simplices $\Pi^{(1)}$ and $\Pi^{(2)}$ in two dimensions. Each deGua Simplex $\Pi^{(k)}$ is a triangle determined by some positive $\boldsymbol{a}^{(k)}$, represented by its extremals $\boldsymbol{a}^{(k)1}, \boldsymbol{a}^{(k)2}$.*

*The resulting Cephoid, i.e., the sum $\Pi = \Pi^{(1)} + \Pi^{(2)}$ of the two triangles in $\mathbb{R}^2_+$ – represented by the right hand side in Figure 2.1 – is a polyhedron with Pareto surface given by two line segments. These line segments are translates of the ones characterizing the DeGua Simplices in the left hand sketch. The translation* *takes place by one of the extremal points of the other deGua Simplex respectively. The extremals* $\boldsymbol{a}^{(1)1}$ *of* $\Pi^{(1)}$ *and* $\boldsymbol{a}^{(2)2}$ *of* $\Pi^{(2)}$ *ad up to a vertex/extremal* $\bar{\boldsymbol{x}} = \boldsymbol{a}^{(1)1} + \boldsymbol{a}^{(2)2}$ *of* $\Pi$. *The normal cone* $\mathcal{K}(\bar{\boldsymbol{x}})$ *at* $\Pi$ *in* $\bar{\boldsymbol{x}}$ *is indicated. This is the intersection of the normal cone* $\mathcal{K}^1$ *at* $\Pi^{(1)}$ *in* $\boldsymbol{a}^{(1)1}$ *and the normal cone* $\mathcal{K}^2$ *at* $\Pi^{(2)}$ *in* $\boldsymbol{a}^{(2)2}$.

*Thus, figure 2.1 shows the most simple version of a Cephoid and demonstrates the essential role of the normal cones of the deGua Simplices involved.*

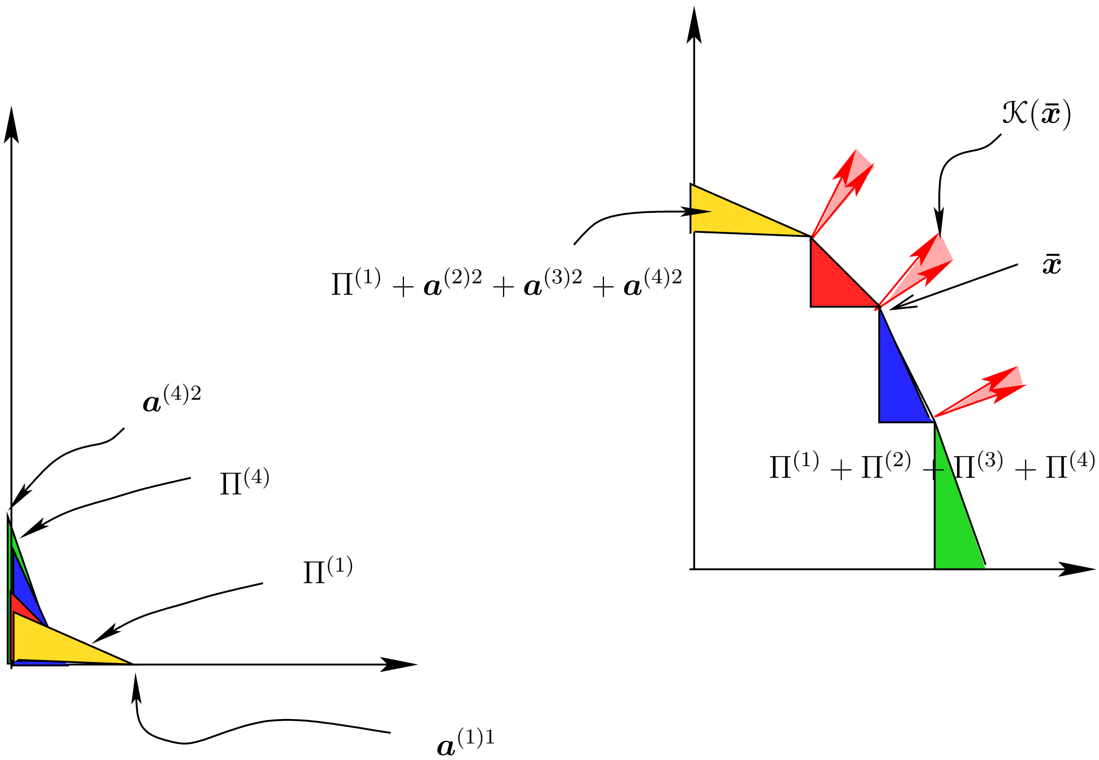


Figure 2.2: Cephoids in $\mathbb{R}^2_+$: Summing 4 deGua Simplices

*Similarly, Figure 2.2 depicts a Cephoid in* $\mathbb{R}^2_+$ *which is a sum of four deGua Simplices. The Pareto surface consists of four line segment; each one of these is a translate of one of the generating Simplices* $\Delta^{\boldsymbol{a}^{(k)}}$. *The translation is performed by extremals of the other deGua Simplices. The vertex/extreme point* $\bar{\boldsymbol{x}}$ *is a sum of vertices/extreme points of the four deGua Simplices involved. E.g., if, in the right hand sketch, we count the deGua Simplices involved according to increasing slope, then we find*

$$\bar{x} = \boldsymbol{a}^{(1)1} + \boldsymbol{a}^{(2)1} + \boldsymbol{a}^{(3)2} + \boldsymbol{a}^{(4)2} \ .$$

*The normal cone* $\mathcal{K}(\bar{\boldsymbol{x}})$ *is the intersection of the normal cones* $\mathcal{K}^{(2)1}$ *(at* $\Pi^{(2)}$ *in* $da^{(2)1}$ *and* $\mathcal{K}^{(3)2}$ *(at* $\Pi^{(3)}$ *in* $da^{(3)2}$.

*This way the generation of Cephoids in* $\mathbb{R}^2$ *appears to be straightforward. .*

*Clearly, the vertices of the Cephoid are obtained (uniquely!) by a sum of vertices of the deGua Simplices involved. The converse is not true. Not every sum of vertices of the deGua Simplices involved results in a vertex of the Cephoid. This is a consequence of the results of Convex Geometry explained in* SECTION *1: the sum of two vertices is a vertex if and only if both vertices admit of a common normal .*

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We demonstrate that *any* convex compact comprehensive polyhedron in $\mathbb{R}^2$ is a sum of triangles as depicted in Figure 2.2. For more than two dimensions, this statement is not true – it turns out that in 3 dimensions convex compact comprehensive polyhedra in general cannot be expected to be Cephoidal.

**Example 2.2.** *Consider a compact convex comprehensive polyhedron* $\Pi \subseteq \mathbb{R}^2_+$ *as in Figure 2.3. Observe that the Pareto surface of this polyhedron consists of finitely many line segments including a finite number of vertices or extremal points. Any such line segment is described by two vertices (cf. Figure 2.3).*

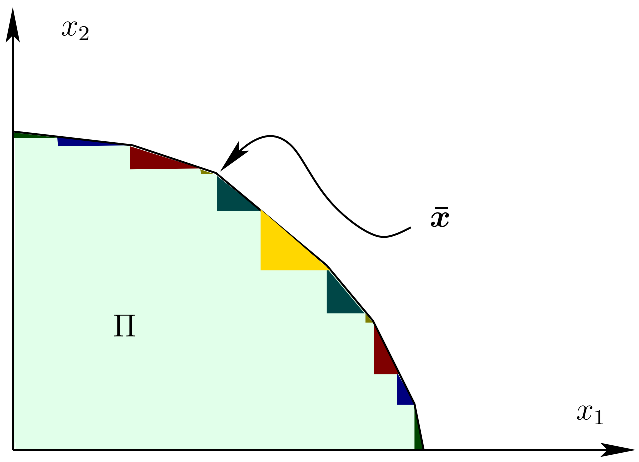


Figure 2.3: A general Cephoid in 2 dimensions

*Let there be* $K$ *different line segments. For line segment* $k \in \boldsymbol{K} = \{1, \ldots, K\}$*, draw a line parallel to the* $x_1$ *axis through the right hand vertex and a line parallel to the* $x_2$ *axis through the left hand vertex of line segment* $k \in \boldsymbol{K}$*. The intersection of these two lines together with the two extremals defines a triangle. This triangle constitutes a translated deGua simplex in two dimensions. Call the lengths of the lower and the left side of this triangle* $a_1^{(k)}$ *and* $a_2^{(k)}$ *respectively.*

*Then define*

$$\boldsymbol{a}^{(k)} \; := \; (a_1^{(k)}, a_2^{(k)}). \tag{1}$$

*Obviously the polyhedron* $\Pi$ *can be written as*

$$\Pi \;=\; \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}. \tag{2}$$

*In other words,* $\Pi$ *is the sum of the triangles (i.e., deGua Simplices) as constructed above. Hence, in a very elementary way,* $\Pi$ *is a cephoid as the sum of the the deGua Simplices involved.*

*If it so happens that the slopes* $\frac{a_1^{(k)}}{a_2^{(k)}}$ *of the line segments are strictly decreasing in* $k$ *(i.e. triangles in Figure 2.3 enumerated from "left to right"), then the Pareto faces of* $\Pi$ *(i.e. the line segments) are given by*

$$\boldsymbol{F}^{(k)} \; := \; \sum_{l<k} a^{(l)1} + \Delta^{\boldsymbol{a}^{(k)}} + \sum_{l>k} a^{(l)2} \quad (k\in\boldsymbol{K}). \tag{3}$$

*Also, the extremal* $\bar{\boldsymbol{x}}$ *of* $\Pi$ *as indicated would be given by*

$$\bar{\boldsymbol{x}} = \sum_{k\le 4} \boldsymbol{a}^{(k)1} + \sum_{k>4} \boldsymbol{a}^{(k)2}. \tag{4}$$

*In general, if the ordering of the slopes is arbitrary, we just have to to employ a suitable permutation generating the correct ordering. We can then write down the analogs to (3) and (4) accordingly.*

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Combining we obtain

**Theorem 2.3.** *A comprehensive compact convex polyhedron in* $\mathbb{R}^2_+$ *is a Cephoid.*

**Example 2.4.** *Now we turn to three dimensions. We continue our preliminary geometrical approach, discussing the sum of two deGua Simplices in* $\mathbb{R}^3$*, say* $\Pi^{\boldsymbol{a}}$ *(blue) and* $\Pi^{\boldsymbol{b}}$ *(red). The sum is the Cephoid* $\Pi = \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}$*. Figure 2.4 shows a version of both the original Simplices and the resulting Cephoid.*

*Now, the normal of* $\Delta^{\boldsymbol{a}}$ *is also a feasible normal for the extremal* $\boldsymbol{b}^{(1)}$ *of* $\Delta^{(b)}$*. Hence, there appears a translate* $\Delta^{\boldsymbol{a}} + \boldsymbol{b}^{(1)}$ *on the Pareto surface* $\partial\Pi$ *of* $\Pi$*. A similar remark applies for* $\Delta^{\boldsymbol{b}} + \boldsymbol{a}^{(2)}$*.*

*However, there appears a new shape on the Pareto surface of* $\Pi$*, a parallelogram or* ***rhombus****.*

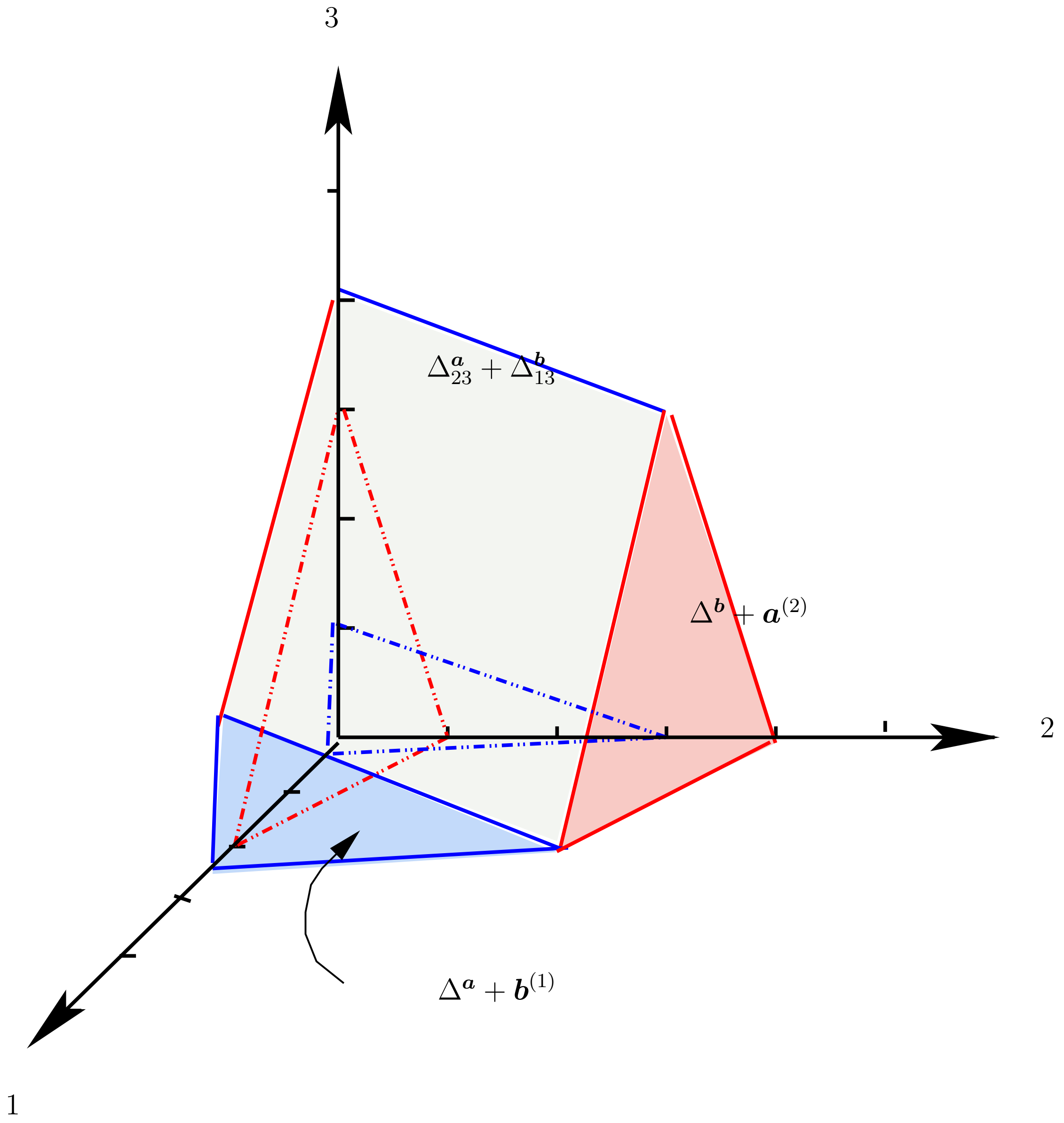


Figure 2.4: The sum of 2 deGua Simplices in $\mathbb{R}^3_+$

*In order to appreciate the situation regarding the rhombus, consider Figure 2.5 which depicts the situation in* $\Pi^{\boldsymbol{a}}$*. First, the normal* $\mathfrak{n}^{\boldsymbol{a}}$ *is the one to the Simplex* $\Delta^{\boldsymbol{a}}$*. There is also a normal* $\mathfrak{n}^{\boldsymbol{a},23}$ *to* $\Delta_{23}^{\boldsymbol{a}}$ *locaten within* $\mathbb{R}^3_{23+}$*. Hence, the normal cone to* $\Delta_{23}^{\boldsymbol{a}}$ *is seen to be* $\mathcal{K}^{\boldsymbol{a},23}$*.*

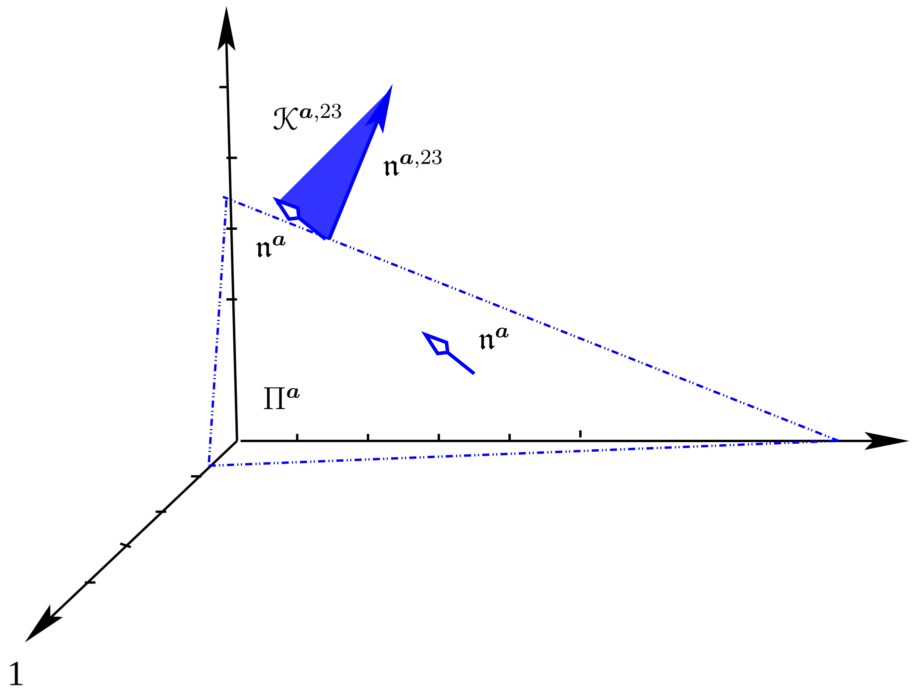


Figure 2.5: Normals to $\Delta^{\boldsymbol{a}}$

*Similarly, we observe the situation on* $\Delta^{\boldsymbol{b}}$ *as depicted in Figure 2.6. Again we start with the normal* $\mathfrak{n}^{\boldsymbol{b}}$ *to* $\Delta^{\boldsymbol{b}}$*. Also a normal* $\mathfrak{n}^{\boldsymbol{b},13}$ *to* $\Delta_{13}^{\boldsymbol{b}}$ *is locaten within* $\mathbb{R}^3_{13+}$*. Therefore, the normal cone to* $\Delta_{13}^{\boldsymbol{b}}$ *is seen to be* $\mathcal{K}^{\boldsymbol{b},13}$*.*

*With some phantasy, the reader realizes that the cones* $\mathcal{K}^{\boldsymbol{a},23}$ *and* $\mathcal{K}^{\boldsymbol{b},13}$ *do have a nonempty intersection which defines a common normal to these two Subsimplices.*

*Therefore, the sum* $\Delta_{23}^{\boldsymbol{a}} + \Delta_{13}^{\boldsymbol{b}}$ *is a Pareto face of* $\Pi$*. The shape of this face is the sum of the two line segments involved, which is the rhombus.*

*There are no further Pareto faces of* $\Pi$*, thus the Pareto surface* $\partial\Pi$ *consists of:*

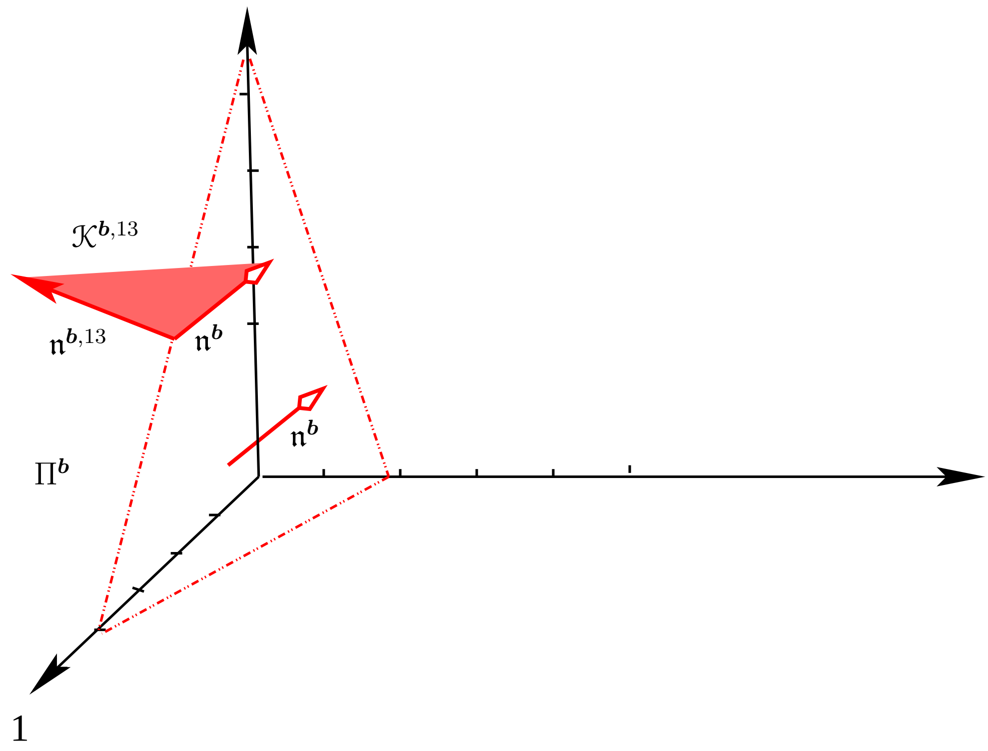


Figure 2.6: Normals to $\Delta^{\boldsymbol{b}}$

1. *Translates of* $\Delta^{\boldsymbol{a}}$ and $\Delta^{\boldsymbol{b}}$,
2. *The rhombus* $\Delta^{\boldsymbol{a}}_{23} + \Delta^{\boldsymbol{b}}_{13}$ .

*Next, a sum of three deGua Simplices is depicted in Figure 2.7. It can be seen (and will later on be clarified by our general theory) that*

1. *Each of the generating Simplices yields a translate on the Pareto surface* $\partial\Pi$ *of the Cephoid.*
2. *Any two Simplices generate a rhombus on the Pareto surface* $\partial\Pi$ *of the Cephoid.*

*We carry this visual argument one step further. Let us add a further deGua simplex to the Cephoid of Figure 2.7 such that its surface has a joint normal with the central vertex of that Cephoid.*

*The result is the Cephoid indicated in Figure 2.8. First we observe the translates of each of the generating Simplices by means of three vertices of the other deGua*

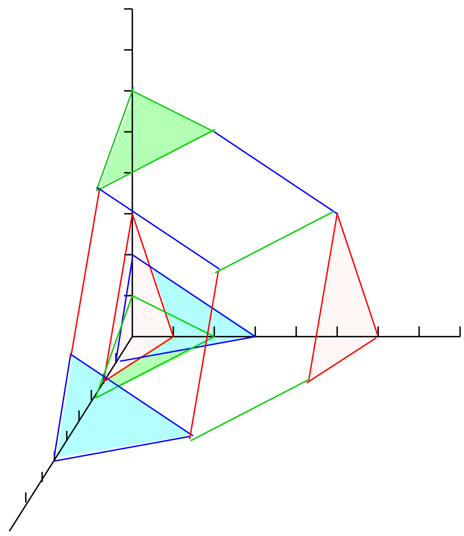

Figure 2.7: The sum of three deGua Simplices

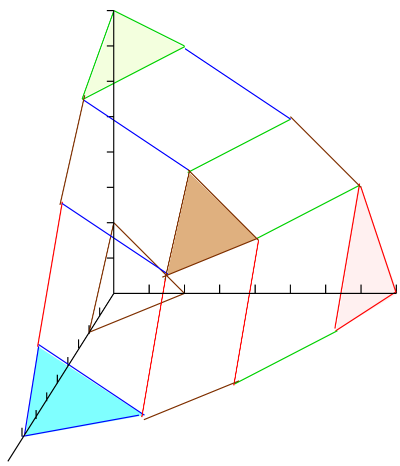

Figure 2.8: Adding a further deGua simplex

*Simplices. Then there is a rhombus generated by each pair of the Simplices involved with suitable vertices of the third and fourth deGua Simplex involved.*

◦ ∼∼∼∼∼∼∼ ◦

The above examples involve deGua Simplices that commonly enjoy a distinctive feature: the deGua Simplices are "non homothetic" in a very strict sense. Not only do they not admit of a joint normal. But also any two Subsimplices admit of at most one joint normal – if any.

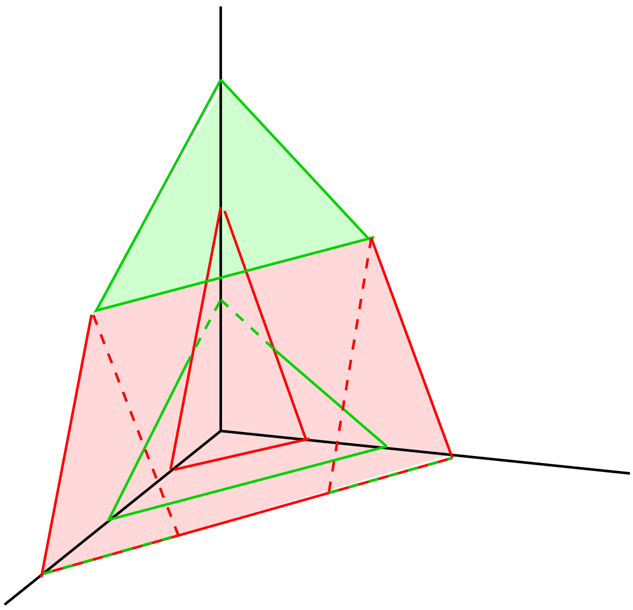

Figure 2.9: A sum of two prisms with degeneracy

This last requirement is a substantial one. What happens by omitting it, is demonstrated in Figure 2.9. Here the two generating summands have Subsimplices in the 12–plane that are homothetic, i.e., translated dilatations of each other, and hence admit of exactly the same normal cone. As a result, there appears a trapezoidal face on the surface of the Cephoid that is the sum of the red Simplex and the green Subsimplex in the 12–plane. Obviously the statement that translates of any generating Simplex appear on the Pareto surface of the Minkowski sum is incorrect in this case. Similarly, it is not true that any two Simplices generate exactly one rhombus.

We consider a family including homothetic deGua (Sub-) Simplices *degenerate*. This applies to the example of Figure 2.9. By contrast, the previous examples (e.g. Figure 2.7, Figure 2.8) will called *nondegenerate*.

We provide a precise definition of the term "nondegeneracy". Essentially one has to make sure that the dimension of the joint normal cones of a family of Subsimplices is obtaine by counting the coordinate indices involved. Right now, we provide a formal definition only, the interpretation of the equations involved in terms of normals of Simplices will be gradually become clear.

Thus,the appropriate version of a nondegenerate family is best captured by the following definition.

**Definition 2.5.** *A family* $\boldsymbol{a}^\bullet = \left\{\boldsymbol{a}^{(k)}\right\}_{k=1}^K$ *of positive vectors is called* ***non-degenerate*** *if the following conditions hold true:*

1. *For any system of nonempty index sets* $\boldsymbol{J}^{(1)}, \ldots, \boldsymbol{J}^{(K)} \subseteq \boldsymbol{I}$ *with*

$$\bigcup_{k \in \boldsymbol{K}} \boldsymbol{J}^{(k)} \ = \ \boldsymbol{I}$$

*the system of linear homogeneous equations in the variables* $x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_K$ *given by*

$$a_i^{(k)} x_i - \lambda_k = 0 \quad (i \in \boldsymbol{J}^{(k)},\ k \in \boldsymbol{K}) \tag{5}$$

*has a space of solutions* $\Pi$ *of dimension*

$$\dim \Pi = n + K - \sum_{k \in \boldsymbol{K}} j_k \tag{6}$$

*with* $j_k = |\boldsymbol{J}^{(k)}|$.

2. *For any* $\boldsymbol{I}^{(0)} \subseteq \boldsymbol{I}$ *the restricted system*

$$\boldsymbol{a}^\bullet\Big|_{\boldsymbol{I}^{(0)}} := \left(\boldsymbol{a}^{(k)}\Big|_{\boldsymbol{I}^0}\right)_{k \in \boldsymbol{K}} \tag{7}$$

*obtained by restricting the vectors to* $\boldsymbol{I}^{(0)}$ *satisfies the condition of* item *1 in* $\mathbb{R}^{\boldsymbol{I}^{(0)}}$.

The term ***nondegenerate*** will also be applied to the corresponding family of deGua Simplices

$$\left\{\Pi^{\boldsymbol{a}^{(k)}}\right\}_{k=1}^K \ =: \ \left\{\Pi^{(k)}\right\}_{k=1}^K \tag{8}$$

as well as to the Cephoid generated by a nondenerate family $\boldsymbol{a}^\bullet$.

**Theorem 2.6.** *A nondegenerate Cephoid is uniquely represented as a sum of (nonhomothetic) deGua Simplices.*

The proof follows from [30] Theorem 3.2.8.

The following sketch of Figure2.10 suggests the general shape of a Cephoid in $\mathbb{R}^3$.

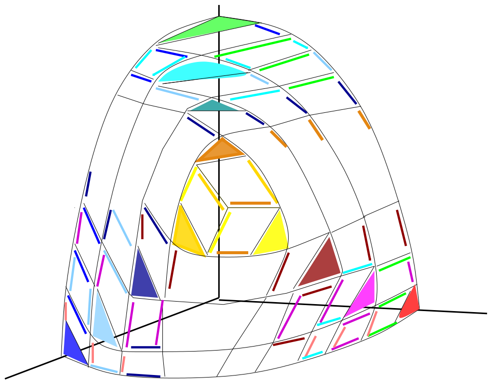

Figure 2.10: The general Cephoid in $\mathbb{R}^3$

For dimensions exceeding 3 the picture gets increasingly complicated. The Pareto surface of a Cephoid involves not just translated Simplices and rhombi. In each dimension new types of polyhedra appear on the surface, being generated as sums of certain Subsimplices of the deGua Simplices involved.

# Chapter 2

# The Structure of $\partial\Pi$: Representation

Viewing the examples of the previous chapter we notice that the structure of the Pareto surface depends essentially on the relations between various subsimplices and their normal cones. More exactly, a Pareto face is provided by a nonempty intersection of the normal cones of the various subsimplices involved in its construction. What is less important is the actual size of the various Pareto faces.

We would like to represent the relative position of the maximal faces by implementing a bijection of the Pareto surface $\partial\Pi \subseteq \mathbb{R}^n$ onto the $K$–fold unit Simplex $K\Delta^{\boldsymbol{e}} \subseteq \mathbb{R}^n$ such that the structure of the Pareto surface is preserved. This means that the partially ordered set of Pareto faces can be uniquely recognized on that Simplex. Since we can represent $K\Delta^{\boldsymbol{e}}$ in $n-1$ dimensions this allows for better insight regarding the Pareto surface for up to 4 dimensions.

Thus, the procedure serves to greatly improve the understanding of the "typical" appearance of the various shapes of possible Pareto surfaces of Cephoids.

Within this Chapter, we describe the appropriate version of "projecting" the surface of a cephoid on the Simplex $K\Delta^{\boldsymbol{e}}$ in a "canonical fashion". We call the result – the "Canonical Representation".

We then proceed by applying our representation to the simple examples induced by Chapter 1.

# 1 The Canonical Representation

The structure exhibited in Figures 2.4, 2.7, and 2.8 reflects a certain position of the Pareto faces; in the three dimensional case we obtain triangles and diamonds. Given a set of normals, the relative size of the faces is less relevant. In what follows we want to emphasize the structure and make it more visible. To this end we construct a mapping of the surface structure of a Cephoid on a suitable positive multiple of the unit Simplex such that both structures are "combinatorically equivalent", i.e., the posets (partially ordered sets) of subfaces are isomorphic (see Ewald [7]).

The poset will be exhibited on a multiple $K\Delta\boldsymbol{e} = \Delta^{K\boldsymbol{e}}$ of the unit simplex $\Delta^{\boldsymbol{e}}$. Naturally, this multiple is generated by a family of copies of the unit Simplex $\Delta^{\boldsymbol{e}}$. Clearly, this family does not satisfy the requirement of nondegeneracy. Hence we formulate a slightly relaxed version of nondegeneracy as follows.

**Definition 1.1.** *A family* $\boldsymbol{a}^{\bullet} = \left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ *of positive vectors is called* ***weakly nondegenerate*** *if there is a partition of* $\boldsymbol{K}$*, say* $\boldsymbol{K} = \bigcup_{\rho=1}^{r} \boldsymbol{L}_{\rho}$*, such that the members of each family* $\left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{L}_\rho}$ *are homothetic and a family* $\left\{\boldsymbol{a}^{(k)}\right\}_{\rho=1,\ldots,r}$ *of representatives of each* $\boldsymbol{L}_\rho$ *is nondegenerate (in the sense of Definition 2.5 of* Chapter *1). In other words, the family is nondegenerate up to homothetic copies.*

In particular, a family of identical copies of the unit deGua Simplex is weakly nondegenerate. The poset of Pareto faces in this case is not uniquely defined *ex ante*. however, we will find a way to induce a poset given the poset of a Cephoid $\Pi$ that satisfies nondegeneracy.

Now we fix a family $\boldsymbol{a}^{\bullet} = \left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ of positive vectors; we focus on the Cephoid $\Pi := \sum_{k\in\boldsymbol{K}} \Pi^{(k)}$ and its Pareto surface $\partial\Pi$.

We take $K$ copies of the vector $\boldsymbol{e} := (1,\ldots,1)$ which we denote by $\boldsymbol{a}^{0(1)},\ldots,\boldsymbol{a}^{0(K)}$. As in Section 1 of Chapter 1 we write $\boldsymbol{a}^{0(k)i} := a_i^{0(k)}\boldsymbol{e}^i$, where $a_i^{0(k)}$ denotes the $i^{th}$ coordinate of $\boldsymbol{a}^{0(k)}$ and $\boldsymbol{e}^i$ is the $i^{th}$ unit vector.

For every $k \in \boldsymbol{K}$ let $\Pi^{0(k)} := \Pi^{\boldsymbol{e}}$ and $\Delta^{0(k)} := \Delta^{\boldsymbol{e}}$ be a copy of the unit deGua Simplex and the unit Simplex respectively. The (homothetic) sums generated are denoted by

$$\Pi^0 := \sum_{k\in\boldsymbol{K}} \Pi^{0(k)} = \Pi^{K\boldsymbol{e}} = K\Pi^{\boldsymbol{e}}$$

and

$$\Delta^0 \;:=\; \sum_{k\in\boldsymbol{K}} \Delta^{0(k)} = \Delta^{K\boldsymbol{e}} = K\Delta^{\boldsymbol{e}}$$

respectively. Trivially we have

$$\partial\Pi^0 = \Delta^0 \ .$$

The family $\boldsymbol{a}^{0\bullet} = \{\boldsymbol{a}^{0(k)}\}_{k\in\boldsymbol{K}}$ is degenerate in the sense of Definition 2.5 of Chapter 1 as all Simplices and subSimplices involved are homothetic. Weak nondegeneracy suffices for our purpose.

We now indicate a procedure to generate a copy of the poset of $\partial\Pi$ on $\Delta^0 = \partial\Pi^0$. First, we generate a "grid" on the surface $\Delta^0 = K\Delta^{\boldsymbol{e}}$ by the set of integer vectors

$$(1) \qquad \mathcal{E}^0 \;:=\; \left\{ \boldsymbol{k} \;=\; (k_1,\ldots,k_n), \;\; k_i \in \mathbb{N}_0 \; (i\in\boldsymbol{I}), \sum_{i\in\boldsymbol{I}} k_i = K \;\right\}.$$

These vectors can be seen as sums of vertices of the Simplices $\Delta^{0(k)}$ in various ways. More precisely, given arbitrary pairwise disjoint sets $\boldsymbol{K}_1,\ldots,\boldsymbol{K}_n$ with $\bigcup_{i\in\boldsymbol{I}} \boldsymbol{K}_i = \boldsymbol{K}$, we obtain a grid vector

$$(2) \qquad \boldsymbol{k} \;=\; \sum_{k\in\boldsymbol{K}_1} \boldsymbol{a}^{0(k)1} + \ldots + \sum_{k\in\boldsymbol{K}_n} \boldsymbol{a}^{0(k)n} \ ,$$

and *all* grid vectors are obtained this way.

With the vertices of $\partial\Pi$ this is different: by nondegeneracy every vertex is a unique sum of certain vertices of the $\Delta^{\boldsymbol{a}^{(k)}}$ involved. (But not every sum of such vertices is necessarily Pareto efficient). Now we make this more precise by defining a mapping $\mathbf{i}^\bullet$ which associates the various vertices of the Simplices involved to a vertex $\boldsymbol{u} \in \partial\Pi$.

**Definition 1.2.** *1. Let $\mathcal{E}^0$ be defined as in (2). Denote the set of vertices of $\partial\Pi$ by $\mathcal{E}$. We define a mapping $\boldsymbol{\kappa} : \mathcal{E} \to \mathcal{E}^0$ as follows:*

*2. Let $\boldsymbol{u} \in \mathcal{E}$. Let*

$$(3) \qquad \mathbf{i}^\bullet \;:=\; \boldsymbol{K} \to \boldsymbol{I}$$

*be defined by the representation of $\boldsymbol{u}$ as the unique sum of vertices of the $\{\Delta^{(k)}\}_{k\in\boldsymbol{K}}$, i.e., by*

$$(4) \qquad \boldsymbol{u} = \boldsymbol{a}^{\mathbf{i}^\bullet} \;:= \sum_{k\in\boldsymbol{K}} \boldsymbol{a}^{(k)\mathbf{i}_k}.$$

3. *Let* $\boldsymbol{u}$ *be a vertex on* $\partial\Pi$ *and let* $\mathbf{i}_\bullet$ *be the corresponding mapping as described by* (3) *and* (4). *Then*

$$\text{(5)} \qquad \boldsymbol{u}^0 \;:=\; \boldsymbol{\kappa}(\boldsymbol{u}) \;:=\; \sum_{k\in\boldsymbol{K}} \boldsymbol{a}^{0(k)\mathbf{i}_k}$$

*is the* ***Canonical Representation*** *of* $\boldsymbol{u}$ *on* $\Delta^0 \;=\; \partial\Pi^0$ .

4. *Let* $\boldsymbol{F}$ *be a face of* $\Delta$ *and let* $\boldsymbol{u}^1,\ldots,\boldsymbol{u}^L$ *be its extremal points. Then the convex hull of the images, i.e.,*

$$\text{(6)} \qquad \boldsymbol{\kappa}(\boldsymbol{F}) \;:=\; \boldsymbol{F}^0 \;:=\; \boldsymbol{CovH}\{\boldsymbol{\kappa}(\boldsymbol{u}^1),\ldots,\boldsymbol{\kappa}(\boldsymbol{u}^L)\},$$

*is the* ***Canonical Representation*** *of* $\boldsymbol{F}$ *on* $\Delta$.

5. *Let* $\mathcal{V}$ *be the poset of faces of* $\Delta$ *and let*

$$\text{(7)} \qquad \mathcal{V}^0 \;:=\; \boldsymbol{\kappa}(\mathcal{V}) \;:=\; \{\boldsymbol{\kappa}(\boldsymbol{F}) \,|\, \boldsymbol{F} \in \mathcal{V}\}$$

*be the collection of images of faces under the mapping* $\kappa$. *Then* $\mathcal{V}^0$ *is the* ***Canonical Representation*** *of* $\mathcal{V}$ *on* $\Delta$.

**Theorem 1.3.** $\mathcal{V}^0$ *is a* ***partially ordered set (poset)*** *which is isomorphic to* $\mathcal{V}$. *Hence* $(\Delta,\mathcal{V})$ *and* $(\Delta^0,\mathcal{V}^0)$ *are combinatorically equivalent.*

**Proof:**

This is a standard procedure in convex geometry (see e.g. Pallaschke and Urbański [20]). The mapping $\boldsymbol{\kappa}$ is bijective between the vertices of $\Delta$ and the appropriate subset of grid vectors as described in equations (1) and (2). The minimum of two faces (whenever it exists) is obtained by taking the intersection of the corresponding two sets of extremal points. Similarly, if the maximum of two faces exists, then it is obtained via the union of the sets of extremal points. Each Representation of a vertex is one hand a vector $\boldsymbol{k}$ as described in (1). On the other hand, given the natural ordering on $\boldsymbol{K} = \{1,\ldots,K\}$, it is described or "labelled" via some function $\mathbf{i}_\bullet$ by $(\mathbf{i}_1,\ldots,\mathbf{i}_K)$.

**q.e.d.**

Somewhat sloppily, we use the term "Canonical projection" and "Canonical Representation" for the mapping $\boldsymbol{\kappa}$ as well as for images under the mapping – or even for triples like $\mathcal{V}^0, \mathcal{V}, bsk$. We consider this construction to be useful for better understanding the Pareto surface $\partial\Pi$ or rather the poset $\mathcal{V}$ of some

Cephoid Π. The reason is that the image is decreased in dimension, i.e., $\mathcal{V}^0$ is located on the $(n-1)$–dimensional Simplex $\Delta^0 = K\Delta^0$. Disregarding the various sizes opens the view for the poset structure. In what follows, we will illustrate this within 3 dimensions. Later on, we will attempt to visualize the surface of 4–dimensional Cephoids on a suitable positive multiple of the unit Simplex of $\mathbb{R}^3$ (a tetrahedron). This will most vividly demonstrate the new type of a Pareto face appearing in 4 dimensions and hence open up the alley to a general understanding of the Pareto surface of a Cephoid.

# 2 Examples, Simple Classfication

We consider non–degenerate Cephoids in 3 dimensions. The shape of the Pareto surface is completely described by presenting all Pareto faces, as the lower dimensional facets are given as (joint) subfaces. To "classify" a Cephoid, we consider the relative size of the summands involved to be irrelevant, it is only the relative position that matters. That is, we refer to the Canonical Representation.

**Example 2.1 ( The Circle ).** *The first example is the sum of three deGua Simplices listed in in Figure 2.7 of* Section *2 repeated here for convenience.*

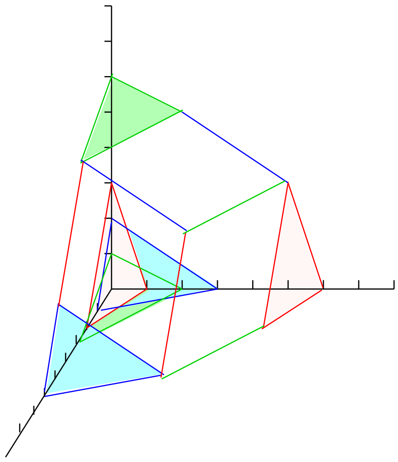

Figure 2.1: Three deGua Simplices: The "Circle"

*To have a name for reference and better mnemotechnial impact, we dubb this Cephoid (or rather the family leading to the same Canonical Representation) "the Circle". The Canonical Representation is rather obviously given by Figure 2.2. The term "clockwise orientation" emphasizes the contrast to Example 2.2 which folllows below and explains the notation.*

*We describe the Pareto surface as follows. First, we use the colors to denote the deGua Simplices involved, so write* $\Pi^{green}, \Delta^{green}$ *instead of* $\Pi^{\boldsymbol{a}}, \Delta^{\boldsymbol{a}}$ *etc. Then, as it suffices to list the coordinates/indices of the subSimplices involved in order to describe Pareto faces by just listing the indices involved; e.g. the rhombus*

(1) $$\Delta^{green}_{\{12\}} + \Delta^{blue}_{\{23\}} + \Delta^{red}_{\{3\}}$$

*is conveniently written*

(2)

| *green* | *blue* | *red* |
|---|---|---|
| 12 | 23 | 3 |

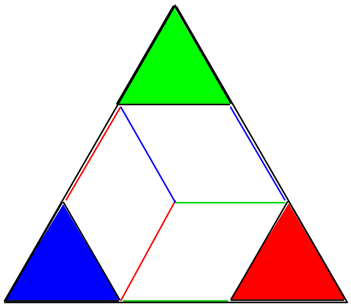

Figure 2.2: The Circle – Clockwise. Canonical Representation

*etc. To identify this polyhedron observe that it is the sum of a green and a blue line, hence a rhombus generated by the green and blue deGua Simplex and translated by the third vertex of the red deGua Simplex. Also, we see that it involves one first coordinate and two copies of the coordinates* $2, 3$ *respectively, hence it is located on the* $23$*–boundary of* $\Pi$ *– which leads to a Canonical Representation so that the image is touching the* $23$*–edge of the image Simplex* $3\Delta^e$*. One can now clearly identify the gree–blue rhombus in Figure 2.2.*

*Now consider the set of Pareto faces*

(3)

| green | blue | red |
|---|---|---|
| 123 | 3 | 3 |
| 12 | 23 | 3 |
| 1 | 123 | 1 |

*The rhombi of green and blue involved are given by*

(4)

| green | blue |
|---|---|
| 123 | 3 |
| 12 | 23 |
| 1 | 123 |

*These rhombi obey a "moving index principle": in each line there is exactly one common index and each subsequent line is obtained from the one above by shifting one index from the left to the right (and canceling the previous double). One can verify that a similar diagram holds for blue vs red and red vs. green. Later on, we will see that there is a systematic behind this feature.*

◦ ∼∼∼∼∼∼∼ ◦

Our presentation so far is predominantly intuitive and based on geometric considerations. A precise description of the Pareto faces will follow later on.

However, at this preliminary stage we point out that a formal description of the Pareto surface will have to be supported by computational methods that serve to generate Pareto faces and the complete Pareto surface by appropriate algorithms.

Again we postpone a precise treatment; algorithms as such will be explained later. However, one can visualize the results – at least for simple examples treated above. The programming language we employ is 𝔸ℙ𝕃 , and the computational results will appear in a TEX environment indicating their origin.

The following starts with the presentation/outprint of the above Example 2.1 in just that context. Our notation is based on the shorthand representation of Pareto faces as explained by (1) and (2). Accordingly, a Cephoid will be represented by a matrix $\boldsymbol{A}$, the rows of which correspond to the vectors $\left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$.

We start with the Cephoid "Circle". The name is now augmented by calling it "clockwise" (i.e., mathematically negative orientation). The term emphasizes the contrast to the subsequent version "Circle – counterclockwise" (i.e., mathematically positive orientation) which is presented below.

Now we write the data of the Cephoid "Circel – clockwise", listing the matrix the complete set of its Pareto faces as obtained via an algorithm in 𝔸ℙ𝕃 .

**Example 2.2 ( The Circle – Results of Algorithmic Treatment ).** *The results of the algorithmic procedure are presented as follows.*

```
THE CEPHOID CIRCLE -- CLOCKWISE ORIENTATION:

                  9  8  1
                  1 14 17
                 11  1 13

PARETO FACES OF CEPHOID CIRCLE -- CLOCKWISE:

     2  ⊛      2  ⊛  1 2 3  ⊛
   1 2  ⊛    2 3  ⊛      3  ⊛
 1 2 3  ⊛      3  ⊛      3  ⊛
     1  ⊛  1 2 3  ⊛      1  ⊛
     1  ⊛    2 3  ⊛    1 3  ⊛
   1 2  ⊛      2  ⊛    1 3  ⊛
```

*The Canonical Representation is the one depicted in Figure 2.2. The reader is obliged to identify the Pareto faces shown in the figure with the algorithmic results as suggested above. In particular, the Pareto faces mentioned in Formula* (3) *appear in the list generated although in a different order (due to the nature of the algorithm employed).*

*A closer inspection of Figure 2.2 reveals that there is indeed an orientation of of the rhombi. This orientation is "clockwise" (mathematically negative) as from each triangle (Pareto face resulting from the original Simplex) the adjacent diamond requires a motion in the clockwise sense.*

*Now, there exists indeed a version of "The Circle" with the reverse orientation – counterclockwise or mathematically positive. This version is (Canonically) Represented by Figure 2.3.*

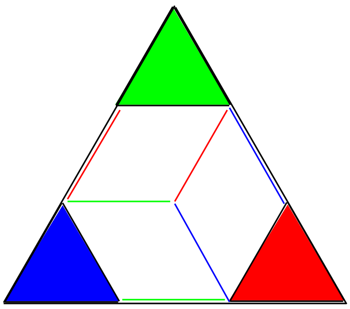

Figure 2.3: The Circle – counterclockwise

*A geometrical inspection shows that the group of maximal facets corresponding to* (3) *is obtained by changing the middle 1 in column red from 1 to 3. Thus we obtain*

(5)

| *green* | *blue* | *red* |
|---|---|---|
| 123 | 3 | 3 |
| 12 | 23 | 3 |
| 1 | 123 | 1, |

*which again reveals the vague idea of the "moving index principle". A complete treatment of this phenomenon and its relation to permutations will be postponed until we have the theory available.*

*Again we indicate the results of an algorithmic treatment of this kind of "Circle – counterclockwise":*

```
THE CEPHOID CIRCLE -- COUNTERCLOCKWISE ORIENTATION:

                17 14  1
                 1  8  9
                13  1 11



PARETO FACES OF CEPHOID CIRCLE -- COUNTERCLOCKWISE:

     2  ⊛       2  ⊛  1 2 3  ⊛
     2  ⊛     2 3  ⊛    1 3  ⊛
 1 2 3  ⊛       3  ⊛      3  ⊛
     1  ⊛  1 2 3  ⊛       1  ⊛
   1 2  ⊛       3  ⊛    1 3  ⊛
   1 2  ⊛     2 3  ⊛      1  ⊛
```

∘ ∼∼∼∼∼∼∼ ∘

**Example 2.3.** *This example reflects an attempt to classify all sums of 3 deGua Simplices in 3 dimensions, for short all $3 \times 3$ Cephoids. We sketch a canonical representation and some representative that was obtained by our algorithm.*

*The first type of Cephoid as given by the Canonical Representation of Figure 2.4 is dubbed the **Windmill** by obvious reasons.*

*A Cephoid (a matrix $\boldsymbol{A} = \left\{\boldsymbol{a}^{(k)}\right\}_{k \in \boldsymbol{K}}$) is given by the following printout:*

```
      THE CEPHOID WINDMILL :

 2 4 2
 3 3 6
 5 1 1
```

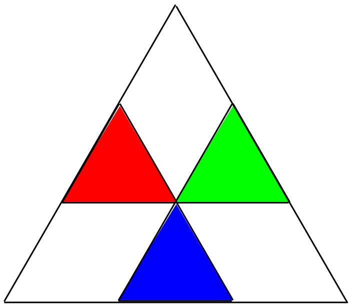

Figure 2.4: The Windmill

```
    PARETO FACES OF CEPHOID WINDMILL :

        2  ⊛     2 3  ⊛     1 2  ⊛
        2  ⊛       3  ⊛   1 2 3  ⊛
      2 3  ⊛       3  ⊛     1 3  ⊛
      1 2  ⊛     1 3  ⊛       1  ⊛
    1 2 3  ⊛       3  ⊛       1  ⊛
        2  ⊛   1 2 3  ⊛       1  ⊛
```

*Some further types of $3 \times 3$ Cephoids are indicated as follows.*

***Inductive types*** *are the result of a sum of 2 deGua Simplices and a third one. Implicitly, a classification of "sums of two" is offered.*

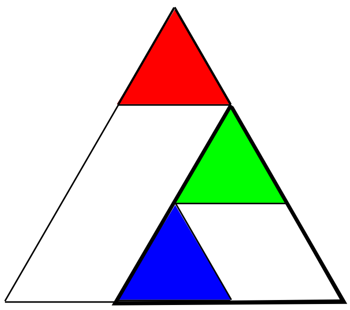

Figure 2.5: $1^{st}$ Inductive type

*The **Saw** is also obtained by an inductive procedure – in two ways.*

*The attempt to construct a simple representative for the Canonical Representation*

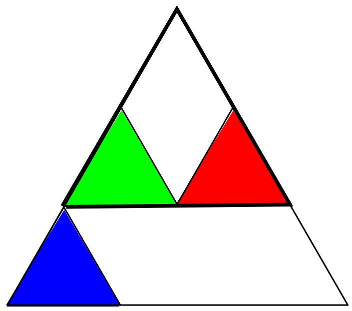

Figure 2.6: $2^{nd}$ Inductive type

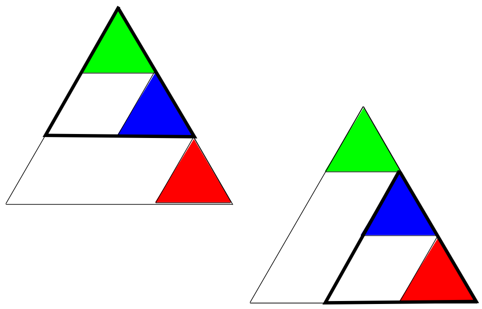

Figure 2.7: The Saw – a $3^{rd}$Inductive type

*via the matrix*

$$A = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 4 & 0 \\ 5 & 2 & 6 \end{pmatrix}$$

*fails, as it results in a programming error. This matrix does not satisfy non–degeneracy. We slightly change it to "Saw100" which is* $A' := 100A + 3$*. Then the algorithmic treatment is successful.*

```
                 THE CEPHOID SAW.

                     2 3 2
                     1 4 0
                     5 2 6

               THE CEPHOID SAW100:

                   203 303 203
                   103 403   3
                   503 203 603

     PARETO FACES OF  CEPHOID SAW100:

     2  ⊛      2  ⊛  1 2 3  ⊛
 1 2 3  ⊛      2  ⊛      3  ⊛
     3  ⊛  1 2 3  ⊛      3  ⊛
     1  ⊛    1 2  ⊛    1 3  ⊛
   1 3  ⊛    1 2  ⊛      3  ⊛
   1 2  ⊛      2  ⊛    1 3  ⊛
```

◦ ∼∼∼∼∼∼ ◦

**Example 2.4.** *The following "sum of 4" has been already introduced in* Chapter *2* Section *2 (see Figure 2.8).*

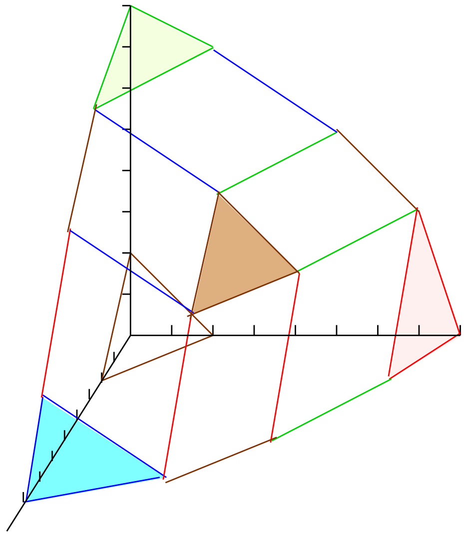

Figure 2.8: A sum of four deGua Simplices – Odot

*This Cephoid is now dubbed* ***Odot*** *as it reflects a (Clockwise) Circle with a point in the center: ⊙. The following shows a concrete example of Cephoid with the shape indicated. Again, the way to obtain this description via an algorithmic treatment is left to future chapters.*

```
THE CEPHIOD ODOT: ONE ADDITIONAL FACE TO THE
CIRCLE CLOCKWISE. THE CIRCLE CLOCKWISE IS:

                9   8   1
                1  14  17
               11   1  13

THE FIRST 3 ROWS OF ODOT ARE TEN TIMES THOSE OF THE
CIRCLE CLOCKWISE -  SLIGHTLY PERTURBATED FOR TO ACHIEVE ND:


                   ODOT
               91   81   11
               12  139  167
              109   10  131
              110  102  123
```

```
PARETO FACES OF ODOT:

     2  ⊛      2  ⊛  1 2 3  ⊛      2  ⊛
     1  ⊛      2  ⊛    1 3  ⊛    1 2  ⊛
     1  ⊛      2  ⊛      3  ⊛  1 2 3  ⊛
     1  ⊛  1 2 3  ⊛      1  ⊛      1  ⊛
     1  ⊛    2 3  ⊛      3  ⊛    1 3  ⊛
     1  ⊛    2 3  ⊛    1 3  ⊛      1  ⊛
 1 2 3  ⊛      3  ⊛      3  ⊛      3  ⊛
   1 2  ⊛      2  ⊛      3  ⊛    2 3  ⊛
   1 2  ⊛      2  ⊛    1 3  ⊛      2  ⊛
   1 2  ⊛    2 3  ⊛      3  ⊛      3  ⊛
```

*One can identify the Pareto faces in Figure 2.8 by inspecting the above list. For example, the first line, the Pareto face given by*

$$2 \quad 2 \quad 123 \quad 2$$

*reflects a triangle translated via the second extremal of all deGua Simplices involved, obviously, this is the deGua Simplex/triangle containing the extremal in the second coordinate direction which is the red one. From which we conclude, that third column represents the contributions of the red deGua Simplex.*

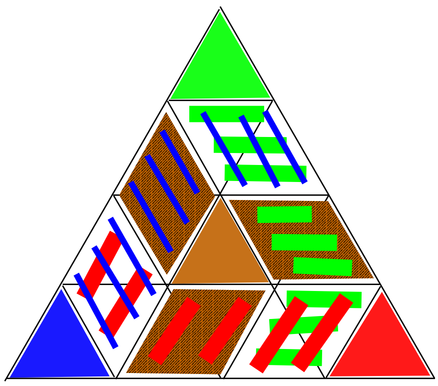

Figure 2.9: Odot – Canonical Representation

*Similarly, the third line in the list, i.e.,*

$$1 \quad 2 \quad 3 \quad 123$$

*reflects a triangle translated to the Pareto Surface via the extremals in all three directions of the deGua Simplices involved – this is necessarily the brown one, i.e.,* *the one added to the Circle Clockwise by the augmenting it with the last line in the matrix* ODOT.

*We conclude that the last column reflects the brown deGua Simplex and its SubSimplices respectively.*

*Figure 2.9 shows the Canonical Representation of Odot. Each DeGua Simplex generates a copy of itself plus a diamond with each of the other ones. E.g., "Blue" provides a sequence of* 3 *diamonds which is indicated by the blue hatched rhombi that also offer a second color accordingly. One recognizes the "orientation" ("clockwise") involved in this example.*

° ~~~~~~~ °

Finally, we show the Canonical Representation of a possible "sum of seven" (Figure 2.10). Of course, the source is not made precise, that is, we do not present a concrete matrix yielding this type of Cephoid. Here again, any two deGua Simplices generate a rhombus on the Pareto surface of the resulting Cephoid. These rhombi are sums of one–dimensional SubSimplices plus vertices of the 5 remaining Simplices. Now consider the path of the rhombi related to the "central" (brown) deGua Simplex. Here, for the first time, we observe a "cephalopodic" structure generated by the Pareto faces involving the central deGua Simplex. A "tentacle" is sent out towards each of the boundaries.

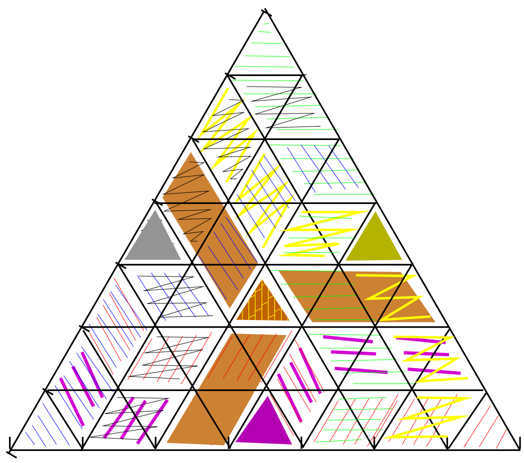

Figure 2.10: Canonical Representation of a sum of 7

# Chapter 3

# Faces and Normals

The Pareto surface of a Cephoid is described by exhibiting the Pareto faces. So far we have used geometrical ideas and algorithmic results (of algorithms not explained) to present a first access to the structure of a Pareto face of a Cephoid. Now we are going to develop formal procedures to simultaneously characterize and compute Pareto faces (and further "outward faces") as well as the corresponding normals. Our procedure starts with a necessary condition to be satisfied by a Pareto face - the Coincidence Theorem. This then opens up the alley to further analytical treatment of the structure of a Cephoid.

# 1 Adjustment of Faces: The Coincidence Theorem

Within this section, we embark on a characterization of the Pareto faces of a Cephoid $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{(k)}$, i.e., "outward faces" of dimension $n-1$. The following Coincidence Theorem lists the properties of such a face and its normal in relation to the subfaces of the deGua Simplices involved and the corresponding normal cones.

A (Pareto) face $\boldsymbol{F}$ is necessarily a Minkowski sum of certain subfaces of the summands which are deGua Simplices. Hence these subfaces are Standard Subsimplices (just written Subsimplices) of lower dimension - possibly just one–dimensional and hence an extremal point.

More precisely, given a Pareto face $\boldsymbol{F}$ of $\partial\Pi$, then for each $k \in \boldsymbol{K}$ there is an index set $\boldsymbol{J}^{(k)} \subseteq \boldsymbol{I}$ and a corresponding Subsimplex $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ of $\Delta^{(k)}$ such that

$$\boldsymbol{F} \;=\; \sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \tag{1}$$

holds true.

**Definition 1.1.** *1. Let $\boldsymbol{F}$ be a Pareto face of a Cephoid $\Pi$ satisfying (1). We call the sets $\boldsymbol{J}^{(k)}$ the **reference sets** and the family $\mathfrak{J} = \left(\boldsymbol{J}^{(k)}\right)_{k\in\boldsymbol{K}}$ the **reference system** of $\boldsymbol{F}$.*

*2. For $i \in \boldsymbol{I}$ define*

$$K_i \;:=\; |\{k \in \boldsymbol{K} \,\big|\, i \in \boldsymbol{J}^{(k)}\}| - 1 \;,$$

*such that $K_i + 1$ is the number of appearances of $i$ within the various reference sets of $\boldsymbol{F}$. We define*

$$\boldsymbol{L} \;:=\; \{l \in \boldsymbol{I} \,|\, K_l \geq 1\} \;,$$

*that is, $\boldsymbol{L} \subseteq \boldsymbol{I}$ contains those coordinates $l \in \boldsymbol{I}$ that appear in at least two of the $\boldsymbol{J}^{(k)}$. Thus, for $l \in \boldsymbol{L}$,*

$$K_l \;:=\; |\{k \in \boldsymbol{K} \,\big|\, l \in \boldsymbol{J}^{(k)}\}| - 1 \geq 1 \;. \tag{2}$$

*The set $\boldsymbol{L}$ is called the **adjustment set**. $L \;:=\; |\boldsymbol{L}|$ denotes the power of the adjustment set.*

Clearly, the reference system defines $\boldsymbol{F}$ uniquely. The adjustment set will serve to determine the normal of $\boldsymbol{F}$. The normalization of the numbers $K_i$ by diminishing the number of appearances by 1 will be justified at once by our results below.

Because of Theorem 1.6 there must be a common normal $\mathfrak{n}$ to the subfaces described by the reference system. This normal is also a normal to the Pareto face under consideration. As the dimension of $\boldsymbol{F}$ is $n-1$, the normal $\mathfrak{n}$ is unique up to a multiple factor.

The following is a precise formulation of the situation induced by a Pareto face. It is a necessary condition resulting from Pareto efficiency.

**Theorem 1.2 (The Reference Theorem).** *Let $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ be a Cephoid with $K \geq 2$ and let $\boldsymbol{F}$ be a Pareto face of $\partial\Pi$. Let $\mathfrak{n}^\star$ be the normal to $\boldsymbol{F}$. Then the following holds.*

1. *For each $k \in \boldsymbol{K}$ there is an index set $\boldsymbol{J}^{(k)}$ and a corresponding Subsimplex $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ of $\Delta^{(k)}$ such that*

$$\boldsymbol{F} \;=\; \sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \tag{3}$$

*holds true. The vector $\mathfrak{n}^\star$ is a also normal to $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ $(k \in \boldsymbol{K})$. Thus in particular, the linear function $\boldsymbol{x} \mapsto \mathfrak{n}^\star\boldsymbol{x}$ attains its maximum relative to $\Delta^{(k)}$ on $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$; we write*

$$\max_{\boldsymbol{x}\in\Delta^{(k)}} \mathfrak{n}^\star\boldsymbol{x} \;=\; \max_{\boldsymbol{x}\in\Delta^{(k)}_{\boldsymbol{J}^{(k)}}} \mathfrak{n}^\star\boldsymbol{x} \;=:\; t_k \;. \tag{4}$$

2. *In view of Definition 1.1 the following holds true.*
   (a) $\sum_{l\in\boldsymbol{L}} K_l = K-1$
   (b) $L = |\boldsymbol{L}| \leq K-1$.
   (c) $\boldsymbol{J}^{(k)} \cap \boldsymbol{L} \neq \emptyset \quad (k \in \boldsymbol{K})$.

3. *Let $\boldsymbol{a}^{(k)}_{\boldsymbol{L}} = \boldsymbol{a}^{(k)}{}_{|\boldsymbol{L}}$ $(k \in \boldsymbol{K})$ be the restriction of $\boldsymbol{a}^{(k)}$ to $\mathbb{R}^n_{\boldsymbol{L}+} = \mathbb{R}^n_{+}{}_{|\boldsymbol{L}}$ and let $\Pi_{\boldsymbol{L}} := \sum_{j\in\boldsymbol{K}} \boldsymbol{a}^{(k)}_{\boldsymbol{L}}$ be the Cephoid generated in $\mathbb{R}^n_{\boldsymbol{L}+}$ by the restricted family $\left\{\boldsymbol{a}^{(k)}_{\boldsymbol{L}}\right\}_{k\in\boldsymbol{K}}$. Then*

$$\Pi_{\boldsymbol{L}} \;=\; \sum_{j\in\boldsymbol{K}} \boldsymbol{a}^{(k)}_{\boldsymbol{L}} \;=\; \sum_{j\in\boldsymbol{K}} \Pi^{(k)} \cap \mathbb{R}^n_{\boldsymbol{L}} \tag{5}$$

*and the intersection*

$$(6)\qquad \boldsymbol{F_L} \;:=\; \boldsymbol{F} \cap \mathbb{R}^n_{\boldsymbol{L}} \;=\; \left(\sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}\right) \cap \mathbb{R}^n_{\boldsymbol{L}} \;=\; \sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \cap \mathbb{R}^n_{\boldsymbol{L}}$$

*is a Pareto face of* $\Pi_{\boldsymbol{L}}$, *hence has dimension* $L-1$.

**Proof:**

**1st STEP** : As $\boldsymbol{F}$ is a Pareto face, we can apply Theorem 1.15 in Ewald [7], see also Theorem 3.1.1 in Pallaschke–Urbanski [20]. Accordingly, there is, for each $k \in \boldsymbol{K}$, a subface $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ of $\Delta^{(k)}$ such that

$$(7)\qquad \boldsymbol{F} \;=\; \sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$$

holds true. Moreover, $\mathfrak{n}^\star$ is a normal to every $\Delta^{(k)}$ $(k \in \boldsymbol{K})$.

**2nd STEP** :

Let $j_k := \;|\boldsymbol{J}^{(k)}|\;\; (k \in \boldsymbol{K})$. Then each summand $\Delta^{(k)}$ contributes a dimension of $j_k-1$ to $\boldsymbol{F}$. Consequently, in order to produce the proper dimension $(n-1)$ of a Pareto face, we must have

$$j_1 - 1 + j_2 - 1 + \cdots + j_K - 1 \geq n-1$$

and by our non-degeneracy assumption we obtain

$$(8)\qquad j_1 - 1 + j_2 - 1 + \cdots + j_K - 1 = n-1$$

or

$$(9)\qquad \sum_{k\in\boldsymbol{K}} j_k = K + n - 1 = n + (K-1) \;\geq\; n+1\ .$$

Now, as

$$(10)\qquad \bigcup_{k\in\boldsymbol{K}} \boldsymbol{J}^{(k)} \;=\; \boldsymbol{I} \;=\; \{1,\ldots,n\},$$

(9) shows that some of the indices $i \in \boldsymbol{I}$ must appear at least twice in the reference sets $\boldsymbol{J}^{(k)}$ so that a total of $K-1$ multiple appearances occurs. Hence the adjustment set $\boldsymbol{L}$ is certainly nonempty. More than that, in view of the definition of the integers $K_l$ we know

$$(11)\qquad l \;\text{ appears } K_l + 1 \text{ times}\ .$$

Consequently it follows from (9), that and (10)

$$K_1 + \cdots + K_L = K - 1 \tag{12}$$

holds true, which is *item* 2a. Obviously, it follows that $L = |\boldsymbol{L}| \leq K - 1$ is the case, that is, we have verified *item* 2b.

**3rdSTEP** : Now we turn to *item* 2c. We have to show that every $\boldsymbol{J}^{(k)}$ contains at least one index $l$ which appears in some other $\boldsymbol{J}^{(i')}$, i.e., verify $\boldsymbol{J}^{(k)} \cap \boldsymbol{L} \neq \emptyset$ for every $k \in \boldsymbol{K}$.

Let us assume that this is not true. Then there exists a $k^* \in \boldsymbol{K}$ with $\boldsymbol{J}^{(k^*)} \cap \boldsymbol{L} = \emptyset$ and this implies that for all $k \in \boldsymbol{K} \setminus \{k^*\}$ we have $\boldsymbol{J}^{(k)} \cap \boldsymbol{J}^{(k^*)} = \emptyset$ and hence $\Delta^{(k)}_{\boldsymbol{J}^{(k)}} \subset \mathbb{R}^{n-j_{k^*}}$ for all $k \in \boldsymbol{K} \setminus \{k^*\}$. Now consider the Cephoid obtained by omitting $k^*$ and $j_{k^*}$, i.e.,

$$\Pi^{[-k^\star]} \;=\; \Pi^\star \;:=\; \sum_{k \in \boldsymbol{K} \setminus \{k^*\}} \Pi^{(k)} \subset \mathbb{R}^{n-j_{k^*}}.$$

By non-degeneracy we may apply the analogue to (9) to $\Pi^\star$. We have $K-1$ members in the family of deGua Simplices and hence the analog formula is

$$\sum_{\substack{k \in \boldsymbol{K} \\ k \neq k^*}} j_k \;=\; (n - j_{k^*}) + (K-1) - 1 \;=\; n + K - j_{k^*} - 2$$

which implies

$$\sum_{k \in \boldsymbol{K}} j_k \;=\; \sum_{\substack{k \in \boldsymbol{K} \\ k \neq k^*}} j_k \;+\; j_{k^*} \;=\; n + K - j_{k^*} - 2 + j_{k^*} = n + K - 2 < n + K - 1,$$

explicitely contradicting (9).

**4thSTEP** : Finally, we turn to the proof of 3.

Clearly

$$\Pi^{(k)}_{\boldsymbol{J}^{(k)}} \cap \mathbb{R}^n_{\boldsymbol{L}} \;=\; \Pi^{(k)}_{\boldsymbol{J}^{(k)} \cap \boldsymbol{L}} \quad \text{as well as } \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \cap \mathbb{R}^n_{\boldsymbol{L}} \;=\; \Delta^{(k)}_{\boldsymbol{J}^{(k)} \cap \boldsymbol{L}}$$

and hence $\dim \left( \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \cap \mathbb{R}^n_{\boldsymbol{L}} \right) = |\boldsymbol{L} \cap \boldsymbol{J}^{(k)}| - 1$. Consequently, the dimension of the polyhedron

$$\sum_{k \in \boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \cap \mathbb{R}^n_{\boldsymbol{L}} \;=\; \sum_{k \in \boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{L} \cap \boldsymbol{J}^{(k)}} \subseteq \boldsymbol{F} \tag{13}$$

is

$$\sum_{k\in\boldsymbol{K}}(|\boldsymbol{L}\cap\boldsymbol{J}^{(k)}|-1) \;=\; \left(\sum_{k\in\boldsymbol{K}}|\boldsymbol{L}\cap\boldsymbol{J}^{(k)}|\right)-K\;. \tag{14}$$

Now, recall the notation in Definition 1.1: any index in $\boldsymbol{L}$ appears $K_l+1$ times. Hence

$$\sum_{k\in\boldsymbol{K}}|\boldsymbol{L}\cap\boldsymbol{J}^{(k)}| \;=\; \sum_{k\in\boldsymbol{K}}(K_l+1) \;=\; (\sum_{k\in\boldsymbol{K}}K_l)+L \;=\; K-1+L\;. \tag{15}$$

Combining (14) and (15) we see that the dimension of the polyhedron (13) is indeed $L-1$. It follows that $\sum_{j\in\boldsymbol{K}}\Delta^{(k)}_{\boldsymbol{J}^{(k)}}\cap\mathbb{R}^n_{\boldsymbol{L}}$ is a face of dimension $L-1$ of the Cephoid $\Pi_{\boldsymbol{L}}$. Since $\sum_{j\in\boldsymbol{K}}\Delta^{(k)}_{\boldsymbol{J}^{(k)}}\cap\mathbb{R}^n_{\boldsymbol{L}}\subseteq\left(\sum_{j\in\boldsymbol{K}}\Delta^{(k)}_{\boldsymbol{J}^{(k)}}\right)\cap\mathbb{R}^n_{\boldsymbol{L}}=\boldsymbol{F}\cap\mathbb{R}^n_{\boldsymbol{L}}$, and $\dim(\boldsymbol{F}\cap\mathbb{R}^n_{\boldsymbol{L}})\leq L-1$, we find that

$$\boldsymbol{F}_{\boldsymbol{L}} \;=\; \boldsymbol{F}\cap\mathbb{R}^n_{\boldsymbol{L}} \;=\; \left(\sum_{j\in\boldsymbol{K}}\Delta^{(k)}_{\boldsymbol{J}^{(k)}}\right)\cap\mathbb{R}^n_{\boldsymbol{L}} \;=\; \sum_{j\in\boldsymbol{K}}\Delta^{(k)}_{\boldsymbol{J}^{(k)}}\cap\mathbb{R}^n_{\boldsymbol{L}}$$

is a maximal face of the Cephoid $\Pi_{\boldsymbol{L}}$ of dimension $L-1$, hence a Pareto face.

**q.e.d.**

**Definition 1.3.** *Let* $\Pi \;=\; \sum_{k=1}^{K}\Pi^{(k)}$ *be a Cephoid with* $K\geq 2$. *Let*

$$\boldsymbol{F} \;=\; \sum_{k=1}^{K}\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$$

*be a Pareto face of* $\Pi$ *with normal* $\mathfrak{n}^\star$. *Let* $\boldsymbol{L}$ *be the adjustment set corresponding to* $\boldsymbol{F}$.

1. *The set*

$$\mathbb{L} \;:=\; \left\{(k,l)\,\middle|\, l\in\boldsymbol{L},\;\boldsymbol{J}^{(k)}\ni l\right\}\subseteq\boldsymbol{K}\times\boldsymbol{I} \tag{16}$$

*is called the set of* ***characteristics*** *of* $\boldsymbol{F}$. *Sloppily we may also refer to the family*

$$\boldsymbol{L}^{(k)} \;:=\; \boldsymbol{L}\cap\boldsymbol{J}^{(k)}\quad(k\in\boldsymbol{K}) \tag{17}$$

*as to the characteristics.*

2. *The linear system of equations in variables* $(c_k, \lambda_l)$, $((k,l) \in \mathbb{L})$ *given by*

$$c_k a_l^{(k)} = \lambda_l \quad ((k,l) \in \mathbb{L}). \tag{18}$$

*is called the **linear adjustment system** (corresponding to* $\boldsymbol{F}$*).*

3. *The Cephoid* $\Pi_{\boldsymbol{L}}$ *given by* (5) *and identified by Theorem 1.2 is called the* $\boldsymbol{L}$*–**reduced** Cephoid (of* $\boldsymbol{F}$*).*

4. *The Pareto face* $\boldsymbol{F}_{\boldsymbol{L}}$ *of* $\Pi_{\boldsymbol{L}}$ *as given by* (6) *and characterized by 1.2 is called the* $\boldsymbol{L}$*–**reduced** Pareto face (of* $\boldsymbol{F}$*).*

**Theorem 1.4** (***The Coincidence Theorem***)**.** *Let* $\Pi = \sum_{k=1}^{K} \Pi^{(k)}$ *be a Cephoid with* $K \geq 2$. *Let*

$$\boldsymbol{F} = \sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$$

*be a Pareto face of* $\Pi$ *with normal* $\mathfrak{n}^\star$. *Let* $\boldsymbol{L}$ *be the adjustment set corresponding to* $\boldsymbol{F}$ *and let* $\mathbb{L}$ *be the characteristics.*

*Then the following holds true.*

1. *The linear adjustment system* (18) *has a positive solution*

$$(\boldsymbol{c}_\bullet, \boldsymbol{\lambda}_\bullet) = \left\{ (c_k, \lambda_l) \right\}_{(k,l) \in \mathbb{L}} \tag{19}$$

*which is unique up to a positive multiple.*

2. $\mathfrak{n}^\star$ *is (up to a positive multiple) exactly the normal of the deGua Simplex*

$$\Pi^\star = \boldsymbol{CovH}\left( \bigcup_{k \in \boldsymbol{K}} c_k \Pi^{(k)} \right) =: \bigvee_{k \in \boldsymbol{K}} c_k \Pi^{(k)} \tag{20}$$

*(see* (10) *in* CHAPTER *1* SECTION *1 for the definition).*

3. *Whenever* $l \in \boldsymbol{L}$ *satisfies* $l \in \boldsymbol{J}^{(k)} \cap \boldsymbol{J}^{(k^*)}$, *then the deGua Simplices* $c_k \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ *and* $c_{k^*} \Delta^{(k^*)}_{\boldsymbol{J}^{(k^*)}}$ *have a joint vertex* $c_k \boldsymbol{a}^{(k)l} = c_{k^*} \boldsymbol{a}^{(k^*)l}$.

*4. The quantities*

$$a_i^\star := \max_{k\in\boldsymbol{K}} c_k \boldsymbol{a}_i^{(k)} > 0, \tag{21}$$

*yield the normal via* $\mathfrak{n}^\star = \left(\frac{1}{a_1^\star}, \ldots, \frac{1}{a_n^\star}\right)$ *up to a positive multiple.*

**Proof:** The normal $\mathfrak{n}^\star$ of $\boldsymbol{F}$ induces the linear function $x \mapsto \mathfrak{n}^\star x$. For $k \in \boldsymbol{K}$, denote by $t_k > 0$ be the maximum of this function on the deGua Simplex $\Pi^{(k)}$. This maximum is attained precisely (because of non-degeneracy) on $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$.

Let

$$t^\star := \max_{k\in\boldsymbol{K}} t_k \tag{22}$$

and

$$c_k := \frac{t^\star}{t_k} > 0 \quad (k \in \boldsymbol{K}). \tag{23}$$

Then, for all $k \in \boldsymbol{K}$, the maximum of the linear function $x \mapsto \mathfrak{n}^\star \cdot x$ relative to the deGua Simplex $c_k\Pi^k$ is equal to $t^\star$ and achieved on $c_k\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$.

Pick $l \in \boldsymbol{L}$ with $l \in \boldsymbol{J}^{(k')} \cap \boldsymbol{J}^{(k'')}$. Then necessarily

$$\mathfrak{n}_l^\star c_{k'} \boldsymbol{a}_l^{(k')} = \mathfrak{n}_l^\star c_{k''} \boldsymbol{a}_l^{(k'')} ,$$

i.e.

$$c_{k'} \boldsymbol{a}_l^{(k')} = c_{k''} \boldsymbol{a}_l^{(k'')} .$$

Hence, there is a positive constant $\lambda_l > 0$ such that for all $k$ with $l \in \boldsymbol{J}^{(k)}$

$$c_k \boldsymbol{a}^{(k)l} = \lambda_l \tag{24}$$

holds true. Consequently, $(\boldsymbol{c}_\bullet, \boldsymbol{\lambda}_\bullet)$ is a solution to the linear system of equations (18).

As the linear function $x \mapsto \mathfrak{n}^\star x$ achieves its maximum $t^\star > 0$ relative to every deGua simplex $c_k\Pi^k$ precisely on $c_k\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$, it follows that it is constant on

$$\Pi^\star = \boldsymbol{CovH}\left(\bigcup_{k\in\boldsymbol{K}} c_k\Pi^{(k)}\right) = \bigvee_{k\in\boldsymbol{K}} c_k\Pi^{(k)}$$

with value $t^\star > 0$. Therefore $\mathfrak{n}^\star$ is a normal of the Simplex $\Pi^\star$.

Define the vector $\boldsymbol{a}^\star = (a_1^\star, \ldots, a_n^\star)$ via

$$a_i^\star := \max_{k \in \boldsymbol{K}} c_k \boldsymbol{a}_i^{(k)} > 0 \quad (i \in \boldsymbol{I}).$$

Then clearly $\Pi^\star = \Delta^{\boldsymbol{a}^\star}$. Consequently, $\mathfrak{n}^\star$ is a positive multiple of the vector $\left(\frac{1}{a_1^\star}, \ldots, \frac{1}{a_n^\star}\right)$. Therefore, all components of $\mathfrak{n}^\star$ are positive and can be computed via the solution of the linear system (18).

Finally, it follows from non–degeneracy that the solution space of the system (18) has dimension 1. For the number of variables is $K + L$. The number of equations is

$$K_1 + 1 + \cdots + K_L + 1 \ = \ K - 1 + L.$$

**q.e.d.**

**Example 1.5.** *We return to our first significant Example 2.4 of* SECTION *2* CHAPTER *1. The Cephoid is $\Pi = \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}$ as recalled in Figure 1.1. Consider the Pareto face $\boldsymbol{F} = \Delta^{\boldsymbol{a}}_{23} + \Delta^{\boldsymbol{b}}_{13}$ which is the sum of two 1–dimensional Simplices. As $\boldsymbol{L} = \{3\}$ the adjustment takes place in a way such that the third coordinate of $c_3\Delta^{\boldsymbol{a}}$ and $c_3\Delta^{\boldsymbol{b}}$ coincide.*

*To see this more clearly, observe that we have $\boldsymbol{L} = \{3\}$ and*

$$\mathbb{L} \ = \ \left\{(k,l) \,\middle|\, l \in \boldsymbol{L}, \boldsymbol{J}^{(k)} \ni l\right\} \ = \ \left\{(k,3) \,\middle|\, 3 \in \boldsymbol{J}^{(k)}\right\} \ = \ \{(1,3),(2,3)\} \ .$$

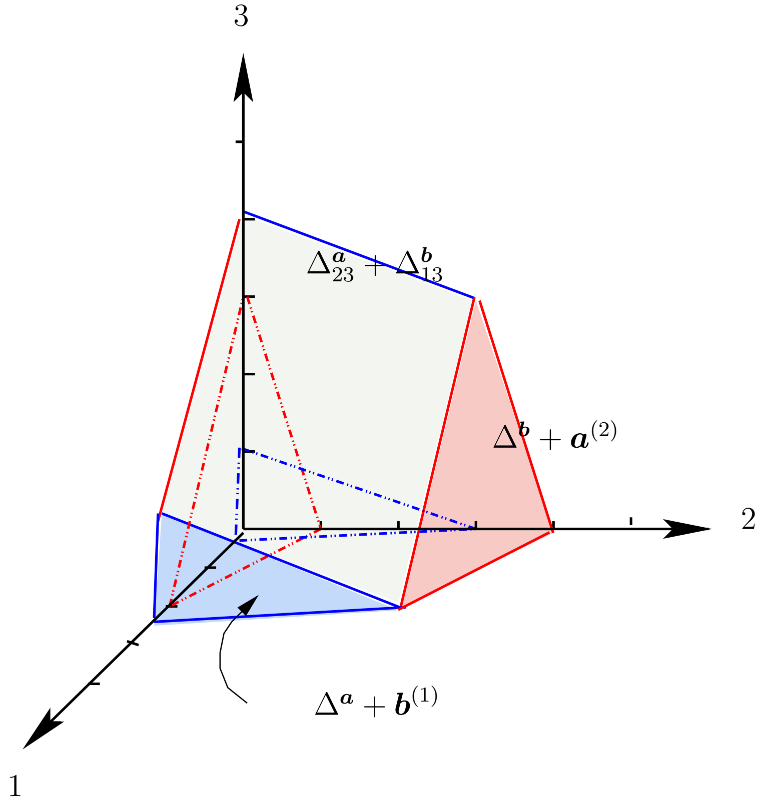


Figure 1.1: The sum of 2 deGua Simplices recalled

*Therefore, the linear adjustment system in variables* $c1, c2, \lambda_3$ *is*

$$c_1 a_3 = \lambda_3 \ , \quad c_2 b_3 = \lambda_3 \quad .$$

*We choose the solution*

$$\bar{c}_1 = b_3, \ \bar{c}_2 = a_3, \ \bar{\lambda}_3 = a_3 b_3 \ .$$

*This leads to adjusted Simplies* $\bar{c}_1 \Delta^{\boldsymbol{a}} = \Delta^{\bar{c}_1 \boldsymbol{a}}$ *with*

$$\bar{c}_1 \boldsymbol{a} = (a_1 b_3, a_2 b_3, a_3 b_3) = (a_1 b_3, a_2 b_3, \bar{\lambda})$$

*and* $\bar{c}_2 \Delta^{\boldsymbol{b}} = \Delta^{\bar{c}_2 \boldsymbol{b}}$ *with*

$$\bar{c}_2 \boldsymbol{b} = (a_3 b_1, a_3 b_2, a_3 b_3) = (a_3 b_1, a_3 b_2, \bar{\lambda}) \ .$$

*Figure 1.2 shows the original two Simplices as well as the two adjusted versions such the third coordinates both equal* $\lambda_3$*.*

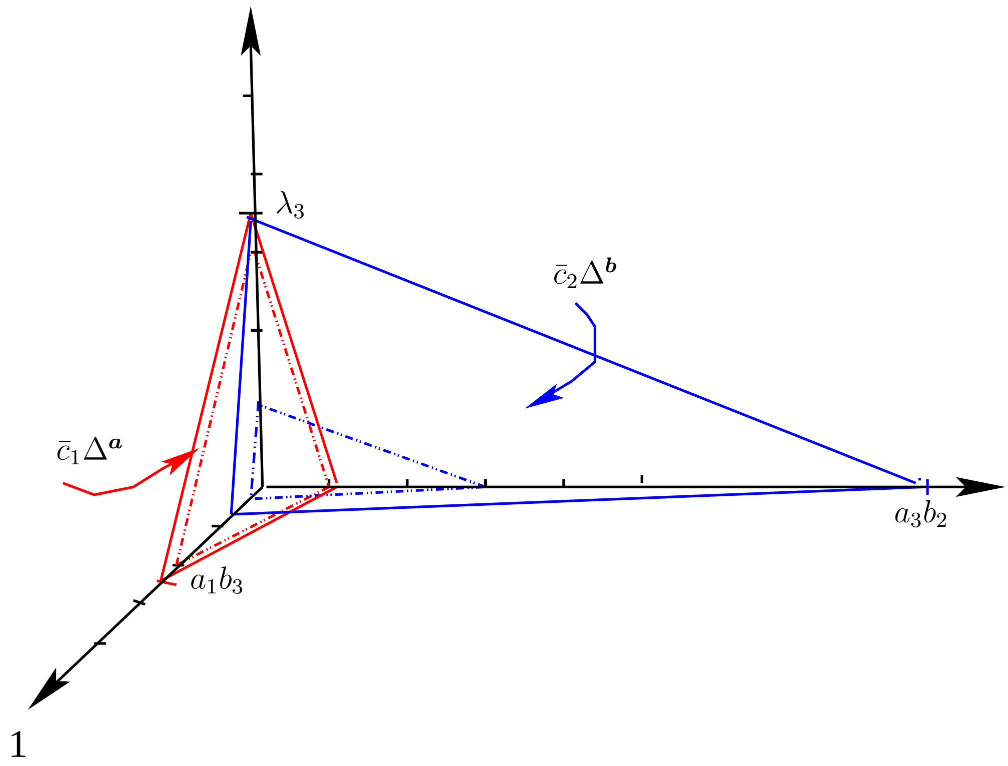


Figure 1.2: Adjusting two Simplices

*Next we proceed ostensively as the quantities* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *have not been defined but by the geometrical sketch. We obtain*

$$a_1^\star \;=\; \max\{a_3b_1, a_1b_3\} \;=\; a_3b_1$$

*(the red adjusted Simplex has the larger* $1^{st}$ *coordinate) and*

$$a_2^\star \;=\; \max\{a_2b_3, a_3b_2\} \;=\; a_2b_3$$

*(the blue adjusted Simplex has the larger* $2^{nd}$ *coordinate). Clearly,* $a_3^\star = \bar{\lambda}_3$*; this is the common length of the two third coordinates of the deGua Simplices involved. Thus,* $\Pi^\star$ *and* $\Delta^\star$ *are spanned by the maximal coordinates of the extremals of the adjusted Simplices as depicted in Figure 1.3. Obviously, the Simplex* $\Delta^\star$ *is also spanned by the adjusted multiples of the two SubSimplices involved, i.e., by* $c_1\Delta_{23}^{\boldsymbol{a}}$ *and* $c_2\Delta_{13}^{\boldsymbol{b}}$*. Figure 1.3 plausibly shows that this Simplex has the same normal as the Pareto face* $\boldsymbol{F}$ *we started out with.*

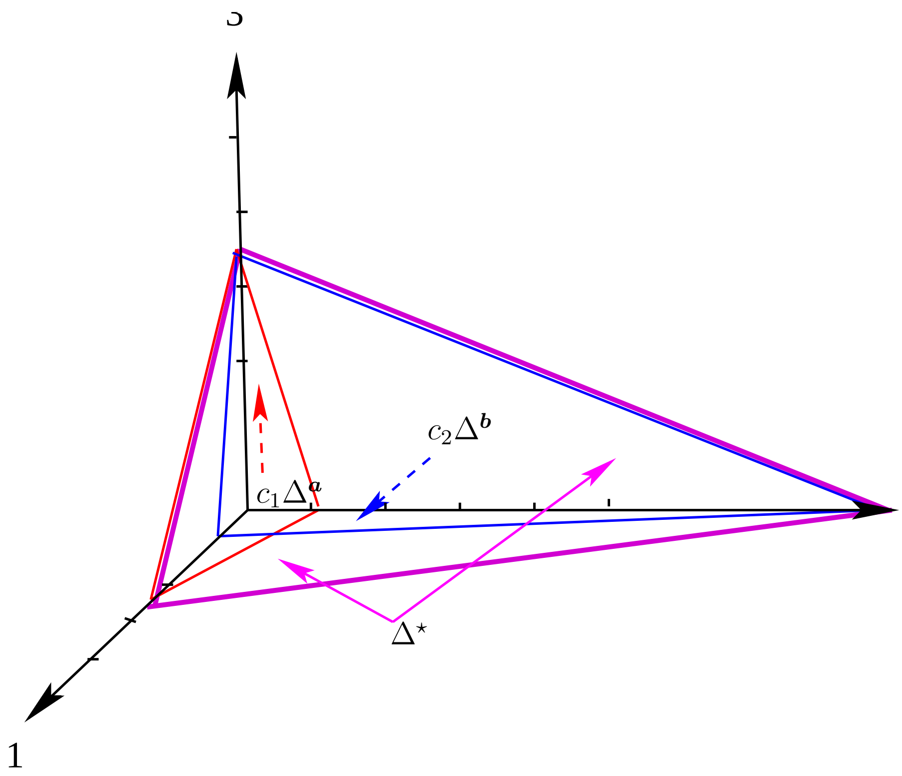


Figure 1.3: Fitting $\Delta^\star$

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**Definition 1.6.** *Let* $\boldsymbol{F} = \sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ *be a Pareto face of a Cephoid* $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{(k)} \subset \mathbb{R}^n_+$*. The (unique up to a positive multiple) positive solution*

$$(\boldsymbol{c}_\bullet, \boldsymbol{\lambda}_\bullet) = \{(c_k, \lambda_l)\}_{(k,l)\in\mathbb{L}} \tag{25}$$

*defines the* ***adjustment coefficients****.*

**Remark 1.7.** *By expanding/shrinking* $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ *via* $c_k$ *the SubSimplices involved in a Pareto face are adjusted in a way such that the linear function* $x \mapsto \mathfrak{n}^\star x$ *takes the joint maximal value* $t^\star$ *on each* $c_k\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$*. Clearly for* $l \in \boldsymbol{L} \cap \boldsymbol{J}^{(\kappa)}$*, one has*

$$a^\star_l = \max_{k\in\boldsymbol{K}} c_k a^{(k)}_i = c_\kappa a^{(\kappa)}_l = \lambda_l \ . \tag{26}$$

*that is, geometrically the coefficients* $\lambda_l$ *reflect the common length of the* $l^{th}$ *coordinate of the adjusted Simplices* $c_\kappa\Delta^{(\kappa)}_{\boldsymbol{J}^{(\kappa)}}$ *with index set* $\boldsymbol{J}^{(\kappa)}$ *containing* $l$*.*

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**Remark 1.8.** *Let* $\Pi = \sum_{k\in\boldsymbol{K}} \boldsymbol{a}^{(k)}$ *be a Cephoid. Let* $l \in \boldsymbol{I}$ *and consider the vector* $\boldsymbol{a}^{(\star)l} := \sum_{k\in\boldsymbol{K}} \boldsymbol{a}^{(k)l}$*. This vector is a multiple of the* $l^{th}$ *unit vector, hence Pareto efficient, i.e., located in* $\partial\Pi$*.*

*Consider a Pareto face* $\boldsymbol{F}$ *such that* $\boldsymbol{a}^{(\star)} \in \boldsymbol{F}$*. In this situation we have* $\boldsymbol{L} = \{l\}$ *and* $K_l = K - 1,\ K_i = 0\ \ (i \neq l)$*. We claim that the Pareto face containing* $\boldsymbol{a}^{(\star)i}$ *is uniquely defined. This is of course a consequence of nondegeneracy; however, we want to elaborate on the situation at this instant.*

*Therefore, consider the situation* $K = 2$ *(the general version is treated in the subsequent Theorem).*

*So let* $\Pi = \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}$*. Fix* $l \in \boldsymbol{I}$ *and suppose that there are two Pareto faces containing* $\boldsymbol{a}^l + \boldsymbol{b}^l$*, then there must be two Pareto faces* $\boldsymbol{F} = \Delta_{\boldsymbol{J}^{\boldsymbol{a}}} + \Delta_{\boldsymbol{J}^{\boldsymbol{b}}}$ *and* $\boldsymbol{F}' = \Delta_{\boldsymbol{J}'^{\boldsymbol{a}}} + \Delta_{\boldsymbol{J}'^{\boldsymbol{b}}}$ *containing* $\boldsymbol{a}^l + \boldsymbol{b}^l$*. These faces must have a joint subface of dimension 1 containing* $\boldsymbol{a}^l + \boldsymbol{b}^l$*, which must be of the form*

$$\Delta^{\boldsymbol{a}}_{\{lj\}} + \Delta^{\boldsymbol{b}}_{\{l\}} \text{ and } \Delta^{\boldsymbol{a}}_{\{l\}} + \Delta^{\boldsymbol{b}}_{\{lj\}}. \tag{27}$$

*For indeed, assume that w.l.g that* $\boldsymbol{J}^{\boldsymbol{a}} \neq \boldsymbol{J}^{\boldsymbol{a}'}$ *and there is an index* $j \neq l,\ j \in \boldsymbol{J}^{\boldsymbol{a}}$ *such that* $j \notin \boldsymbol{J}^{\boldsymbol{a}'}$*. As* $j \notin \boldsymbol{L}$ $|\boldsymbol{J}^{\boldsymbol{a}}| \geq |\boldsymbol{J}^{\boldsymbol{b}}|$ *and hence* $|\boldsymbol{J}'^{\boldsymbol{a}}| \leq |\boldsymbol{J}'^{\boldsymbol{b}}|$*. As* $j \notin \boldsymbol{L}$ *necessarily* $j \in \boldsymbol{J}'^{\boldsymbol{b}}$ *which is (27).*

*The one dimensional subfaces described by (27) constitute and edge at* $\boldsymbol{a}^l + \boldsymbol{b}^l$ *located in* $\boldsymbol{F}$ *and* $\boldsymbol{F}'$ *respectively, hence located in* $\partial\Pi$*. The second vertex of this edge is* $\boldsymbol{a}^j + \boldsymbol{b}^l$ *and* $\boldsymbol{a}^l + \boldsymbol{b}^j$ *respectively. Necessarily, these edges are parallel line segments, hence for some real* $\lambda$ *we have*

$$(\boldsymbol{a}^j + \boldsymbol{b}^l) - (\boldsymbol{a}^l + \boldsymbol{b}^l) = \lambda\left((\boldsymbol{a}^l + \boldsymbol{b}^j) - (\boldsymbol{a}^l + \boldsymbol{b}^l)\right)$$

*i.e.*

$$\boldsymbol{a}^j - \boldsymbol{a}^l = \lambda(\boldsymbol{b}^j - \boldsymbol{b}^l)$$

*hence* $(a_j, a_l) = \lambda(b_j, b_l)$ *, which contradicts our nondegeneracy assumption (Definition 2.5 in* Section *1* Chapter *2).*

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The following Theorem provides the appropriate generalization referring to the adjustment characteristics.

**Theorem 1.9.** *Let* $\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{(k)}$ *be a Cephoid and let* $\boldsymbol{F}$ *and* $\boldsymbol{F}'$ *be Pareto faces with adjustment sets* $\boldsymbol{L}$ *and* $\boldsymbol{L}'$*. Let* $\mathbb{L}$ *and* $\mathbb{L}'$ *be the characteristics.*

*Then*

1. $\mathbb{L} \neq \mathbb{L}'$
2. *The characteristics or equivalently the subface* $\boldsymbol{F_L}$ *uniquely determine a Pareto face* $\boldsymbol{F}$*.*
3. *Hence* $\boldsymbol{F}_L$ *and* $\boldsymbol{F}_{L'}$ *are different and the intersection satisfies*

$$\boldsymbol{dim}\ (\boldsymbol{F_L} \cap \boldsymbol{F_{L'}}) < \min\{L, L'\} ,$$

*i.e., the assignment of a Pareto face to the L–dimensional subface generated by the adjustment indices is unique.*

**Proof:**

The characteristics of a Pareto face, i.e., set $\mathbb{L}$, determine the linear adjustment system which in turn determines the normal. As the normal determines a Pareto face, it follows that so does $\mathbb{L}$.

**q.e.d.**

# 2 Adjacent Faces

Some considerations as presented for Pareto faces can be repeated for lower dimensional (outward) faces. Thus, the concepts of reference system, adjustment set and characteristics are obviously not depending on the maximality of a face in $\partial\Pi$.

In particular, this holds true for a face $\boldsymbol{F}^\star \subseteq \partial\Pi$ of dimension $n-2$. Whenever $\boldsymbol{F}^\star$ is a subset of a Pareto face $\boldsymbol{F}$, the reference systems $\mathfrak{J}$ and $\mathfrak{J}^\star$ of $\boldsymbol{F}$ and $\boldsymbol{F}^\star$ will differ by exactly one index. More precisely:

**Theorem 2.1.** *Let $\boldsymbol{F} \in \partial\Pi$ be a Pareto face of a Cephoid $\Pi = \sum_{k\in\boldsymbol{K}} \boldsymbol{a}^{(k)}$ with reference system $\mathfrak{J}$ and adjustment set $\boldsymbol{L}$. Also, let Let $\boldsymbol{F}^\star \in \partial\Pi$ be a subface of $\boldsymbol{F}$ with dimension $n-2$ with reference system $\mathfrak{J}^\star$ and adjustment set $\boldsymbol{L}^\star$. Then the following holds true:*

1. *There is $\kappa \in \boldsymbol{K}$ and $j \in \boldsymbol{J}^{(\kappa)}$ such that*

$$\boldsymbol{J}^{\star(\kappa)} = \boldsymbol{J}^{(\kappa)} \setminus \{j\}\ , \quad \boldsymbol{J}^{\star(k)} = \boldsymbol{J}^{(k)} \quad (k \in \boldsymbol{K} \setminus \{\kappa\}). \tag{1}$$

2. *$\boldsymbol{L}^\star \subseteq \boldsymbol{L}$, $\quad \mathbb{L} = \mathbb{L}^\star \setminus \{(\kappa, j)\}$ .*

3. 

$$|\boldsymbol{L}| - 1 \leq |\boldsymbol{L}^\star| \leq |\boldsymbol{L}|, \quad |\mathbb{L}^\star| = |\mathbb{L}| - 1. \tag{2}$$

**Definition 2.2.** *Let $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ be a Cephoid. We call two Pareto faces $\boldsymbol{F}$ and $\boldsymbol{F}'$ (with dimension $(n-1)$) of $\partial\Pi$ **adjacent** whenever their intersection $\boldsymbol{F}^\star = \boldsymbol{F} \cap \boldsymbol{F}' \subseteq \partial\Pi$ is a subface in $\partial\Pi$ with dimension $(n-2)$.*

Now we can compare the reference system and the adjustment indices of two adjacent Pareto faces.

**Theorem 2.3 (The Neighborhood Theorem).** *Let $\Pi = \sum_{k\in\boldsymbol{K}} \boldsymbol{a}^{(k)}$ be a cephoid and let $\boldsymbol{F}, \boldsymbol{F}'$ be adjacent Pareto faces with an $(n-2)$–dimensional common subface $\boldsymbol{F}^\star = \boldsymbol{F} \cap \boldsymbol{F}' \subseteq \partial\Pi$. Let $\mathfrak{J}$ and $\mathfrak{J}'$ be the reference systems and let $\boldsymbol{L}$ and $\boldsymbol{L}'$ be the adjustment sets. Then there exist indices $p, q \in \boldsymbol{K}$, $p \neq q$, and $i_0, i_1 \in \boldsymbol{I}$, $i_0 \neq i_1$, with $i_0 \in \boldsymbol{L}$, $i_1 \in L'$, such that the following holds:*

$$\begin{array}{rcl} \boldsymbol{J}^{(k)} & = & \boldsymbol{J}'^{(k)} \quad (k \neq p, q) \\ \boldsymbol{J}^{(p)} & = & \boldsymbol{J}'^{(p)} \cup \{i_0\} \\ \boldsymbol{J}'^{(q)} & = & \boldsymbol{J}^{(q)} \cup \{i_1\}\ . \end{array} \tag{3}$$

*Consequently, the characteristics of* $\boldsymbol{F}$,$\boldsymbol{F}'$, *and* $\boldsymbol{F}^{\star}$ *satisfy*

$$\text{(4)} \qquad \mathbb{L} \;=\; \mathbb{L}^{\star} \cup \{(p, i_0)\} \quad , \quad \mathbb{L}' \;=\; \mathbb{L}^{\star} \cup \{(q, i_1)\}$$

**Proof:**

**1<sup>st</sup>STEP** : Let

$$\text{(5)} \qquad \boldsymbol{F}^{\star} \;=\; \boldsymbol{F} \cap \boldsymbol{F}' \;=\; \sum_{k \in \boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)} \cap \boldsymbol{J}'^{(k)}}$$

denote the common subface of $\boldsymbol{F}$ and $\boldsymbol{F}'$, dimension is $n-2$. Counting indices and referring to nondegeneracy (the same argument as in the proof of Theorem 1.2 $2^{nd}STEP$, formula (8)) we obtain

$$\sum_{k \in \boldsymbol{K}} \mid \boldsymbol{J}^{(k)} \cap \boldsymbol{J}'^{(k)} \mid = n - 2 + K \ .$$

As the corresponding sums for the two faces yield $n-1+K$, we must necessarily find $p, q$ such that

$$\text{(6)} \qquad \begin{aligned} \boldsymbol{J}^{(p)} = \quad & \left(\boldsymbol{J}^{(p)} \cap \boldsymbol{J}'^{(p)}\right) \cup \{i_0\} \quad = \quad \boldsymbol{J}'^{(p)} \cup \{i_0\} = \quad \boldsymbol{J}^{\star(p)} \cup \{i_0\} \\ \boldsymbol{J}'^{(q)} = \quad & \left(\boldsymbol{J}^{(q)} \cap \boldsymbol{J}'^{(q)}\right) \cup \{i_1\} \quad = \quad \boldsymbol{J}^{(q)} \cup \{i_1\} = \quad \boldsymbol{J}^{\star(q)} \cup \{i_1\} \ . \end{aligned}$$

Now $p = q$ is not possible as we would have $\boldsymbol{J}^{(p)} \;=\; \widetilde{\boldsymbol{J}}'^{(p)}$. This would imply equal reference systems for both faces, hence they would coincide. Therefore $p \neq q$.

**3<sup>rd</sup>STEP** : Assume that $i_0 \notin \boldsymbol{L}$ and $i_1 \notin \boldsymbol{L}'$ is the case. Then we have $\mathbb{L} \;=\; \mathbb{L}'$. As the system $\mathbb{L}$ determines $\boldsymbol{F}$ uniquely, it would follow that $\boldsymbol{F} = F'$ holds true. On the other hand, assume e.g. $i_0 \in \boldsymbol{L}, i_1 \notin \boldsymbol{L}'$. Then for the characteristics, we have $\mathbb{L}' \subseteq \mathbb{L}$. Then it would follow that all equations of the linear adjustment system generated by $\mathbb{L}'$ appear in the linear adjustment system generated by $\mathbb{L}$ as well. But both system must have maximal *rank,* i.e., generate a solution space of dimension 1. Evidently, the two systems have the same solution space, in which case the normals coincide. Hence again we would find $\boldsymbol{F} \;=\; \boldsymbol{F}'$, which cannot happen. Hence $i_0 \in \boldsymbol{L},\; i_1 \in \boldsymbol{L}'$ is true. Finally $i_0 \neq i_1$ for otherwise $\boldsymbol{L} = \boldsymbol{L}'$ again would imply $\boldsymbol{F} = \boldsymbol{F}'$.

**q.e.d.**

**Corollary 2.4.** *Let* $\Pi$ *be a cephoid and let* $\boldsymbol{F}, \boldsymbol{F}'$ *be adjacent Pareto faces. Let* $i_0, i_1$ *and* $p, q$ *be given by* (6) *via Theorem 2.3.*

*Also, let* $\boldsymbol{L}^\star$ *and* $\mathbb{L}^\star$ *denote the adjustment set and the characteristics of* $\boldsymbol{F}^\star = \boldsymbol{F} \cap \boldsymbol{F}'$. *Then we have*

1. *The normal cone to* $\boldsymbol{F}^\star$ *is the convex hull of the normal rays*

$$\mathfrak{N}^\star = \boldsymbol{CovH}\left\{\{t\mathfrak{n} \mid t \geq 0\} \ , \ \{t\mathfrak{n}' \mid t \geq 0\}\right\} \tag{7}$$

   *spanned by the normals* $\mathfrak{n}$ *and* $\mathfrak{n}'$ *of* $\boldsymbol{F}$ *and* $\boldsymbol{F}'$ *respectively,*

2. $\mathfrak{N}^\star$ *is the positive part of two–dimensional subspace of* $\mathbb{R}^n$ *of solutions to the linear adjustment system corresponding to* $\boldsymbol{F}^\star$*, which is*

$$c_k a_l^{(k)} = \lambda_l \quad ((k,l) \in \mathbb{L}^\star). \tag{8}$$

   *in variables* $(c_k, \lambda_l)$, $((k,l) \in \mathbb{L}^\star)$ . *Adding one of the equations*

$$c_p a_{i_0}^{(p)} = \lambda_{i_0} \quad or \quad c_q a_{i_1}^{(q)} = \lambda_{i_q}$$

   *to* (8) *results in the extremal rays corresponding to the normal cones of* $\boldsymbol{F}$ *and* $\boldsymbol{F}'$ *respectively.*

3. *The extremals* $\mathfrak{n}$ *and* $\mathfrak{n}'$ *of the normal cone of* $\boldsymbol{F}^\star$ *(i.e., the normals to* $\boldsymbol{F}$ *and* $\boldsymbol{F}'$ *respectively) can be normalized so that both coincide on* $\{i \in \boldsymbol{I} \setminus \{i_0, i_1\}\}$. *Then, w.r.t. coordinates* $i_0, i_1$ *we have*

$$\begin{array}{rl} & \qquad\quad i_0 \qquad\qquad\quad i_1 \\ \mathfrak{n}: & (\ldots, \lambda_{i_0}, \ldots \quad \ldots, a_{i_1}^\star, \ldots) \\ \mathfrak{n}': & (\ldots, a_{i_0}^\star, \ldots \quad \ldots, \lambda_{i_1}, \ldots) \ . \end{array} \tag{9}$$

**Proof:** Concerning the last statement we observe that we can choose the normalization $t^\star$ to b equal for both $\boldsymbol{F}$ and $\boldsymbol{F}'$. Then the terms $\lambda_{i_0}, \lambda_{i_1}$ result from Remark 1.7

**q.e.d.**

So far we have considered the $(n-2)$–dimensional intersection $\boldsymbol{F}^\star$ of two adjacent faces $\boldsymbol{F}$ and $\boldsymbol{F}'$. Now we start out with some $(n-2)$ dimensional subface $\boldsymbol{F}^\star$ of some Pareto face $\boldsymbol{F}$ and ask for a Pareto face that possibly containes $\boldsymbol{F}^\star$ and serves as the adjacent neighbor of $\boldsymbol{F}$. However, observe that $\boldsymbol{F}$ can be a boundary face hence has no adjacent neighbor containing $\boldsymbol{F}^\star$. Thus, for the moment, we slightly change our viewpoint and include maximal faces that are not Pareto faces.

To this end, we repeat Definition 1.3 in SECTION 1 CHAPTER1, slightly rephrased.

**Definition 2.5.** *For* $i_0 \in \boldsymbol{I}$ *and* $\boldsymbol{a} \in \mathbb{R}^n$ *let* $\boldsymbol{a}^{(-i_0)} = \boldsymbol{a}_{\mid \mathbb{R}^n_{\boldsymbol{I}\setminus\{i_0\}}}$ *denote the projection on* $\mathbb{R}^n_{\boldsymbol{I}\setminus\{i_0\}}$. *Let* $\Pi^{(-i_0)}$ *denote the Cephoid generated by the family*

$$\left\{\boldsymbol{a}^{(k)(-i_0)}\right\}_{k\in\boldsymbol{K}} . \tag{10}$$

*Then* $\Pi^{(-i_0)}$ *is the* $i_0$*–face of* $\Pi$.

**Theorem 2.6.** *Let* $\boldsymbol{F}$ *be a Pareto face of a cephoid* $\Pi$ *with reference system* $\mathcal{J} = \left\{\boldsymbol{J}^{(k)}\right\}_{k\in\boldsymbol{K}}$ *and let* $\mathfrak{n}$ *be the normal to* $\boldsymbol{F}$. *Let* $(p, i_0) \in (\boldsymbol{K}, \boldsymbol{I})$ *satisfy* $i_0 \in \boldsymbol{J}^{(p)}, i_0 \notin \boldsymbol{L}$ *(hence* $|\boldsymbol{J}^{(p)}| \geq 2$*). Define*

$$\boldsymbol{F}^\star := \sum_{k\in\boldsymbol{K}-p} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} + \Delta^{(p)}_{\boldsymbol{J}^{(p)}\setminus\{i_0\}}. \tag{11}$$

*Then then dimension of* $\boldsymbol{F}^\star$ *is* $(n-2)$. $\boldsymbol{F}^\star = \boldsymbol{F} \cap \Pi^{(-i_0)} \subseteq \partial\Pi^{(-i_0)}$ *is a Pareto face of* $\Pi^{(-i_0)}$. $\boldsymbol{F}^\star$ *is as well an* $(n-2)$ *dimensional subface of* $\boldsymbol{F}$ *and the second extremal to the normal cone of of* $\boldsymbol{F}^\star$ *is*

$$\mathfrak{n}^\star = \mathfrak{n} - \mathfrak{n}_{i_0}\boldsymbol{e}^{i_0} = (\mathfrak{n}_1, \ldots, 0, , \ldots, \mathfrak{n}_n)$$

**Proof:** $\boldsymbol{F}^\star \subseteq \boldsymbol{F}$ is obvious; clearly the dimension of $\boldsymbol{F}^\star$ is $(n-2)$ and vectors in $\boldsymbol{F}^\star$ have a zero at coordinate $i_0$. Thus we have to specify the normal cone.

However, the normal cone of $\Pi^{(-i_0)}$ (viewed in $\mathbb{R}^n$) is spanned by $\boldsymbol{e}^{i_0}$ hence the one of $\boldsymbol{F}^\star$ is spanned by $\mathfrak{n}$ and $\boldsymbol{e}^{i_0}$, or equivalently by $\mathfrak{n}$ and $\mathfrak{n}^\star$.

**q.e.d.**

**Remark 2.7.** *There are two ways to view* $\boldsymbol{F}^\star$ *and the normal* $\mathfrak{n}^\star$. *We can employ the above Theorem 2.6 in which we compute the normal of* $\boldsymbol{F}$, *then the normal cone of* $\boldsymbol{F}^\star$ *and finally, employ a projection argument. That is, we first work the machinery to compute the normal of Theorems 1.2 and 2.4 for the Cephoid* $\Pi$ *in* $\mathbb{R}^n$ *and view* $\boldsymbol{F}^\star$ *as an* $(n-1)$*–dimensional subface of* $\partial\Pi$.

*On the other hand, we can first apply the projection and then set up the machinery of Theorems 1.2 and 2.4 for* $\Pi^{(-i_0)}$. *Then we view* $\boldsymbol{F}^\star$ *as a Pareto face of* $\partial\Pi^{(-i_0)}$ *in* $\mathbb{R}^n_{\boldsymbol{I}\setminus\{i_0\}}$. *Note that* $\mathbb{L}^\star = \mathbb{L}$ *as* $i_0 \notin \boldsymbol{L}$.

*That is, with*

$$\begin{aligned} proj^{-i_0} &:= \mathbb{R}^n \to \mathbb{R}^n \\ proj^{-i_0}(\boldsymbol{x}) &:= \boldsymbol{x}_{\mid \mathbb{R}^n\setminus\{i_0\}} \end{aligned}$$

*the following diagram is commutative*

$$
(12) \qquad \begin{array}{ccccccc}
\Pi \to & & \boldsymbol{F},\ \boldsymbol{F}^\star & \to & \mathbb{L} = \mathbb{L}^\star & \to & \mathfrak{n},\ \mathfrak{n}^\star \\
proj^{-i_0} \downarrow & & & & & & \downarrow proj^{-i_0} \\
\Pi^{(-i_0)} \to & & \boldsymbol{F}^\star & \to & \mathbb{L}^\star & \to & \mathfrak{n}^\star \mid_{\mathbb{R}^n \setminus \{i_0\}} .
\end{array}
$$

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Now alternatively (to Theorem 2.6), consider the case that the $(n-2)$ dimensional subface $\boldsymbol{F}^\star \subseteq \boldsymbol{F}$ is obtained by omitting an index $l_0 \in \boldsymbol{L}$ from the reference system of $\boldsymbol{F}$. Then there must be a Pareto face $\boldsymbol{F}'$ adjacent to $\boldsymbol{F}$ such that $\boldsymbol{F}^\star = \boldsymbol{F} \cap \boldsymbol{F}^\star$ is the case.

**Theorem 2.8.** *Let $\boldsymbol{F}$ be a Pareto face of a cephoid $\Pi$ with reference system $\mathfrak{J}$. Let $(p, l_0) \in \mathbb{L}$ such that $|\boldsymbol{J}^{(p)}| \geq 2$. Let*

$$
(13) \qquad \boldsymbol{F}^\star \ := \sum_{k \in \boldsymbol{K} \setminus \{p\}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} + \Delta^{(p)}_{\boldsymbol{J}^{(p)} \setminus \{l_0\}}
$$

*be an $(n-2)$–dimensional subface of $\boldsymbol{F}$. Then there is some Pareto face $\widetilde{\boldsymbol{F}}$ of $\Pi$ such that*

$$
\boldsymbol{F}^\star \ = \ \boldsymbol{F} \cap \widetilde{\boldsymbol{F}} \ .
$$

**Proof:**

The normal cone to $\boldsymbol{F}^\star$ is described by Theorem 2.6. The one extremal of the normal cone of $\boldsymbol{F}^\star$ is provided by the normal $\mathfrak{n}$ of $\boldsymbol{F}$.Let $\widetilde{\mathfrak{n}}$ be the second extremal of this cone. By the nondegeneracy assumption, this extremal can either have exactly one zero coordinate $\widetilde{\mathfrak{n}}_i$ (in which case $\boldsymbol{F}^\star$ is located in the corresponding $\partial\Pi^{(-i)}$), or else it is positive. In the latter case the theorem is verified. However, if the neighboring face is a boundary face in the sense of Theorem 2.6, then we have seen that the characteristics satisfy $\mathbb{L} = \mathbb{L}^\star$ (Remark 2.7). This is not compatible with the representation of $\boldsymbol{F}^\star$ by (11) as $l_0 \in \mathbb{L}$ is missing in the reference system of $\boldsymbol{F}^\star$.

**q.e.d.**

# 3 Extremal Points

At this stage, we insert some remarks concerning the (Pareto efficient) extremal points or ***vertices*** of a Cephoid. As we focus mainly on the Pareto faces, we do this in passing; however we want to mention the topic as it constitutes an alternative way of describing a Cephoid.

A compact convex polyhedron can be described by indicating all maximal faces, that is intersections of supporting hyperplanes with the polyhedron of maximal dimensions. In our present setup concerning a Cephoid $\Pi$ we distinguish between Pareto faces and maximal faces that are not Pareto efficient, that is, intersections of $\Pi$ with an $(n-1)$ dimensional subspace $\mathbb{R}^n_{\boldsymbol{I}\setminus\{i_0\}}$ for some $i_0 \in \boldsymbol{I}$. Such an intersection is, in turn, a Cephoid in its own right, namely the one generated by the same family of positive vectors, but restricted to $\mathbb{R}^n_{\boldsymbol{I}\setminus\{i_0\}}$; we have denoted this Cephoid by $\Pi^{(-i_0)}$. Needless to say that in case of a Cephoid, the Pareto faces are sufficient to provide a complete description of the polyhedron.

Alternatively, one can establish a list of the extremals, in our case vertices and $\mathbf{0}$. The polyhedron is then obtained as the convex hull of its extremals (Minkowski, Caratheodory, Krein–Milman). In a sense, both procedures are *dual* to each other. Also, for a complete description it suffices to indicate the vertices.

Without explicit proofs (which are more or less provided by our previous results) we list the following statements concerning the Extremal Points of a Cephoid $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}} = \sum_{k\in\boldsymbol{K}} \Pi^{(k)}$.

1. Every vertex of $\Pi$ is of the form

   $$(1) \qquad \boldsymbol{a}^{i_1,\ldots,i_K} = \boldsymbol{a}^{(1)i_1} + \ldots + \boldsymbol{a}^{(K)i_K} = a^{(1)}_{i_1}\boldsymbol{e}^{i_1} + \ldots + a^{(K)}_{i_K}\boldsymbol{e}^{i_K}$$

   with $i_k \in \boldsymbol{I} \quad (k \in \boldsymbol{K})$.

2. A sum $\boldsymbol{a}^{i_1,\ldots,i_K}$ as in (1) is a vertex if and only if it is Pareto efficient.

3. Let

   $$(2) \qquad \boldsymbol{F} = \sum_{k\in\boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$$

   be a Pareto face of $\Pi$ with reference system $\mathfrak{J} = \left\{\boldsymbol{J}^{(k)}\right\}_{k\in\boldsymbol{K}}$.

   Then, for each selection $i_1 \in \boldsymbol{J}^{(1)}, \ldots, i_K \in \boldsymbol{J}^{(K)}$, the vector $\boldsymbol{a}^{i_1,\ldots,i_K}$ as in (2) is a vertex of $\Pi$.

4. In view of nondegeneracy, every extremal point $\boldsymbol{a}^{i_1,\ldots,i_K}$ belongs to exactly $n$ maximal faces of $\Pi$. If $n > K$ holds, then necessarily not all of these maximal faces are Pareto efficient. Rather some of them are boundary faces of the form $\Pi^{(-i_0)}$.

5. To every vertex $\boldsymbol{a}^{i_1,\ldots,i_K}$ of $\Pi$ there exists a normal cone of dimension $n$ which is the intersection of the normal cones at the various DeGua Simplices $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ in the extremals $\boldsymbol{a}^{(k)i_k}$ $(k \in \boldsymbol{K})$.

# 4 The Sum of Two deGua Simplices: Cylinders

Presently we apply our results to the case $K = 2$. We consider the sum $\Pi = \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}$ two deGua Simplices with Pareto surface $\partial\Pi$. This case is treated at length as it provides a first insight into the nature of Pareto faces and thus suggests the path for further developments.

**Theorem 4.1.** *Let $\boldsymbol{F}$ be a Pareto face of $\Pi$. Then there exists a unique $l \in \boldsymbol{I}$ such that $\boldsymbol{a}^l + \boldsymbol{b}^l \in \boldsymbol{F}$. On the other hand, for every $i \in \boldsymbol{I}$ the vertex $\boldsymbol{a}^i + \boldsymbol{b}^i$ of $\partial\Pi$ is contained in one and only one Pareto face $\boldsymbol{F}$.*

*Moreover, for every Pareto face $\boldsymbol{F}$ with corresponding $l \in \boldsymbol{I}$ as above, the following hold true:*

1. *The reference system $\mathfrak{J}$ consists of two reference sets $\boldsymbol{J}^1, \boldsymbol{J}^2 \subseteq \boldsymbol{I}$ such that*

   $$\boldsymbol{F} \;=\; \Delta^{\boldsymbol{a}}_{\boldsymbol{J}^1} + \Delta^{\boldsymbol{b}}_{\boldsymbol{J}^2} \tag{1}$$

   *with*

   $$|\boldsymbol{J}^1| + |\boldsymbol{J}^2| = n + 1 \ .$$

   *and*

   $$\boldsymbol{J}^1 \cap \boldsymbol{J}^2 = \{l\}. \tag{2}$$

   *That is, the adjustment set is $\boldsymbol{L} = \{l\}$. The characteristics can be written*

   $$\boldsymbol{L} \;=\; \{(1,l),(l,2)\} \;=\; \{(\boldsymbol{a},l),(l,\boldsymbol{b})\} \tag{3}$$

   *with an obvious abuse of notation for $K = 2$.*

2. *Accordingly, there are positive constants $c_{\boldsymbol{a}}, c_{\boldsymbol{b}}$ such that $c_{\boldsymbol{a}}\Pi^{\boldsymbol{a}}$ and $c_{\boldsymbol{b}}\Pi^{\boldsymbol{b}}$ have exactly one common vertex; this is the the vertex*

   $$c_{\boldsymbol{a}}\boldsymbol{a}^l = c_{\boldsymbol{b}}\boldsymbol{b}^l \ . \tag{4}$$

3. *The normal $\mathfrak{n}^\star$ of $\boldsymbol{F}$ is (up to a positive multiple) exactly the normal of the deGua Simplex*

   $$\Delta^\star \;=\; c_{\boldsymbol{a}}\Pi^{\boldsymbol{a}} \vee c_{\boldsymbol{b}}\Pi^{\boldsymbol{b}} \ . \tag{5}$$

**Proof:** Clearly follows from Theorem 1.2, Theorem 1.4, and Remark 1.8 specified for $K = 2$.

**q.e.d.**

We can now exactly describe the Pareto surface of the sum of two polyhedra. To this end, we introduce the following notation. Let $\prec$ be a total ordering of $\boldsymbol{I}$. We denote by

$$T_j^{\prec} \;:=\; \{i \in \boldsymbol{I} \,|\, i \prec j\} \cup \{j\} \tag{6}$$

the set of predecessors of $j \in \boldsymbol{I}$ including $j$ with respect to $\prec$. Similarly, let

$$S_j^{\prec} \;:=\; \{i \in \boldsymbol{I} \,|\, j \prec i\} \cup \{j\} \tag{7}$$

denote the set of successors of $j$ including $j$. Clearly

$$S_j^{\prec} \cap T_j^{\prec} \;=\; \{j\} \quad (j \in \boldsymbol{I})$$

holds. Combining we have the following

**Theorem 4.2.** *1. The sum of two deGua Simplices in* $\mathbb{R}^n$ *has exactly* $n$ *Pareto faces. Each Pareto face is uniquely identified by a vertex* $\boldsymbol{a}^l + \boldsymbol{b}^l$ *containing it.*

2. *For any sum of two deGua Simplices* $\Pi = \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}$ *there exists uniquely an ordering* $\prec$ *of* $\boldsymbol{I}$ *such that the Pareto faces are exactly described by*

$$\boldsymbol{F}^{\prec i} \;:=\; \Delta^{\boldsymbol{a}}_{S_i^{\prec}} + \Delta^{\boldsymbol{b}}_{T_i^{\prec}} \quad (i \in \boldsymbol{I}). \tag{8}$$

3. *In view of nondegeneracy, there are exactly* $n!$ *"types" of sums of two deGua Simplices. Each type corresponds to an ordering of* $\boldsymbol{I}$ *such that the Pareto faces are given by equation* (8).

**Proof:** Each Pareto face $\boldsymbol{F}$ of the surface contains exactly one vertex $\boldsymbol{a}^l + \boldsymbol{b}^l$ of $\Delta$ for some $l \in \boldsymbol{I}$ and, the other way around, for every $i \in \boldsymbol{I}$, there is exactly one face containing $\boldsymbol{a}^i + \boldsymbol{b}^i$. Thus , there is a bijection between vertices and Pareto faces.

Now the Neighborhood Theorem (Theorem 2.3) and the subsequent results of SECTION 2 require that each Pareto face is either a boundary face (with one adjacent neighbor) or has two adjacent neighbors.

For $l \in \boldsymbol{I}$, let $\boldsymbol{F}^l$ denote the Pareto face featuring $\boldsymbol{L} = \{l\}$. Now define an ordering $\prec$ of $\boldsymbol{I}$ in a way such that each face $\boldsymbol{F}^l$ is adjacent to two its neighbors in the ordering. That is $l, l'$ are neighbors in the ordering $\prec$ if and only if $\boldsymbol{F}^l, \boldsymbol{F}^{l'}$ are adjacent.

Now, viewing the Neighborhood Theorem 2.3 (in particular formula (6) of the proof), we observe that the transition from the reference set $\boldsymbol{J}^{l\boldsymbol{a}}$ of $\boldsymbol{F}^l$ to the the reference set $\boldsymbol{J}^{l'\boldsymbol{a}}$ of $\boldsymbol{F}'$ w.l.g is performed by

$$\boldsymbol{J}^{l'\boldsymbol{a}} \;=\; \boldsymbol{J}^{l\boldsymbol{a}} \cap \{l'\}$$

(or vice versa for $\boldsymbol{b}$). Thus, $\boldsymbol{J}^{l\boldsymbol{a}}$ contains all the predecessors of $l$ including $l$ with respect to the ordering $\prec$ which is as well the ordering of the Pareto faces. That is, we have indeed

$$\boldsymbol{F}^l \;=\; \boldsymbol{F}^{\prec l}$$

in the sense of (8).

**q.e.d.**

Thus for any two deGua Simplices the total ordering that generates the Pareto faces as in formula (8) is uniquely defined. Therefore, we come up with a formal definition.

**Definition 4.3.** 1. *For any sum of two deGua Simplices* $\Pi = \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}$ *we shall say that* $\prec$ *as defined via Theorem 4.2 is* ***ordering*** $\partial\Pi$.

2. *Let* $\{\boldsymbol{a}^{(k)}\}_{k\in\boldsymbol{K}}$ *be a family of positive vectors. Let* $k^\star, k' \in \boldsymbol{K}$ *and let*

$$\Pi^{(k^\star k')} \;:=\; \Pi^{(k^\star)} \;+\; \Pi^{(k')}\;.$$

*Let* $\prec$ *be the ordering of* $\partial\Pi^{(k^\star k')}$ *and let the predecessors and successors be defined by* (6) *and* (7)*. Then, for* $k, k' \in \boldsymbol{K}$ *we define*

$$\boldsymbol{F}^{kk';\prec i} \;:=\; \Delta^{(k)}_{S_i^\prec} + \Delta^{(k')}_{T_i^\prec} \quad (i \in \boldsymbol{I}). \tag{9}$$

The second *item* will be will be useful in generating translates of the faces of each pair of a family $\{\boldsymbol{a}^{(k)}\}_{k\in\boldsymbol{K}}$ on the Pareto surface of the corresponding Cephoid, see later in Theorem 4.9.

**Remark 4.4.** *Consider the case that the ordering* $\prec$ *is the natural one, i.e.,* $1 \prec 2 \prec 3 \prec, \ldots, \prec n$*. Then we can indicate the Pareto faces using the reference system, that is, by listing the reference sets* $S_j^\prec, T_j^\prec$ $(j \in \boldsymbol{I})$ *according to formula*

(8) *as follows.*

$$
\begin{array}{lr}
1 & 1234\dots n \\
12 & 234\dots n \\
123 & 34\dots n \\
\cdots & \cdots \\
123\dots n & n
\end{array}
\tag{10}
$$

*Obviously, a reference system and hence each face is obtained from its neighbor by canceling the double appearing index (i.e., the on in* $\boldsymbol{L} = \{l\}$*)on the right side and instead moving the next index to the left so that it appears double. Sloppily we refer to this procedure as to the* ***moving index principle*** *for one index. Later on, there will be a generalization for more than one index.*

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**Example 4.5.** *Subsequently, we demonstrate concrete examples obtained by computational methods that will be explained later (*CHAPTER *8). The order in which the faces are listed is reversed due to the algorithm employed, yet, these are the examples referring to Remark 4.4, thus demonstrating the "moving index principle". The second example ("Cephoid CEPERN") emerges from the first one by permuting the columns to the reverse order.*

```
CEPHOID ⋆CEPERM⋆:

 13 4 2 2
  2 3 3 6

FACES OF CEPHOID ⋆CEPERM⋆:

 1 2 3 4  ⊛        4  ⊛
   1 2 3  ⊛      3 4  ⊛
     1 2  ⊛    2 3 4  ⊛
       1  ⊛  1 2 3 4  ⊛

PERMUTING ⋆CEPERM⋆ -- CEPHOID ⋆CEPERN⋆:

 2 2 4 13
 6 3 3  2

FACES OF CEPHOID ⋆CEPERN⋆:

 1 2 3 4  ⊛        1  ⊛
   2 3 4  ⊛      1 2  ⊛
     3 4  ⊛    1 2 3  ⊛
       4  ⊛  1 2 3 4  ⊛
```

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**Example 4.6.** *Again we consider our introductory simple Example 2.4 in* Section *2* Chapter *1. See also Example 1.5, we repeat the sketch in in Figure 4.1.*

*We observe that* $\boldsymbol{a}^{(1)} + \boldsymbol{b}^{(1)}$ *determines* $\boldsymbol{F}^1 = \Delta^{\boldsymbol{a}} + \boldsymbol{b}^1$. *Similarly* $\boldsymbol{a}^{(2)} + \boldsymbol{b}^{(2)}$ *determines* $\boldsymbol{F}^2 = \Delta^{\boldsymbol{b}} + \boldsymbol{a}^2$. *Finally the third face* $\boldsymbol{F}^3 = \Delta^{\boldsymbol{a}}_{\{23\}} + \Delta^{\boldsymbol{b}}_{\{13\}}$ *corresponds to* $\boldsymbol{a}^{(3)} + \boldsymbol{b}^{(3)}$. *Hence we obtain a list of the Pareto faces as follows:*

$$
\begin{aligned}
\boldsymbol{F}^{\prec 2} &= \Delta^{\boldsymbol{a}}_2 + \Delta^{\beta}_{231} \\
\boldsymbol{F}^{\prec 3} &= \Delta^{\boldsymbol{a}}_{23} + \Delta^{\boldsymbol{b}}_{31} \\
\boldsymbol{F}^{\prec 1} &= \Delta^{\boldsymbol{a}}_{231} + \Delta^{\boldsymbol{b}}_{1} \; .
\end{aligned}
\tag{11}
$$

*Now, using our notation explained above we obtain the list of reference systems*

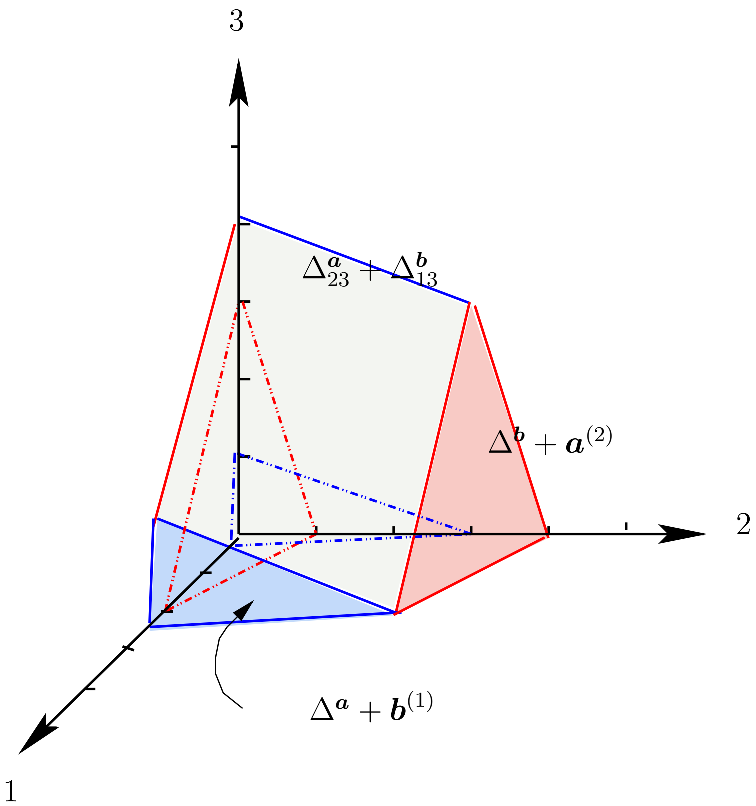


Figure 4.1: The familiar sum of 2 deGua Simplices

*as follows:*

$$(12) \qquad \begin{array}{ll} 2 & 231 \\ 23 & 31 \\ 231 & 1 \end{array}$$

*Clearly the underlying ordering is indicated by* $\prec \; = \; (2,3,1)$. *At each face* $\boldsymbol{F}^{l,\prec}$ *the decisive index/coordinate appears double (in both reference sets) and exhibits the coordinate* $l \in \boldsymbol{L}$ *which governs the adjustment.*

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**Example 4.7.** *Now consider the case* $K = 2$ *and* $n = 4$. *A Cephoid as well as its Pareto surface are polyhedra in* $\mathbb{R}^4$. *However, the canonical representation maps both structures into* $\mathbb{R}^3$. *Indeed the Simplex in which the canonical representation appears is* $2\Delta^{\boldsymbol{e}}$, *the two–fold unit Simplex. This can be sketched in* $\mathbb{R}^2$ *as a tetrahedron spanned by the vectors* $2\boldsymbol{e}^i \quad (i \in \{1,2,3,4\})$. *Now, within this representation, the Pareto faces are either translates of one of the generating deGua Simplices or else a sum of a line segment and a triangle.*

*Consider the sketch of a canonical representation, as depicted in Figure 4.2. Here* $\Pi^{\boldsymbol{a}}, \Delta^{\boldsymbol{a}}$ *are painted in blue and* $\Pi^{\boldsymbol{b}}, \Delta^{\boldsymbol{b}}$ *are painted in red. We tacitly assume that* $\boldsymbol{a} = \boldsymbol{a}^{(1)}, \boldsymbol{b} = \boldsymbol{a}^{(2)}$, *this way inducing an enumeration of the family of vectors generating the Cephoid* $\Pi = \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}} = \Pi^{\boldsymbol{a}^{(1)}} + \Pi^{\boldsymbol{a}^{(2)}}$.

*The left hand Figure in 4.2 corresponds to the ordering* $\prec$ *given by* $2341$. *For, the Pareto faces are given by*

$$(13) \qquad \begin{aligned} \boldsymbol{F}^{\prec 2} &= \Delta^{\boldsymbol{a}}_{2} + \Delta^{\boldsymbol{b}}_{2341} \\ \boldsymbol{F}^{\prec 3} &= \Delta^{\boldsymbol{a}}_{23} + \Delta^{\boldsymbol{b}}_{341} \\ \boldsymbol{F}^{\prec 4} &= \Delta^{\boldsymbol{a}}_{234} + \Delta^{\boldsymbol{b}}_{41} \\ \boldsymbol{F}^{\prec 1} &= \Delta^{\boldsymbol{a}}_{2341} + \Delta^{\boldsymbol{b}}_{1} \; . \end{aligned}$$

*For every* $l \in \boldsymbol{I}$ *the coordinate* $l$ *appears twice in the reference system*

$$\mathfrak{J}^{(l)} = \left\{ \boldsymbol{J}^{l(1)}, \boldsymbol{J}^{l(2)} \right\}$$

*(i.e.,* $\boldsymbol{L} = \{l\}$*) while simultaneously the face* $\boldsymbol{F}^{\prec l}$ *contains the vector* $2\boldsymbol{e}^l$.

*To be precise, what we see in these sketches are the canonical projections, hence normalized to a unit length for each generating Simplex and SubSimplex respectively. The essential feature is the relative location or more precisely, the lattice structure of the Pareto surface.*

*E.g, in terms of the Cephoid* $\Pi$, *the SubSimplex* $\Delta^{\boldsymbol{a}}_{234}$ *yields a basis and the SubSimplex* $\Delta^{\boldsymbol{b}}_{41}$ *yields a line segment generating a* ***cylinder*** $\boldsymbol{F}^{\prec 4} = \Delta^{\boldsymbol{a}}_{234} + \Delta^{\boldsymbol{b}}_{41}$.

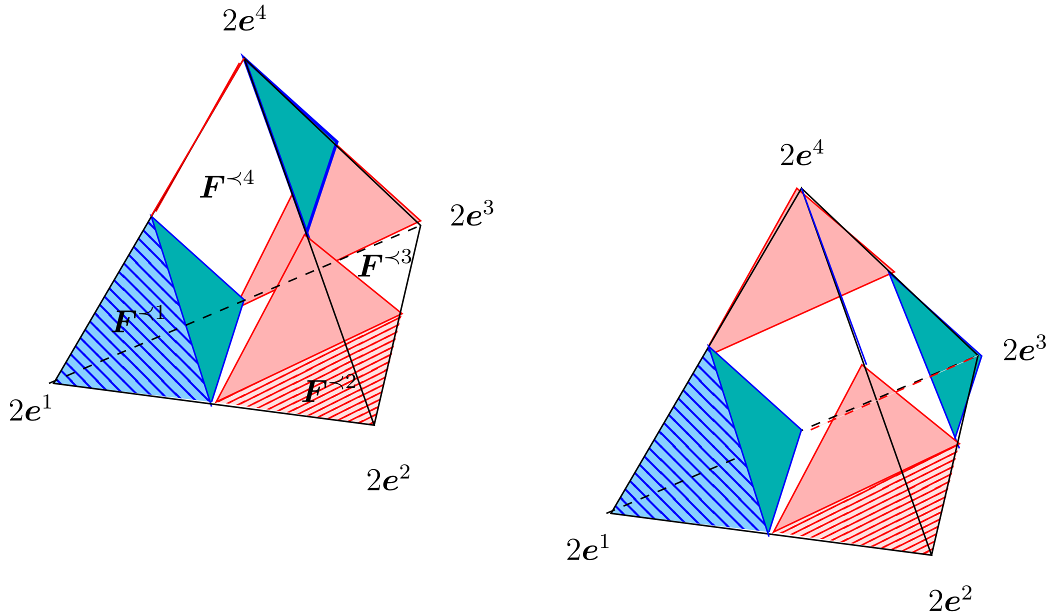


Figure 4.2: The sum of two deGua Simplices for $n = 4$

*The vertices of this cylinder are*

$$\boldsymbol{a}^2 + \boldsymbol{b}^4 = (0, a_2, 0, b_4), \ = \ \boldsymbol{a}^3 + \boldsymbol{b}^4 = (0, 0, a_3, b_4), \boldsymbol{a}^4 + \boldsymbol{b}^4 = (0, 0, 0, a_4 + b_4) \ .$$
$$\boldsymbol{a}^2 + \boldsymbol{b}^1 = (b_1, a_2, 0, 0), \ = \ \boldsymbol{a}^3 + \boldsymbol{b}^1 = (b_1, 0, a_3, 0), \boldsymbol{a}^4 + \boldsymbol{b}^1 = (b_1, 0, 0, a_4) \ .$$

*Projected into the canonical representation, we obtain the copies*

$$\boldsymbol{e}^2 + \boldsymbol{e}^4 = (0, 1, 0, 1), = \boldsymbol{e}^3 + \boldsymbol{e}^4 = (0, 0, 1, 1), \boldsymbol{e}^4 + \boldsymbol{e}^4 = (0, 0, 0, 2)$$
$$\boldsymbol{e}^2 + \boldsymbol{e}^1 = (1, 1, 0, 0), = \boldsymbol{e}^3 + \boldsymbol{e}^1 = (1, 0, 1, 0), \boldsymbol{e}^4 + \boldsymbol{e}^1 = (1, 0, 0, 1)$$

*These are the vertices of the cylinder in the uppermost corner of the left-hand sketch in Figure 4.2.*

*The reader may verify that the right-hand side version of Figure 4.2 corresponds to the ordering* 2431.

∘ ~~~~~~ ∘

Based on our observations we continue by exhibiting a few facts regarding a general sum of $K$ prisms. This will be a useful preparation for the general discussion of cephoids to be presented later on. First of all, it is easy to see that translates of all deGua Simplices will appear on the surface. The case $K = 2$ treated so far shows this clearly, but it is a general feature for all Cephoids.

**Theorem 4.8.** *Let* $\{\boldsymbol{a}^{(k)}\}_{k\in\boldsymbol{K}}$ *be a (non–degenerate) family of positive vectors. Let* $\kappa \in \boldsymbol{K}$*. Then, for every* $l \in \boldsymbol{K} \setminus \{\kappa\}$ *the Simplex* $\Delta^{(\kappa)}$ *has a joint normal with exactly one vertex of* $\Delta^{(l)}$*.*

**Proof:** The normals that belong to all the vertices of $\Delta^{(\kappa)}$ span the nonnegative orthant $\mathbb{R}^n_+$. If two of these normals are joint to the one of $\Delta^{(l)}$, then the normal cone of a two dimensional subface of $\Delta^{(l)}$ equals the corresponding one of $\Delta^{(\kappa)}$, which we have ruled out by nondegeneracy.

**q.e.d.**

**Theorem 4.9** (**The Translation Theorem**)**.** *Let* $\{\boldsymbol{a}^{(k)}\}_{k\in\boldsymbol{K}}$ *be a family of (non–degenerate) positive vectors and let* $\Pi := \sum_{k\in\boldsymbol{K}} \Pi^{(k)}$ *be the Cephoid generated. Then the following holds.*

1. *For each* $\kappa \in \boldsymbol{K}$ *there appears a translate of* $\Delta^{(\kappa)}$ *on* $\partial\Pi$*. Precisely, for* $k \in \boldsymbol{K} \setminus \{\kappa\}$ *there exists (uniquely)* $i_k \in \boldsymbol{I}$ *with*

   $$(14)\quad \boldsymbol{\Delta}^{\{\kappa\}} = \boldsymbol{F}^{\kappa;i_\bullet} := \sum_{k\in\boldsymbol{K}\setminus\{\kappa\}} \Delta^{(k)}_{i_k} + \Delta^{(\kappa)} = \sum_{k\in\boldsymbol{K}\setminus\{\kappa\}} \boldsymbol{a}^{(k)i_k} + \Delta^{(\kappa)} \subseteq \partial\Pi\ .$$

   *(we use the notation* $\boldsymbol{\Delta}^{\{\kappa\}}$ *whenver there is no need to mention the unique sequence* $i_\bullet := (i_k)_{k\in\boldsymbol{K}\setminus\{\kappa\}}$ *explicitely)*

2. *Let* $k^\star, k' \in \boldsymbol{K}$ *and let* $\Pi^{k^\star,k'} = \Pi^{(k^\star)} + \Pi^{(k')}$*. Let* $\prec$ *be the ordering of* $\partial\Pi^{k^\star,k'}$*. Let* $i \in \boldsymbol{I}$ *and* $\prec$ *be be such that* $\boldsymbol{F}^{k^\star k';\prec i}$ *is a Pareto face of* $\Pi^{k^\star,k'}$ *as constructed in Theorem 4.2, formula* (8) *(see Definition 4.3, formula* (9)*). Then there appears a translate of* $\boldsymbol{F}^{k^\star k';\prec i}$ *on* $\partial\Pi$*.*

   *Precisely, for* $k \in \boldsymbol{K} \setminus \{k^\star, k'\}$ *there exists (uniquely)* $i_k \in \boldsymbol{I}$ *with*

   $$(15)\quad \begin{aligned} \boldsymbol{F}^{\{k^\star k'\}} = \boldsymbol{F}^{k^\star k';\prec i;i_\bullet} &:= \sum_{k\in\boldsymbol{K}\setminus\{k^\star k'\}} \Delta^{(k)}_{i_k} + \boldsymbol{F}^{k^\star k';\prec i} \\ &= \sum_{k\in\boldsymbol{K}\setminus\{k^\star k'\}} \boldsymbol{a}^{(k)i_k} + \boldsymbol{F}^{k^\star k';\prec i} \subseteq \partial\Pi\ . \end{aligned}$$

(where $\prec$ and $i_\bullet := (i_k)_{k\in\boldsymbol{K}\setminus\{\kappa^\star k'\}}$ are omitted if appropriate ).

**Proof:**

The proof employs obvious generalizations of the one for Theorem 4.8.

**q.e.d.**

Thus each pair $\Pi^{k^\star}, \Pi^{k'}$ transports all the Pareto faces of its sum to the Pareto surface $\partial\Pi$ of the total sum $\Pi$. It is rather clear that this situation will be generalized in a way such that any triple of Cephoids taken from the family will transport all its faces to $\partial\Pi$ etc. Clearly, this opens the path for inductive procedures to get hold of the structure of $\partial\Pi$ through analyzing Pareto faces of subfamilies.

Now we specify for the particular case that the face $\boldsymbol{F}^{k^\star k';\prec i}$ is the sum of an edge/line segment and an $(n-2)$ dimensional Simplex. We call such a sum a *cylinder*, a notation we have sloppily used previously. In view of Theorem 4.2 and Definition 4.3 this means that the Pareto face in question is of the form

$$\boldsymbol{F}^{k^\star k';\prec l} \;:=\; \Delta^{(k^\star)}_{S_l^\prec} + \Delta^{(k')}_{T_l^\prec} \quad (l \in \boldsymbol{I}). \tag{16}$$

such that, for $j \in \boldsymbol{I}$ with $j \prec l$ we have

$$S_l^\prec = \{j,l\}\ ,\quad T_l^\prec = \boldsymbol{I} \setminus \{j\}\ .$$

that is, in the ordering $\prec$ of $\partial\Pi^{k^\star,k'}$ the coordinat $j$ is first and $l$ is second. Also $\boldsymbol{L} = \{l\}$ is the adjustment set of the face (16). Now we reformulate our results as while varying the pairs taken from $\boldsymbol{K}$ follows.

**Corollary 4.10.** *Let $\left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ be a family of (nondegenerate) positive vectors and let $\Pi \;:=\; \sum_{k\in\boldsymbol{K}} \Pi^{(k)}$ be the Cephoid generated. Then the following holds true.*

1. *For every pair $k^\star, k' \in \boldsymbol{K}, k^\star \neq k'$, there exists uniquely $j, l \in \boldsymbol{I}$ (the first two indices in the ordering of $\Pi^{k^\star,k'}$) such that the Simplices*

   $$\Delta^{(k^\star)}_{jl} \quad \text{and} \quad \Delta^{(k')}_{\boldsymbol{I}\setminus\{j\}}$$

   *admit of a joint normal $\mathfrak{n}^{\star\prime}$. Also, there are positive coefficients $c_{k^\star}, c_{k'}$ such that the deGua Simplices $c_k\Pi^{(k)}$ and $c_{k'}\Pi^{k'}$ are adjusted so that the vertices in direction $l$ are equal, i.e., $c_k^\star \boldsymbol{a}^{(k^\star)l} \;=\; c_{k'}\boldsymbol{a}^{(k')l}$. These coefficients are determined by the Coincidence Theorem 1.4.*

2. *$\mathfrak{n}^{\star\prime}$ is the normal of of the Simplex*

   $$\bigvee \left\{ c_k\Pi^{(k)}, c_{k'}\Pi^{k'} \right\}$$

   *as well as the normal of the Pareto face $\boldsymbol{F}^{k^\star k';jl} \;:=\; \Delta^{(k^\star)}_{jl} + \Delta^{(k')}_{\boldsymbol{I}\setminus\{j\}}$ of the sum*

   $$\Pi^{k^\star k'} \;=\; c_{k^\star}\Pi^{(k^\star)} + c_{k'}\Pi^{k'}$$

3. *For every* $k \in \boldsymbol{K} - \{k^\star, k'\}$ *there exists a unique* $i_k \in \boldsymbol{I}$ *such that* $\Pi^{(k)}$ *admits of a joint normal with* $\Pi^{k^\star k'}$ *in* $\boldsymbol{a}^{(k)i_k}$*; this is the coordinate* $i_k$ *for which* $\mathfrak{n}^{\star\prime}$ *is admitted as a normal in* $\boldsymbol{a}^{(k)i_k}$*.*

4. *Hence, for every pair* $k^\star k' \in \boldsymbol{K}, k^\star \neq k'$*, there exist uniquely* $j, l \in \boldsymbol{I}, j \neq l$*, and a sequence* $i_\bullet = (i_k)_{k \neq k^\star, k'}$ *such that*

$$(17)\quad \boldsymbol{C}^{k^\star k'} \;:=\; \Delta^{(k^\star)}_{jl} + \Delta^{(k')}_{\boldsymbol{I}\setminus\{j\}} + \sum_{k \neq k^\star k'} \boldsymbol{a}^{(k)i_k} \;=\; \boldsymbol{F}^{k^\star k'; \prec l} + \sum_{k \neq k^\star k'} \boldsymbol{a}^{(k)i_k}$$

*is a Pareto face of* $\Pi$*. If* $\prec$ *is the ordering of* $\Pi^{(k^\star k')}$*, then* $j \prec l$ *are the first indices in this ordering and the Pareto face can also be written*

$$\boldsymbol{C}^{k^\star k'} \;=\; \boldsymbol{F}^{k^\star k'; \prec l} + \sum_{k \neq k^\star k'} \boldsymbol{a}^{(k)i_k}$$

Observe that the basis of a cylinder is a $n-2$ dimensional subface of $\Pi^{k'}$ while the height is line segment from $\Pi^{k^\star}$.

**Definition 4.11.** *Let* $\Pi$ *be a Cephoid. The Pareto face* $\boldsymbol{C}^{k^\star k'}$ *described by Corollary 4.10 is the* ***cylinder*** *generated by* $k^\star$ *and* $k'$*. In view of* (17) *we call* $\Delta^{(k')}_{\boldsymbol{I}\setminus\{j\}}$ *the* ***basis*** *and* $\Delta^{(k^\star)}_{jl}$ *the* ***height*** *of the cylinder.*

# 5 The Tentacles

Each Simplex $\Delta^{(k^\star)}$ produces a copy/translate $\boldsymbol{\Delta}^{\{k^\star\}}$ on the Pareto surface of a Cephoid $\Pi$. The detailed description is presented by $\boldsymbol{\Delta}^{\{k^\star\}}$ as defined in formula (14) of Theorem 4.9 of SECTION 4.

The same holds true for the cylinders $\Pi^{k^\star k'}$ generated by each pair of the family $\left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$; here we refer to $\boldsymbol{C}^{k^\star k'}$ as given by (17) of SECTION 4.

Now the family of cylinders has a vivid geometrical interpretation: a well defined "path" of cylinders connects any $(n-2)$ dimensional SubSimplex of $\boldsymbol{F}^{k^\star;i_\bullet}$ with the boundary of $\partial\Pi$. Viewing this for the various subsimplices, one gets the impression of "arms" or "tentacles" stretching from the deGua copy towards the boundary. This feature generates the appearance of an "cephalopod" and motivates the name "Cephoid" we have chosen.

In what follows we make this more precise; we start up with the appropriate definition.

**Definition 5.1.** *Let* $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ *be a Cephoid. For some* $\kappa \in \boldsymbol{K}$ *let*

$$\boldsymbol{\Delta}^{\{\kappa\}} = \sum_{k\in\boldsymbol{K}\setminus\{\kappa\}} \boldsymbol{a}^{(k)i_k} + \Delta^{(\kappa)} \subseteq \partial\Pi$$

*be the translate of the deGua Simplex* $\Delta^{(\kappa)}$ *according to* (14). *Define, for* $j \in \boldsymbol{I}$,

$$\boldsymbol{K}_j^{(\kappa)} \;:=\; \boldsymbol{K} \setminus (\{\kappa\} \cup \{k \in \boldsymbol{K} \mid i_k = j\}) \tag{1}$$

*Then the set of cylinders*

$$\mathbb{T}_j^{(\kappa)} \;:=\; \left\{\boldsymbol{C}^{k^\star\kappa} \,\middle|\, k^\star \in \boldsymbol{K}_j^{(\kappa)}\right\} \tag{2}$$

*is the* ***tentacle system*** *generated by* $\Delta^{(k')}$.

Let us try to imagine the geometrical meaning: given $\kappa$, $\Delta^{(\kappa)}$, and the corresponding deGua translate $\boldsymbol{\Delta}^{\{\kappa\}}$, we consider for $j \in \boldsymbol{I}$ the $(n-2)$ dimensional subface

$$\boldsymbol{F}^{(\kappa)\{-j\}} \;=\; \boldsymbol{F}^{\{-j\}} \;:=\; \sum_{k\in\boldsymbol{K}\setminus\{\kappa\}} \boldsymbol{a}^{(k)i_k} + \Delta^{(\kappa)}_{\boldsymbol{I}\setminus\{j\}} \tag{3}$$

. This subface is an $(n-2)$ dimensional Simplex which employs the same family of basis vectors as $\boldsymbol{\Delta}^{\{\kappa\}}$, that is, the family $\boldsymbol{a}^{\star\bullet} = \left\{\boldsymbol{a}^{(k)i_k}\right\}_{k\in\boldsymbol{K}}$.

Intuitively for each $k \in \boldsymbol{K}_j^{(\kappa)}$, one can imagine that the sum in (7) indicates that the Simplex is moved by one step from the basis vector $K\boldsymbol{e}^j$ towards the boundary $K\Delta^{\boldsymbol{e}^{\{-j\}}}$.

In particular, whenever $\boldsymbol{K}_j^{(\kappa)} = \emptyset$, then all of the vectors $\boldsymbol{a}^{(k)i_k}$ satisfy $\boldsymbol{a}_j^{(k)i_k} = 0$, hence $\boldsymbol{F}^{(\kappa)\{-j\}}$ has just zero coordinates at $j$ and hence constitutes a boundary subface of $\boldsymbol{\Delta}^{\{\kappa\}}$.

**Theorem 5.2 (The Tentacles).** *For every $\kappa \in \boldsymbol{K}$ and $j_0 \in \boldsymbol{I}$ the tentacle system $\mathbb{T}_{j_0}^{(\kappa)}$ generates a coherent path of cylinders connecting $\boldsymbol{F}^{\{-j_0\}}$ of $\boldsymbol{\Delta}^{\{\kappa\}}$ with the boundary $\Pi^{-j_0}$. These cylinders are*

$$\boldsymbol{C}^{\{k_1,\kappa\}}\ ,\ \boldsymbol{C}^{\{k_2,\kappa\}}, \ldots, \boldsymbol{C}^{\{k_\star,\kappa\}}, \tag{4}$$

*where*

$$\{k_1, k_2, \ldots, k_\star\} \ = \ \boldsymbol{K}_{j_0}^{(\kappa)} \ := \ \{k \in \boldsymbol{K} \setminus \{\kappa\} \,|\, i_k = j_0\}\ . \tag{5}$$

*We refer to the system as well as to the path of cylinders as to a* ***tentacle****.*

**Proof:**

**1$^{\text{st}}$STEP** : If

$$\boldsymbol{\Delta}^{\{\kappa\}} \ = \sum_{k\in\boldsymbol{K}\setminus\{\kappa\}} \boldsymbol{a}^{(k)i_k} + \Delta^{(\kappa)} \tag{6}$$

is a boundary face, then there is nothing to prove (and $\boldsymbol{K}_{j_0}^{(\kappa)} = \emptyset$). Assume that the subface

$$\boldsymbol{F}^{\{-j_0\}} \ := \sum_{k\in\boldsymbol{K}\setminus\{\kappa\}} \boldsymbol{a}^{(k)i_k} + \Delta_{\boldsymbol{I}\setminus\{j_0\}}^{(\kappa)} \tag{7}$$

is not located on the boundary. Then not all vectors under the sum can have zero $j_0$–coordinates, hence

$$\boldsymbol{K}_0 \ := \ \boldsymbol{K}_{j_0}^{(\kappa)} \ = \ \{k \in \boldsymbol{K} \setminus \{\kappa\} \,|\, i_k = j_0\} \neq \emptyset\ .$$

Abbreviate

$$\boldsymbol{a}^{\star 0} \ := \sum_{k\in\boldsymbol{K}\setminus(\boldsymbol{K}_0\cup\{\kappa\})} \boldsymbol{a}^{(k)i_k}\ , \tag{8}$$

this is the part of the sum in (7) that has 0 at coordinate $j_0$. Then $\boldsymbol{\Delta}^{\{\kappa\}}$ is written

$$\boldsymbol{\Delta}^{\{\kappa\}} \ = \ \boldsymbol{a}^{\star 0} + \sum_{k\in\boldsymbol{K}_0} \boldsymbol{a}^{(k)j_0} + \Delta^{(\kappa)}$$

while the subface is written

$$F^{\{-j_0\}} \;=\; a^{\star 0} + \sum_{k\in K_0} a^{(k)j_0} + \Delta^{(\kappa)}_{I\setminus\{j_0\}} \;. \tag{9}$$

Now we apply the Neighborhood Theorem (2.3). As $\boldsymbol{\Delta}^{\{\kappa\}}$ is not a boundary face, there must be a Pareto face adjacent to $\boldsymbol{\Delta}^{\{\kappa\}}$ such that the intersection is exactly $F^{\{-j_0\}}$. Necessarily this Pareto face must be obtained by summing a line segment to $F^{\{-j_0\}}$.

We claime that this line segment must originate from some $k_1 \in K_0$ and must be of the form $\Delta^{(k_1)}_{j_0 l_1}$ for some $l_1 \in I \setminus \{j_0\}$ such that the Pareto face we have reached is of the form

$$C^{\{k_1,\kappa\}} \;=\; a^{\star 0} + \sum_{k\in K_0\setminus\{k_1\}} a^{(k)j_0} + \Delta^{(k_1)}_{j_0 l_1} + \Delta^{(\kappa)}_{I\setminus\{j_0\}} \tag{10}$$

Indeed, to see this compare (9) and (10). One of the vectors $a^{(k)j_0}$ listed in the sum is necessarily also a vertex of the line segment, for $F^{\{-j_0\}}$ is a subface of $\boldsymbol{\Delta}^{\{\kappa\}}$ as well as of $C^{\{k_1,\kappa\}}$. This way we have found a Pareto face adjacent to the starting deGua translate; this is the cylinder generated by the pair $\Pi^{k_1}$ and $\Pi^{\kappa}$.

**2ndSTEP** :

Now we continue by constructing the next adjacent face, if any. To this end, we first of all focus on the subface of $C^{\{k_1,\kappa\}}$ given by

$$F^{\{-k_1\}} \;:=\; a^{\star 0} + \sum_{k\in K_0\setminus\{k_1\}} a^{(k)j_0} + a^{(k_1)l_1} + \Delta^{(\kappa)}_{I\setminus\{j_0\}}$$

which one obtains by means of the second vertex of the above line segment.If this is not a boundary face, then again there is the next neighbor which is

$$C^{\{k_2,\kappa\}} a^{\star 0} + \sum_{k\in K_0\setminus\{k_1,k_2\}} a^{(k)j_0} + a^{k_1 l_1} + \Delta^{(k_2)}_{l_2 j_0} + \Delta^{(\kappa)}_{I\setminus\{j_0\}}$$

with some $k_2 \in K_0$ and $l_2 \in I \setminus \{j_0\}$. The argument is the same as above, based on the conectivity of the subface $F^{\{-k_1\}}$. Clearly $l_2$ is the adjustment index corresponding to $\Delta^{(k_2)}_{l_2 j_0}$ and $\Delta^{(\kappa)}_{I\setminus\{j_0\}}$.

**3rdSTEP** :

Continuing this process, we obtain a sequence of Pareto faces (and subfaces), each one adjacent to its predecessor and each one being a cylinder generated by $\Delta^{(\kappa)}_{\boldsymbol{J}^{(\kappa)}}$ and some $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ and some $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ with $k \in \boldsymbol{K}_0$ being taken from the remainders in $\boldsymbol{K}_0$ that have not been employed. The adjustment index $l_k$ is always different from $j_0$.

Finally, when $\boldsymbol{K}_0$ is exhausted we have obtained the last cylinder

$$\boldsymbol{a}^{\star 0} + \sum_{k \in \boldsymbol{K}_0 \setminus \{k_\star\}} \boldsymbol{a}^{(k) l_k} + \Delta^{(k_\star)}_{l_\star j_0} + \Delta^{(\kappa)}_{\boldsymbol{I} \setminus \{j_0\}}$$

with the last subface

$$\boldsymbol{F}^{\{k_*\}} := \boldsymbol{a}^{\star 0} + \sum_{k \in \boldsymbol{K}_0 \setminus \{k_\star\}} \boldsymbol{a}^{(k) l_k} + \boldsymbol{a}^{(k_\star) l_\star} + \Delta^{(\kappa)}_{\boldsymbol{I} \setminus \{j_0\}} = \boldsymbol{a}^{\star 0} + \sum_{k \in \boldsymbol{K}_0} \boldsymbol{a}^{(k) l_k} + \Delta^{(\kappa)}_{\boldsymbol{I} \setminus \{j_0\}}$$

where $k_\star$ is the last element of $\boldsymbol{K}_0$ in the order of construction and $l_\star$ chosen accordingly (that is, $l^\star \neq j_0$ is the the element in the last adjustment set, the one of $\Delta^{(k_\star)}_{l_\star j_0}$ and $\Delta^{(\kappa)}_{\boldsymbol{I} \setminus \{j_0\}}$).

Now, none of the vectors under the sum does have a positive coordinate $j_0$ as always $l_k \neq j_0$ ; neither has the vector $\boldsymbol{a}^{\star 0}$. Consequently, $\boldsymbol{F}^{\{k_*\}}$ is a boundary subface, i.e., $\boldsymbol{F}^{\{k_*\}} \subseteq \Pi^{-j_0}$.

**q.e.d.**

**Remark 5.3.** *The vector* $\boldsymbol{a}^{\star 0} = \boldsymbol{a}^{(\kappa) \star j_0}$ *defined in* (8) *or more precisely by*

$$\boldsymbol{a}^{(\kappa) \star 0} := \sum_{k \in \boldsymbol{K} \setminus (\boldsymbol{K}^{(\kappa)}_{j_0} \cup \{\kappa\})} \boldsymbol{a}^{(k) i_k} , \tag{11}$$

*indicates the distance of* $\boldsymbol{\Delta}^{\{\kappa\}}$ *to the boundary* $\Pi^{-j_0}$. *For, each vector* $\boldsymbol{a}^{(k) i_k}$ *with* $i_k \neq j_0$ *moves the deGua translate* $\boldsymbol{\Delta}^{\{\kappa\}}$ *one step further in direction of the boundary* $\{\boldsymbol{x} \mid x_{j_0} = 0\}$. *Vaguely speaking, in "barycentric coordinates" the* $j_0$*–vertex of that deGua translate has coordinate*

$$\sum_{k \in \boldsymbol{K} \setminus \boldsymbol{K}^{(\kappa)}_{j_0}} \boldsymbol{a}^{(k) i_k} = \sum_{k \in \boldsymbol{K} \setminus (\boldsymbol{K}^{(\kappa)}_{j_0} \cup \{\kappa\})} \boldsymbol{a}^{(k) i_k} + \boldsymbol{a}^{(\kappa) j_0}$$

*which is* 0 *if that translate is located in the* $j_0$ *vertex of* $\partial\Pi$ *and* $\sum_{k \in \boldsymbol{K}} a^{(k)}_j$ *whenever it is located on the opposite boundary of* $\partial\Pi$.

*This is even more clearly seen in the framework of the canonical representation. In* $K\Delta^{\boldsymbol{e}}$ *the hyperplanes* $\{\boldsymbol{x} \mid x_{j_0} = t\}$ *represent points of equal distance to the boundary* $\{\boldsymbol{x} \,\big|\, x_{j_0} = 0\}$, *that is, to* $K\Delta^{\boldsymbol{e}^{-j_0}}$ *for some* $j_0 \in \boldsymbol{I}$.

*For example, consider the grid on $K\Delta^{\boldsymbol{e}}$ for $n = 3, K = 7$ as depicted in figure 5.1. We choose $j_0 = 2$, so the lines $\{x_{j_0} = t\}$ are parallel to the basis $\{x_2 = 0\}$ and represent points that are $t$ steps away from that basis.*

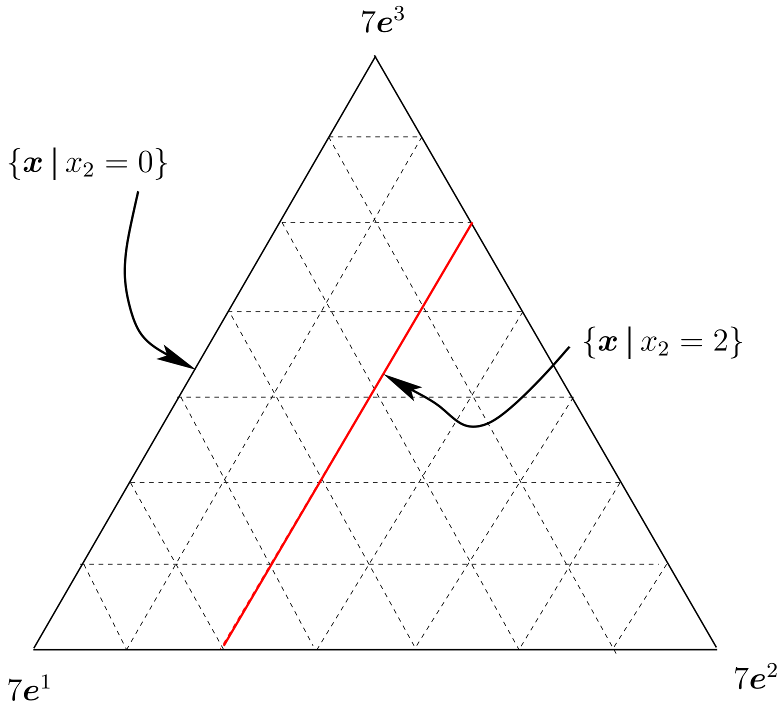


Figure 5.1: The Grid, $n = 3$, $K = 7$

*Next consider Figure 5.2.*

*The translate $\boldsymbol{\Delta}^{\{\kappa\}}$ is the brown triangle in the center. Its second vertex is 3 steps away from the boundary $\{x_2 = 0\}$, the opposite line segment 2 steps. We find a tentacle of 2 brown cylinders (rhombi in this case) connecting that line segment to the boundary $\{x_{j_0} = 0\}$. These are the rhombi $\boldsymbol{C}^{1\kappa}$ created by $\Delta^{(\kappa)}$ and $\Delta^{(1)}$ (blue lines) and $\boldsymbol{C}^{2\kappa}$ created by $\Delta^{(\kappa)}$ and $\Delta^{(2)}$ (grey).*

*Note the tentacle of $\Delta^{(1)}$ (the image of) which (under the canonical projection) is located in the vertex $7\boldsymbol{e}^1$ of $7\Delta^{\boldsymbol{e}}$. The only tentacle is the one leaving at the basis opposite to this vertex and stretching all the way through six rhombi until it reaches the opposite boundary of $K\Delta^{\boldsymbol{e}}$. Of course, the rhombus $\boldsymbol{C}^{1\kappa}$ is exactly the crossing point of both arms.*

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**Remark 5.4 (The Cephalopodic Structure).** *The above construction explains the name of "Cephoid" chosen for our topic: the translate $\Delta^{\{\kappa\}}$ of a deGua*

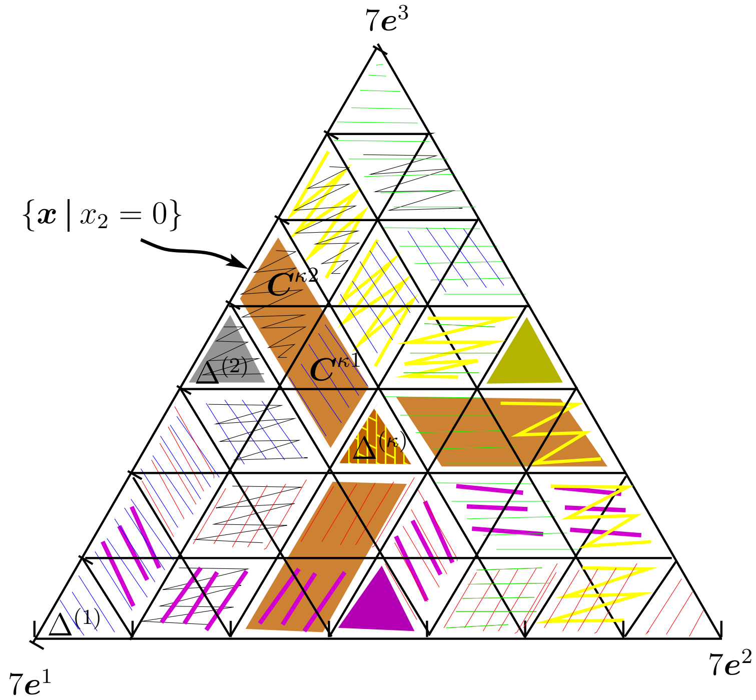


Figure 5.2: Tentacles in 3 Dimensions, $K = 7$

*Simplex on the Pareto surface of a Cephoid appears in a shape resembling the center/head of a cephalopod. There are arms/tentacles stretching from this center towards each boundary. Each tentacle – represented by the system* $\mathbb{T}_j^{\{\kappa\}}$ *in direction* $j$ *– connects the center with the boundary* $\{x \mid x_j = 0\}$.

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We finish this section by adding two observations regarding the number of cylinders and the size of the tentacles in a Cephoid.

**Corollary 5.5.**

1. *For* $n = 3$ *any pair of deGua Simplices generates one cylinder (rhombus in this case), hence the number of cylinders is* $\frac{K(K-1)}{2} = \binom{K}{2}$.

2. *For* $n > 3$, *any pair of deGua Simplices generates two cylinders, hence there are* $K(K-1)$ *cylinders on* $\partial\Pi$.

3. *Let* $\kappa \in \boldsymbol{K}$. *The number of cylinders in the tentacle* $\mathbb{T}_{j_0}^{\kappa}$ *is*

$$\left|\mathbb{T}_{j_0}^{\kappa}\right| = \left|\boldsymbol{K}_{j_0}^{(\kappa)}\right| = \left|\{k \in \boldsymbol{K} \setminus \{\kappa\} \mid i_k = j_0\}\right| .$$

# 6 The Sum of Three DeGua Simplices: Blocks

Analogously to the presentation in SECTION 4 this section is devoted to the special treatment of the case $K = 3$, i.e., the sum of three deGua Simplices in detail. Fortunately, we can still visualize the situation for n=4. Indeed, while a Cephoid $\Pi$ is a subset of $\mathbb{R}^4_+$, its Pareto surface $\partial P$ is three dimensional. Especially, when we concentrate on the canonical representation, then we consider a multiple of the unit Simplex in $\mathbb{R}^3_+$. As we are capable of viewing this by sketches in the two-dimensional paper space $\mathbb{R}^2$ we can visualize and explain some features that are new in comparison to the situation within the "sum of two" realm treated in SECTION 4.

For a start consider the case $n = 3$. We write $\Pi = \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}} + \Pi^{\boldsymbol{c}}$, as usual $\partial\Pi$ is the Pareto surface of $\Pi$. Now, $\boldsymbol{K}$ contains just three indices. For any a Pareto face $\boldsymbol{F}$ of $\Pi$ any reference set must contain an element of the adjustment set $\boldsymbol{L}$.

Next, a Pareto face cannot have a reference system $\mathcal{J} = \{\{12\}\{13\}\{23\}\}$, for then $\boldsymbol{L} = \{123\}$ would contradict $|\boldsymbol{L}| \leq K - 1 = 2$ ( *item* 2a of the Reference Theorem 1.2).

Clearly, this leaves only two choices for the adjustment set $\boldsymbol{L}$: with suitable indices we have either $\boldsymbol{L} = \{l\}$ or $\boldsymbol{L} = \{l_0, l_1\}$. Accordingly, we reformulate the Coincidence Theorem for this particular case.

**Theorem 6.1.** *Let $K = n = 3$. Let $\boldsymbol{F} \subseteq \partial\Pi$ be a Pareto face of $\Pi$. Then*

1. *Either there exists uniquely $l \in \boldsymbol{I}$ such that $\boldsymbol{a}^l + \boldsymbol{b}^l + \boldsymbol{c}^l \in \boldsymbol{F}$ holds true. That is, $\boldsymbol{L} = \{l\}$ and $\boldsymbol{F}$ contains a unique multiple of a unit–vector. Thus, with suitable $\boldsymbol{J}^{(k)} \subseteq \boldsymbol{I}$ $(k = 1, 2, 3)$*

   $$\boldsymbol{F} = \Delta^{\boldsymbol{a}}_{\boldsymbol{J}^{(1)}} + \Delta^{\boldsymbol{b}}_{\boldsymbol{J}^{(2)}} + \Delta^{\boldsymbol{c}}_{\boldsymbol{J}^{(3)}} \tag{1}$$

   *such that $l \in \cap_{k=1}^3 \boldsymbol{J}^{(k)}$.*

2. *Or else there is a unique pair $l_1, l_2 \in \boldsymbol{I}$ such that $\boldsymbol{L} = \{l_1, l_2\}$. That is, with suitable $\boldsymbol{J}^{(k)} \subseteq \boldsymbol{I}$ $(k = 1, 2, 3)$ we have*

   $$\boldsymbol{F} = \Delta^{\boldsymbol{a}}_{\boldsymbol{J}^{(1)}} + \Delta^{\boldsymbol{b}}_{\boldsymbol{J}^{(2)}} + \Delta^{\boldsymbol{c}}_{\boldsymbol{J}^{(3)}} \tag{2}$$

   *such that $\{l_1, l_2\} \subseteq \boldsymbol{J}^{(k)}$ holds true for one $k$ while the other two index sets contain either $l_2$ or $l_2$ and not both.*

*Moreover,* $\boldsymbol{F} \cap \Delta_{\{l_1,l_2\}}$ *is a nondegenerate interval located within the relative interior of* $\Delta_{\{l_1,l_2\}}$.

3. *There are positive constants* $c_{\boldsymbol{a}}, c_{\boldsymbol{b}}$ *and* $c_{\boldsymbol{c}}$ *(unique up to a positive multiple) such that the normal* $\mathfrak{n}^\star$ *of* $\boldsymbol{F}$ *is (up to a positive multiple) exactly the normal of the deGua Simplex*

$$\Delta^\star = c_{\boldsymbol{a}}\Pi^{\boldsymbol{a}} \vee c_{\boldsymbol{b}}\Pi^{\boldsymbol{b}} \vee c_{\boldsymbol{c}}\Pi^{\boldsymbol{c}} . \tag{3}$$

**Example 6.2.** *Let* $n = K = 3$. *Then, by Theorem 4.9, we obtain* 3 *translates of the generating simplices. Also, by Corollary 5.5 we obtain* 3 *diamonds/rhombi on the Pareto Surface* $\partial\Pi$ *of* $\Pi$. *More Pareto faces are not feasible by our above introductory remarks.*

*The tentacle system generated by (the translate of) each Simplex consists of two rhombi/diamonds. Any two Simplices share exactly one diamond. As there are no further Pareto faces, the number of faces is always* $3 + 3 = 6$.

*This enhances our understanding of the examples presented preliminarily in Section 2 of Chapter 2. E.g., reconsider Examples 2.1 and 2.2 of Chapter 2, i.e., the* ***Circle****. Our graphic representations rests on the canonical projection. Thus the lattice structure of the faces of* $\partial\Pi$ *is projected on the Simplex* $3\Delta^{\boldsymbol{e}}$. *We sketch* $\Delta^{\boldsymbol{a}}$ *in green,* $\Delta^{\boldsymbol{b}}$ *in blue and* $\Delta^{\boldsymbol{c}}$ *in red.*

*The result is Figure 6.1. the* ***Circle****. To interpret this, recall that the Pareto faces of a sum of two are reflected by a set of orderings and the "moving index principle" (Remark 4.4). For the Circle, the orderings are indeed "cyclic" as they are induced by the cyclic subgroup of permutations of three elements. We find*

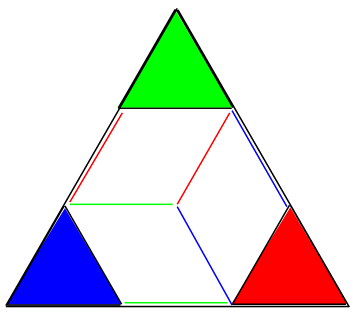

Figure 6.1: The circle

*the orderings*

$$123 \quad 231 \quad 312 \tag{4}$$

*for* $\boldsymbol{a}$ vs. $\boldsymbol{b}$, $\boldsymbol{a}$ vs. $\boldsymbol{c}$, *and* $\boldsymbol{b}$ vs. $\boldsymbol{c}$.

*More precisely, the sum of $\Delta^{\boldsymbol{a}}$ and $\Delta^{\boldsymbol{b}}$ has three Pareto faces which are described by*

(5)

| $\boldsymbol{a}$ | $\boldsymbol{b}$ |
|---|---|
| 123 | 3 |
| 12 | 23 |
| 1 | 123 |

*As there a three deGua Simplices involved, we obtain the complete description of the Pareto faces generated by $\boldsymbol{a}$ and $\boldsymbol{b}$ by adding a suitable vertex of $\boldsymbol{c}$ which yields the following list.*

(6)

| $\boldsymbol{a}$ | $\boldsymbol{b}$ | $\boldsymbol{c}$ |
|---|---|---|
| 123 | 3 | 3 |
| 12 | 23 | 1 |
| 1 | 123 | 1 |

*To verify/read this result, observe that (in terms of the canonical representation) $\Delta^{\boldsymbol{a}}$ ("green") is indeed located at the vertex $3\boldsymbol{e}^3 = \boldsymbol{e}^3 + \boldsymbol{e}^3 + \boldsymbol{e}^3$ corresponding to $\boldsymbol{L} = \{3\}$. Also, $\Delta^{\boldsymbol{b}}$ ("blue") is located at the vertex $3\boldsymbol{e}^1 = \boldsymbol{e}^1 + \boldsymbol{e}^1 + \boldsymbol{e}^1$ corresponding to $\boldsymbol{L} = \{1\}$. Moreover, we see that the common cylinder/rhombus/diamond is $\Delta^{\boldsymbol{a}}_{12} + \Delta^{\boldsymbol{b}}_{23}$ shifted to the Pareto surface by $\boldsymbol{c}^1$ (corresponding to $\boldsymbol{e}^1$ within the framework of the canonical projection). Thus, $\boldsymbol{L} = \{1, 2\}$; the diamond generated by green $\boldsymbol{a}$ and blue $\boldsymbol{b}$ is the one having a one–dimensional intersection exactly with the Subsimplex $3\Delta^{\boldsymbol{e}}_{12} = (3\Delta^{e} \cap \{x_3 = 0\})$.*

*Next , a similar diagram holds for $\boldsymbol{b}$ vs. $\boldsymbol{c}$. we have*

(7)

| $\boldsymbol{b}$ | $\boldsymbol{c}$ |
|---|---|
| 231 | 1 |
| 23 | 31 |
| 2 | 231 |

*which is augmented to*

(8)

| $\boldsymbol{b}$ | $\boldsymbol{c}$ | $\boldsymbol{a}$ |
|---|---|---|
| 231 | 1 | 1 |
| 23 | 31 | 2 |
| 2 | 231 | 2 |

*Finally $\boldsymbol{c}$ vs. $\boldsymbol{a}$ yields*

(9)

$$\begin{array}{rl} \boldsymbol{c} & \boldsymbol{a} \\ 312 & 2 \\ 31 & 12 \\ 3 & 312 \end{array}$$

*which is augmented to*

(10)

$$\begin{array}{rcc} \boldsymbol{c} & \boldsymbol{a} & \boldsymbol{b} \\ 312 & 2 & 2 \\ 31 & 12 & 2 \\ 3 & 321 & 3 \end{array}$$

*Listing all three diagrams as induced by the* 3 *permutations we obtain the complete structure of* $\partial\Pi$*. The three diagrams list 9 faces, but each Simplex appears twice, so we have indeed* 6 *Pareto faces.*

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We have explained at length how to transfer our our lists into the sketch of Figure 6.1. It is also worth exercising the other way around: looking at the sketch one may want to identify e.g. the correct representation of the diamond corresponding to $\boldsymbol{c}$ (red) and $\boldsymbol{a}$ (green) . One observes that the generating Subsimplices are $\Delta^{\boldsymbol{c}}_{31}$ and $\Delta^{\boldsymbol{a}}_{12}$ as the line segments defining the green–red diamond are parallel to the line segments $3\Delta^{\boldsymbol{e}}_{31}$ and $3\Delta^{\boldsymbol{e}}_{12}$. To find the augmenting vector we observe that the diamond in question has a non–trivial interval in common with $3\Delta^{\boldsymbol{e}}_{13}$, which means that 2 is the missing index in $\boldsymbol{L}$, thus necessary the diamond is

$$\Delta^{\boldsymbol{c}}_{31} + \Delta^{\boldsymbol{a}}_{12} + \Delta^{\boldsymbol{b}}_{3} \quad \text{– or in shorthand as above –} \quad \{31\}\{12\}\{3\}\ .$$

**Example 6.3.** *For the next example, we revisit the Windmill represented by Figure 2.4. See also the preliminary treatment in Section 2 of Chapter 2. This cephoid involves*

*the three orderings*

(11) 132, 321, *and* 213

*between the three pairs which refer to the acyclic subgroup of permutations of three elements. Again a complete description has to involve a suitable vertex of the third deGua Simplex.*

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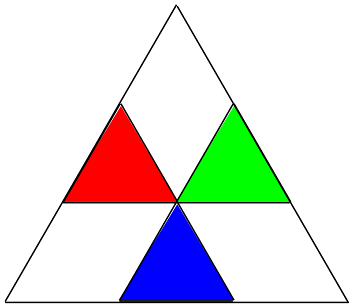

Figure 6.2: The windmill

Next we turn to the case $n = 4, K = 3$.

We can still depict the canonical representation (not the Pareto surface) as the projection is a structure located on the Simplex spanned by the vectors $3\boldsymbol{e}^i \quad (i \in \{1,2,3,4\})$ of $\mathbb{R}^4$. This Simplex is 3–dimensional and has the shape of a tetrahedron in $\mathbb{R}^3$.

From Theorem 4.9, we know that there are three translates of the three Simplices involved on $\partial\Pi$.

In addition, each deGua Simplex involved generates two cylinders, one with each of the other two deGua Simplices involved. There is also the tentacle system (Definition 5.1, Theorem 5.2) which generates tentacles consisting of cylinders. Thus, we find immediately 9 Pareto faces.

In addition to these, there appears exactly one additional Pareto face which has the shape of a parallelepiped or for short a ***block***. We formulate this as follows.

**Lemma 6.4.** *Let $\Pi$ be a cephoid with $n = 4$ and $K = 3$. Then $\partial\Pi$ has exactly 10 faces. There are 3 translates of the generating deGua Simplices, 6 cylinders, and one parallelepiped or block. The block is necessarily of the shape*

$$\Delta^{\boldsymbol{a}}_{\boldsymbol{J}^a} + \Delta^{\boldsymbol{b}}_{\boldsymbol{J}^b} + \Delta^{\boldsymbol{c}}_{\boldsymbol{J}^c} \tag{12}$$

*with $|\boldsymbol{J}^a| = |\boldsymbol{J}^b| = |\boldsymbol{J}^c| = 2$.*

**Proof:** For any Pareto face the adjustment set $\boldsymbol{L}$ has at most 2 elements. Therefore, any Pareto face contains either a vertex or cuts the interior of an $(n-2)$ dimensional boundary, via the subface $\boldsymbol{F_L}$. (cf. the Reference Theorem 1.2). In the framework of the canonical representation this means that any image of a Pareto face either contains a threefold basis vector $3\boldsymbol{e}^i$ or else cuts properly into an edge of $3\Delta^{\boldsymbol{e}}$ (the intersection is an interval).

On the other hand, each edge intersects exactly 3 Pareto faces.

Indeed, for $j_0 \in \boldsymbol{I}$ consider the Pareto faces of the Cephoid $\Pi^{(-j_0)}$. The tentacle system of $\Pi$ for each deGua Simplex involved shows an arm that ends on $\Pi^{(-j_0)}$, thus one finds the bases of the final cylinders in the tentacle system. That is, the Pareto faces of $\partial\Pi^{(-j_0)}$ are generated by the projected deGua Simplices $\Pi^{(k)(-j_0)}$. Consequently, any 2–dimensional edge shows exactly 3 intersections of Pareto faces in $\partial\Pi^{(-j_0)}$, hence 3 intersections of Pareto faces in $\partial\Pi$.

Now, the 4–dimensional unit Simplex (and its multiples) has 4 vertices and 6 edges. Each vertex is contained in exactly one Pareto face and each edge contains one additional line segment which is the intersection with a further Pareto face of $\partial\Pi$. Consequently, we obtain 10 Pareto faces. As all cylinders and translates of deGua Simplices are already listed, the tenth face necessarily involves an edge from each of the deGua Simplices.

**q.e.d.**

In the following, we describe a situation using the canonical representation. We find exactly 3 translate of deGua Simplices, 6 cylinders, and one block. These examples reflect only a geometrical sketch, at this stage we do not present a precise numerical description. Later we present the numerical treatment of such Cephoids.

**Example 6.5.** *The first example can be seen as a circle of* 3 *in* $\mathbb{R}^4$. *The canonical representation of this polyhedron is presented in Figure 6.3. Viewing the* 124*–Subsimplex we observe the structure of the* Circle *as described in Example 6.2.*

*Accordingly, the translates of the Simplices are located in the corners* $3\boldsymbol{e}^l$ $(l = 1,2,4)$ *of* $\partial\Pi$. *These Pareto faces show an adjustment set* $\boldsymbol{L} = \{l\}$ *for some* $l = 1,2,4$. *The fourth Pareto face containing the vertex* $3\boldsymbol{e}^3$ *(hence* $\boldsymbol{L} = \{3\}$ *is a cylinder with a red basis and a green height. All the other cylinders and the block have an adjustment set* $\boldsymbol{L}$ *with* 2 *elements leading to a proper one–dimensional intersection with some edge of the* 3*–fold unit Simplex. In particular, the block is seen to be*

$$\Delta_{12}^{\boldsymbol{a}} + \Delta_{23}^{\boldsymbol{b}} + \Delta_{34}^{\boldsymbol{c}} \ , \tag{13}$$

*such that* $\boldsymbol{L} = \{23\}$. *We list the three orderings referring to each sum of two*

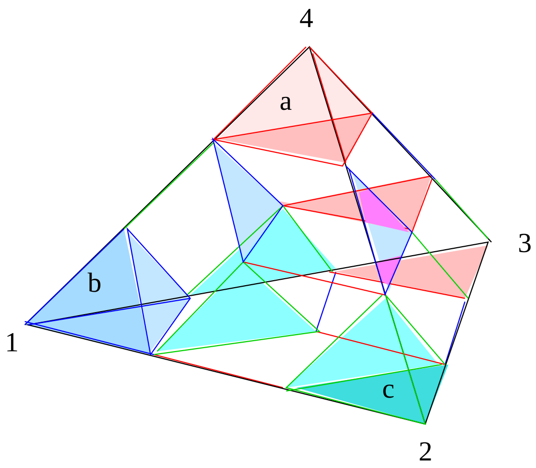


Figure 6.3: A circle of 3 deGua Simplices in 4 dimensions.

*deGua Simplices, these are*

(14)

| $\boldsymbol{a}$ $\boldsymbol{b}$ | $\boldsymbol{b}$ $\boldsymbol{c}$ | $\boldsymbol{c}$ $\boldsymbol{a}$ |
|---|---|---|
| 1234 | 2341 | 4312 |

*This is not complete representation, but the orderings appear again in the full description of all Pareto faces, where a suitable vertex is added to each of the faces generated. As it turns out, the single block corresponds to a further ordering, this one is already suggested by formula* (13) *to be* 1234*. Thus, a full description of all Pareto faces is given as follows:*

(15)

| $\boldsymbol{a}$ | $\boldsymbol{b}$ | $\boldsymbol{c}$ | $\boldsymbol{b}$ | $\boldsymbol{c}$ | $\boldsymbol{a}$ | $\boldsymbol{c}$ | $\boldsymbol{a}$ | $\boldsymbol{b}$ |
|---|---|---|---|---|---|---|---|---|
| 1234 | 4 | 4 | 2341 | 1 | 1 | 4312 | 2 | 2 |
| 123 | 34 | 4 | 234 | 41 | 1 | 431 | 12 | 2 |
| 12 | 234 | 4 | 23 | 341 | 1 | 43 | 312 | 3 |
| 1 | 1234 | 1 | 2 | 2341 | 2 | 4 | 4312 | 4 |

| $\boldsymbol{a}$ | $\boldsymbol{b}$ | $\boldsymbol{c}$ |
|---|---|---|
| 12 | 23 | 34 |

*Generalizing this in an obvious way we obtain:*

**Theorem 6.6.** *Let $n = 4$ and $K = 3$. Let $(\boldsymbol{a}^{(k)})_{k=1}^{K}$ denote a (non–degenerate) family of positive vectors in and let*

$$\Pi \;=\; \sum_{k=1}^{K} \Pi^{\boldsymbol{a}^{(k)}} \;=\; \sum_{k=1}^{K} \Pi^{(k)}$$

*be the Cephoid generated. Then the Pareto faces of $\Pi$ are given as follows:*

1. *There are three orderings* $\overset{kk^\star}{\prec}$, *each one referring to a pair of deGua Simplices* $\Pi^{(k)}, \Pi^{(k^\star)}$ $(k, k^\star \in \boldsymbol{K})$, *which yield the Pareto faces in the corresponding sum of these* two *deGua Simplices* .

2. *To each of these Pareto faces there corresponds a unique vertex of the third deGua Simplex such that the result is a Pareto face of* $\Pi$.

3. *There is a further ordering representing exactly one block. This block is uniquely defined by either one of the following requirements:*

   (a) *The block covers exactly the missing vertex or interval in an edge that is* not *covered by the above faces constructed from the sums of two.*

   (b) *The block is adjacent to at least one face generated by each of the sums of two.*

***Proof:*** *This is an obvious result.* ***q.e.d.***

*Going back to the table* (15) *in Example 6.5, we can deduce from the upper set of three matrices (representing the sums of two plus a vertex) that the interior interval of edge* 23 *is not covered by a Pareto face and that indeed the edge* 23 *does not intersect any translate of* $\Delta^{\boldsymbol{b}}$. *As the edge* 23 *intersects a translate of* $\Delta^{\boldsymbol{a}}_{12}$ *and of* $\Delta^{\boldsymbol{c}}_{24}$ *it is clear that* $\Delta^{\boldsymbol{b}}_{23}$ *is the missing edge.*

*As for the second argument, observe that*

$$\begin{array}{ccc} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ 12 & 234 & 4 \end{array} \tag{16}$$

*is adjacent to the block. This cylinder stems from the sum of* $\Delta^{\boldsymbol{a}}$ and $\Delta^{\boldsymbol{b}}$. *It is, by the way, also adjacent to its predecessor*

$$\begin{array}{ccc} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ 123 & 34 & 4 \end{array} \tag{17}$$

*which precedes within the same ordering, as the third vertex (i.e.* 4*) does not change. Similar, if we look to the second ordering (referring to* $\boldsymbol{b}$ and $\boldsymbol{c}$*), then we observe that*

$$\begin{array}{ccc} \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} \\ 1 & 23 & 341 \end{array} \tag{18}$$

*is adjacent to the block as well as to its predecessor in the ordering.*

*Finally, let us look to the third ordering, the one defined by* $\boldsymbol{a}$ and $\boldsymbol{c}$. *Here indeed the block has* two *neighbors which are*

(19)

| $\boldsymbol{a}$ | $\boldsymbol{b}$ | $\boldsymbol{c}$ |
|---|---|---|
| 21 | 2 | 134 |
| 213 | 3 | 34 |

*These two have been adjacent as far as the sum of $\Pi^{\boldsymbol{a}}$ and $\Pi^{\boldsymbol{c}}$ was concerned. But the unique vertex of $\boldsymbol{b}$ that renders these Pareto faces to become Pareto faces of $\Pi$ changes from 2 to 3, so they are no longer adjacent but both adjacent to the block.*

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**Example 6.7.** *We provide two additional sketches of Cephoids which are a sum of three deGua Simplices in $\mathbb{R}^4$.*

*The first sketch (Figure 6.4) shows two translates of deGua Simplices located at the corners, thus $\boldsymbol{F}^{blue\ 1}$ and $\boldsymbol{F}^{yellow\ 3}$ have reference sets $\boldsymbol{L} = \{1\}$ and $\boldsymbol{L} = \{3\}$. The translate of "red" is $\boldsymbol{F}^{red\ 14}$ with $\boldsymbol{L} = \{14\}$.*

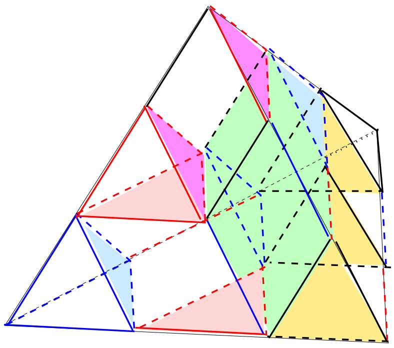

Figure 6.4: A further sum of 3 deGua simplices

*The Pareto faces at vertices $3\boldsymbol{e}^2$ and $3\boldsymbol{e}^4$ are cylinders. The block is*

$$\Delta_{24}^{blue} + \Delta_{23}^{red} + \Delta_{14}^{yellow} \ .$$

*Finally, we consider the Cephoid in Figure 6.5. The interesting feature is the structure of the 3-dimensional subfaces. They resemble two examples for $3 \times 3$ Cephoids known to us. The Subsimplex $\Delta_{124}$ (in the canonical representation) shows the Windmill and the Subsimplex $\Delta_{234}$ reflects the Circle. Thus, Figure 6.5 is called "**the Marriage of a Windmill and a Circle**". The block is*

$$\Delta_{34}^{blue} + \Delta_{23}^{red} + \Delta_{14}^{green} \ ,$$

*so similar to the one as in Figure 6.4. The formal description is as follows.*

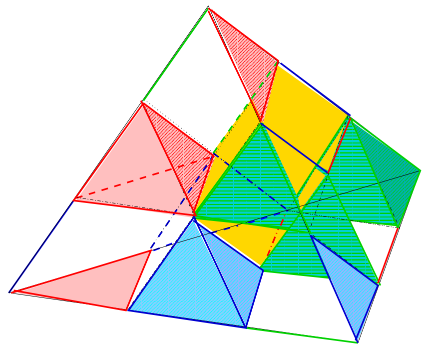

Figure 6.5: The Marriage of a Windmill and a Circle

```
THE CEPHOID *MARRIAGE*:


                  1 1 19 5
                  2 4 13 2
                  6 3  2 3

PARETO FACES OF CEPHOID MARRIAGE:

       3  ⊛        3  ⊛  1 2 3 4  ⊛
     3 4  ⊛      2 3  ⊛      1 4  ⊛
       4  ⊛    2 3 4  ⊛      1 4  ⊛
   2 3 4  ⊛        2  ⊛      1 2  ⊛
     3 4  ⊛        2  ⊛    1 2 4  ⊛
       3  ⊛      2 3  ⊛    1 2 4  ⊛
   1 3 4  ⊛      1 2  ⊛        1  ⊛
       4  ⊛  1 2 3 4  ⊛        1  ⊛
     3 4  ⊛    1 2 3  ⊛        1  ⊛
 1 2 3 4  ⊛        2  ⊛        1  ⊛
```

◦ ~~~~~~~ ◦

We are now in the position to comprehensively describe the case $K = 3$ as follows.

**Theorem 6.8.** *Let $K = 3$ and $K \leq n - 1$. Then a Cephoid $\Pi$ has $n + \binom{n}{2}$ Pareto faces.*

**Proof:** Consider the canonical representation.

There are $\binom{n}{2}$ edges of the deGua Simplex $\Delta^{Ke}$ used for the representation and each of them has a proper cut with exactly 3 Pareto faces. On the other hand, each Pareto face is either a 1–face (hence contains a vertex) or a 2–face (hence intersects exactly one edge properly and contains no vertex. Thus, the total number of faces is indeed $n + \binom{n}{2}$, i.e., the number of vertices plus the number of edges in the canonical representation.

**q.e.d.**

**Theorem 6.9.** *Let* $\Pi$ *be a sum of* 3 *DeGua Simplices in* $\mathbb{R}^n_+$ *and assume that no block contains a vertex. Then* $\Pi$ *is characterized by* 4 *orderings. Three orderings correspond to each pair of deGua Simplices. These generate all together* $(n-3)$ *Pareto faces according to the moving index principle for one index (see Remark 4.4). A further order which is connecting all three deGua Simplices generates* $\binom{n-3}{2}$ *faces according to the moving index principle.*

**Proof:**

By Theorem 4.2 any two deGua Simplices generate an ordering and hence $n$ Pareto faces of their sum according to the moving index principle (Remark 4.4). Each of these generates a Pareto face of the sum of three deGua Simplices $\Pi$ when combined with a proper vertex of the third deGua Simplex (Theorem 4.8, Theorem 4.9). Clearly, the three translates of the deGua Simplices (each one with a suitable vertex of the other two) appear twice within this scheme, hence the total number of Pareto faces that correspond to pairs of two deGua Simplices equals $3n - 3$.

In view of Theorem 6.8, the number of the remaining Pareto faces is then

$$n + \binom{n}{2} - 3(n-1) = \binom{n-2}{2}$$

These Pareto faces have to be sums involving at least an edge from each Simplex, hence the size of each index set $\boldsymbol{J}^{(k)}$ $k = 1, 2, 3$ is at least two. As they have to be neighbors each of them has to be obtained from another one by the neighborhood theorem (Theorem 2.3, SECTION 2, CHAPTER3). Thus, the two common indices have to be moved according to the moving index principle.

The number of reference sets triplets $\boldsymbol{J}^{(1)}, \boldsymbol{J}^{(2)}, \boldsymbol{J}^{(3)}$ to be obtained by the moving index principle is indeed $\binom{n-2}{2}$. To see this, consider the natural ordering $1, 2, \ldots, n$. Then, first of all, there is one triplet of reference sets of the type

$$1, 2, \ldots, n-2 \quad * \quad n-2, n-1 \quad * \quad n-1, n \ .$$

Next, we have two triplets involving the first $(n-3)$ for the first index set; these are

$$1, 2, \ldots, n-3 \quad * \quad n-3, n-2, n-1 \quad * \quad n-1, n,$$

and

$$1, 2, \ldots, n-3 \quad * \quad n-3, n-2, \quad * \quad n-2, n-1, n \ ;$$

i.e., we obtain two triplets by moving the second index.

similarly, we will obtain three triplets fixing the first $(n-4)$ indices etc.

Thus we have

$$1 + 2 + 3 + \ldots + n - 3 \ = \ \binom{n-2}{2}$$

systems which exactly generate the missing number of Pareto faces.

**q.e.d.**

**Example 6.10.** *For $n = 7$, assuming that the ordering is the natural one, the blocks are suggested by the moving index principle for two indices as follows:*

| | | |
|---|---|---|
| 12345 | 56 | 67 |
| | | |
| 1234 | 456 | 67 |
| 1234 | 45 | 567 |
| | | |
| 123 | 3456 | 67 |
| 123 | 345 | 567 |
| 123 | 34 | 4567 |
| | | |
| 12 | 23456 | 67 |
| 12 | 2345 | 567 |
| 12 | 234 | 4567 |
| 12 | 23 | 34567 |

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# Chapter 4

# Duality

We introduce a notion of duality for Cephoids. Quite naturally, duality for Cephoids is established by duality of the matrices involved. Given a Cephoid represented by a matrix $\left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$, the dual Cephoid is obtained by the transpose of that matrix.

The idea is then extended to Pareto faces. Again we define the appropriate canonical version of duality. Then there is a natural bijection mapping the faces of a Cephoid onto the dual Pareto faces of the dual Cephoid.

All operations are rather straightforward induced by just interchanging $\boldsymbol{I}$ and $\boldsymbol{K}$.

# 1 Duality: Cephoids and Pareto Faces

A Cephoid is provided by a family of positive vectors or, equivalently, by a positive matrix, the rows of which represent the various deGua Simplices. The dual Cephoid to be introduced now is provided by the transposed matrix. More precisely, we supply the following definition.

**Definition 1.1.** *Let* $\boldsymbol{a}^{\bullet} = \left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ *be a family of positive vectors and* $\Pi = \Pi^{\boldsymbol{a}^{\bullet}} = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ *be the Cephoid generated. Put* $\bar{a}_k^{(i)} := a_i^{(k)} (i \in \boldsymbol{I},\ k \in \boldsymbol{K})$. *We call the family*

$$(1) \qquad \left(\bar{\boldsymbol{a}}^{(i)}\right)_{i\in\boldsymbol{I}}$$

*the **dual family** and the Cephoid*

$$(2) \qquad \bar{\Pi} \;=\; \Pi^{\bar{\boldsymbol{a}}^{\bullet}} = \sum_{i\in\boldsymbol{I}} \Pi^{\bar{\boldsymbol{a}}^{(i)}}$$

*the **dual Cephoid**.*

More detailed, $\left(\Pi, \bar{\Pi}\right)$ constitutes a dual pair. Yet, it is convenient to speak of the "primal" and "dual" Cephoid despite the fact that each is "the dual" of the other one. If the "primal" family $\boldsymbol{a}^{\bullet}$ is regarded as a matrix , then the "dual" family is represented by the transposed matrix $(\bar{a}_k^{(i)})_{i\in\boldsymbol{I},k\in\boldsymbol{K}}$. Within this context we assume nondegeneracy. The notion is extended to hold true simultaneously for the primal and dual Cephoid simultaneously.

We continue by immediately introducing duality for Pareto faces.

**Definition 1.2.** *Let* $\boldsymbol{F}$ *be a Pareto face of a Cephoid* $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ *and let* $\mathfrak{J} = \left(\boldsymbol{J}^{(k)}\right)_{k\in\boldsymbol{K}}$ *be the reference system. Define, for* $i \in \boldsymbol{I}$

$$(3) \qquad \bar{\boldsymbol{J}}^{(i)} \;:=\; \left\{k \in \boldsymbol{K} \,\middle|\, i \in \boldsymbol{J}^{(k)}\right\}.$$

*Then we call*

$$(4) \qquad \bar{\mathfrak{J}} \;=\; \left(\bar{\boldsymbol{J}}^{(i)}\right)_{i\in\boldsymbol{I}}$$

*the **dual reference system**.*

Clearly we have, for any $k \in \boldsymbol{K}$

$$(5) \qquad J^{(k)} \;=\; \left\{i \in \boldsymbol{I} \,\middle|\, k \in \bar{\boldsymbol{J}}^{(i)}\right\},$$

so $(\mathfrak{J}, \bar{\mathfrak{J}})$ again constitute a dual pair. Now we introduce

**Definition 1.3.** *Let* $\boldsymbol{F}$ *be a Pareto face of* $\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$. *Then*

$$\mathbb{J} := \{(k,i) \,\big|\, i \in \boldsymbol{J}^{(k)}\} = \{(i,k) \,\big|\, k \in \bar{\boldsymbol{J}}^{(i)}\} \tag{6}$$

*is called the* ***cross–reference system*** *of* $\boldsymbol{F}$.

Obviously, $\mathbb{J}$ yields both families, the reference system and it's dual simultaneously as cuts in coordinate directions. As a consequence, we have

$$n + K - 1 = \sum_{k \in \boldsymbol{K}} |\boldsymbol{J}^{(k)}| = |\mathbb{J}| = \sum_{i \in \boldsymbol{I}} |\bar{\boldsymbol{J}}^{(i)}| \quad . \tag{7}$$

We continue by

**Definition 1.4.**

$$\begin{aligned} \bar{\boldsymbol{L}} &:= \left\{k \in \boldsymbol{K} \,\middle|\, k \textit{ is in at least two different } \bar{\boldsymbol{J}}^{(i)}\right\} \\ &= \left\{k \in \boldsymbol{K} \,\middle|\, \boldsymbol{J}^{(k)} \textit{ contains at least two different indices } i\right\} \\ &= \left\{k \in \boldsymbol{K} \,\middle|\, |\boldsymbol{J}^{(k)}| \geq 2\right\} \end{aligned} \tag{8}$$

*is the* ***dual adjustment set****.*

The analogous property of the ("primal") adjustment system reads now

$$\begin{aligned} \boldsymbol{L} &:= \left\{i \in \boldsymbol{I} \,\middle|\, i \text{ is in at least two of the } \boldsymbol{J}^{(k)}\right\} \\ &= \left\{i \in \boldsymbol{I} \,\middle|\, |\bar{\boldsymbol{J}}^{(i)}| \geq 2\right\}. \end{aligned} \tag{9}$$

Recalling the notation for the characteristics

$$\mathbb{L} := \left\{(k,l) \,\middle|\, l \in \boldsymbol{L},\ \boldsymbol{J}^{(k)} \ni l\right\} = \left\{(k,l) \,\middle|\, l \in \boldsymbol{L}^{(k)}\right\} , \tag{10}$$

we obtain the dual version

$$\begin{aligned} \bar{\mathbb{L}} &:= \left\{(i,s) \,\middle|\, s \in \bar{\boldsymbol{L}},\ \bar{\boldsymbol{J}}^{(i)} \ni s\right\} = \left\{(i,s) \,\middle|\, s \in \bar{\boldsymbol{L}}^{(i)}\right\} \\ &= \left\{(i,s) \,\middle|\, i \in \boldsymbol{J}^{(s)}, \left|\boldsymbol{J}^{(s)}\right| \geq 2\right\} , \end{aligned} \tag{11}$$

which is of course the ***dual characteristics*** w.r.t.$\boldsymbol{F}$.

Now we identify the dual face to some Pareto face $\boldsymbol{F}$ of $\Pi$. As it turns out, the linear adjustment system as defined for the primal face supplies the normal for the dual face *immediately*.

**Theorem 1.5.** *Let* $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ *be a Cephoid and* $\overline{\Pi}$ *its dual. Let* $\boldsymbol{F}$ *be a Pareto face of* $\Pi$ *with reference system* $\mathfrak{J}$*. Let* $\overline{\mathfrak{J}}$ *be the dual reference system.*

*Next, let*

$$(\boldsymbol{c}^\star, \boldsymbol{\lambda}^\star) \;=\; (c_k^\star, \lambda_l^\star)_{(k,l)\in\mathbb{L}} \tag{12}$$

*be a solution of the linear adjustment system corresponding to* $\boldsymbol{F}$ *(see Definition 1.3 of* SECTION *1* CHAPTER *3).*

*Then*

$$\bar{\boldsymbol{F}} \;:=\; \sum_{i\in\boldsymbol{I}} \bar{\Delta}^{(i)}_{\bar{\boldsymbol{J}}^{(i)}}. \tag{13}$$

*is a Pareto face of* $\bar{\Pi}$ *with adjustment set* $\bar{\boldsymbol{L}}$ *and normal* $\boldsymbol{c}^\star$*.*

**Proof:**

We return to the situation in SECTION 1 of CHAPTER 3.

Let $\mathfrak{n}^\star$ denote the normal of $\boldsymbol{F}$; then we know that the function $\boldsymbol{x} \mapsto \mathfrak{n}^\star\boldsymbol{x}$ attains its maximal value – say $t_k$ – relative to the Simplex $\Delta^{(k)}$ exactly on the Subsimplex $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$. Moreover, the joint maximal value $t^\star$ is attained on every $c_k^\star\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ (with a suitable choice of $c_k^\star$, say $c_k^\star = \frac{t^\star}{t_k}$).

Consequently, we have

$$\mathfrak{n}^\star c_k^\star \boldsymbol{a}^{(k)i} \quad \begin{cases} = & t^\star \quad ((k,i)\in\mathbb{J}) \\ < & t^\star \quad ((k,i)\notin\mathbb{J}) \end{cases}$$

which can as well be written

$$n_i^\star a_i^{(k)} c_k^\star \quad \begin{cases} = & t^\star \quad ((k,i)\in\mathbb{J}) \\ < & t^\star \quad ((k,i)\notin\mathbb{J}) \end{cases} \quad . \tag{14}$$

Equivalently we have

$$c_k^\star \overline{a}_k^{(i)} n_i^\star \quad \begin{cases} = & t^\star \quad ((k,i)\in\mathbb{J}) \\ < & t^\star \quad ((k,i)\notin\mathbb{J}) \end{cases}$$

which is also

$$\boldsymbol{c}^\star n_i^\star \overline{\boldsymbol{a}}^{(i)k} \quad \begin{cases} = & t^\star \quad (k\in\bar{\boldsymbol{J}}^{(i)}) \\ < & t^\star \quad (k\notin\bar{\boldsymbol{J}}^{(i)}) \end{cases} \quad . \tag{15}$$

Now, equation (15) shows that, for each $i \in \boldsymbol{I}$, the function $\boldsymbol{y} \mapsto \boldsymbol{c}^\star \boldsymbol{y}$ attains its maximal value $t^\star$ relative to $n_i^\star \overline{\Delta}^{(i)}$ exactly on $n_i^\star \overline{\Delta}^{(i)}_{\bar{\boldsymbol{J}}^{(i)}}$. Thus, $\boldsymbol{c}^\star$ is normal to

$$(16) \qquad \widehat{\overline{\Delta}} := \bigvee_{i \in \boldsymbol{I}} n_i^\star \overline{\Delta}^{(i)} .$$

**2ndSTEP** : Since $\boldsymbol{J}^{(k)} \neq \emptyset$ for all $k \in \boldsymbol{K}$, there is, for any $k \in \boldsymbol{K}$, some $i \in \boldsymbol{I}$ such that $i \in \boldsymbol{J}^{(k)}$ holds true. Therefore

$$\bigcup_{i \in \boldsymbol{I}} \bar{\boldsymbol{J}}^{(i)} = \bigcup_{i \in \boldsymbol{I}} \left\{ k \in \boldsymbol{K} \,\middle|\, i \in \boldsymbol{J}^{(k)} \right\} = \boldsymbol{K}$$

Now, as $\widehat{\overline{\Delta}}$ is spanned by $n_i^\star \overline{\Delta}^{(i)}_{\bar{\boldsymbol{J}}^{(i)}}$, we conclude that the dimension is $\boldsymbol{dim}\ \widehat{\overline{\Delta}} = K-1$, that is, the Simplex $\widehat{\overline{\Delta}}$ has maximal dimension. Write $\bar{r}_i := |\bar{\boldsymbol{J}}^{(i)}|$ $(i \in \boldsymbol{I})$. Then we have for the dimension of the spanning Subsimplices

$$(17) \qquad \begin{aligned} \sum_{i \in \boldsymbol{I}} \boldsymbol{dim}\ \overline{\Delta}^{(i)}_{\bar{\boldsymbol{J}}^{(i)}} &= \sum_{i \in \boldsymbol{I}} (|\bar{\boldsymbol{J}}^{(i)}| - 1) \\ &= (\sum_{i \in \boldsymbol{I}} \bar{r}_i) - n = (n + K - 1) - n = K - 1 \quad , \end{aligned}$$

where the second equation follows from $|\boldsymbol{I}| = n$ and the third one from equations (7).

**3rdSTEP** : The function $\boldsymbol{y} \mapsto \boldsymbol{c}^\star \boldsymbol{y}$ takes its maximal value relative to $\overline{\Delta}^{(i)}$ exactly on $\overline{\Delta}^{(i)}_{\bar{\boldsymbol{J}}^{(i)}}$; this value is $\frac{t^\star}{n_i^\star}$ for $i \in \boldsymbol{I}$. Therefore it is seen that

$$(18) \qquad \bar{\boldsymbol{F}} = \sum_{i \in \boldsymbol{I}} \bar{\Delta}^{(i)}_{\bar{\boldsymbol{J}}^{(i)}} .$$

as specified in (13) is a face of $\bar{\Pi}$ with normal $\boldsymbol{c}^\star$. We will establish that it is a Pareto face.

First of all, we show that $|\bar{\boldsymbol{J}}^{(i)} \cap \bar{\boldsymbol{J}}^{(j)}| \leq 1$ for all $i \neq j$. Assume that, on the contrary, we have $r, s \in \bar{\boldsymbol{J}}^{(1)} \cap \bar{\boldsymbol{J}}^{(2)}$ for some $r \neq s$. In view of (14) we obtain the following equations:

$$\begin{aligned} n_r^\star a_r^{(1)} c_1^\star &= n_s^\star a_s^{(1)} c_1^\star \\ n_r^\star a_r^{(2)} c_2^\star &= n_s^\star a_s^{(2)} \ .c_2^\star \end{aligned}$$

Dividing both equations we obtain

$$\frac{a_r^{(1)} c_1^\star}{a_r^{(2)} c_2^\star} = \frac{a_s^{(1)} c_1^\star}{a_s^{(2)} c_2^\star} ,$$

that is,

$$\frac{a_r^{(1)}}{a_r^{(2)}} = \frac{a_s^{(1)}}{a_s^{(2)}} ,$$

contradicting nondegeneracy.

Consequently, all Subsimplices $\overline{\Delta}^i_{\bar{\boldsymbol{J}}^{(i)}}$ are located in pairwise orthogonal subspaces. This implies

$$\textbf{\textit{dim}} \left( \sum_{i \in \boldsymbol{I}} \overline{\Delta}^{(i)}_{\bar{\boldsymbol{J}}^{(i)}} \right) = \sum_{i \in \boldsymbol{I}} \textbf{\textit{dim}}\ \overline{\Delta}^{(i)}_{\bar{\boldsymbol{J}}^{(i)}} = K - 1 , \tag{19}$$

meaning that $\bar{\boldsymbol{F}}$ is indeed a Pareto face.

**q.e.d.**

**Definition 1.6.** *Let* $\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ *be a Cephoid and* $\overline{\Pi}$ *its dual. Let* $\boldsymbol{F}$ *be a Pareto face of* $\Pi$ *with reference system* $\mathfrak{J}$ *and let* $\overline{\boldsymbol{F}}$ *be the dual face with* $\overline{\mathfrak{J}}$ *as the dual reference system.*

*Recall the linear adjustment system with respect to the face* $\boldsymbol{F}$ *which is*

$$c_k a_l^{(k)} = \lambda_l \quad ((k,l) \in \mathbb{L}). \tag{20}$$

*The **dual linear adjustment system** (dual to* $\boldsymbol{F}$ *or* $\mathfrak{J}$ *or* (20)*) is the linear system of equations in variables* $(n_\bullet, \mu_\bullet)$

$$\overline{a}_s^{(i)} n_i = \mu_s \quad ((i,s) \in \overline{\mathbb{L}}). \tag{21}$$

*Using only primal terms, this system is written*

$$a_i^{(s)} n_i = \mu_s \quad \Big( (i,s) \in \boldsymbol{I} \times \boldsymbol{K}, i \in \boldsymbol{J}^{(s)}, \mid \boldsymbol{J}^{(s)} \mid \geq 2 \Big) \tag{22}$$

**Remark 1.7.** *Analogously to the situation in Theorem 1.5, every solution* $\mathfrak{n}^\star$ *of the system* (21) *(or* (22)*) provides a normal to the primal face* $\boldsymbol{F}$. The adjustment coefficients of the primal face constitute the normal of the dual face and vice versa. *Thus, the system* (21) *directly serves to compute the normal of the primal face.*

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**Remark 1.8.** *The set $\bar{\boldsymbol{L}}$ is the adjustment set for $\bar{\boldsymbol{F}}$. We write $\bar{L} := |\bar{\boldsymbol{L}}|$. Now let $s \in \bar{\boldsymbol{L}}$, say $s \in \bar{\boldsymbol{J}}^{(i_0)} \cap \bar{\boldsymbol{J}}^{(i_1)}$ for suitable $i_0, i_1 \in \boldsymbol{I}$. Then the vertex*

$$n^\star_{i_0} \bar{\boldsymbol{a}}^{(i_0)s} = n^\star_{i_1} \bar{\boldsymbol{a}}^{(i_1)s} \tag{23}$$

*is common to the deGua Subsimplices $n^\star_{i_0} \bar{\Pi}^{\bar{\boldsymbol{a}}^{(i_0)}}_{\bar{\boldsymbol{J}}^{(i_0)}}$ and $n^\star_{i_1} \bar{\Pi}^{\bar{\boldsymbol{a}}^{(i_1)}}_{\bar{\boldsymbol{J}}^{(i_1)}}$. Now recall that, for $i \in \boldsymbol{I} \setminus \boldsymbol{L}$, the set $\bar{\boldsymbol{J}}^{(i)}$ consists of just one element. Therefore, using* (7) *and writing $\bar{r}_i := |\bar{\boldsymbol{J}}^{(i)}|$, we obtain*

$$\begin{aligned} \sum_{i \in \boldsymbol{L}} \bar{r}_i = & \sum_{i \in \boldsymbol{I}} \bar{r}_i - \sum_{i \in \boldsymbol{I} \setminus \boldsymbol{L}} \bar{r}_i = \sum_{i \in \boldsymbol{I}} \bar{r}_i - \sum_{i \in \boldsymbol{I}, \bar{r}_i = 1} \bar{r}_i \\ = & \ (n + K - 1) - (n - L) = K + L - 1 \end{aligned} \tag{24}$$

*or*

$$\sum_{i \in \boldsymbol{L}} |\bar{\boldsymbol{J}}^{(i)}| = K + L - 1 = \sum_{k \in \boldsymbol{K}} |\boldsymbol{L}^{(k)}|, \tag{25}$$

*the last equation is formula* (15) *of* , SECTION *1 of* CHAPTER *3.* .

*The analog equation connecting the primal reference sets with the dual adjustment sets in size is based on the definition $\bar{\boldsymbol{L}}^{(i)} := \bar{\boldsymbol{L}} \cap \bar{\boldsymbol{J}}^{(i)}$ $(i \in \boldsymbol{I})$ and reads*

$$\sum_{k \in \bar{\boldsymbol{L}}} |\boldsymbol{J}^{(k)}| = n + \bar{L} - 1 = \sum_{i \in \boldsymbol{I}} |\bar{\boldsymbol{L}}^{(i)}|. \tag{26}$$

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**Corollary 1.9.** *Let $(\Pi, \bar{\Pi})$ be a dual pair. Let $\boldsymbol{F}$ and $\widetilde{\boldsymbol{F}}$ be adjacent maximal faces of $\Pi$. Then the dual faces $\bar{\boldsymbol{F}}$ and $\bar{\widetilde{\boldsymbol{F}}}$ are adjacent.*

**Proof:** Let $\mathfrak{J} = \left(\boldsymbol{J}^{(k)}\right)_{k \in \boldsymbol{K}}$ and $\widetilde{\mathfrak{J}} = \left(\boldsymbol{J}^{(k)}\right)_{k \in \boldsymbol{K}}$ be the reference systems to $\boldsymbol{F}$ and $\widetilde{\boldsymbol{F}}$ respectively. By the Neighborhood Theorem there are indices $k_0, l_0 \in \boldsymbol{K}$ as well as $p, q \in \boldsymbol{I}$ such that $p \notin \boldsymbol{J}^{(k_0)}, q \in \widetilde{\boldsymbol{J}}^{(k_0)}$

$$\widetilde{\boldsymbol{J}}^{(k_0)} = \boldsymbol{J}^{(k_0)} \cup \{p\} \ , \quad \widetilde{\boldsymbol{J}}^{(l_0)} = \boldsymbol{J}^{(l_0)} \setminus \{q\} \tag{27}$$

while for all indices $k \in \boldsymbol{K}, k \neq k_0, l_0$ the reference sets $\boldsymbol{J}^{(k)}$ and $\widetilde{\boldsymbol{J}}^{(k)}$ coincide. are equal. Inspection of Definition 1.2 shows that

$$\bar{\widetilde{\boldsymbol{J}}}^{(p)} = \bar{\boldsymbol{J}}^{(p)} \cup \{k_0\} \ , \quad \bar{\widetilde{\boldsymbol{J}}}^{(q)} = \bar{\boldsymbol{J}}^{(q)} \setminus \{l_0\} \tag{28}$$

while for all $i \neq p, q$ the reference sets $\bar{\boldsymbol{J}}^{(i)}$ and $\bar{\bar{\boldsymbol{J}}}^{(i)}$ coincide. From this it follows that $\bar{\boldsymbol{F}}$ and $\bar{\bar{\boldsymbol{F}}}$ are adjacent. **q.e.d.**

As it turns out, the complete lattice structure of $\partial\Pi$ is *not* preserved during the transition to the dual. We will see that within the examples of the next section.

# 2 Duality: Examples

We return to our earlier examples to review them in the light of duality theory.

**Example 2.1.** *We start out by revisiting Example 4.7 of* SECTION *4* CHAPTER *3 which deals with* $K = 2$, *i.e., the "sum of two". Again we write* $\Pi = \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}$. *The Pareto surface* $\partial\Pi$ *is completely described by an ordering or permutation* $\prec$. *Thus, for some* $i_0 \in \boldsymbol{I}$ *a typical Pareto face is of the shape*

$$\boldsymbol{F}^{\prec i_0} \;=\; \Delta^{\boldsymbol{a}}_{\{i|i\preceq i_0\}} + \Delta^{\boldsymbol{b}}_{\{i|i_0\preceq i\}}. \tag{1}$$

*The reference system for* $\boldsymbol{F}^{\prec i_0}$ *is therefore*

$$\left\{\boldsymbol{J}^{(1)}, \boldsymbol{J}^{(2)}\right\} = \{\{i|i \preceq i_0\}, \{i|i_0 \preceq i\}\}\,.$$

*The dual Cephoid is located in two dimensions, hence it is of the shape indicated in* CHAPTER *1,* SECTION *2 by Figure 2.3. Thus, the Pareto surface consists of family of line segments. The canonical representation results in the one–dimensional Simplex* $n\Delta^{\boldsymbol{e}} \;=\; n\Delta^{(1,1)}$.

*Now fix some Pareto face* $\boldsymbol{F}^{\prec i_0}$ *as in* (1). *Then the corresponding dual reference system is* $\overline{\mathfrak{J}} = \left\{\bar{\boldsymbol{J}}^{(i)}\right\}_{i\in\boldsymbol{I}}$ *given by*

$$\bar{\boldsymbol{J}}^{(i)} = \{k \in \boldsymbol{K} \,\big|\, i \in \boldsymbol{J}^{(k)}\}, \quad (i \in \boldsymbol{I}) \tag{2}$$

*which specifies to*

$$\bar{\boldsymbol{J}}^{(i)} = \{1\} \quad (i \prec i_0) \text{ and } \bar{\boldsymbol{J}}^{(i)} = \{2\} \quad (i_0 \prec i) \tag{3}$$

*while for* $i \in \boldsymbol{I} \setminus \{i_0\}$ *while for* $i = i_0$ *we obtain*

$$\bar{\boldsymbol{J}}^{(i)} = \{1, 2\}. \tag{4}$$

*Hence the dual face to* $\boldsymbol{F}^{\prec i_0}$ *is*

$$\bar{\boldsymbol{F}}^{\prec i_0} = \sum_{i\prec i_0} \bar{\boldsymbol{a}}^{(1)i} + \Delta^{(i_0)}_{\{1,2\}} + \sum_{i_0\prec i} \bar{\boldsymbol{a}}^{(2)i}\;. \tag{5}$$

*We realize that* $\bar{\boldsymbol{F}}^{\prec i_0}$ *is the line segment* $\Delta^{(i_0)}_{\{1,2\}}$ *translated to the Pareto surface of* $\overline{\Pi}$ *by the appropriate axis vectors. Thus, the Pareto surface of* $\bar{\Pi}$ *is a linear curve with line segments being the translates of the various* $\Delta^{(i_0)}_{\{1,2\}}$ . *If* $i_0$ *is the first w.r.t.* $\prec$, *then the Pareto face*

$$\bar{\boldsymbol{F}}^{\prec i_0} = \Delta^{(i_0)}_{\{1,2\}} + \sum_{i\neq i_0} \bar{\boldsymbol{a}}^{(2)i}\;.$$

*is the "uppermost" line segment, i.e., the first in the ordering induced by the slope when we begin with the smallest slope (in absolute value). Thus, it is seen that* $\prec$ *also represents the ordering of the line segments within the Pareto surface of the dual Cephoid according to slope.*

*We specify this example a bit more: consider Figure 4.2 which represents a case* $K = 2, n = 4$. *We repeat the sketch of the canonical representation in Figure 2.1 of* CHAPTER *3.*

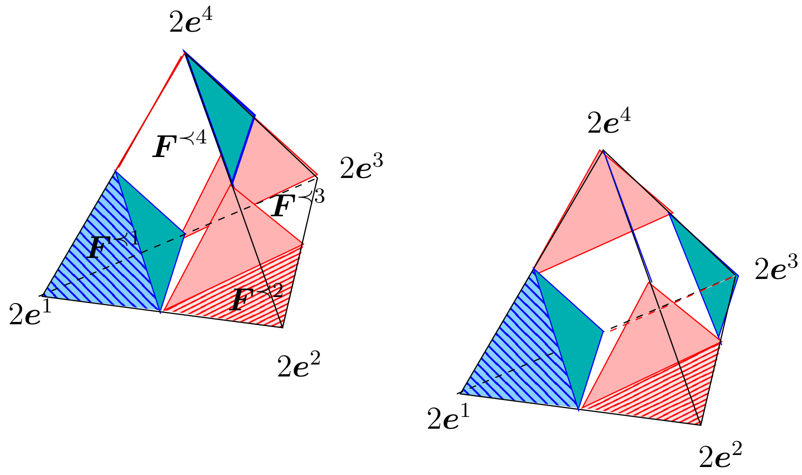


Figure 2.1: The sum of two prisms for $n = 4$

*Assuming that the translate of* $\Delta^{\boldsymbol{a}}$ *occupies the first vertex of the sum (i.e.,* $2\boldsymbol{e}^1$*), and the translate of* $\Delta^{\boldsymbol{b}}$ *the second one, the left hand version of Figure 2.1 corresponds to the ordering* $\prec \; = (2341)$. *The* 3*-dimensional faces are given by*

$$
\begin{array}{rcl}
\boldsymbol{F}^{\prec 2} & = & \Delta^{\boldsymbol{a}}_{2} + \Delta^{\boldsymbol{b}}_{2341} \\
\boldsymbol{F}^{\prec 3} & = & \Delta^{\boldsymbol{a}}_{23} + \Delta^{\boldsymbol{b}}_{341} \\
\boldsymbol{F}^{\prec 4} & = & \Delta^{\boldsymbol{a}}_{234} + \Delta^{\boldsymbol{b}}_{41} \\
\boldsymbol{F}^{\prec 1} & = & \Delta^{\boldsymbol{a}}_{2341} + \Delta^{\boldsymbol{b}}_{1} \; .
\end{array} \tag{6}
$$

*The ordering* $\prec$ *represents the neighborhood structure of the four faces simultaneously indicating the unique extremal vector* $\boldsymbol{c}^i = \boldsymbol{a}^i + \boldsymbol{b}^i$ *assigned to a face. If we start with* $\boldsymbol{F}^{\prec 2}$ *containing* $\boldsymbol{c}^2$*, then the unique neighbor is* $\boldsymbol{F}^{\prec 3}$ *containing* $\boldsymbol{c}^3$ *etc.. Thus, while running through the extremals* $\boldsymbol{c}^i$ *according to* $\prec$ *one also passes from one face to it's neighbor.*

*The same situation prevails with respect to the dual Cephoid* $\bar{\Pi}$. *The dual face to* $\boldsymbol{F}^{\prec 2}$ *i.e., generated by the reference system* $\mathfrak{J} = \{\{2\}, \{2341\}\}$ *is* $\bar{\boldsymbol{F}}^{\prec 2}$ *which is given by*

$$
\bar{\mathfrak{J}} \;=\; \{\{2\}, \{12\}, \{2\}, \{2\}\}
$$

*i.e.,*

$$\bar{\boldsymbol{F}}^{\prec 2} = \Delta^{(1)}_{\{2\}} + \Delta^{(2)}_{\{12\}}\} + \Delta^{(3)}_{\{2\}} + \Delta^{(4)}_{\{2\}}$$

*this is a translate of* $\bar{\Delta}^{(2)}_{\{12\}}$ *by means of* $\boldsymbol{a}^{(1)2} + \boldsymbol{a}^{(3)2} + \boldsymbol{a}^{(4)2}$*. Similarly,*

$$\bar{\boldsymbol{F}}^{\prec 3} = \Delta^{(1)}_{\{2\}} + \Delta^{(2)}_{\{1\}} + \Delta^{(3)}_{\{12\}} + \Delta^{(4)}_{\{2\}}$$

*is a translate of* $\Delta^{(3)}_{\{12\}}$*. The further two dual faces are.*

$$\bar{\boldsymbol{F}}^{\prec 4} = \Delta^{(1)}_{\{2\}} + \Delta^{(2)}_{\{1\}} + \Delta^{(3)}_{\{1\}} + \Delta^{(4)}_{\{12\}}$$

*and*

$$\bar{\boldsymbol{F}}^{\prec 1} = \Delta^{(1)}_{\{12\}} + \Delta^{(2)}_{\{1\}} + \Delta^{(3)}_{\{1\}} + \Delta^{(4)}_{\{1\}} \; .$$

*The Pareto surface* $\partial\bar{\Pi}$ *is sketched together with its canonical representation in Figure 2.2. When we start in the uppermost face and run through the faces*

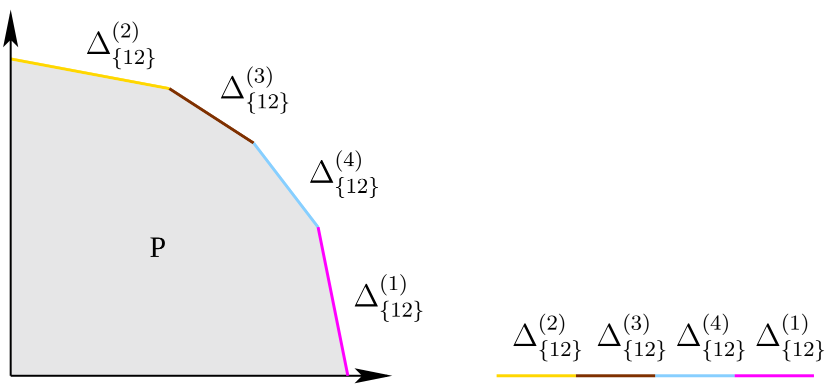


Figure 2.2: The dual surface and its canonical representation

*according to* $\prec$*, then we pass all faces in downwards direction.*

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**Example 2.2.** *Next we recall the"Marriage of a Windmill and a Circle" as introduced in Example 6.7 of* Section *6,* Chapter *3, see Figure 6.5. The canonical representation (with* $n = 4, K = 3$*) is repeated hereby (Figure 2.3). There is also a POV version (Figure 2.4).*

*We use* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ *for the primal family assuming that* $\Delta^{(\boldsymbol{a})}$ *corresponds to "blue",* $\Delta^{(\boldsymbol{b})}$ *corresponds to "red", and* $\Delta^{(\boldsymbol{c})}$ *corresponds to "green". Then the following is a list of the Pareto faces.*

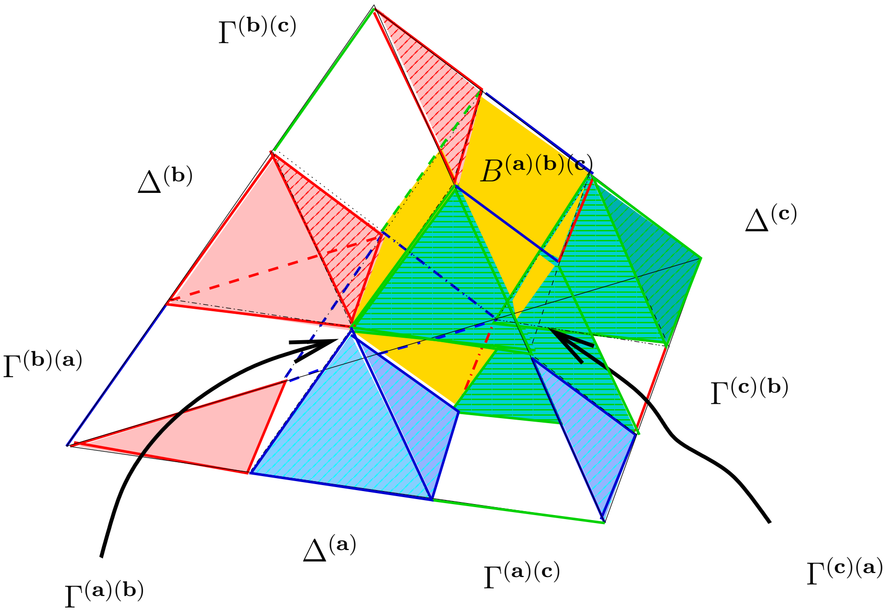


Figure 2.3: The Marriage of a Windmill and a Circle

(7)

| *Name* | $\boldsymbol{J}^{(\boldsymbol{a})}$ | $\boldsymbol{J}^{(\boldsymbol{b})}$ | $\boldsymbol{J}^{(\boldsymbol{c})}$ |
|---|---|---|---|
| $\Delta^{(\boldsymbol{a})}$ | $\{1234\}$ | $\{2\}$ | $\{1\}$ |
| $\Gamma^{(\boldsymbol{a})(\boldsymbol{c})}$ | $\{234\}$ | $\{2\}$ | $\{12\}$ |
| $\Gamma^{(\boldsymbol{a})(\boldsymbol{b})}$ | $\{134\}$ | $\{23\}$ | $\{1\}$ |
| | | | |
| $\Delta^{(\boldsymbol{b})}$ | $\{4\}$ | $\{1234\}$ | $\{1\}$ |
| $\Gamma^{(\boldsymbol{b})(\boldsymbol{c})}$ | $\{4\}$ | $\{234\}$ | $\{14\}$ |
| $\Gamma^{(\boldsymbol{b})(\boldsymbol{a})}$ | $\{14\}$ | $\{123\}$ | $\{1\}$ |
| | | | |
| $\Delta^{(\boldsymbol{c})}$ | $\{3\}$ | $\{3\}$ | $\{1234\}$ |
| $\Gamma^{(\boldsymbol{c})(\boldsymbol{b})}$ | $\{3\}$ | $\{23\}$ | $\{124\}$ |
| $\Gamma^{(\boldsymbol{c})(\boldsymbol{a})}$ | $\{34\}$ | $\{2\}$ | $\{124\}$ |
| | | | |
| $B^{(\boldsymbol{a})(\boldsymbol{b})(\boldsymbol{c})}$ | $\{34\}$ | $\{23\}$ | $\{14\}$ |

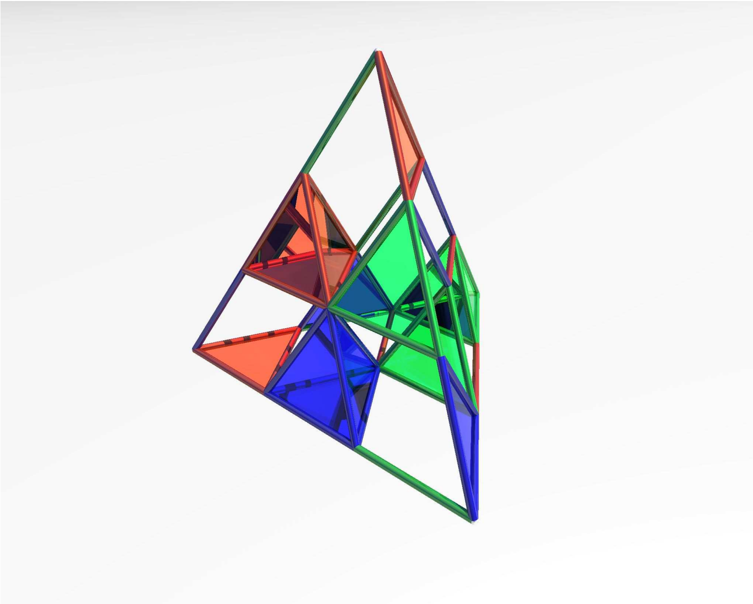

Figure 2.4: The Marriage - POV Version

*The formal presentation based on an algorithmic result is also copied from* Section *6,* Chapter *3.*

```
THE CEPHOID *MARRIAGE*:

                   1 1 19 5
                   2 4 13 2
                   6 3  2 3

PARETO FACES OF CEPHOID MARRIAGE:

       3  ⊛        3  ⊛  1 2 3 4  ⊛
     3 4  ⊛      2 3  ⊛      1 4  ⊛
       4  ⊛    2 3 4  ⊛      1 4  ⊛
   2 3 4  ⊛        2  ⊛      1 2  ⊛
     3 4  ⊛        2  ⊛    1 2 4  ⊛
       3  ⊛      2 3  ⊛    1 2 4  ⊛
   1 3 4  ⊛      1 2  ⊛        1  ⊛
       4  ⊛  1 2 3 4  ⊛        1  ⊛
     3 4  ⊛    1 2 3  ⊛        1  ⊛
 1 2 3 4  ⊛        2  ⊛        1  ⊛
```

*Now consider the dual Cephoid focusing on Figure 2.3. The dual Cephoid is the sum of 4 DeGua Simplices in 3 dimensions. We denote the dual family by* $\left\{\bar{\boldsymbol{a}}^{(i)}\right\}_{i\in\boldsymbol{I}}$ *The canonical representation is given by the following sketch.*

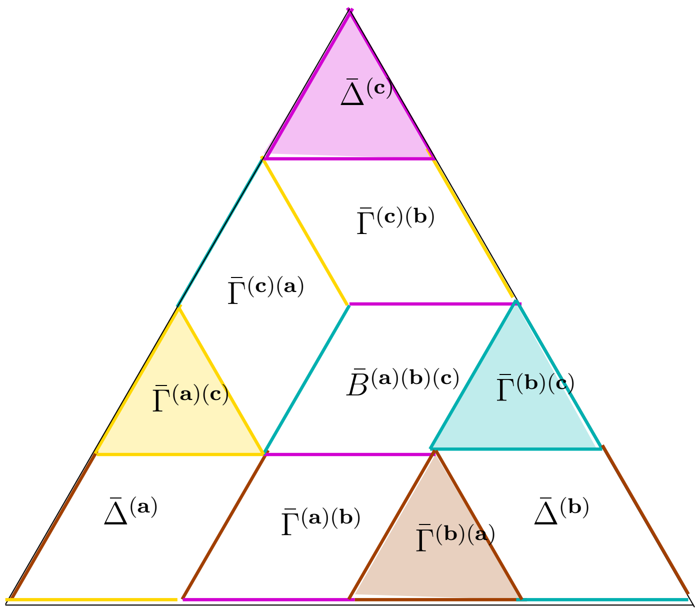


Figure 2.5: The Dual Marriage

*The Pareto faces are listed in the same order as their primal counterparts and* *indicated accordingly. We obtain the following image.*

(8)

| *Name* | $\overline{\boldsymbol{J}}^{(1)}$ | $\overline{\boldsymbol{J}}^{(2)}$ | $\overline{\boldsymbol{J}}^{(3)}$ | $\overline{\boldsymbol{J}}^{(4)}$ |
|---|---|---|---|---|
| $\bar{\Delta}^{(\boldsymbol{a})}$ | $\{13\}$ | $\{12\}$ | $\{1\}$ | $\{1\}$ |
| $\bar{\Gamma}^{(\boldsymbol{a})(\boldsymbol{c})}$ | $\{3\}$ | $\{123\}$ | $\{1\}$ | $\{1\}$ |
| $\bar{\Gamma}^{(\boldsymbol{a})(\boldsymbol{b})}$ | $\{13\}$ | $\{2\}$ | $\{12\}$ | $\{1\}$ |
| | | | | |
| $\bar{\Delta}^{(\boldsymbol{b})}$ | $\{23\}$ | $\{2\}$ | $\{2\}$ | $\{12\}$ |
| $\bar{\Gamma}^{(\boldsymbol{b})(\boldsymbol{c})}$ | $\{3\}$ | $\{2\}$ | $\{2\}$ | $\{123\}$ |
| $\bar{\Gamma}^{(\boldsymbol{b})(\boldsymbol{a})}$ | $\{123\}$ | $\{2\}$ | $\{2\}$ | $\{1\}$ |
| | | | | |
| $\bar{\Delta}^{(\boldsymbol{c})}$ | $\{3\}$ | $\{3\}$ | $\{123\}$ | $\{3\}$ |
| $\bar{\Gamma}^{(\boldsymbol{c})(\boldsymbol{b})}$ | $\{3\}$ | $\{23\}$ | $\{12\}$ | $\{3\}$ |
| $\bar{\Gamma}^{(\boldsymbol{c})(\boldsymbol{a})}$ | $\{3\}$ | $\{23\}$ | $\{1\}$ | $\{13\}$ |
| | | | | |
| $\bar{B}^{(\boldsymbol{a})(\boldsymbol{b})(\boldsymbol{c})}$ | $\{3\}$ | $\{2\}$ | $\{12\}$ | $\{13\}$ |

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# Chapter 5

# The Recursive Structure

For a Cephoid $\Pi$, the maximal (but not Pareto efficient) faces $\Pi^{(-i)}$ $(i \in \boldsymbol{I})$ obtained by intersecting $\Pi$ with the $(n-1)$ dimensional boundaries of the positive orthant each constitute a Cephoid in $\mathbb{R}^n_{\boldsymbol{I}\setminus\{i\}+}$ which inherits the Pareto efficient structure. On the other hand, the characteristics and in particular the "$\boldsymbol{L}$–reduced" version of some Pareto face (see Definition 1.3 in Chapter 3) determine that Pareto face completely. Therefore, we expect that every Pareto face is described by its lower dimensional boundary faces.

This way we obtain a recursive structure that organizes the Pareto surface of a Cephoid. This structure is now exhibited more precisely. As a first result, we will be able to enumerate the faces and to provide a first algorithm for computing the faces.

# 1 Universal Quantities

First of all within this section we provide the means to enumerate the Pareto faces of a Cephoid. We show that the total number of Pareto faces of a Cephoid is a quantity that depends on $K$ and $n$ only - independent on the particular choice of the family $\boldsymbol{a}^{\bullet} = \left\{\boldsymbol{a}^{(k)}\right\}_{k \in \boldsymbol{K}}$. We start by identifying the minimal subspace a face of a Cephoid is "rooted" in.

**Definition 1.1.** *Let* $\Pi$ *be a Cephoid and let* $\boldsymbol{F}$ *be a Pareto face of* $\Pi$. *Let* $\boldsymbol{J} \subseteq \boldsymbol{I}$.

1. *We say that* $\boldsymbol{F}$ *has a* ***proper*** $\boldsymbol{J}$*–**cut*** *if*

$$(1) \qquad dim(\boldsymbol{F} \cap \mathbb{R}^{+}_{\boldsymbol{J}}) \;=\; |\boldsymbol{J}| - 1.$$

2. *Proper cuts are ordered by inclusion. A* ***minimal proper cut*** *is a proper cut with minimal dimension.*
3. $\boldsymbol{F}$ *is called* $l$*–**based** (or just an* $l$*–face) if the dimension of the minimal proper cut is* $l-1$.

That is, the intersection of the Pareto face $\boldsymbol{F}$ with a boundary subspace is the Pareto face of the Cephoid in that subspace generated by the restrictions $\left\{\boldsymbol{a}^{(k)}_{\boldsymbol{J}}\right\}_{k \in \boldsymbol{K}}$ (see Theorem 1.2 and Theorem 1.4 in Chapter 3). Hence, that intersection has the full dimension of a surface relative to the boundary subspace.

Verbally, a 1–face contains a vertex, a 2–face cuts properly into a 2 dimensional subspace of $\mathbb{R}^{n}_{+}$ but does not contain a vertex, etc. E.g., we know that for $K = 2$ every face is 1–face (Theorem 4.2 of Chapter 3). An n-face or n-based Pareto face is one that is properly contained in $\mathbb{R}^{n}_{+}$ but does not touch a lower-dimensional boundary subspace.

The results of Section 1 Chapter 3 can be reformulated as follows.

**Lemma 1.2.** *Let* $\boldsymbol{F}$ *be a Pareto face of a Cephoid* $\Pi$. *Then* $\boldsymbol{F}$ *is an* $l$*–based face for some* $l \leq \min\{K-1, n-2\}$. *The boundary* $\mathbb{R}^{+}_{\boldsymbol{L}}$ *of* $\mathbb{R}^{n}_{+}$ *that yields the minimal proper cut is uniquely defined by the adjustment set* $\boldsymbol{L}$ *(Definition 1.1* Chapter *3).*

The proof is an immediate consequence of the Coincidence Theorem 1.4 Chapter 3.

**Remark 1.3.** *The projection of a de Gua Simplex $\Delta^{(k)}$ onto some subspace yields the corresponding Subsimplex. Restricting the summation to a subspace amounts to adding de Gua Simplexes within this subspace and generating a cephoid of lower dimension. In general, Pareto faces (actually all kind of faces) can disappear by the restriction to lower dimensions. However, if a Pareto face intersects a subspace of lower dimension properly, then the intersection is a Pareto face of the restricted Cephoid.*

*In particular, for $K \leq n-1$, consider a Pareto face with adjustment set set $\boldsymbol{L}$. If the restriction to some lower dimensional $\mathbb{R}_{\boldsymbol{J}}$ respects $\boldsymbol{L}$ (i.e., $\boldsymbol{L} \subseteq \boldsymbol{J}$), then $\boldsymbol{F} \cap \mathbb{R}^{+}_{\boldsymbol{J}}$ is indeed a Pareto face. Then, $\boldsymbol{F} \cap \mathbb{R}^{+}_{\boldsymbol{J}}$ is indeed an $l$–based face with the same set of boundary indices.*

*The recursive procedure is essentially based on this property of Cephoids: $l$–based faces appear already in lower dimensions, hence can be enumerated and characterized recursively.*

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**Definition 1.4.** *Let $\boldsymbol{a}^{\bullet} = \left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ be an (n.d.) family of positive vectors and let $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ be the Cephoid generated. The number of Pareto faces of $\Pi$ is denoted by $f(K,n) = f(K,n)^{\boldsymbol{a}^{\bullet}}$. The number of $n$–based faces is denoted by $h(K,n) = h(k,n)^{\boldsymbol{a}^{\bullet}}$.*

The upper script $^{\boldsymbol{a}^{\bullet}}$ will be be necessary until we have verified that it can indeed be omitted. In what follows we do not always write it (for clarity), but it is always thought to be carried along.

**Remark 1.5.** *For $K \leq n$ we know that $h^{\boldsymbol{a}^{\bullet}}(K,n) = 0$ as every Pareto face cuts properly into an $(n-2)$–dimensional subface. Indeed, This follows from the construction of the $\boldsymbol{L}$–reduced subface (Definition 1.3 and Theorem 1.2) which yields a dimension of the $\boldsymbol{L}$–reduced Cephoid $\Pi_{\boldsymbol{L}}$ of dimension $\boldsymbol{dim}\ (\boldsymbol{F} \cap \mathbb{R}^{n}_{\boldsymbol{L}}) \leq L-1 \leq K-2$.*

*For example, the Cephoid "Odot" (Figure 2.8 CHAPTER 2) is an example with $K=3$ and $n=4$ with one "interior" Pareto face.*

*Also note that $f^{\boldsymbol{a}^{\bullet}}(K,1) = h^{\boldsymbol{a}^{\bullet}}(K,1) = 1$ and $f^{\boldsymbol{a}^{\bullet}}(K,2) = K$, $h^{\boldsymbol{a}^{\bullet}}(K,2) = K-2$ $(K \geq 2)$ holds true immediately.*

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**Lemma 1.6.** *For every $K \in \mathbb{N}$*

$$f^{\boldsymbol{a}^{\bullet}}(K,n) = \sum_{l=1}^{\min\{K,n\}} \binom{n}{l} h^{\boldsymbol{a}^{\bullet}}(K,l). \tag{2}$$

**Proof:** Consider the case $K \leq n-1$. We collect the faces according to the minimal subface they are sharing a proper cut with. In view of the Coincidence Theorem, the dimension of such a subface is at most $K-1$. Each Pareto face is represented uniquely by its minimal proper cut (cf. Remark 1.3). Therefore the number of $(K-1)$–based faces of $\Delta$ can be obtained by counting the $(K-1)$–based faces in each of the $\binom{n}{K-1}$ restrictions of of $\Delta$ with dimension $K-1$ etc.

The second formula follows by Proposition 1.2.

**q.e.d.**

On the other hand, if we know the total number of faces for some dimension $n$, then we can compute the number of "interior" faces by subtracting all faces that properly cut into some boundary face, formally:

**Corollary 1.7.** *For $K \geq n$*

$$h^{\boldsymbol{a}^\bullet}(K,n) = f^{\boldsymbol{a}^\bullet}(K,n) - \left(\binom{n}{n-1} h^{\boldsymbol{a}^\bullet}(K,n-1) + \ldots n h^{\boldsymbol{a}^\bullet}(K,1)\right). \tag{3}$$

Now we are in the position to prove:

**Theorem 1.8.** *The number of Pareto faces is universal, i.e., there is a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that, for any (n.d.) family $\left\{\boldsymbol{a}^{(k)}\right\}_{k \in \boldsymbol{K}}$ of positive vectors in $\mathbb{R}^n$ it follows that*

$$f(K,n) \;=\; f(K,n)^{\boldsymbol{a}^\bullet} \;. \tag{4}$$

**Proof:**

For $n=2$ the number of line segments is always $K$ (Example 2.2 Chapter 1). For $K=2$ the number of Pareto faces is $n$ by Theorem 4.2 Chapter 3. Actually, one statement follows from the other one by duality, compare Example 2.1 Chapter 4.

We proceed by induction in $n$.

Let $K \leq n$. In view of formula (2), we can compute the number $f(K,n)$ by means of the numbers

$$h(K,K-1),\; h(K,K-2),\; \ldots,\; h(K,1). \tag{5}$$

because of $K-1 < K \leq n$ the second arguments are $l < n$

The numbers $h(K,l)$ in turn can be computed successively in terms of the number $f(K,l)$ and $h(k,l')$ for $l' < l < n$ via formula (3) of Corollary 1.7. Thus $h(K,l)$ as used in (5) is universal. Hence, $f(K,n)$ can be computed recursively using numbers that – by induction – are universal.

Finally, for $n \leq K$, the result follows by duality.

**q.e.d.**

## 2 The Number of Pareto Faces

Now we are going to provide the exact shape of the function $f$ that indicates the number of Pareto faces. As we know that it is universal, i.e., does not depend on the particular choice of the family $\boldsymbol{a}^{\bullet}$, it suffices to compute it for some representative family and the corresponding n.d. Cephoid. To this end, we start by constructing a family of "'Test Cephoids". The members of this family are specified by a particular location of all subfaces added by (say) the deGua Simplex $\Delta^{(K)}$ to some Pareto face. We want all these subfaces of $\Delta^{(K)}$ to be located on the boundary $\mathbb{R}^n_{\boldsymbol{I}\setminus\{n\}}$, hence Pareto faces generating faces of $\Pi^{(-n)}$ as well.

**Remark 2.1.** *To enlighten the situation we have in mind recall that, for a Cephoid* $\Pi$ *and some Pareto face* $\boldsymbol{F} = \sum_{k\in\boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ *of* $\Pi$ *with normal* $\mathfrak{n}^{\boldsymbol{F}}$*, the following are equivalent.*

1. $n \in \boldsymbol{J}^{(K)}, n \notin \boldsymbol{L}$.
2. $\boldsymbol{F} \cap \Pi^{(-n)}$ *is a Pareto face of* $\Pi^{(-n)}$.
3. *The normal cone of* $\boldsymbol{F}$ *has an extremal which is the normal of* $\boldsymbol{F} \cap \Pi^{(-n)}$.

*This follows from the Reference Theorem 1.2 and and the Coincidence Theorem 1.4 as discussed in* Section*1 of* Chapter *3.*

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The following Lemma illustrates the idea extensively in the case of a sum of two deGua Simplices. The generalization is then obvious.

**Lemma 2.2.** *1. Let* $\boldsymbol{a}$ *be a positive vector and let* $\Pi^{\boldsymbol{a}}$ *be the coresponding deGua Simplex. Then there exists an open set of postive vectors* $\boldsymbol{b}$ *such that for any face*

$$\boldsymbol{F} = \Delta^{\boldsymbol{a}}_{\boldsymbol{J}^{\boldsymbol{a}}} + \Delta^{\boldsymbol{b}}_{\boldsymbol{J}^{\boldsymbol{b}}}$$

*of* $\Pi = \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}$ *with* $|\boldsymbol{J}^{\boldsymbol{b}}| \geq 2$ *it follows that* $n \notin \boldsymbol{L}$.

*2. Let*

$$\Pi^{-K} = \sum_{k\in\boldsymbol{K}\setminus\{K\}} \Pi^{\boldsymbol{a}^{(k)}} \tag{1}$$

*be a Cephoid. Then there exists an open set of vectors* $\boldsymbol{b}^{(K)}$ *generating deGua Simplices* $\Delta^{(K)}$ *such that the following holds true:*

*For any Pareto face*

(2)
$$\boldsymbol{F} = \sum_{k\in\boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \ \textit{of}\ \Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$$

*with*

$$|\boldsymbol{J}^{(K)}| \geq 2 \ \textit{and}\ n \in \boldsymbol{J}^{(K)}$$

*it follows that* $n \notin \boldsymbol{L}$.

**Proof: 1^st^STEP** :

By Theorem 4.2 the Pareto faces of $\Pi$ correspond to an ordering $\prec$ of $\boldsymbol{I}$. If $n$ is the last index with respect to this ordering, then the Pareto faces are indicated by

(3)
$$\begin{array}{lr} i & ijl\dots n \\ ij & ijl\dots n \\ ijl & l\dots n \\ \dots & \dots \\ ijl\dots n & n \end{array}$$

(inspect Remark 4.4). That is, all Pareto faces with the exeption of the last one are characterized by some $\boldsymbol{L} = \{i\}$ with $i \neq n$. Obviously, these are all Pareto faces with $|\boldsymbol{J}^{(\boldsymbol{b})}| \geq 2$. Hence, given $\boldsymbol{a}$, all positive vectors $\boldsymbol{b}$ (such that $\Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}$ is n.d.) can be decomposed into classes corresponding to some $i_0 \in \boldsymbol{I}$ being the last index with respect to $\prec$, say

(4) $$\boldsymbol{B}_{i_0} := \{\boldsymbol{b} \,|\, i_0 \text{ is last w.r.t. } \prec \text{ induced by } \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}\}\ .$$

Thus, e.g., (3) reflects orderings such that the corresponding vectors $\beta$ are located in $\boldsymbol{B}_n$

By symmetry reasons, all $\boldsymbol{B}_{\iota_0}$ are open sets with positive Lebesgue measure, in particular, this holds true for the set $\boldsymbol{B}_n$.

Now we have a precise method of describing this open set as follows. In view of Example 2.1 the ordering $\prec$ is exactly represented by the ordering of the faces of the dual Cephoid $\overline{\Pi}$ according to the slopes of the dual faces (i.e., line segments). The slope of line segment $\overline{\Delta}^{(i)}$ is ( in absolute value, i.e.,omitting the sign)

(5) $$\frac{\overline{a}_2^{(i)}}{\overline{a}_1^{(i)}} = \frac{a_i^{(2)}}{a_i^{(1)}} = \frac{b_i}{a_i}\ .$$

The "last" segment is the one with maximal slope (absolutely), hence we obtain

$$\boldsymbol{B}_{i_0} \;=\; \left\{ \boldsymbol{b} \;\middle|\; \frac{b_{i_0}}{a_{i_0}} > \frac{b_i}{a_i} \;\; (i \in \boldsymbol{I} \setminus \{i_0\}) \right\} . \tag{6}$$

Thus, the set of all vectors $\boldsymbol{b}$ as claimed by our Lemma is

$$\boldsymbol{B}_{n} \;=\; \left\{ \boldsymbol{b} \;\middle|\; \frac{b_{n}}{a_{n}} > \frac{b_i}{a_i} \;\; (i \in \boldsymbol{I} \setminus \{n\}) \right\} , \tag{7}$$

which is an open set with positive Lebesgue measure as required.

We note that $\mathfrak{n}^n = (a_n, b_n)$ is the normal to the line segment $\overline{\Delta}^{(n)}$. Thus (7) reflects the fact that $\overline{\Delta}^{(n)}$ has no $\boldsymbol{a}^{(i)2}$ in common with its normal cone, hence no $\boldsymbol{a}^{(i)2}$ appears in the translation

$$\sum_{i \in \boldsymbol{I} \setminus \{n\}} \overline{\boldsymbol{a}}^{(i)1} + \overline{\Delta}^{(n)}_{12} \;=\; \sum_{i \in \boldsymbol{I} \setminus \{n\}} \overline{\boldsymbol{a}}^{(i)1} + \overline{\Delta}^{(n)}_{12} \; .$$

**2$^{\text{nd}}$STEP** :

To generalize this to the case of a sum of $\Pi^{(\star K)}$ of $K-1$ deGua Simplices and an additional DeGua Simplex $\Delta^{(K)}$, one observes that the normals/slopes of all subfaces of $\Delta^{(K)}$ have to satisfy certain inequalities corresponding to (7), simultaneously for all $\Delta^{(k)}\;\; k \in \boldsymbol{K} \setminus \{K\}$.

**q.e.d.**

**Theorem 2.3 (The Test Cephoid).** *There is an open set of families $\left\{\boldsymbol{a}^{(k)}\right\}_{k \in \boldsymbol{K}}$ such that the resulting Cephoids $\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ satisfy condition* (2).

Figure 2.1 indicates the Canonical Representation of a Test Cephoid for $n = 3$. The DeGua Simplex $\Delta^{(K)}$ is drawn in red, two further deGua Simplices (and their tentacles) are indicated. The Pareto faces involving some $\Delta^{(K)}_{\boldsymbol{J}}$ with $|\boldsymbol{J}| \geq 2$ occupy the boundary $\{x \mid x_3 = x_n = 0\}$.

Figure 2.2 suggests that, by induction, we can construct a family of Cephoids such that the Pareto faces involving some $\Delta^{(k)}$ with $\Delta^{(k)}_{\boldsymbol{J}}, |\boldsymbol{J} \geq 2|$ occupy exactly the "$k^{th}$ layer" of the representation.

Now we can compute the number of faces of a Cephoid which is a sum of $K$ deGua Simplices in $n$ dimensions. We know that this number is universal, hence it suffices to compute it for the class of Test Cephoids.

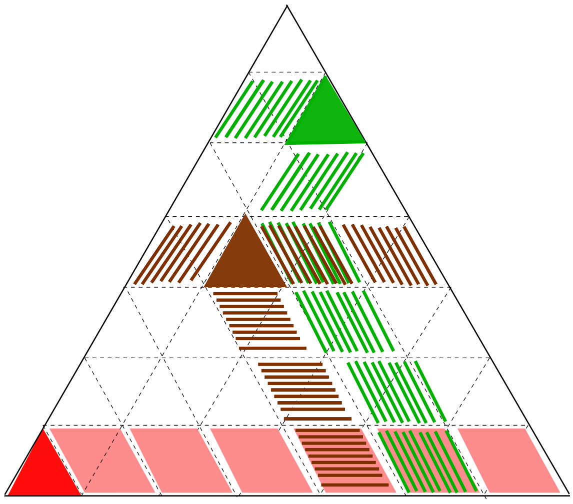

Figure 2.1: A Test Cephoid for $n = 3$

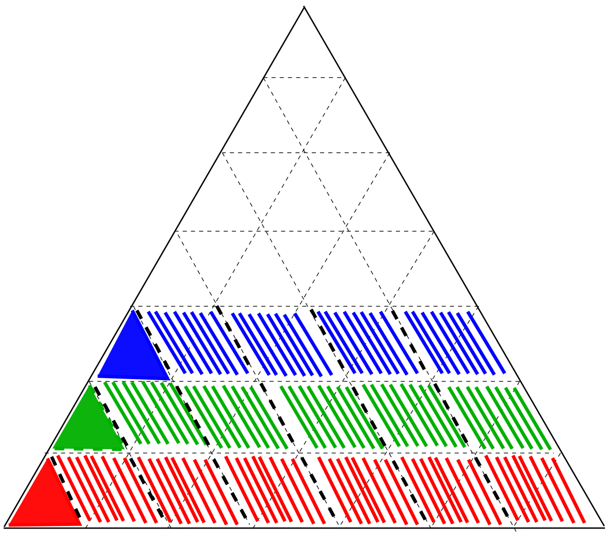

Figure 2.2: A Test Cephoid with layers according to $\boldsymbol{K}$

**Theorem 2.4.** *Let $K \geq 2$ and let $\{\boldsymbol{a}^{(k)}\}_{k\in\boldsymbol{K}}$ be a family of positive vectors in $\mathbb{R}^n$ generating a Test Cephoid*

$$\Pi \; = \; \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}} \subset \mathbb{R}^n_+$$

*as in Theorem 2.3.*

*Then the number $f(K,n)$ of Pareto faces of $\Pi$ is universal and satisfies the following difference equation:*

$$\begin{array}{lcl} f(K,2) & = & K \;(K\geq 2) \quad , \quad f(2,n) = n \quad (n \geq 2) \\ f(K,n) & = & f(K-1,n) + f(K,n-1) \quad (K,n\geq 3) \end{array} \tag{8}$$

**Proof:**

Let $\Pi^{[-K]}$ be the Cephoid

$$\Pi^{[-K]} \; := \sum_{k\in\boldsymbol{K}\setminus\{K\}} \Pi^{\boldsymbol{a}^{(k)}} \subset \mathbb{R}^n_+$$

generated by the first $K-1$ members of the family. A Pareto face of $\Pi$ is given via

$$\boldsymbol{F} \; = \; \sum_{k=1}^{K-1} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} + \Delta^{(K)}_{\boldsymbol{J}^{(K)}}$$

We partition the set $\mathcal{F}$ of all Pareto faces of $\Pi$ into

$$\mathcal{F} = \mathcal{F}^{[-K]} \cup \mathcal{F}^{\star} = \left\{ \boldsymbol{F} \,\middle|\, |\boldsymbol{J}^{(K)}| = 1 \right\} \cup \left\{ \boldsymbol{F} \,\middle|\, |\boldsymbol{J}^{(K)}| > 1 \right\}.$$

Now, for any $\boldsymbol{F} \in \mathcal{F}^{[-K]}$ the summand from $\Delta^{(K)}$ consists of some basis vector $\boldsymbol{a}^{(K)i} = a^{(K)}_i \boldsymbol{e}^i$ only and there is a unique face

$$\boldsymbol{F}^{[-K]} \; := \; \sum_{k=1}^{K-1} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$$

of the Cephoid $\Pi^{[-K]}$ corresponding to $\boldsymbol{F}$.

On the other hand, let $\boldsymbol{F} \in \mathcal{F}^{\star}$, then $|\boldsymbol{J}^{(K)}| \geq 2$ and by construction $n \notin \boldsymbol{L}$. That is, $n$ is *not* contained in any further $\boldsymbol{J}^{(k)}$ $(k \neq K)$. Hence, there corresponds uniquely the Pareto face

$$\boldsymbol{F}^{\star} \; := \; \boldsymbol{F}_{|\mathbb{R}_{\boldsymbol{I}\setminus\{n\}}} \; = \; \sum_{k=1}^{K-1} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} + \Delta^{(K)}_{\boldsymbol{J}^{(K)}\setminus\{n\}}$$

which is a Pareto face of

$$\Pi^{(-n)} \;:=\; \Pi_{|\mathbb{R}_{\boldsymbol{I}\setminus\{n\}}} \,.$$

The correspondence is obviously bijective. Therefore we obtain

$$
\begin{aligned}
f(K,n) \;=\; |\mathcal{F}| \;&=\; |\mathcal{F}^{[-K]}| + |\mathcal{F}^{\star}| \\
&=\; |\{\text{Pareto faces of } \Pi^{[-K]}\}| + |\{\text{Pareto faces of } \Pi^{(-n)}| \\
&=\; f(K-1,n) + f(K,n-1)\ .
\end{aligned} \tag{9}
$$

The result holds true generally because the function $f$ is universal, hence the computation for the Test Cephoid suffices.

**q.e.d.**

Inspect Figures 2.1 and 2.2 once again. In both cases, one can nicely see, that the number of Pareto faces involving $K$ (i.r. "red") is exactly the number of Pareto faces for $n = 2$, i.e., the number of line segments with the same number of DeGua Simplices involved. Also, the remaining Pareto faces are exactly those in 3 dimensions but with $K-1$ (not "red", that is) DeGua Simplices involved.

Next, we compute the function $f$ in a closed form as follows.

**Theorem 2.5.** *The number of faces of an (n.d.) Cephoid given as a sum of $K$ deGua Simplices in dimension $n$ is*

$$f(K,n) \;=\; \binom{n+K-1}{n} \;-\; \binom{n+K-2}{n}. \tag{10}$$

**Proof:** Observe that generally for natural numbers $k,n$ one has

$$\binom{n}{k} = \binom{n-1}{k} \;+\; \binom{n-1}{k-1}\ .$$

Therefore, the difference equation:

$$
\begin{aligned}
f(K,2) \;&=\; K\ (K\geq 2)\quad ,\quad f(2,n) = n\ \ (n\geq 2) \\
f(K,n) \;&=\; f(K-1,n) + f(K,n-1)\ \ (K,n\geq 3)
\end{aligned} \tag{11}
$$

yields the function as indicated because of

$$
\begin{aligned}
f(K-1,n) + f(K,n-1) \;=\; &\binom{n+K-2}{n} - \binom{n+K-3}{n} \\
&+\binom{n+K-2}{n-1} - \binom{n+K-3}{n-1} \\
&= f(K,n),
\end{aligned}
$$

**q.e.d.**

# 3 Computating the Pareto faces

We continue by pointing out an algorithm that provides the Pareto faces of a Cephoid recursively. We consider families of positive vectors, always assumed to be n.d..

$$\boldsymbol{a}^{\bullet} := \left(\boldsymbol{a}^{(k)}\right)_{\kappa \in \boldsymbol{K}} \tag{1}$$

Let $\mathcal{P}(\bullet)$ denote the operation of the power set to a set. Consider the mapping

(2)

$$\mathcal{F}(K, n; \star) \ : \ \left\{\boldsymbol{a}^{\bullet} := \left(\boldsymbol{a}^{(k)}\right)_{\kappa \in \boldsymbol{K}} \,\middle|\, \boldsymbol{a}^{\bullet} \text{ is a positive n.d. family}\right\} \to \mathcal{P}\left((\mathcal{P}(\boldsymbol{I}))^{\boldsymbol{K}}\right)$$

$$\boldsymbol{a}^{\bullet} \to \left\{\mathcal{J} = (\boldsymbol{J}^{(1)}, \ldots, \boldsymbol{J}^{(K)}) \,\middle|\, \mathcal{J} \text{ is the reference system of a Pareto face of } \boldsymbol{a}^{\bullet}\right\}$$

which associates with a set of positive vectors in $\mathbb{R}^n_+$ the finite set of reference systems corresponding to the Pareto faces of $\Pi$. According to Theorem 2.5 we know that $|\mathcal{F}(K, n; \boldsymbol{a}^{\bullet})| = f(K, n)$ can be recursively computed, independently of $\boldsymbol{a}^{\bullet}$. We now indicate that a recursive computation can as well be obtained for the function $\mathcal{F}$ – of course depending on the particular family $\boldsymbol{a}^{\bullet}$.

We start with $n = 2$. In this case the Pareto surface consists of line segments Let $2 \leq K \in \mathbb{N}$ and let $\boldsymbol{a}^{\bullet} = \left\{\boldsymbol{a}^{(k)}\right\}_{k \in \boldsymbol{K}}$ be a family. Assume that the slopes of line segments $\frac{a_1^{(k)}}{a_2^{(k)}}$ are strictly decreasing in $k$. Then the Pareto faces of $\Pi$ are given by

$${}^2\boldsymbol{F}^{(k)} := \sum_{l<k} a^{(l)1} + \Delta^{\boldsymbol{a}^{(k)}} + \sum_{l>k} a^{(l)2} \ . \tag{3}$$

The corresponding reference sets are given by

$${}^2\mathcal{J}^{(k)} := \left\{ \begin{array}{rcccrcr} \boldsymbol{J}^{(1)} & = & \ldots & = & \boldsymbol{J}^{(k-1)}, & = & \{1\}, \\ & & & & \boldsymbol{J}^{(k)} & = & \{1,2\}, \\ \boldsymbol{J}^{(k+1)} & = & \ldots & = & \boldsymbol{J}^{(K)} & = & \{2\}. \end{array} \right\} \tag{4}$$

This way, all faces are completely described. For short – this way a Pareto face will appear in A Programming Language – we write ${}^2\boldsymbol{F}^{(k)}$

$${}^2\boldsymbol{F}^{(k)} := \begin{array}{ccccccc} 1 & 1 & \ldots & 12 & \ldots & 1 & 1 \\ 1 & 2 & \ldots & k & \ldots & (K-1) & K \end{array} \tag{5}$$

The actual shape of this quantity depends on the Programming Language – in APL it is just a $K$–vector.

Thus, for $n = 2$, we have a simple algorithm to describe all faces, a method which works also as a first step in s recursive procedure .

For completeness, we formulate this as a

**Theorem 3.1 (Begin of recursion).** *Let $n = 2$. Then the Pareto faces are translated line segments. If the slopes of the line segments are strictly decreasing in $k$, then Formulae* (3) *and* (4) *or* (5) *yield a description of the Pareto faces, i.e.,*

$$\mathcal{F}(K,2;\boldsymbol{a}^{\bullet}) \;=\; \left\{ {}^{2}\mathfrak{J}^{(k)} \,\middle|\, k \in \boldsymbol{K} \right\} \tag{6}$$

*If the ordering is not the one according to $1, \ldots, K$, then one can apply an appropriate permutation.*

**Remark 3.2.** *For $K = 2$ we can compute all faces utilizing a duality argument. Clearly, one can also apply the discussion centering around Theorem 4.2 and in particular Remark 4.4 of* CHAPTER *3 but the results will be related in a natural way. As duality will be used within our development of an algorithm, we formulate the following theorem as part of the algorithmic treatment.*

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**Theorem 3.3 (Computing The Dual Cephoid).**
*Regard the family $\left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ as a $K \times n$ matrix. Then the dual family $\left\{\bar{\boldsymbol{a}}^{(i)}\right\}_{i\in\boldsymbol{I}}$ is the transpose of this matrix. The Pareto faces of the dual family are dual to the primal Pareto faces in the sense of Theorem 1.5 of* CHAPTER *4.*

**Remark 3.4.** *Let $K \geq n$. Let $\boldsymbol{F}$ be a Pareto face of $\Pi^{\boldsymbol{a}^{\bullet}}$ and let $\mathfrak{J}$ be the corresponding reference system. Then there is at least one $k_0 \in \boldsymbol{K}$ such that $\boldsymbol{J}^{(k_0)}$ is a singelton, i.e. for some $i_0 \in \boldsymbol{I}$ we have*

$$\boldsymbol{J}^{(k)} \;=\; \left\{ \boldsymbol{a}^{(k)i_0} \right\}. \tag{7}$$

*(with $\boldsymbol{a}^{(k)i_0} \;=\; a_{i_0}^{(k_0)} \boldsymbol{e}^{i_0}$ ).*

*This follows from the Reference Theorem 1.2, see also Theorem 4.8 of* CHAPTER 3*; also consult formula* (8) *in* CHAPTER *3 which shows that the dimension of a Pareto* *face (i.e. $n-1$) will be exceeded if each $\boldsymbol{J}^{(k)}$ in the reference set has a size $j_k$ exceeding* 2.

*Consequently, there is a Pareto face $\boldsymbol{F}^{[-k_0]}$ of*

$$\Pi^{[-k_0]} := \sum_{k\in\boldsymbol{K}\setminus\{k_0\}} \Pi^{\boldsymbol{a}^{(k)}}$$

*such that*

$$\boldsymbol{F} \;=\; \boldsymbol{F}^{[-k_0]} + \boldsymbol{a}^{(k_0)i_0}$$

*holds true.*

*The coordinate $i_0$ is found following the argument e.g. given in Theorem 4.8 in* Chapter 3*: The normals to $\Delta^{(k_0)}$ at all vertices $\boldsymbol{a}^{(k_0)i}$ $(i\in\boldsymbol{I})$ of $\Delta^{(k_0)}$ span $\mathbb{R}^n_+$. Therefore, the normal to $\boldsymbol{F}^{[-k_0]}$ belongs to one (and only one) of the normal cones of these vertices.*

*The vector $\boldsymbol{a}^{(k_0)i_0}$ such that the normal computed above is located within the normal cone to $\Delta^{(K)}$ at $\boldsymbol{a}^{(K)i_0}$. However, as the mapping $\boldsymbol{x}\mapsto \mathfrak{n}\boldsymbol{x}$ attaines its maximum on $\boldsymbol{F}^{[-k]}$ and $\boldsymbol{F}$ has the same normal $\mathfrak{n}$, it is clear that $i_0$ is determined as the maximizer of $\{\boldsymbol{a}^{(K)i} \,\big|\, i\in\boldsymbol{I}\}$, that is*

$$\mathfrak{n}\boldsymbol{a}^{(K)i_0} \;=\; \max_{i\in\boldsymbol{I}} \mathfrak{n}\boldsymbol{a}^{(K)i} \tag{8}$$

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In order to cast this into an algorithm, we formulate this again within three theorems:

**Theorem 3.5** (**Computing the normal**)**.** *Computing the normal of a Pareto face $\boldsymbol{F}$ is done by Theorem 1.4 of* Chapter *3, that is, given the adjustment set $\boldsymbol{L}$ of $\boldsymbol{F}$ one has to solve the linear adjustment system* (18) *and then compute the normal according to Formula* (21) *in the Coincidence Theorem 1.4 of* Chapter *3.*

**Theorem 3.6** (**Translating a Pareto face of K − 1** )**.** *Let $k_0\in\boldsymbol{K}$ and let $\boldsymbol{F}^{[-k_0]}$ be a Pareto face of $\Pi^{[-k_0]}$. Let $\mathfrak{n}^{[-k_0]}$ be the normal to this face.*

*Choose $i_0\in\boldsymbol{I}$ to be the maximizer of $\mathfrak{n}^{[-k_0]}_i a^{(k_0)}_i$, that is*

$$\mathfrak{n}^{[-k_0]}_{i_0} a^{(k_0)}_{i_0} \;=\; \max_{i\in\boldsymbol{I}} \mathfrak{n}^{[-k_0]}_{i_0} a^{(k_0)}_{i_0} \tag{9}$$

*then*

$$\boldsymbol{F}^{[-k_0]} + \boldsymbol{a}^{(k_0)i_0} \;=\; \boldsymbol{F}^{[-k_0]} + \Delta^{(k_0)}_{i_0}$$

*is a Pareto face of $\Pi$.*

**Theorem 3.7** (**The Case K ≥ n**). *Assume $K \geq n$ .*

1. *Compute all faces $\boldsymbol{F}^{[-k_0]}$ of $\Pi^{[-k_0]}$ $(k_0 \in \boldsymbol{K})$ according to Recursion for $K-1$ deGua Simplices.*

2. *For each such face $\boldsymbol{F}^{[-k_0]}$ compute the normal $\mathfrak{n}^{[-k_0]}$ according to Theorem 3.5*

3. *For each $k_0 \in \boldsymbol{K}$ choose $i_0 \in \boldsymbol{I}$ according to Theorem 3.6.*

$$\mathfrak{n}_{i_0}^{[-k_0]} a_{i_0}^{(k_0)} \;=\; \max_{i \in \boldsymbol{I}} \mathfrak{n}_i^{[-k_0]} a_i^{(k_0)} \tag{10}$$

4. *For each $k_0 \in \boldsymbol{K}$, list the Pareto face*

$$\boldsymbol{F}^{[-k_0]} + \boldsymbol{a}^{(k_0)i_0} \;=\; \boldsymbol{F}^{[-k_0]} + \Delta_{i_0}^{(k_0)}$$

*in $\mathcal{F}(K, n; \boldsymbol{a}^{\bullet})$.*

**Remark 3.8.** *The procedure explained in Theorem 3.7 is very time and space consuming, thus slows down the computation considerably. Preferably, one restricts the computation to the case $K = n$ and computes all cases $K > n$ by using the dual Cephoid. By Theorem 3.3, this involves just transposing the matrix which represents the family $\{\boldsymbol{a}^{(k)}\}_{k \in \boldsymbol{K}}$.*

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**Remark 3.9.** *Finally we consider the case $K \leq n-1$. Now any Pareto face $\boldsymbol{F}$*

$$\boldsymbol{F} \;=\; \sum_{k=1}^{K} \Delta_{\boldsymbol{J}^{(k)}}^{(k)} \tag{11}$$

*has an adjustment set $\boldsymbol{L}$ with $|\boldsymbol{L}| \leq K-1 \leq n-2$. Now consider the Cephoid $\Pi_{\boldsymbol{L}} := \sum_{j \in \boldsymbol{K}} \boldsymbol{a}_{\boldsymbol{L}}^{(k)}$ generated in $\mathbb{R}_{\boldsymbol{L}}^n$ by the restricted family $\left\{\boldsymbol{a}_{\boldsymbol{L}}^{(k)}\right\}_{k \in \boldsymbol{K}}$. Then*

$$\Pi_{\boldsymbol{L}} \;=\; \sum_{j \in \boldsymbol{K}} \Pi^{(k)} \cap \mathbb{R}_{\boldsymbol{L}}^n$$

*features the Pareto face*

$$\boldsymbol{F}_{\boldsymbol{L}} \;=\; \sum_{k=1}^{K} \Delta_{\boldsymbol{J}^{(k)} \cap \boldsymbol{L}}^{(k)} \ .$$

*The adjustment set* $\boldsymbol{L}$ *generates the normal* $\mathfrak{n}^{[\boldsymbol{L}]}$ *(to* $\boldsymbol{F}$ *as well), so we can construct the Pareto face* $\boldsymbol{F}$ *my choosing, for each* $k \in \boldsymbol{K}$*, the maximizers of the mapping* $\boldsymbol{x} \mapsto \mathfrak{n}^{[\boldsymbol{L}]}\boldsymbol{x}$ *over* $\Delta^{(k)}$*, i.e., those* $i_{\in}\boldsymbol{I}$ *which satisfy*

$$\mathfrak{n}^{[\boldsymbol{L}]}_{i_0} a^{(k)}_{i_0} = \max_{i \in \boldsymbol{I}} \mathfrak{n}^{[\boldsymbol{L}]}_{i} a^{(k_0)}_{i}, \tag{12}$$

*that is,*

$$\boldsymbol{J}^{(k)} = \{ i_0 \in \boldsymbol{I} \mid i_0 \text{ satisfies } (12) \} \tag{13}$$

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Combining we obtain:

**Theorem 3.10** (**The Case K < n**)**.** *Assume* $k < n$*. For any* $i_1, i_2 \in \boldsymbol{I}$ *let* $\boldsymbol{I}^1 := \boldsymbol{I} \setminus \{i_1, i_2\}$ *and let* $\Pi_{\boldsymbol{I}^1}$ *be* $(n-2)$*– dimensional boundary Cephoid generated by the family* $\left\{ \boldsymbol{a}^{(k)}_{\boldsymbol{I}^1} \right\}_{k \in \boldsymbol{K}}$*. Let* $\boldsymbol{F}^1$ *be a Pareto face of* $\Pi_{\boldsymbol{I}^1}$ *and let* $\boldsymbol{L}$ *be the adjustment system. Compute the normal* $\mathfrak{n}^{[\boldsymbol{L}]}$ *of this face according to Theorem 3.5. Then the* $\boldsymbol{F}$ *as defined via* (12) *and* (13) *is a Pareto face of* $\Pi$*. All Pareto faces of* $\Pi$ *are being obtained by running through all pairs* $i_1, i_2$ *(but some may appear multiply).*

# Chapter 6

# The Reference Vector

A Pareto face $\boldsymbol{F}$ of a Cephoid corresponds uniquely to the family of Subsimplices of the various DeGua Simplices involved. This family is the reference system $\mathscr{J} = \left\{\boldsymbol{J}^{(k)}\right\}_{k \in \boldsymbol{K}}$. The reference vector is the list of cardinalities of those Subsimplices; essentially it indicates their dimensions. Given the recursive structure we have now available, we shall prove that the correspondence between reference vectors and Pareto faces is bijective.

# 1 The Reference Vector

**Definition 1.1.** *Let* $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ *be a Cephoid and let*

$$\boldsymbol{F} \;=\; \sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$$

*be a Pareto face of* $\Pi$. *We write* $r_k \;:=\; |\boldsymbol{J}^{(k)}| \quad (k \in \boldsymbol{K})$ *and call* $\boldsymbol{r} = (r_1, \ldots, r_K)$ *the* ***reference vector*** *of* $\boldsymbol{F}$.

We shall show that, for any Cephoid $\Pi$, the correspondence between reference vectors and Pareto faces is a bijective mapping. To this end, we tentatively introduce the notion of a "reference code" as a vector which *a priori* does not stem from a face of a Cephoid.

**Definition 1.2.** *Let* $n, K \in \mathbb{N}$. *A positive vector* $\boldsymbol{r} \;=\; (r_1, \ldots, r_K) \in \mathbb{N}^K$ *is said to be a* $(K, n)$*–**reference code** if*

$$1 \leq r_k \leq n \quad (k \in \boldsymbol{K}) \tag{1}$$

*and*

$$\sum_{\kappa=1}^{K} r_k \;\leq\; K + n - 1 \tag{2}$$

*holds. A reference code* $\boldsymbol{r}$ *is* ***maximal*** *if an equation prevails in* (2).

Thus, a reference vector of a Pareto face of a Cephoid is a maximal reference code.

**Theorem 1.3 (The Bijection Theorem).** *Let* $\boldsymbol{a}^\bullet \;=\; \left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ *be a nondegenerate family of positive vectors in* $\mathbb{R}^n$. *Let* $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{(k)}$ *be the Cephoid generated by* $\boldsymbol{a}^\bullet$. *Then, for every maximal* $(K, n)$*–reference code* $\boldsymbol{r}$, *there exists uniquely a Pareto face* $\boldsymbol{F}$ *of* $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{(k)}$ *with reference system*

$$\left\{\boldsymbol{J}^{(k)}\right\}_{k\in\boldsymbol{K}} \tag{3}$$

*such that*

$$|\boldsymbol{J}^{(k)}| = r_k \quad (k \in \boldsymbol{K}) \;, \tag{4}$$

*i.e.,* $\boldsymbol{r}$ *is the reference vector of* $\boldsymbol{F}$.

**Proof: 1[st]STEP** : For $n = 2$ the Theorem is obvious. For $K = 2$ the Theorem follows from Theorem 4.2 of CHAPTER 3 – or else by duality.

Now we proceed by induction.

**2[nd]STEP** :

First of all, assume $K \leq n-1$. Let $\boldsymbol{F}$ be a Pareto face and let

$$\boldsymbol{F} \;=\; \sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$$

be the representation via the reference system. The reference vector is denoted by $\boldsymbol{r} = (r_1, \ldots, r_l, \ldots, r_K)$. Also, let $\boldsymbol{L}$ be the set of adjustment indices. As $L = |\boldsymbol{L}| \leq n-2$, there are at least two indices, say 1 and $n$, that do not belong to $\boldsymbol{L}$. As a consequence $\boldsymbol{F}|_{\mathbb{R}_{\boldsymbol{I}-1}}$ and $\boldsymbol{F}|_{\mathbb{R}_{\boldsymbol{I}-n}}$ are Pareto faces of $\Pi^{(-1)}$ and $\Pi^{(-n)}$ as generated by the families $\boldsymbol{a}^{\bullet}|_{\mathbb{R}_{\boldsymbol{I}-1}}$ and $\boldsymbol{a}^{\bullet}|_{\mathbb{R}_{\boldsymbol{I}-n}}$ respectively. The reference vectors are $(r_1, \ldots, r_\kappa - 1, \ldots, r_K)$ and $(r_1, \ldots, r_l - 1, \ldots, r_K)$ with suitable $\kappa, l \in \boldsymbol{K}$ such that $r_\kappa, r_l \geq 2$. By induction, these reference vectors uniquely determine the reference systems

$$\left\{\boldsymbol{J}^{(k)} \setminus \{1\}\right\}_{k \in \boldsymbol{K}} \quad , \quad \left\{\boldsymbol{J}^{(k)} \setminus \{n\}\right\}_{k \in \boldsymbol{K}}$$

of two Pareto faces of $\Pi^{(-1)}$ and $\Pi^{(-n)}$ respectively. Hence the reference system

$$\left\{\boldsymbol{J}^{(k)}\right\}_{k \in \boldsymbol{K}}$$

of $\boldsymbol{F}$ is uniquely determined by $\boldsymbol{r}$. This shows, that there is at most one face corresponding to a reference code, provided $K \leq n-1$ holds true.

But for $K \geq n$ we know that every maximal face is the sum of at most $n-1$ subfaces of the $\Delta^{(k)}$ plus a number of vertices from the remaining ones. By the above argument, with respect to the $n-1$ faces that yield reference sets of size at least 2, these reference sets are uniquely defined. The remaining vertices, however, are uniquely defined as well.

Thus, a reference code defines a Pareto face uniquely, if at all.

**3[rd]STEP** :

On the other hand, given a family $\boldsymbol{a}^{\bullet}$ and the cephoid $\Pi$ generated, let $\mathcal{F}(K, n)$ be the set of Pareto faces of $\Pi$. ( $\mathcal{F}$ is the "listing function" of the Pareto faces). Let $\Pi|_{-n}$ be the cephoid generated by the family $\boldsymbol{a}|_{\mathbb{R}_{\boldsymbol{I}-n}}$ of vectors projected onto $\mathbb{R}_{\boldsymbol{I}-n}$ and let $\mathcal{F}(K, n-1)$ denote the family of its Pareto faces.

Similarly, let $\Pi^{(-K)} = \sum_{k=1}^{K-1} \Pi^{(k)}$ be the sum of the first $K-1$ deGua Simplices and let $\mathcal{F}(K-1,n)$ denote the system of Pareto faces of this cephoid. The induction hypothesis applies to both cephoids constructed.

Now, let $\boldsymbol{F} \in \mathcal{F}(K,n)$ and let $\boldsymbol{r}$ be its reference vector. First of all, assume that $r_K = 1$ is the case, that is, $\boldsymbol{F}$ consists of a Pareto face $\boldsymbol{F}^{(-K)}$ of $\Pi^{(-K)}$ plus a vertex of $\Delta^{(K)}$. By induction, the Pareto face $\boldsymbol{F}^{(-K)}$ is uniquely defined by $(r_1, \ldots, r_{K-1})$ and the remaining vertex of$\Delta^{(K)}$ is uniquely defined as well. On the other hand, every Pareto face $\Pi^{(-K)}$ together with a suitable unique vertex of $\Delta^{(K)}$ yields a face in $\mathcal{F}(K-1,n)$. Thus, $\mathcal{F}(K-1,n)$ and $\left\{ \boldsymbol{F} \in \mathcal{F}(K,n) \,\middle|\, r_K = |\boldsymbol{J}^{(K)}| = 1 \right\}$ are bijectively mapped into each other in a canonical way.

**4<sup>th</sup>STEP** :

Next, let $\boldsymbol{F} \in \mathcal{F}(K,n)$ be such that $r_K = |\boldsymbol{J}^{(K)}| \geq 2$ is true. By induction, there is a unique Pareto face, say $\boldsymbol{F}^{\star} \in \mathcal{F}(K, n-1)$ of $\Pi|_{\mathbb{R}^{\boldsymbol{I}-n}}$ that corresponds to the reference vector $(r_1, \ldots, r_K - 1)$. By the first step, we conclude that

$$\left| \left\{ \mathcal{F}(K,n) \,\middle|\, |\boldsymbol{J}^{(K)}| \geq 2 \right\} \right| \leq |\mathcal{F}(K, n-1)| \tag{5}$$

holds. But by Theorem 2.4 of Section 2 Chapter 5 we know that $|\mathcal{F}(K,n)| = f(k,n)$ satisfies

$$|\,\mathcal{F}(K,n)\,| \;=\; |\,\mathcal{F}(K-1,n)\,| + |\;\mathcal{F}(K,n-1).\,|$$

Hence, equation prevails in formula (5) and hence there is indeed for every maximal code $\boldsymbol{r}$ a Pareto face that has $\boldsymbol{r}$ as its reference vector.

**q.e.d.**

**Corollary 1.4.** *Let $\boldsymbol{a}^{\bullet} = \left\{ \boldsymbol{a}^{(k)} \right\}_{k \in \boldsymbol{K}}$ be an (n.d.) family of positive vectors in $\mathbb{R}^n$ and let $\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{(k)}$ be the Cephoid generated*

1. *Then, for every $k \in \boldsymbol{K}$ and every $i \in \boldsymbol{I}$ there is a bijection $\mathcal{P}^{(i)}$ which maps*

$$\left\{ \boldsymbol{F} \in \mathcal{F}(K,n) \,\middle|\, |\boldsymbol{J}^{(k)}| = 2 \right\}$$

*on*

$$\mathcal{F}^{(-i)}(K, n-1) \;:=\; \left\{ \boldsymbol{F} \,\middle|\, \boldsymbol{F} \text{ is a Pareto face of } \boldsymbol{a}^{\bullet}|_{\mathbb{R}_{\boldsymbol{I}\setminus\{-i\}}} \right\}$$

*This bijection is obtained by associating with any Pareto face $\boldsymbol{F}$ with reference code $\boldsymbol{r}$, $r_k \geq 2$, the Pareto face on $\partial\Pi^{(-i)} := \partial\Pi_{\,|\,\mathbb{R}^{\boldsymbol{I}\setminus\{i\}}}$ defined via $\boldsymbol{r} - \boldsymbol{e}^k$.*

2. *There is a bijection of* $\mathcal{F} := \{\boldsymbol{F} | \boldsymbol{F}$ *is a Pareto face of* $\Pi\}$ *onto the set of maximal* $(K, n)$*–reference codes, i.e., the set of vectors satisfying*

$$1 \leq r_k \leq n \quad (k \in \boldsymbol{K}) \tag{6}$$

*and*

$$\sum_{\kappa=1}^{K} r_k \;=\; K + n - 1 \tag{7}$$

We now drop the notion of a "reference code" and henceforth refer to any vector of natural number satisfying (6) and (7) as to a $(K, n)$–reference vector or just a reference vector if the context is obvious. Note that a $(K, n)$–reference vector can also be seen as a $(K - 1, n + 1$–reference vector or as a $(K + 1, n - 1)$–reference vector.

## 2 An Algorithm via Reference Vectors

The Bijection Theorem 1.3 in a straightforward way induces a further algorithmic procedure for computing the Pareto faces of a Cephoid. The idea is to run through a list of the reference vectors and, to each reference vector, compute the corresponding face.

Again the procedure is recursive. It is based on the idea that, for $K \geq n$, a reference vector has at least one coordinate that equals 1. We sketch the argument as it essentially consists of a reformulation of known facts.

**Lemma 2.1.** *For $K \geq n$ any $(K, n)$–reference vector has at least one coordoinate equal to* 1*.*

*This is an immediate Consequence of the corresponding theorem for Pareto faces, see e.g. Remark 3.4 and references to the Coincidence Theorem and the Reference theorem. Of course, it is much easier to directly deduce this from the definition of a reference vector, say from* (7)*.*

*Indeed, if a reference vector satisfies $r_k \geq 2$ holds true for all $k \in \boldsymbol{K}$, then we have*

$$n + K - 1 = \sum_{k \in \boldsymbol{K}} r_k \geq 2K,$$

*that is, $n - 1 \geq K$.*

Let us denote the set of reference vectors by $\mathcal{R}(K, n)$. Recall the function $f(\bullet, \bullet)$ as defined in Theorem 1.4 and specified in Theorem (2.4) of CHAPTER 5. $f$ denotes the number of faces of a Cephoid – universally, i.e., independent of the generating family. Then we have

**Theorem 2.2.** *For $(K, n) \in \mathbb{N} \times \mathbb{N}$ we have*

(1) $\mathcal{R}(K, n) = \{(\boldsymbol{r}^\star, 1) \mid \boldsymbol{r}^\star \in \mathcal{R}(K-1, n)\} \cup \left\{\widehat{\boldsymbol{r}} + \boldsymbol{e}^K \;\middle|\; \widehat{\boldsymbol{r}} \in \mathcal{R}(K, n-1)\right\}$ .

*Moreover,*

$$|\mathcal{R}(K, n)| = f(K, n) \qquad (2)$$

**Proof:** Easy: because of

$$\mathcal{R}(K, n) = \{\boldsymbol{r} \mid \boldsymbol{r} \in \mathcal{R}(K, n) \mid r_K = 1\} \cup \{\boldsymbol{r} \mid \boldsymbol{r} \in \mathcal{R}(K, n) \mid r_K \geq 2\}$$

the first statement is seen immediately. The second follows at once as the recursive equations (8), SECTION 2 CHAPTER 5 defining $f$ obviously holds true for the number of reference vectors as well.

**q.e.d.**

**Remark 2.3.** *The recursive structure for the reference vectors is a much easier matter than the one for Pareto faces and their numbers. As we have seen, Theorem 2.2 can be proved without mentioning the results of* CHAPTER *5, it is solely a statement referring to certain vectors in $\mathbb{R}^K$. Moreover, for the reference vectors, this theorem provides immediate access to a recursive construction as formula* (1) *defines the set $\mathcal{R}(K,n)$ in terms of the two sets of a lower level of the recursion. This fact we use to construct a recursive procedure for the construction of Pareto faces as well.*

*As for the concrete description of the set of reference vectors, it is useful to recall that – say in $\mathbb{APL}$ – the set $\mathcal{R}(K,n)$ ought to be represented as a matrix. For the sake of clarity call this matrix $\mathbf{R}(K,n)$; the* ***reference matrix****. Formally we* define *that matrix recursively as follows.*

∘ ∼∼∼∼∼∼∼ ∘

We use $\mathbb{N}_1 := \mathbb{N} \setminus \{1\}$.

**Definition 2.4.** *Let $(K,n) \in \mathbb{N}_1 \times \mathbb{N}_1$. Define $\mathbf{R}(K,n)$ recursively as follows:*

*For $n = 2$:*

$$(3) \qquad \mathbf{R}(K,2) \;=\; \begin{pmatrix} 2 & 1 & 1 & \dots & 1 & 1 \\ 1 & 2 & 1 & \dots & 1 & 1 \\ 1 & 1 & 2 & \dots & 1 & 1 \\ & & & \dots & & \\ 1 & 1 & 1 & \dots & 2 & 1 \\ 1 & 1 & 1 & \dots & 1 & 2 \end{pmatrix} \in \mathbb{R}^K \times \mathbb{R}^K \quad (K \in \mathbb{N}_1),$$

*For $K = 2$:*

$$(4) \qquad \mathbf{R}(2,n) \;=\; \begin{pmatrix} K & 1 \\ K-1 & 2 \\ K-2 & 3 \\ & \dots \\ 2 & K-1 \\ 1 & K \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}^2 \quad (K \in \mathbb{N}_1),$$

*For* $K, n \geq 3$*:*

$$
\begin{aligned}
\mathbf{R}(K,n) &= \begin{pmatrix} \begin{pmatrix} \mathbf{R}(K-1,n) & \begin{matrix}1\\1\\ \cdot\\ \cdot\\ \cdot\\ 1\end{matrix}\end{pmatrix} \\ \mathbf{R}(K,n-1) + \begin{pmatrix}1\\1\\ \cdot\\ \cdot\\ \cdot\\ 1\end{pmatrix}\end{pmatrix} \\
&= \begin{pmatrix}
r_1^\star & ,\dots, & r^\star_{(K-1)}\ , & 1 \\
 & \cdots & & \\
 & \cdots & & \\
 & \cdots & & \\
r_1^{\star\prime} & ,\dots, & r^{\star\prime}_{(K-1)}\ , & 1 \\
\widehat{r}_1 & ,\dots & \dots\dots & \widehat{r}_K + 1 \\
 & \cdots & \dots\dots & \\
 & \cdots & \dots\dots & \\
 & \cdots & \dots\dots & \\
\widehat{r}_1^{\prime} & ,\dots & \dots\dots & \widehat{r}_K^{\prime} + 1
\end{pmatrix}
\end{aligned}
\tag{5}
$$

Within our algorithmic approach, we consider the case $K \geq n$ and treat the alternative $K < n$ via an application of the Duality Theorem as in the algorithm of Section 3 of Chapter 5.

Now, given some family $\left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ and the resulting Cephoid $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$, consider the case that $r_K = 1$. This corresponds to all vectors in the upper half of the matrix $\mathbf{R}(K,n)$.

Then the vector $(r_1, \dots, r_{(K-1)})$ is a reference vector for the cephoid

$$\Pi^{[-K]} := \sum_{k\in\boldsymbol{K}\setminus\{K\}} \Delta^{(k)} \; .$$

Therefore, we can obtain the Pareto face $\boldsymbol{F}^{[-K]}$ of $\Pi^{[-K]}$ corresponding to this reference vector by a recursive procedure as $\Pi^{[-K]}$ is a sum of $(K-1)$ deGua simplices in $\mathbb{R}^n$. It remains to construct the Pareto face of $\Pi$ that results from adding a (unique) vertex $\boldsymbol{a}^{(K)i_0}$ of $\Delta^{(K)}$ to $\boldsymbol{F}^{[-K]}$. This vertex is at once computed according to Theorem 3.6 of Chapter 5.

Thus, adding $\boldsymbol{a}^{(K)i_0}$ to $\boldsymbol{F}^{[-(K-1)]}$ yields a Pareto face of $\Pi$ which has the correct reference vector $\boldsymbol{r}$.

Next consider all vectors $\boldsymbol{r}$ wit $r_{(K-1)} = 1$ that have *not been covered so far*, i.e., those in the lower half of the matrix $\mathbf{R}(K, n)$. In order to obtain the corresponding Pareto faces, we first compute the Pareto faces of

$$\Pi^{[-(K-1)]} \ := \sum_{k \in \boldsymbol{K} \setminus \{(K-1)\}} \Delta^{(k)} \ .$$

Then we list the reference vectors (of length $(K-1)$) to these Pareto faces. Eliminate those that correspond to reference vectors that *have already been covered*, i.e., take only those faces that correspond to reference vectors that have not been covered in the first step – apart from the 1 at coordinate $(K-1)$.

Now proceed as above: Construct the Pareto face of $\Pi$ that results from adding a (unique) vertex $\boldsymbol{a}^{(K-1)i_1}$ of $\Delta^{(K-1)}$ to $\boldsymbol{F}^{[-(K-1)]}$. This vertex is at once computed according to Theorem 3.6 of Chapter 5. Adding $\boldsymbol{a}^{(K-1)i_1}$ to $\boldsymbol{F}^{[-(K-1)]}$ yields a Pareto face of $\Pi$ which has the correct reference vector $\boldsymbol{r}$.

Next, proceed to all reference vectors with coordinate $r_{(K-2)} = 1$ that have not been covered so far etc.

Combining these considerations within the following Theorem we obtain an algorithm for computing all Pareto faces of a Cephoid via the reference vectors.

**Theorem 2.5.** *Let* $\boldsymbol{a}^{\bullet} \left\{\boldsymbol{a}^{(k)}\right\}_{k \in \boldsymbol{K}}$ *be a family of positive vectors and let* $\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ *be the Cephoid generated. The set* $\mathcal{F}(K, n; \boldsymbol{a}^{\bullet})$ *of Pareto of* $\Pi$ *is obtained as follows.*

**1$^{\text{st}}$STEP** :

*If* $n = 2$ *or* $K = 2$*, then apply the Theorem* **"Beginn of recursion"***,i.e., Theorem 3.1 of* Chapter *5.*

**2$^{\text{nd}}$STEP** : *If* $K < n$*, then apply the dualization process twice as described in the Theorem* **"Computing the Dual"** *, i.e., 3.3 of* Chapter *5.*

**4$^{\text{th}}$STEP** : *Assume* $K \geq n$*. Compute the reference matrix* $\mathbf{R}(K, n)$*. For* $k_0 \in \boldsymbol{K}$ *take the subset* $\boldsymbol{K} \setminus \{k_0\}$ *with* $k-1$ *elements.*

**5$^{\text{th}}$STEP** :

*Beginn the recursion ("backwards") with* $k_0 = K$ *and proceed until* $k_0 = 1$*. Compute all faces of* $\Pi^{[-K]}$ *corresponding to the family* $\left\{\boldsymbol{a}^{(k)}_{k \in \boldsymbol{K} - \{K\}}\right\}$.

*To any such face* $\boldsymbol{F}^{[-K]}$*, compute the normal* $\mathfrak{n}^{[-K]}$ *by the Theorem* **"Computing the Normal"***, i.e., by Theorem 3.5 of* CHAPTER *5.*

**6<sup></sup>**

**6thSTEP** : *Apply Theorem 3.6 of* CHAPTER *5 (***"Translating a Pareto face"** *). This way, find the unique vertex* $\boldsymbol{a}^{(K)i_0}$ *of* $\Delta^{(K)}$ *which admits* $\mathfrak{n}^{[-K]}$ *as a normal to* $\Delta^{(K)}$*. Then* $\boldsymbol{F}^{[-K]} + \boldsymbol{a}^{(K)i_0}$ *is a face of the family* $\Pi$*.*

**7thSTEP** :

*Eliminate all reference vectors from the matrix* $\mathbf{R}(K, n)$ *with coordinate* $r_K = 1$*, i.e., the first group in formula* $(5)$*. Call the resulting matrix* $\mathbf{R}^{[-K]}$*.*

**8thSTEP** : *Assume now the procedure has been performed for* $K, K-1, \ldots, k_0 + 1$*. The remaining matrix of reference vectors is* $\mathbf{R}^{[-(k_0+1)]}$*.*

**9thSTEP** :

*Compute all Pareto faces of the family* $\left\{\boldsymbol{a}^{(k)}_{k \in \boldsymbol{K}-\{k_0\}}\right\}$*. Compute the* $(K-1, n)$*–reference vectors of all these faces. Augment these vectors by a coordinate* $r_{k_0} = 1$ *to obtain* $(K, n)$ *reference vectors. Reduce the set of Pareto faces to those the reference vectors of which appear in* $\mathbf{R}^{[-(k_0+1)]}$*.*

**10thSTEP** :

*To any remaining face* $\boldsymbol{F}^{[-k_0]}$*, compute the normal* $\mathfrak{n}^{[-k_0]}$ *by the Theorem* **"Computing the Normal"***, i.e., by Theorem 3.5 of* CHAPTER *5.*

**11thSTEP** :

*Apply the Theorem* **"Translating a Pareto face"** *, i.e., Theorem 3.6 of* CHAPTER *5. That is, find the unique vertex* $\boldsymbol{a}^{(k_0)i_0}$ *of* $\Delta^{(k_0)}$ *which admits* $\mathfrak{n}^{[-k_0]}$ *as a normal to* $\Delta^{(k_0)}$*. Then* $\boldsymbol{F}^{[-k_0]} + \boldsymbol{a}^{(k_0)i_0}$ *is a Pareto face of* $\Pi$*.*

**12thSTEP** :

*Eliminate all reference vectors from the matrix* $\mathbf{R}^{[-(k_0+1)]}$ *with coordinate* $r_{k_0} = 1$*. Call the resulting matrix* $\mathbf{R}^{[-k_0]}$*. Proceed with the recursion as in the* **8thSTEP** :

# Chapter 7

# Graphs on the Pareto Surface

The partially ordered set (the poset) of Pareto faces - the Pareto surface - has extensively been discussed in Chapter 2. We have predominantly focused on the Pareto faces and their neighborhood structure. Now we want to exhibit an even more detailed picture of the Pareto surface.

To this end, we exhibit the graphical properties of reference sets. Also, we introduce the graph representing the poset, its nodes, and links as well as the relation to the reference vectors.

# 1 The Reference Graph

We have intensely discussed necessary conditions for Pareto faces. In particular properties of the reference system, of the adjustment set, and the way the corresponding linear adjustment system determines the normal.

Now we want to investigate *sufficient* conditions. We start out with a family $\boldsymbol{a}^{\bullet} = \left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ generating a Cephoid

$$\Pi \;=\; \Pi^{\boldsymbol{a}^{\bullet}} = \sum_{k\in\boldsymbol{K}} \Pi^{(k)}$$

with $\Pi^{(k)} \;=\; \Pi^{\boldsymbol{a}^{(k)}}(k \in \boldsymbol{K})$. Given a family of subsets of $\boldsymbol{I}$, say

$$\mathcal{J} = \left\{\boldsymbol{J}^{(k)}\right\}_{k\in\boldsymbol{K}},$$

we assign to every index set $\boldsymbol{J}^{(k)}$ the Subsimplex

$$\Delta^{(k)}_{\boldsymbol{J}^{(k)}} \;=\; \Delta^{(\boldsymbol{a}^{(k)})}_{\boldsymbol{J}^{(k)}} = \boldsymbol{CovH}\left(\left\{\boldsymbol{a}^{(k)l}\right\}_{l\in\boldsymbol{J}^{(k)}}\right) \subseteq \Delta^{\boldsymbol{a}^{(k)}} = \Delta^{(k)}.$$

Now we look for conditions that ensure these Subsimplices to be the summands of a Pareto face

$$\boldsymbol{F} = \sum_{k\in\boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \tag{1}$$

of the cephoid $\Pi$

To specify these conditions it is first of all necessary to exhibit some further properties of a reference system associated with a Pareto face. On the other hand, we want to treat some of those properties in a more general way. We start by a definition that lists the obvious requirements.

**Definition 1.1.** *Let*

$$\mathcal{J} = \left\{\boldsymbol{J}^{(k)}\right\}_{k\in\boldsymbol{K}}$$

*be a family of subsets of* $\boldsymbol{I}$. $\mathcal{J}$ *is called an* ***admissible system*** *if the following conditions are satisfied:*

1. $\displaystyle\bigcup_{k\in\boldsymbol{K}} \boldsymbol{J}^{(k)} \;=\; \boldsymbol{I}$

2. $\displaystyle\sum_{k\in\boldsymbol{K}} |\boldsymbol{J}^{(k)}| \;=\; \sum_{k\in\boldsymbol{K}} j_k \;=\; K + n - 1\;.$

3. *For any two different indices $k, l \in \boldsymbol{K}$ the sets $\boldsymbol{J}^{(k)}$ and $\boldsymbol{J}^{(l)}$ contain at most one common index.*

4. *For every index $k \in \boldsymbol{K}$ there exists an index $k' \in \boldsymbol{K}$ with $k \neq k'$ and* $\left|\left(\boldsymbol{J}^{(k)} \cap \boldsymbol{J}^{(k')}\right)\right| = 1.$

Thus, the reference system of a face of a cephoid is admissible.

For every admissible system $\mathcal{J}$ we denote by $\boldsymbol{L} \subseteq \boldsymbol{I}$ the set of indices that appear in at least two of the members $\boldsymbol{J}^{(k)}$ of the family. $\boldsymbol{L}$ is called the set of ***critical indices*** (corresponding to $\mathcal{J}$). Accordingly,

$$\boldsymbol{L}_k \; := \; \boldsymbol{L} \cap \boldsymbol{J}^{(k)} \tag{2}$$

defines the ***critical system***

$$\mathcal{L} = \left\{\boldsymbol{L}^{(k)}\right\}_{k \in \boldsymbol{K}} \quad . \tag{3}$$

The critical system obviously inherits the defining properties from its parental admissible system, i.e., we have:

1. $\displaystyle\bigcup_{k \in \boldsymbol{K}} \boldsymbol{L}^{(k)} \; = \; \boldsymbol{L}$

2. For any two different indices $k, k' \in \boldsymbol{K}$ the sets $\boldsymbol{L}^{(k)}$ and $\boldsymbol{L}^{(k')}$ contain at most one common index.

3. For every index $k \in \boldsymbol{K}$ there exists an index $k' \in \boldsymbol{K}$ with $k \neq k'$ and $\left|\left(\boldsymbol{L}^{(k)} \cap \boldsymbol{L}^{(k')}\right)\right| = 1.$

We use this a motivation to define abstract systems $\mathcal{L}$ with these properties.

**Definition 1.2.** *Let $\boldsymbol{L} \subseteq \boldsymbol{I}$ and let*

$$\mathcal{L} = \left(\boldsymbol{L}^{(k)}\right)_{k \in \boldsymbol{K}} \quad .$$

*be a system of subsets of $\boldsymbol{L}$. We say that $\mathcal{L}$ is $\boldsymbol{L}$–**admissible** if the conditions* $1., 2.,$ *and* $3.$ *are satisfied.*

Thus, the critical system of an admissible set is $\boldsymbol{L}$–admissible with respect to the set $\boldsymbol{L}$ of critical indices.

We wish to associate a graph to an admissible $\boldsymbol{L}$–system as follows.

**Definition 1.3.** *The (undirected)* ***graph associated*** *to an admissible* $\boldsymbol{L}$*–system* $\mathcal{L}$ *is the pair*

$$(\mathcal{L}, \mathcal{E}) \tag{4}$$

*given as follows. The* ***nodes*** *of the graph are the elements of the family* $\mathcal{L}$*. An* ***edge*** *or arc of the graph is a pair* $\boldsymbol{E} = (\boldsymbol{L}_k, \boldsymbol{L}_{k'})$ *such that* $\boldsymbol{L}_k \cap \boldsymbol{L}_{k'} \neq \emptyset$ *holds true. Colloquially we say that* $\boldsymbol{L}_k$ *and* $\boldsymbol{L}_{k'}$ *are connected if* $\boldsymbol{E} = (\boldsymbol{L}_k, \boldsymbol{L}_{k'})$ *is an edge.*

As graph as defined above may have cycles, i.e., in our case a sequence of nodes $\boldsymbol{L}^{(k_1)}, \boldsymbol{L}^{(k_2)}, \ldots, \boldsymbol{L}^{(k_T)}$ such that , for any $t \in \{1, \ldots, T-1\}$ the nodes $\boldsymbol{L}^{(k_t)}$ and $\boldsymbol{L}^{(k_{t+1})}$ are connected and $\boldsymbol{L}^{(k_1)} = \boldsymbol{L}^{(k_T)}$ is the case. We call a cycle ***proper*** if the same index $l \in \boldsymbol{L}$ is involved in each edge, i.e., if

$$\boldsymbol{L}^{(k_t)} \cap \boldsymbol{L}^{(k_{t+1})} = \{l\} \tag{5}$$

holds true for some $l \in \boldsymbol{L}$ and all $t \in \{1, \ldots, T-1\}$. Otherwise, we call the cycle ***improper***.

Now we are in the position to proceed with a refinement of our above definition.

**Definition 1.4.** *An* $\boldsymbol{L}$*–admissible family of index sets*

$$\mathcal{L} = \left\{ \boldsymbol{L}^{(k)} \right\}_{k \in \boldsymbol{K}}$$

*is called a* ***pre–adjustment system*** *if the following conditions are satisfied:*

1. $L := |\boldsymbol{L}| \leq K - 1$ *holds true.*
2. $\sum_{k \in \boldsymbol{K}} |\boldsymbol{L}^{(k)}| =: \sum_{k \in \boldsymbol{K}} L_k = K + L - 1$ .
3. *There are at least two indices* $k^*, k^\circ$ *such that* $|\boldsymbol{L}^{k^*}| = |\boldsymbol{L}^{k^\circ}| = 1$ *holds true. That is, the associated graph has at least two* ***boundary nodes****.*
4. *The associated graph* $(\mathcal{L}, \mathcal{E})$ *is connected.*
5. *The associated graph* $(\mathcal{L}, \mathcal{E})$ *has no improper cycles.*

*An admissible family of index sets*

$$\mathcal{J} = \left\{ \boldsymbol{J}^{(k)} \right\}_{k \in \boldsymbol{K}}$$

*is called a* ***pre–reference system*** *if the critical set* $\boldsymbol{L}$ *induces a critical system* $\mathcal{L}$ *that is a pre–adjustment system. The corresponding* ***linear pre–adjustment system*** *is the linear system of equations formed in analogy to* (18) *of* SECTION, *1* CHAPTER *3.*

A reference system resulting from a Pareto face has the properties listed above. Indeed, a reference system induces a set $\boldsymbol{L}$ of adjustment indices as well as an adjustment system which is $\boldsymbol{L}$–admissible. The associated graph is called the ***adjustment graph***. Now we have

**Lemma 1.5.** *Let* $\boldsymbol{F}$ *be a Pareto face of a cephoid* $\Pi = \Pi^{\boldsymbol{a}^{\bullet}}$. *Then the adjustment graph has no improper cycles.*

**Proof:** If the graph has an improper cycle, then the linear adjustment system admits of the trivial solution only. More precisely, let (w.l.o.g)

$$\boldsymbol{L}^{(1)}, \boldsymbol{L}^{(2)}, \ldots \boldsymbol{L}^{(\kappa)}, \boldsymbol{L}^{(1)}$$

constitute an improper cycle. Then we find indices $l_1, l_2, \ldots, l_\kappa$ such that

$$l_1 \in \boldsymbol{L}^{(1)} \cap \boldsymbol{L}^{(2)},\ l_2 \in \boldsymbol{L}^{(2)} \cap \boldsymbol{L}^{(3)},\ \ldots,\ l_\kappa \in \boldsymbol{L}^{(\kappa)} \cap \boldsymbol{L}^{(1)}$$

holds true. Consider the following subsystem of the linear adjustment system, given by

$$\begin{array}{rcl} c_1 \boldsymbol{a}_{l_1}^{(1)} & = & \lambda_{l_1} \\ c_2 \boldsymbol{a}_{l_1}^{(2)} & = & \lambda_{l_1} \\ c_2 \boldsymbol{a}_{l_2}^{(2)} & = & \lambda_{l_2} \\ \cdots & \cdots & \cdots \\ c_\kappa \boldsymbol{a}_{l_{\kappa-1}}^{(\kappa)} & = & \lambda_{l_{\kappa-1}} \\ c_\kappa \boldsymbol{a}_{l_\kappa}^{(\kappa)} & = & \lambda_{l_\kappa} \\ c_\kappa \boldsymbol{a}_{l_\kappa}^{(1)} & = & \lambda_{l_\kappa}. \end{array} \tag{6}$$

This is a system with $2\kappa$ variables and $2\kappa$ equations. If we write $a_{ik} := a_{l_i}^{(k)}$

just for the moment, the coefficient matrix is

$$
(7)\qquad
\begin{array}{c}
\begin{array}{ccccccccccc}
1 & & & & \kappa & & \kappa+1 & & & & 2\kappa
\end{array}\\
\left(\begin{array}{cccccc:cccccc}
a_{11} & & & & & & 1 & & & & & \\
 & a_{21} & & & & & 1 & & & & & \\
 & a_{22} & & & & & & 1 & & & & \\
 & & a_{23} & & & & & 1 & & & & \\
 & & & \cdots & & & & & \cdots & & & \\
 & & & \cdots & & & & & \cdots & & & \\
 & & & & a_{\kappa\kappa-1} & & & & & 1 & & \\
 & & & & a_{\kappa\kappa} & & & & & & 1 & \\
a_{1\kappa} & & & & & & & & & & 1 &
\end{array}\right)
\end{array}.
$$

We claim that the matrix (7) has full rank. To see this, subtract an $a_{1\kappa}$–multiple of the last column from the first column and, thereafter, omit the last column and the last row. Next, add an $\frac{a_{1\kappa}}{a_{\kappa\kappa}}$–multiple of column $\kappa$ to column 1. Then, the last row contains the entry $a_{\kappa\kappa}$ only. Hence (7) has full rank if and only if the following matrix (8)

$$
(8)\qquad
\begin{array}{c}
\begin{array}{ccccccccccc}
1 & & & & \kappa-1 & & \kappa & & & & 2(\kappa-1)
\end{array}\\
\left(\begin{array}{cccccc:cccccc}
a_{11} & & & & & & 1 & & & & & \\
 & a_{21} & & & & & 1 & & & & & \\
 & a_{22} & & & & & & 1 & & & & \\
 & & a_{23} & & & & & 1 & & & & \\
 & & & \cdots & & & & & \cdots & & & \\
 & & & \cdots & & & & & \cdots & & & \\
 & & & & a_{\kappa-1\kappa-2} & & & & & 1 & & \\
 & & & & a_{\kappa-1\kappa-1} & & & & & & 1 & \\
\frac{a_{1\kappa}a_{\kappa\kappa-1}}{a_{\kappa\kappa}} & & & & & & & & & & 1 &
\end{array}\right)
\end{array}
$$

has full rank. By induction, we see that (7) has full rank indeed.

**q.e.d.**

**Lemma 1.6.** *Let* $\boldsymbol{F}$ *be a Pareto face of a cephoid* $\Pi = \Pi^{\boldsymbol{a}^{\bullet}}$. *Then the adjustment graph is connected.*

**Proof:** The proof runs quite analogously to the one of the previous Lemma 1.5. If the adjustment graph can be decomposed into two disjoint graphs, each part admits of an independent solution of the linear adjustment system. Hence the solutions span a linear space of dimension at least two – in which case the normal is not uniquely defined up to a constant. So the Lemma follows from the Coincidence Theorem 1.4 of Chapter 3.

**q.e.d.**

**Lemma 1.7.** *Let* $\boldsymbol{F}$ *be a Pareto face of a cephoid* $\Pi = \Pi^{\boldsymbol{a}^\bullet}$. *Then the adjustment graph has at least two boundary nodes.*

The proof is obvious because the adjustment graph has no improper cycles.

**Corollary 1.8.** *Let* $\boldsymbol{a}^\bullet = \left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ *be a family of positive vectors. Let* $\boldsymbol{F}$ *be a Pareto face of the corresponding Cephoid* $\Pi$. *Then the reference system defining* $\boldsymbol{F}$ *is a pre–reference system. The adjustment system is a pre–adjustment system.*

Clearly, to any pre–adjustment system that arises from a pre–reference system we may associate the polyhedron

$$\boldsymbol{F}_{\boldsymbol{L}} := \sum_{k\in\boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{L}^{(k)}}. \tag{9}$$

Now we have

**Theorem 1.9.** *Let* $\mathfrak{J} = \left\{\boldsymbol{J}^{(k)}\right\}_{k\in\boldsymbol{K}}$ *be a family of subsets of* $\boldsymbol{I}$. *Then*

$$\boldsymbol{F} = \sum_{k\in\boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \tag{10}$$

*is a Pareto face of the Cephoid* $\Pi$ *generated by* $\boldsymbol{a}^\bullet$ *if and only if the following holds.*

1. $\mathfrak{J}$ *is a pre–adjustment system.*
2. *The solution* $(\boldsymbol{c}^\star, \lambda^\star)$ *to the linear pre–adjustment system of equations satisfies*

$$\begin{array}{rcl} c^\star_k a^{(k)}_l & = & \lambda^\star_l \ \ ((k,l)\in\mathbb{L}) \\ & \geq & c^\star_{k'} a^{(k')}_l ((k,l)\notin\mathbb{L}). \end{array} \tag{11}$$

**Proof:** The inequalities in *item 2* ensure that the vector $\boldsymbol{n}^\star = \left(\frac{1}{a^\star_1}, \ldots, \frac{1}{a^\star_n}\right)$ constructed via

$$a^\star_i := \max_{k\in\boldsymbol{K}} c^\star_k a^{(k)}_i \ (i\in\boldsymbol{I}) \tag{12}$$

constitutes a linear function that achieves its maximum relative to $\Delta^{(k)}$ exactly on $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$, thus is a normal to $\Pi$ and, clearly, the normal to $\boldsymbol{F}$. **q.e.d.**

## 2 The Pareto surface as a Graph

We return to the topics discussed in SECTION 5, recall in particular the tentacle system of SECTION 5 of CHAPTER 3.

The tentacle system exhibited in CHAPTER 3 focusses on a deGua Simplex, say $\Delta^{(\kappa)} = \Delta^{\boldsymbol{a}^{(\kappa)}}$ for some $\kappa \in \boldsymbol{K}$, which is a summand of a Cephoid $\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$. For such a summand a translate $\Delta^{\{\kappa\}}$ appears on the Pareto surface $\partial\Pi$ of $\Pi$ (the Translation Theorem 4.9 of CHAPTER 3). This translate is the center of a system of tentacles $\mathbb{T}_i^{(\kappa)}$ $(i \in \boldsymbol{I})$. Each of these tentacle systems connects the translate $\Delta^{\{\kappa\}}$ with the corresponding boundary

$$\Pi^{(-i)} \;=\; \partial\Pi_{\,|\,\mathbb{R}^{\boldsymbol{I}\setminus\{i\}}}$$

via a the system of its cylinders (Theorem " The Tentacles" 5.2 of Chapter 3 , see also Remark 5.4 about the cephalopodic structure).

Now we exhibit the structure of the Pareto surface $\partial\Pi$ of a Cephoid $\Pi$ in more detail. To this end observe that the above description of the center and tentacles of a "cephalopodic" structure is at best represented by introducing the notion of a *graph*.

I will then be seen that *every* face generates a Pareto tentacle system. By this, we mean a well–defined system of Pareto faces connected by the adjacency relation. More precisely, we are talking about connected subgraphs of the system of all Pareto faces.

For the present discussion, we include also boundary faces of a Cephoid $\Pi$ that are not Pareto efficient, i.e., the faces

$$\Pi^{(-i)} \;=\; \Pi \cap \{\boldsymbol{x} \in \Pi \,\big|\, x_i = 0\} \quad (i \in \boldsymbol{I})\;. \tag{1}$$

which can as well be seen as the Cephoids generated by the restrictions

$$\left\{\boldsymbol{a}^{(k)}\;_{|\;\mathbb{R}^n_{\boldsymbol{I}\setminus\{i\}}}\right\}_{k\in\boldsymbol{K}}\;.$$

These are maximal faces of $\Pi$ but not Pareto faces.

For the construction of the graph we use the *nodes* given by the maximal faces of $\partial\Pi$ which have a dimension $(n-1)$. The edges are provided by the $(n-2)$–dimensional subfaces of maximal faces that are common to two faces. So any two nodes are connected by an edge whenever the two faces under consideration do have a common $(n-2)$ dimensional subface. That is, whenever two Pareto faces are *adjacent* in the sense of Definition 2.2 of CHAPTER 3. Or else, whenever some Pareto face has an $(n-2)$– dimensional intersection with some non Pareto efficient maximal face $\Pi^{(-i)}$.

For short we denote the $(n-2)$ dimensional subfaces of a Pareto face by the term *P–Subface*. We do have special versions of P–subfaces. E.g., the cylinder bases $\Delta^{(k^\star)}_{jl}$ as exhibited in Corollary 4.10 and Definition 4.11 of CHAPTER 3. Also we point to the P–Subfaces appearing in the tentacle system, i.e., in Formula (3) of SECTION 5, CHAPTER 3.

This justifies the following

**Definition 2.1.** *Let* $\Pi = \sum_{k\in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ *be a Cephoid.*

1. *A **P–subface** is an* $(n-2)$ *dimensional Pareto efficient subface of a Pareto face of* $\Pi$.
2. *An **edge** is a P–subface that is the intersection of two maximal faces.*
3. *A **node** is a maximal face.*
4. *For* $i_0 \in \boldsymbol{I}$ *an* $i_0$*–**boundary node** is a maximal face* $\Pi^{(-i_0)}$.
5. *The Pareto graph or **P–graph** of* $\Pi$ *is the graph* $\mathcal{P} = (\mathcal{V}, \mathcal{E})$ *with nodes*

$$\mathcal{V} := \{\boldsymbol{F} \mid \boldsymbol{F} \text{ is a node }\} = \{\boldsymbol{F} \mid \boldsymbol{F} \text{ is a Pareto face of } \partial\Pi\}$$

*and edges*

$$\mathcal{E} := \{\boldsymbol{E} \mid \boldsymbol{E} \text{ is a node }\} = \{\boldsymbol{E} \mid \boldsymbol{E} \text{ is a P–subface common to two Pareto faces }\}.$$

Two nodes are connected by an edge if – as maximal faces – they are adjacent; the edge connecting them is the joint P–subface. If a maximal face $\boldsymbol{F}$ (a node) is $i_0$–boundary, then there is the unique subface $\boldsymbol{E}^{(-i_0)}$ connecting $\boldsymbol{F}$ with the graph $\partial\Pi^{(-i_0)}$ as an edge.

**Remark 2.2.** *We extent the notion of a reference vector to P–subfaces as well in a canonical fashion. For, in view of the Neighborhood Theorem 2.3, and in particular Formula* (5), *we know that a P–subface has a representation*

$$\boldsymbol{E} = \sum_{k\in\boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \tag{2}$$

*such that* $r_k := |\Delta^{(k)}_{\boldsymbol{J}^{(k)}}|$ $(k \in \boldsymbol{K})$ *satisfies*

(3) $$\sum_{k\in\boldsymbol{K}} r_k = K + n - 2 \ ,$$

*that is.* $\boldsymbol{r} = (r_1, \ldots, r_K)$ *is a* $(K, (n-1))$ *reference vector. Naturally, the correspondence between* $(K, (n-1))$*–reference vectors and P–subfaces is* not *bijective. Rather consider the following Theorem as a list of possibilities.*

◦ ∼∼∼∼∼∼∼ ◦

We list some properties of the P–graph. All of these have appeared in varying contexts in previous chapters.

**Theorem 2.3.** *Let* $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ *be a Cephoid and let* $\boldsymbol{F} = \sum_{k\in\boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ *be a Pareto face of* $\Pi$. *Let* $\boldsymbol{r} = (r_1, \ldots, r_K)$ *be the reference vector of* $\boldsymbol{F}$.

1. *If, for some* $p \in \boldsymbol{K}$ *we have* $r_p \geq 2$, *then there is* $q \in \boldsymbol{K}$ *and a Pareto face* $\boldsymbol{F}'$ *such that* $\boldsymbol{F}$ *and* $\boldsymbol{F}'$ *are adjacent and the reference vector for* $\boldsymbol{F}'$ *is* $\boldsymbol{r} = (r_1, \ldots, r_q + 1, \ldots, (r_p - 1), \ldots, r_K)$.

   *The edge* $\boldsymbol{E}$ *connecting the nodes* $\boldsymbol{F}$ *and* $\boldsymbol{F}'$ *is the P–subface* $\boldsymbol{E} = \boldsymbol{F} \cap \boldsymbol{F}'$ *with reference vector* $\boldsymbol{r}^0 = (r_1, \ldots, r_q, \ldots, r_p - 1, \ldots, r_K)$.
2. *Also, if for some* $p \in \boldsymbol{K}$, *we have* $r_p \geq 2$, *then for every* $i_0 \in \boldsymbol{I}$ *there is a P–subface* $\boldsymbol{E}^{(-i_0)} \in \mathbb{R}^n_{\boldsymbol{I}\setminus\{i_0\}}$ *with reference vector* $\boldsymbol{r}^0 = (r_1, \ldots, r_q, \ldots, r_p - 1, \ldots, r_K)$ *which is a Pareto face of* $\partial\Pi^{(-i_0)} = \partial\Pi_{|\mathbb{R}^n_{\boldsymbol{I}\setminus\{i_0\}}}$ .
3. *A Pareto face* $\boldsymbol{F}$ *is connected to an* $i_0$*–boundary node if and only if* $i_0 \notin \boldsymbol{L}$ *and there is some* $p \in \boldsymbol{K}$ *such that* $i_0 \in \boldsymbol{J}^{(p)}$ *and* $|\boldsymbol{J}^{(p)}| \geq 2$ .
4. *On the other hand, if for some* $p \in \boldsymbol{K}$, *we have* $r_p = 1$, *then there is some* $i_1 \in \boldsymbol{I}$ *such that* $\boldsymbol{F}$ *is the translate of some face* $\boldsymbol{F}'$ *of the Cephoid* $\Pi^{[-p]} = \sum_{k\in\boldsymbol{K}\setminus\{p\}} \Pi^{\boldsymbol{a}^{(k)}}$, *i.e.,*

   $$\boldsymbol{F} = \boldsymbol{F}' + \boldsymbol{a}^{(p)i_1} \ , \quad \boldsymbol{F}' = \sum_{k\in\boldsymbol{K}\setminus\{p\}} \Delta^{(k)}_{\boldsymbol{J}'^{(k)}} \ .$$

   *Then* $\boldsymbol{F}$ *is* not *an* $i_1$*–boundary node. All other edges of* $\boldsymbol{F}$ *are those of* $\boldsymbol{F}'$ *seen as a node in the P–graph of* $\Pi^{[-p]}$.

**Proof:** Consider the representation $\boldsymbol{F} = \sum_{k\in\boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ of $\boldsymbol{F}$ via the reference sets.

**1^st^STEP** : As to *item* 1, for $r_p = |\boldsymbol{J}^{(p)}| \geq 2$ there is some index $i_0 \in \boldsymbol{J}^{(p)}$ which is an adjustment index, i.e., $i_0 \in \boldsymbol{L}$ and hence

$$\boldsymbol{E} \;=\; \sum_{k \in \boldsymbol{K} \setminus \{p\}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \;+\; \Delta^{(p)}_{\boldsymbol{J}^{(p)} \setminus \{i_0\}}$$

is an edge which necessarily connects some other Pareto face; the rest is done as in the Neighborhood Theorem 2.3.

**2^nd^STEP** : Next, *item* 2 follows from the Bijection Theorem 1.3 applied to $\partial \Pi^{(-i_0)}$.

**3^rd^STEP** : As to *item* 3, it is clear that in this case

$$\boldsymbol{E} \;=\; \sum_{k \in \boldsymbol{K} \setminus \{p\}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \;+\; \Delta^{(p)}_{\boldsymbol{J}^{(p)} \setminus \{i_0\}}$$

is a Pareto face of $\partial \Pi^{(-i_0)}$ with the same adjustment set $\boldsymbol{L}$.

**4^th^STEP** : Finally, to *item* 4. In this case, one has $1 = r_p = |\boldsymbol{J}^{(p)}|$, hence $\boldsymbol{F}$ is a translate of some face $\boldsymbol{F}' = \sum_{k \in \boldsymbol{K} \setminus \{p\}}$ of the Cephoid $\Pi^{[-p]}$ which is generated by the family

$$\left\{\boldsymbol{a}^{(k)}\right\}_{k \in \boldsymbol{K} \setminus \{p\}} .$$

Necessarily, the translation takes place by some vector $\boldsymbol{a}^{(p)i_1}$ with $i_1 \in \boldsymbol{L}$. **q.e.d.**

Next, we exhibit a further version of the ‘ ‘Cephalopodic structure”.

**Definition 2.4.** *Let* $\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ *be a Cephoid and let* $\boldsymbol{F} = \sum_{k \in \boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ *be a Pareto face of* $\Pi$. *Let* $\mathfrak{J}$ *be th reference system and* $\boldsymbol{r}$ *the reference vector.*

*Let* $p \in \boldsymbol{K}$ *be such that* $r_p = |\boldsymbol{J}^{(p)}| \geq 2$.

1. *fork* $\in \boldsymbol{K}$ *the reference vector*

   (4) $$\boldsymbol{r} - \boldsymbol{e}^p + \boldsymbol{e}^k$$

   *is called the* ***p–k–vector*** *of* $\boldsymbol{F}$ *(*$\boldsymbol{r}$ *is the p–p–vector).*

2. *The node (Pareto face) corresponding to the p–k–vector* $\boldsymbol{r} - \boldsymbol{e}^p + \boldsymbol{e}^k$ $(k \in \boldsymbol{K})$ *according to the Bijection Theorem is denoted by* $\boldsymbol{F}^{-p,+k}$ *and called the* ***p–k–node*** *derived from* $\boldsymbol{F}$.

3. *For* $i \in \boldsymbol{I}$, *the P–subface located in* $\partial \Pi^{(-i)} := \partial \Pi_{\,|\, \mathbb{R}^n_{\boldsymbol{I} \setminus \{i\}}}$ *corresponding to the reference vector* $\boldsymbol{r} - \boldsymbol{e}^p$ *is denoted by* $\boldsymbol{E}^{p,(-i)}$ *(or* $\boldsymbol{E}^{p,(-i)}(\boldsymbol{F})$ *if applicable).*

4. *An edge (P–subface)* $\boldsymbol{F}^0$ *of some p–k–face* $\boldsymbol{F}^{-p,+k}$ *with reference vector* $\boldsymbol{r} - \boldsymbol{e}^K$ *is called a* ***p–k–edge****.*

5. *The* $\boldsymbol{P^p}$***–graph*** $\mathcal{P}^p$ *of* $\boldsymbol{F}$ *is the subgraph of* $\mathcal{P}$ *given by the p–nodes and the connecting edges; denoted* $\mathcal{P}^p = (\mathcal{V}^p, \mathcal{E}^p)$ *(or* $\boldsymbol{E}^{p,(-i)}$ *if applicable).*

Our aim is to show that the family of $\boldsymbol{p}$–$\boldsymbol{k}$–faces constitute a simple connected graph without circles which terminates exactly on the $(n-2)$–dimensional boundary of $\partial\Pi$.

**Lemma 2.5.** *There are* $K$ *p–k––faces (including* $\boldsymbol{F} = \boldsymbol{F}^{-p,+p}$*). For* $i_0 \in \boldsymbol{I}$*, there exists a p–k–face* $\widehat{\boldsymbol{F}}$ *such that* $\widehat{\boldsymbol{F}} \cap \mathbb{R}^n_{(\boldsymbol{I}\setminus\{i_0\})} = \boldsymbol{E}^{p,(-i_0)}(\boldsymbol{F})$*.*

**Proof:** The existence of $\partial\Pi^{(-i_0)}$ follows as in *item* 2. There has to be a Pareto face including this P–subface as a boundary face which necessarily has a reference vector augmented at one coordinate.

**q.e.d.**

**Lemma 2.6.** *Let* $\boldsymbol{F}^{-p,+q}$ *be a p–q–face. Then the number of p–q–edges of* $\boldsymbol{F}^{-p,+q}$ *is at least* 2*. More precisely, the number of edges is*

$$\begin{array}{rcll} r_q & \geq & 2 & , \quad \textit{for } q = p, \ \textit{i.e., for } \boldsymbol{F} = \boldsymbol{F}^{-p,+p} \\ r_q + 1 & \geq & 2 & , \quad \textit{for } q \neq p \end{array} \tag{5}$$

**Proof:** Consider the case $q \neq p$. Let $\left\{\widehat{\boldsymbol{J}}^{(k)}\right\}_{k\in\boldsymbol{K}}$ denote the reference system of $\boldsymbol{F}^{-p,+q}$. Then $|\widehat{\boldsymbol{J}}^{(q)}| = r_q + 1 \geq 2$ holds true. For every $i_0 \in \widehat{\boldsymbol{J}}^{(q)}$ the system

$$\left\{\widehat{\boldsymbol{J}}^{(k)}\right\}_{k\in\boldsymbol{K}-q}, \quad \widehat{\boldsymbol{J}}^{(q)} \setminus \{i_0\} \tag{6}$$

constitutes a reference system defining an P–subface of $\boldsymbol{F}^{-p,+q}$ which is a p–q–edge.

Similarly, for $p = q$, given the reference system $\mathcal{J} = \left\{\boldsymbol{J}^{(k)}\right\}_{k\in\boldsymbol{K}}$ of $\boldsymbol{F} = \boldsymbol{F}^{-p,+q}$ we see that for $i_0 \in \boldsymbol{J}^{(p)}$ the system

$$\left\{\boldsymbol{J}^{(p)}\right\}_{k\in\boldsymbol{K}-p}, \quad \boldsymbol{J}^{(p)} \setminus \{i_0\} \tag{7}$$

constitutes a reference system defining an P–subface of $\boldsymbol{F}$ which is a p–p–edge.

**q.e.d.**

**Lemma 2.7.** *Let* $\boldsymbol{F}^{-p,+q}$ *be a* $p$–$q$–*face and let* $\widehat{\boldsymbol{F}}^0$ *be* $p$–$q$–*edge. Let* $i_0 \in \boldsymbol{I}$ *such that the reference system of* $\widehat{\boldsymbol{F}}^0$ *is given by* (6) *or* (7) *respectively. If* $i_0$ *is* not *an adjustment index of* $\boldsymbol{F}^{-p,+q}$, *then* $\widehat{\boldsymbol{F}}^0 = \mathcal{P}^{i_0}(\boldsymbol{F})$, *that is,* $\boldsymbol{F}$ *is connected to the* $i_0$–*boundary node* $\Pi^{(-i_0)}$.

**Proof:** See *item* 3 of Theorem 2.3. **q.e.d.**

**Lemma 2.8.** *Let* $\bar{\boldsymbol{F}} = \boldsymbol{F}^{-p,+q}$ *and* $\widehat{\boldsymbol{F}} = \boldsymbol{F}^{-p,+l}$ *be different* $p$–*–*faces. Let* $\bar{\boldsymbol{J}}^{(p)}$, $\widehat{\boldsymbol{J}}^{(p)}$ *denote the reference set corresponding to* $\Delta^{(p)}$ *for each of these Pareto faces respectively. If* $|\bar{\boldsymbol{J}}^{(p)} \setminus \widehat{\boldsymbol{J}}^{(p)}| \geq 2$ *or* $|\widehat{\boldsymbol{J}}^{(p)} \setminus \bar{\boldsymbol{J}}^{(p)}| \geq 2$ *holds true, the* $\bar{\boldsymbol{F}}$ *and* $\widehat{\boldsymbol{F}}$ *are not adjacent.*

**Proof:** Obvious by the Neighborhood Theorem (Theorem 2.3 of CHAPTER 3).

**q.e.d.**

**Theorem 2.9.** *Let* $\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ *be a Cephoid. Let*

$$\boldsymbol{F} = \sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$$

*be a Pareto face with* $r_k = |\boldsymbol{J}^{(k)}|$ $(k \in \boldsymbol{K})$ *and* $r_p \geq 2$. *The* $P^p$–*graph* $\mathcal{P}^p$ *of* $\boldsymbol{F}$ *is a simple connected graph without cycles, that is, a tree.* $\boldsymbol{F}$ *is the root of this tree.*

1. *For every* $i_0 \in \boldsymbol{I}$ *there is a path from* $\boldsymbol{F}$ *to the* $i_0$–*boundary node* $\Pi^{(-i_0)}$ *via the (last) boundary edge* $\boldsymbol{E}^{p,(-i_0)}(\boldsymbol{F})$.
2. *The number of appearances of* $i_0$ *within the reference system of each of the nodes along the path from* $\boldsymbol{F}$ *to* $\Pi^{(-i_0)}$ *decreases by* 1 *at each consecutive node.*

**Proof:** Pick some $i_0 \in \boldsymbol{I}$.

**1^st^STEP** :

First of all, we consider the case that $i_0 \in \boldsymbol{J}^{(p)}$ holds true. As $r_p \geq 2$, we can in this case construct the edge ($(n-1)$–dimensional subface)

$$\mathring{\boldsymbol{F}}^{i_0} = \sum_{k \in \boldsymbol{K} \setminus \{p\}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} + \Delta^{(K)}_{\boldsymbol{J}^{(p)} \setminus \{i_0\}} .$$

If $i_0 \notin \boldsymbol{L}$, then this edge connects $\boldsymbol{F}$ to the $\Pi^{(i_0)}$. If so, we are done.

**2ndSTEP** :

Alternatively, (still assuming $i_0 \in \boldsymbol{J}^{(p)}$) let $i_0 \in \boldsymbol{L}$ and hence $\mathring{\boldsymbol{F}}^{i_0}$ is not a $i_0$–boundary edge. According to Neighborhood Theorem 2.3 and Theorem 2.8 of CHAPTER 3, there exists $\kappa \in \boldsymbol{K}$ and an adjacent face $\widetilde{\boldsymbol{F}}$ with reference system $\widetilde{\mathcal{J}}$ changed exactly at positions $q$ and $\kappa$, i.e.,

$$\widetilde{\boldsymbol{J}}^{(q)} = \boldsymbol{J}^{(q)} \setminus \{i_0\} \;\;,\;\; \widetilde{\boldsymbol{J}}^{(\kappa)} = \boldsymbol{J}^{(\kappa)} \cup \{i_1\}$$

for a suitable $i_1 \in \boldsymbol{I}$. The reference vector is

$$\widetilde{\boldsymbol{r}} = (r_1, \ldots, r_\kappa + 1, \ldots, r_p - 1, \ldots, r_K) \;.$$

and hence $\widetilde{\boldsymbol{F}} = \boldsymbol{F}^{-p,+\kappa}$ is an (adjacent) p–$\kappa$–face. The adjustment set is $\widetilde{\boldsymbol{L}}$ and the number of appearances of $i_0$ has been diminished by 1. Also, according to the Neighborhood Theorem, we have $i_1 \in \widetilde{\boldsymbol{L}}$.

**3rdSTEP** :

If it so happens that $i_0 \in \widetilde{\boldsymbol{J}}^{(\kappa)}$, then we proceed once again as in the $1^{st}$ and $2^{nd}STEP$: Either $i_0 \notin \widetilde{\boldsymbol{L}}$, then $\widetilde{\boldsymbol{F}}$ is connected to the $i_0$–boundary and we are done. Or else, we remove $i_0$ from $\widetilde{\boldsymbol{J}}^{(\kappa)}$ an add a suitable index to some $\widetilde{\boldsymbol{J}}^{(\gamma)}$. Then we have reached a further p–$\gamma$–face $\widehat{\boldsymbol{F}} = \boldsymbol{F}^{-p+\gamma}$ at which the number of appearances of $i_0$ is reduced by 1. Clearly $\widehat{\boldsymbol{F}}$ is adjacent to $\widetilde{\boldsymbol{F}}$. The reference vector is

$$\widehat{\boldsymbol{r}} = (r_1, \ldots, r_\gamma + 1, \ldots, r_\kappa, \ldots, r_p - 1, \ldots, r_K) \;;$$

obviously $\widehat{\boldsymbol{F}} = \boldsymbol{F}^{-p+\gamma}$ is a $\pi$–$\gamma$–node adjacent to $\widetilde{\boldsymbol{F}} = \boldsymbol{F}^{-p,+\kappa}$.

Thus, at this stage, we have reduced the number of appearances of $i_0$ by two and we have done this by running through adjacent p–$\star$–nodes consecutively.

**4thSTEP** :

Alternatively to the $3^{rd}STEP$ suppose now that $i_0 \notin \widetilde{\boldsymbol{J}}^{(\kappa)}$ holds true.

Then we need more steps to construct a face adjacent to $\widetilde{\boldsymbol{F}}$ which is a p–$\star$–node and yet has the number of appearances of $i_0$ diminished by 1.

To this end we focus on $i_1 \in \widetilde{\boldsymbol{J}}^{(\kappa)} \cap \boldsymbol{L}$ as specified in the $3^{rd}STEP$. We can remove $i_1$ from $\boldsymbol{J}^{(\kappa)}$ and add another index to some $\boldsymbol{J}^{(\lambda)}$; the result is a p–$\lambda$–face $\boldsymbol{F}^\star$ with reference vector

$$\boldsymbol{r}^\star = (r_1, \ldots, r_\lambda + 1, \ldots, r_\kappa, \ldots, r_p - 1, \ldots, r_K)$$

and reference sets changed at positions $\lambda$ and $\kappa$, that is

$$\boldsymbol{J}^{\star(\kappa)} = \boldsymbol{J}^{(\kappa)} \setminus \{i_1\} \quad , \quad \boldsymbol{J}^{\star(\lambda)} = \boldsymbol{J}^{(\lambda)} \cup \{i_2\}$$

but also

$$i_0 \notin \boldsymbol{J}^{\star(\kappa)} \ . \tag{8}$$

Now, any index $i \in \widetilde{\boldsymbol{J}}^{(\kappa)} \cap \widetilde{\boldsymbol{L}}$ can play the role of $i_1$, i.e., there are more indices feasible for the above procedure besides $i_1$. We then perform the procedure for all of them.

This way we construct one or more paths (of $\mathcal{P}$) at which the number of appearances of $i_0$ is constant but one can always proceed along some p–$\gamma'$–face.

**5<sup>th</sup>STEP** :

The path can never return to $\boldsymbol{F}$ because any other path leaving $\boldsymbol{F}$ is characterized by a missing index $i_9 \neq i_0$ instead of $i_0$. No face with missing index $i_9$ is a neighbor to a face with missing index $i_0$ by Lemma 2.8. By a similar argument, the path cannot return to another face met during the construction.

Whenever a path branches off as in the $4^{th}STEP$ , then we can follow all branches and the resulting paths.

As there are only finitely many p–⋆–vectors, the index $i_0$ must eventually occur. That is, at some stage we find a face $\widetilde{\widetilde{\boldsymbol{F}}}$ with reference vector

$$\widetilde{\widetilde{\boldsymbol{r}}} = (r_1, \ldots, r_\nu + 1, \ldots, r_\mu \ldots, r_\kappa, \ldots, r_p - 1, \ldots, r_K)$$

such that $i_0 \in \widetilde{\widetilde{\boldsymbol{J}}}^{(\nu)}$ holds true. We can, therefore, proceed by removing $i_0$ from $\widetilde{\widetilde{\boldsymbol{J}}}^{(\nu)}$ and replacing it by some $i_7 \notin \widetilde{\widetilde{\boldsymbol{J}}}^{(\mu)}$ for a suitable $\mu$, i.e., by turning to some node $\widehat{\widehat{\boldsymbol{F}}}$ such that

$$\widehat{\widehat{\boldsymbol{J}}}^{(\nu)} = \widetilde{\widetilde{\boldsymbol{J}}}^{(\nu)} \setminus \{i_0\} \quad , \quad \widehat{\widehat{\boldsymbol{J}}}^{(\mu)} = \widetilde{\widetilde{\boldsymbol{J}}}^{(\mu)} \cup \{i_7\} \ .$$

implies a reference vector

$$\widehat{\widehat{\boldsymbol{r}}} = (r_1, \ldots, r_\mu + 1, \ldots, r_\nu, \ldots, r_\kappa, \ldots, r_p - 1, \ldots, r_K)$$

Observe now, that $\widehat{\widehat{\boldsymbol{F}}}$ is *also* adjacent to $\widetilde{\boldsymbol{F}}$ that we constructed in the $3^{rd} STEP$ and left in the $4^{th} STEP$. Indeed $i_1$ has been removed from $\widetilde{\boldsymbol{J}}^{(\kappa)}$ in the $4^{th} STEP$ and position $\kappa$ has not been changed thereafter. On the other hand, position $\nu$ is reached the first time. That is we have $i_1 \notin \widehat{\widehat{\boldsymbol{J}}}^{(\kappa)}$ and $i_7 \notin \widetilde{\boldsymbol{J}}^{(\mu)}$ and

hence

$$\widehat{\widehat{\boldsymbol{J}}}^{(\kappa)} = \widetilde{\boldsymbol{J}}^{(\kappa)} \setminus \{i_1\} \;,\; \widehat{\widehat{\boldsymbol{J}}}^{(\mu)} = \widetilde{\boldsymbol{J}}^{(\mu)} \cup \{i_7\} \;.$$

Therefore, the p–$\nu$–node $\widehat{\widehat{\boldsymbol{F}}}$ is adjacent to $\widetilde{\boldsymbol{F}}$ and again has the number of appearances of $i_0$ diminished by one.

So now we are again in the situation of the $1^{st}$ and $2^{nd} STEP$: Either $i_0 \notin \widehat{\widehat{\boldsymbol{L}}}$, then $\widehat{\widehat{\boldsymbol{F}}}$ is connected to the $\Pi^{(-i_0)}$ and we are done. Or else, the number of appearances of $i_0$ in the face $\widehat{\widehat{\boldsymbol{F}}}$ can reduced by 1 once again.

**6<sup></sup>**

Obviously we can proceed in this way, reducing the appearances of $i_0$ until $i_0 \notin \widehat{\widehat{\widehat{\boldsymbol{L}}}}$ is the case. Then we have found a face with an edge $\boldsymbol{E}^{p,(-i_0)}(\boldsymbol{F})$. Hence the path constructed in the p–$\mathcal{P}$ graph connects $\boldsymbol{F}$ and the boundary node $\Pi^{(-i_0)}$.

Thus the p–$\mathcal{P}$ graph has boundary nodes at each $\partial\Pi^{(-i)}$ $(i \in \boldsymbol{I})$, all of them being connected without loops and circles via the central node $\boldsymbol{F}$.

**q.e.d.**

**Remark 2.10.** *We can regard the number of appearances of some $i_0$ in the components $\Delta^{(k)}_{\boldsymbol{J}^{(k)}}$ of $\boldsymbol{F}$ as a measure for the distance of the node $\boldsymbol{F}$ to the $i_0$–boundary $\Pi^{(-i_0)}$ as this number is diminished by each step along the connecting branch ("tentacle") of the p–$\mathcal{P}$ graph. See the corresponding statement in Remark 5.3 of* Chapter *3.*

∘ ∼∼∼∼∼∼∼ ∘

**Example 2.11.** *Compare the "Cephalopodic Structure" as explained in Remark 5.4 of* Chapter *3. We copy Figure 5.2. If the central $\boldsymbol{F}$ equals $\Delta^{(\kappa)}$, then obviously the tentacles as described in Theorem 5.2 represent the paths leading from the central node towards the boundary; in this case a path from the center to the boundary is exactly described by a tentacle. As the "depths" or "distances"(i.e., the number of appearances) w.r.t. the various $i_0$ add up to the number $K$ of DeGua*

*Simplices involved, it is clear that the total number of faces with all possible paths to the boundaries adds up to $K$.*

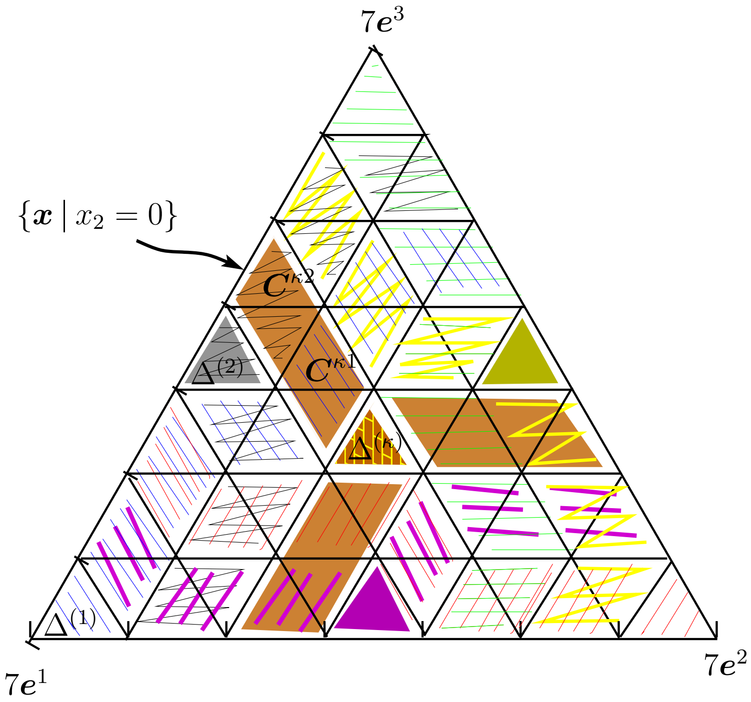


Figure 2.1: The $\boldsymbol{P}$–Graph in 3 dimensions–canonically represented

*However, Theorem 2.9 deals with an arbitrary Pareto face as the center, not just a translate of a DeGua Simplex. So, if in Figure 2.1 we chose e.g. the Pareto face $\boldsymbol{C}^{\kappa 1}$ as the central node $\boldsymbol{F}$ of our considerations, then clearly a path to the boundary $\{\boldsymbol{x} \mid x_2 = 0\}$, i.e., to $\Pi^{(-2)}$ connects directly while a path to the boundarie faces $\Pi^{(i_1)}$ and $\Pi^{(i_2)}$ has to move via $\Delta^{(\kappa)}$. From which it follows that both of the lattter paths proceed jointly until $\Delta^{(\kappa)}$ is reached. Hence the total number of nodes in all possible paths exceeds $K$ – while the number of p–⋆–nodes is obviously $K$ (Lemma 2.5).*

∘ ∼∼∼∼∼∼ ∘

# 3 Identifying Boundary Subfaces

Let $\left\{ \boldsymbol{a}^{(k)} \right\}_{k\in\boldsymbol{K}}$ be a family of positive vectors and let $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ be the Cephoid generated. For $i_0 \in \boldsymbol{I}$ consider the Cephoid $\Pi^{(-i_0)}$ generated by the restricted family $\left\{ \boldsymbol{a}^{(k)}|_{\mathbb{R}^n_{\boldsymbol{I}\setminus\{\iota_0}} \right\}_{k\in\boldsymbol{K}}$. A Pareto face $\boldsymbol{F}^{(-i_0)}$ of this Cephoid is also a P–subface of $\Pi$ and as such an $i_0$ boundary edge.

The $(K, n-1)$–reference vector $\boldsymbol{r}^{(-)}$ of $\boldsymbol{F}^{(-i_0)}$ identifies this Pareto face uniquely when all considerations are restricted to $\mathbb{R}^{(-i_0)}$. But for $i_1 \neq i_0$ the same reference vector also defines a unique Pareto face of the corresponding Cephoid $\Pi^{(-i_1)}$. The number of Pareto faces is in each case $f(K, n-1)$ and the reference vector provides a means to identify them.

Geometrically, these Pareto faces look alike in each boundary, as the number $r_k$ of subfaces of each $\Delta^{(k)}$ is equal. However their position within the Pareto surface of the Cephoids $\Pi^{(-i_0)}$ or $\Pi^{(-i_1)}$ respectively may change considerably as the corresponding restricted Cephoids do have differing data.

Now, the graph $\mathcal{P}^p$ as developed in Section 3 provides a method to establish the identification also via some path of the Pareto graph $\mathcal{P}$.

**Theorem 3.1.** *Let $\Pi$ be a cephoid. For $i_0 \in \boldsymbol{I}$ let $\boldsymbol{E}^{i_0}$ a boundary $\boldsymbol{P}$–subface located in $\Pi^{(-i_0)}$, i.e., a Pareto face of $\Pi^{(-i_0)}$. Let $\boldsymbol{r}^-$ be the reference vector of $\boldsymbol{E}^{i_0}$. Then, for any $\iota_1 \in \boldsymbol{I},\ i_1 \neq i_0$, there is a sequence*

$$\boldsymbol{F}^{\{1\}}, \boldsymbol{F}^{\{2\}}, \ldots, \boldsymbol{F}^{\{p\}}$$

*of Pareto faces of $\Pi$ with the following properties.*

1. *$\boldsymbol{E}^{i_0}$ is a $\boldsymbol{P}$–subface of $\boldsymbol{F}^{\{1\}}$*
2. *Any two consecutive elements of the sequence $\boldsymbol{F}^{\{q\}}, \boldsymbol{F}^{\{q+1\}}$ are adjacent and the common edge ($\boldsymbol{P}$-subface) has reference vector $\boldsymbol{r}^-$.*
3. *$\boldsymbol{F}^{\{p\}}$ is adjacent to $\Pi^{\{-i_1\}}$.*
4. *The link $\boldsymbol{E}^{i_1}$ between $\boldsymbol{F}^{\{p\}}$ and $\Pi^{\{-i_1\}}$ has reference vector $\boldsymbol{r}^-$.*

**Proof:**

Let $\boldsymbol{F}^{\{1\}}$ be the Pareto face of $\Pi$ containing $\boldsymbol{E}^{i_0}$ as a $\boldsymbol{P}$–subface. In Theorem 2.9 replace $i_0$ by $i_1$. Then follow the path towards the $i_1$ boundary edge as provided. **q.e.d.**

In the following example, the subface $\boldsymbol{E}^{i_0}$ can be nicely identified by the colors. For, if we assign colors to the generating DeGua Simplices (blue, red, green, and yellow in the example), then a subface with a certain reference vector $\boldsymbol{r}^-$ can exactly be followed by identifying colors and dimensions. E.g., if $r_k^- = r_{blue}^- = 2$, then this means that there is a blue edge/line segment of dimension 1 involved in every $\boldsymbol{P}$–subface with $\boldsymbol{r}^-$ as reference vector.

**Example 3.2.** *The cephoid "FourFour" is a sum of four prisms in four dimensions. It is given by the matrix*

$$\boldsymbol{A} = \begin{pmatrix} 701 & 502 & 303 & 104 \\ 205 & 116 & 1007 & 128 \\ 139 & 110 & 611 & 512 \\ 67 & 230 & 444 & 777 \end{pmatrix} \tag{1}$$

*Figure 3.1 shows the canonical representation within the Simplex* $4\Delta^{\boldsymbol{e}}$.

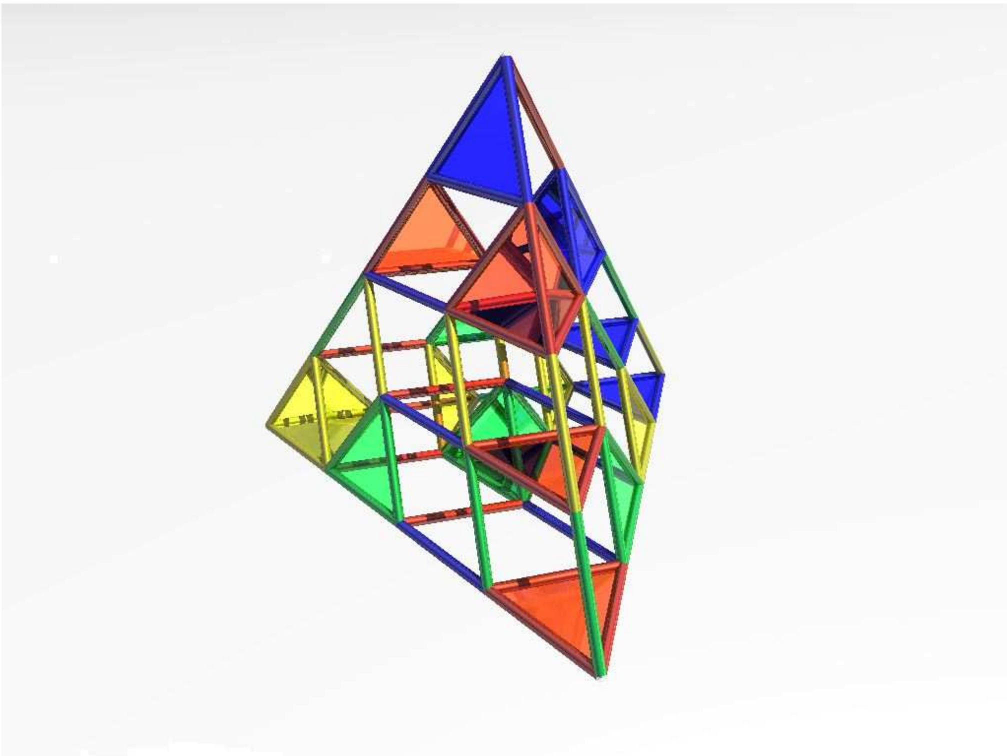

Figure 3.1: The canonical representation of FourFour

*The faces of "FourFour" are represented as follows.*

CEPHOID FOURFOUR

| | | | |
|---|---|---|---|
| 701 | 502 | 303 | 104 |
| 205 | 116 | 1007 | 128 |
| 139 | 110 | 611 | 512 |
| 67 | 230 | 444 | 777 |

FACES OF CEPHOID FOURFOUR

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | ⊛ | 1 3 | ⊛ | 1 4 | ⊛ | 2 4 | ⊛ |
| 1 2 | ⊛ | 3 | ⊛ | 3 4 | ⊛ | 2 4 | ⊛ |
| 1 | ⊛ | 3 | ⊛ | 1 3 4 | ⊛ | 2 4 | ⊛ |
| 1 | ⊛ | 3 | ⊛ | 1 3 | ⊛ | 2 3 4 | ⊛ |
| 1 2 3 | ⊛ | 3 | ⊛ | 3 | ⊛ | 3 4 | ⊛ |
| 2 | ⊛ | 1 2 3 | ⊛ | 4 | ⊛ | 2 4 | ⊛ |
| 1 2 | ⊛ | 3 | ⊛ | 3 | ⊛ | 2 3 4 | ⊛ |
| 1 | ⊛ | 1 | ⊛ | 1 | ⊛ | 1 2 3 4 | ⊛ |
| 1 | ⊛ | 1 3 | ⊛ | 1 | ⊛ | 2 3 4 | ⊛ |
| 1 2 | ⊛ | 1 3 | ⊛ | 4 | ⊛ | 2 4 | ⊛ |
| 1 2 3 4 | ⊛ | 3 | ⊛ | 4 | ⊛ | 4 | ⊛ |
| 1 2 4 | ⊛ | 3 4 | ⊛ | 4 | ⊛ | 4 | ⊛ |
| 1 2 3 | ⊛ | 3 | ⊛ | 3 4 | ⊛ | 4 | ⊛ |
| 2 | ⊛ | 1 2 3 | ⊛ | 2 4 | ⊛ | 2 | ⊛ |
| 2 | ⊛ | 1 2 3 4 | ⊛ | 4 | ⊛ | 4 | ⊛ |
| 1 2 | ⊛ | 3 | ⊛ | 2 3 4 | ⊛ | 2 | ⊛ |
| 1 | ⊛ | 1 3 | ⊛ | 1 2 4 | ⊛ | 2 | ⊛ |
| 1 2 | ⊛ | 1 3 4 | ⊛ | 4 | ⊛ | 4 | ⊛ |
| 1 | ⊛ | 3 | ⊛ | 1 2 3 4 | ⊛ | 2 | ⊛ |
| 1 2 | ⊛ | 1 3 | ⊛ | 2 4 | ⊛ | 2 | ⊛ |

*The computational result does not list the colours. In order to identify them, observe e.g. the (translate of) the yellow deGua Simplex. It is located in the first vertex* $4\boldsymbol{e}^1$ *of the above (canonical) representation. There is exactly one Pareto face in our list which is*

(2)
$$\boldsymbol{\Delta}^g : \begin{array}{cccc} \textit{blue} & \textit{red} & \textit{green} & \textit{yellow} \\ 1 & 1 & 1 & 1234 \end{array}$$

*which carries the full deGua Simplex at position 4. Hence we derive that "yellow" is listed in the fourth position. The others are obtained similarly.*

*Now we exhibit a path. Consider the block without yellow edges - shown as the last Pareto face in our above list.*

(3) $$\boldsymbol{B}^y : \begin{array}{cccc} \textit{blue} & \textit{red} & \textit{green} & \textit{yellow} \\ 12 & 13 & 24 & 2 \end{array} ,$$

*the reference vector of which is* $(2,2,2,1)$. *This block we choose to be* $\boldsymbol{F}^{\{1\}}$. *We choose* $p = red$, *the p–k–vectors are then given by*

$$(2,1,2,1) + \boldsymbol{e}^k \quad (k \in \boldsymbol{K}) \ .$$

*the P–subfaces suggested by the reference vector* $(2,1,2,1)$. *The block* $\boldsymbol{B}^y$ *has the square* $\boldsymbol{F}^3$ *as the boundary node at* $\partial\Pi^{(-3)}$ *(the left front side of the tetrahedron). This in our above notation is the* $\boldsymbol{P}$*–subface* $\boldsymbol{E}^{i_0}$. *Here, the description is*

(4) $$\boldsymbol{F}^3 : \begin{array}{cccc} \textit{blue} & \textit{red} & \textit{green} & \textit{yellow} \\ 12 & 1 & 24 & 2 \end{array} ,$$

*and the reference vector of is* $\boldsymbol{r}^- = (2,1,2,1)$.

*The adjacent face* $\boldsymbol{F}^{\{2\}}$ *is the cylinder that consists of a green triangle and a blue line segment; it is given by*

(5) $$\boldsymbol{Z}^{gr,b} : \begin{array}{cccc} \textit{blue} & \textit{red} & \textit{green} & \textit{yellow} \\ 12 & 3 & 234 & 2 \end{array} .$$

*The Link* $\boldsymbol{E}^2 \ = \ \boldsymbol{B}^y \cap \boldsymbol{Z}^{gr,b}$ *has the correct reference code* $(2,1,2,1)$.

*At* $\boldsymbol{Z}^{gr,b}$ *the path has two branches, the boundary basis* $\boldsymbol{E}^4$ *at the lower subsimplex is part of the cylinder. Thus, the path from* $\Pi{-}3$ *to* $\Pi^{-4}$ *(the lower base subsimplex) ends here and the identification between the blue–green squares is completed.*

*If we follow the second path, we reach a block without red. This one is difficult to recognize, it consists of three line segments of blue, green, and yellow color and is described by*

(6) $$\boldsymbol{B}^r : \begin{array}{cccc} \textit{blue} & \textit{red} & \textit{green} & \textit{yellow} \\ 12 & 3 & 34 & 24 \end{array} .$$

*The final face is the cylinder consisting of a blue triangle and a green line segment. It is described by*

(7) $$\boldsymbol{Z}^{b,gr} : \begin{array}{cccc} \textit{blue} & \textit{red} & \textit{green} & \textit{yellow} \\ 123 & 3 & 34 & 4 \end{array} .$$

*This cylinder has boundary subfaces* $\boldsymbol{E}^1$ *(at the right front side) and* $\boldsymbol{E}^2$ *(at the rear side of the tetrahedron), thus we have found all* $\boldsymbol{P}$*–subfaces* $\boldsymbol{E}^i, \ i \in \boldsymbol{K}$.

*Figure 3.2 shows* all *paths as indicated by the above sequence. The four maximal faces have been isolated from Figure 3.1. The P–subface corresponding the reference vector* $\boldsymbol{r}^- = (2,1,2,1)$ *in each Pareto face consists of a square with blue and green line segments as edges. One can nicely identify such a P–subface within*

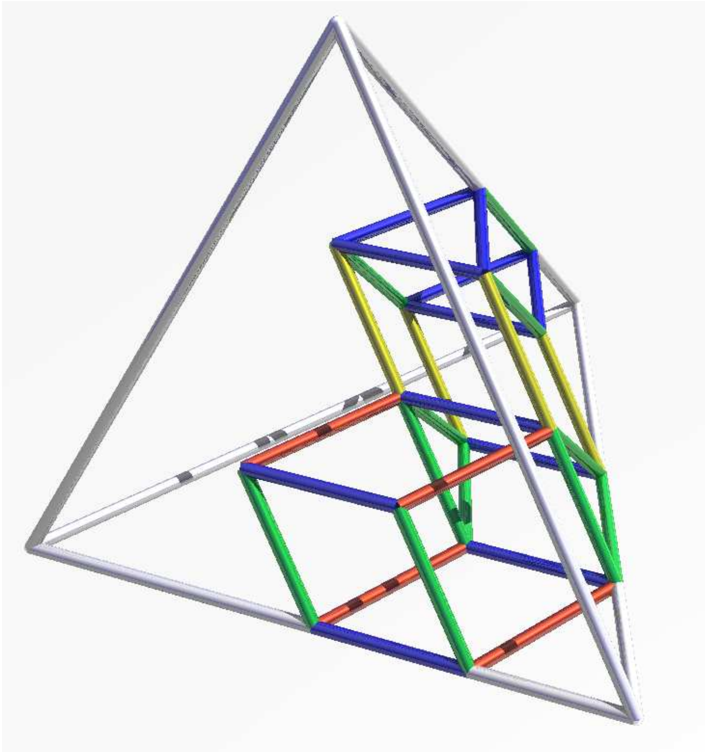

Figure 3.2: The Paths of Example 6.8

*each of the* 4 *Subsimplices of the Simplex* $4\Delta^{\boldsymbol{e}}$ *(the white tetrahedron)*

◦ ∼∼∼∼∼∼ ◦

# Chapter 8

# Computing Faces: $\mathbb{APL}$

Within our present framework, an algorithm is a Mathematical object; more or less the description of a recursively defined function the domain of which is the set of positive matrices interpreted as Cephoids. Algorithms in this sense have been documented in CHAPTERS 5 and 6.

To implement such an algorithm as a working program in a computer language ($\mathbb{APL}$ in our case) is of course a different task. For the curiosity of the IT–minded reader we want to exhibit a glimpse into the programs written along the guidelines of those previous Chapters.

Therefore we present a choice of programs resulting from algorithms for the computation of faces and apply them to particular examples. The programming is performed in $\mathbb{APL}$ ; indeed we have used results of these programs in previous chapters in the context of examples.

Those interested in a concrete set of programs and subroutines (a "workspace" in $\mathbb{APL}$ ) might consider contacting the author.

# 1 Algorithms in 𝔸ℙ𝕃

We start with a (partial) description of an 𝔸ℙ𝕃 workspace named "CEPHALGF" - that is a collection of functions and variables to be employed for computing the data of a Cephoid. Essentially these data are the Pareto face – represented by its collection of reference systems – and the corresponding set of reference vectors.

```
                       DESCRIPTION OF WS 'CEPHALGF'

                              CONTENTS

                 WS TO COMPUTE ALL FACES OF A CEPHOID

      **********************************
       THIS WS CREATED DEC 10 2011 TO DOCUMENT A VERSION
          OF THE FIRST FACE-GENERATING ALGORITHM
      **********************************

.)
A CEPHOID IS REPRESENTED BY A K×N MATRIX. THE ROWS OF THIS MATRIX
CORRESPOND TO THE DE GUA SIMPLICES INVOLVED. THAT IS, A[I;] REFLECTS
A SIMPLEX SPANNED BY VECTORS A^(K) = [0,...,A[K;I],0...,0] (I=1,...,N)
IN R^N_+ FOR K=1,...,KK.

EXAMPLE: CEPHPOID *BLUE* IS A SUM OF 3 VECTORS IN R^8

     BLUE
 110 130 150 170 190 210 230
  55  33  77  99  11   1  49
  98 112 133  14  67  44  23


∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘
.)
```

A FACE OF A CEPHOID IS REPRESENTED BY A N×K MATRIX. THIS MATRIX REPRESENTS THE REFERENCE FAMILY OF THE FACE, EACH ROW CORRESPONDING TO AN INDEX I ϵ II AND EACH COLUMN POINTING TO A DE GUA SIMPLEX K. THUS, COLUMN K REFLECTS THE INDEX SET J^(K) ⊂ II OF THE REFERENCE FAMILY DESCRIBING THE FACE.

EXAMPLE: BLUEF7 IS THE SUM OF TWO UNIT VECTORS E^4 AND THE FULL SIMPLEX DELTA^(3)

```
BLUEF7
 0 0 1
 0 0 2
 0 0 3
 4 4 4
 0 0 5
 0 0 6
 0 0 7
```

FACES CAN BE VISUALIZED BY FUNCTION ⋆ WR ⋆ WHICH WRITES THE REFERENCE FAMILY IN A MORE DIRECT WAY. THE REFERENCE SET IS IMMEDIATELY VISUALIYED.

EXAMPLE:

```
      WR BLUEF7
 4  ⊛  4  ⊛  1 2 3 4 5 6 7  ⊛
```

∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘
 .)
A FAMILY OF FACES IS A VECTOR, THE COORDINATES OF WHICH ARE INCLUSIONS OF THE MEMBERS OF THAT FAMILY. THUS, WE HAVE ρ VV134FACEN = 10 AND ρ VV134FACEN[1] = ZILDE WHILE ρ ⊃VV134FACEN[1] = 3 4 , AS THE INCLUDED 1 ST COOERDINATE IS A 3 4 MATRIX NAMELY

```
                    ⊃VV134FACEN[1]
                        1 1 1 1
                        0 0 0 2
                        0 0 0 3
```

THAT IS

WR ⊃VV134FACEN[1]

```
 1  ⊛  1  ⊛  1  ⊛  1 2 3  ⊛
```

NOW, THE WHOLE FAMILY CAN BE VISUALIZED VIA THE FUNCTION ⋆WRY⋆ AS FOLLOWS:

```
      WRY VV134FACEN
     1  ⊛      1  ⊛      1  ⊛  1 2 3  ⊛
     1  ⊛      2  ⊛    1 2  ⊛    2 3  ⊛
     1  ⊛      2  ⊛  1 2 3  ⊛      3  ⊛
     1  ⊛    1 2  ⊛    1 3  ⊛      3  ⊛
     1  ⊛  1 2 3  ⊛      3  ⊛      3  ⊛
     1  ⊛    1 2  ⊛      1  ⊛    2 3  ⊛
 1 2 3  ⊛      2  ⊛      3  ⊛      3  ⊛
   1 3  ⊛    2 3  ⊛      3  ⊛      3  ⊛
   1 2  ⊛      2  ⊛    2 3  ⊛      3  ⊛
   1 2  ⊛      2  ⊛      2  ⊛    2 3  ⊛
```

THIS VERSION REFLECTS THE COMPLETE PARETO SURFACE VIA THE REFERENCE SETS INVOLVED.

∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘

```
.)
  FUNCTION ⋆REFVECMAT⋆

  SYNTAX
             RS ← REFVECMAT P
⍝ ⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆
⍝ P IS A FAMILY OF FACES; TYPICALLY ALL FACES OF
⍝ A CEPHOID, HENCE A DESCRIPTION OF THE SURFACE OF
⍝ THAT CEPHOID. THAT IS, P IS A VECTOR THE COORDINATES
⍝ OF WHICH ARE INCLUDED N × K MATRICES REPRESENTING
⍝ FACES. THE FUNCTION ⋆REFVECMAT⋆ YIELDS THE
⍝ CORRESPONDING REFERENCE VECTORS. AS EACH FACE SUPPLIES
⍝ A REFERENCE VECTOR, THE RESULT IS A MATRIX THE
⍝ ROWS OF WHICH PROVIDE THE REFERENCE VECTORS OF THE
⍝ FACES OF THE FAMILY P.
⍝ ⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆
```

EXAMPLE, REFERRING TO THE ABOVE FAMILY ⋆VV134FACEN⋆ :

```
 REFVECMAT VV134FACEN
 1 1 1 3
 1 1 2 2
 1 1 3 1
 1 2 2 1
 1 3 1 1
 1 2 1 2
 3 1 1 1
 2 2 1 1
 2 1 2 1
 2 1 1 2
```

∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘

.)
FUNCTIONS ⋆LINCREASE⋆ AND ⋆FAMINCREASE⋆
SYNTAX:

```
MA ← I LINCREASE FACE


⍝ ⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆
⍝ THIS FUNCTION INCREASES ALL ENTRIES IN ROWS R ≥ I
⍝ OF ⋆ FACE ⋆
⍝ BY 1 AND LEAVES ALL ENTRIES IN ROWS R ≤ I UNCHANGED.
⍝ THUS, WHEN APPLIED TO A FACE OF A (K,N) CEPHOID,
⍝ THE RESULT IS NOT A FACE BUT YIELDS INGREDIENTS FOR
⍝ A FACE OF AN (K,N+1) CEPHOID.
⍝ ⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆
```

EXAMPLE:

```
BLUEF25
 0 1 1
 0 0 2
 0 3 0
 0 4 0
 5 0 5
 6 0 0
 0 7 0

        2 LINCREASE BLUEF25
```

```
 0 1 1
 0 0 3
 0 4 0
 0 5 0
 6 0 6
 7 0 0
 0 8 0
```

SIMILARLY (THERE IS NO 2 IN THE FOLLOWING FAMILY):

```
    WRY 2 FAMINCREASE VV134FACEN
     1  ⊛      1  ⊛      1  ⊛  1 3 4  ⊛
     1  ⊛      3  ⊛    1 3  ⊛    3 4  ⊛
     1  ⊛      3  ⊛  1 3 4  ⊛      4  ⊛
     1  ⊛    1 3  ⊛    1 4  ⊛      4  ⊛
     1  ⊛  1 3 4  ⊛      4  ⊛      4  ⊛
     1  ⊛    1 3  ⊛      1  ⊛    3 4  ⊛
 1 3 4  ⊛      3  ⊛      4  ⊛      4  ⊛
   1 4  ⊛    3 4  ⊛      4  ⊛      4  ⊛
   1 3  ⊛      3  ⊛    3 4  ⊛      4  ⊛
   1 3  ⊛      3  ⊛      3  ⊛    3 4  ⊛
∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘
```

.)
FUNCTION ⋆ FACEFROMLMA ⋆

SYNTAX:
FACE ← LMA FACEFROMLMA A

```
⍝ CONSTRUCTS A FACE FROM THE ADJUSTMENTINDICES.
⍝ MATRIX ⋆LMA⋆ REFLECTS AN ADJUSTMENTSYSTEM. THUS,
⍝ N IS THE SAME AS IN MATRIX ⋆A⋆ REPRESENTING THE CEPHOID
⍝ INVOLVED. THE NUMBER OF ROWS IS POSSIBLY SMALLER THAN K.
⍝ THEREFORE
⍝
⍝ ⋆ (LMATRIX  (⊃BLUEFACES[9]))FACEFROMLMA BLUE ⋆
⍝ AND
⍝ ⋆ ⊃BLUEFACES[9] ⋆
⍝ ARE IDENTICAL.
```

EXAMPLE:

```
     ⊃ BLUEFACES [9]
 0 0 1
 0 0 2
 0 0 3
 4 4 0
 0 0 5
 6 0 6
 7 0 0

    LMATRIX ⊃ BLUEFACES [9]
 4 4 0
 6 0 6

(LMATRIX  (⊃BLUEFACES[9]))FACEFROMLMA BLUE
 0 0 1
 0 0 2
 0 0 3
 4 4 0
 0 0 5
 6 0 6
 7 0 0
************************************************************
```

∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘

```
.)
FUNCTION  ⋆ ADJUSTCOEF ⋆

SYNTAX:
C ← FACE ADJUSTCOEF A

⍝ **********************************************************
⍝ COMPUTES THE ADJUSTMENT COEFFICIENTS OF A FACE ⋆FACE⋆
⍝ OF A CEPHOID THAT IS GIVEN BY A MATRIX A.
⍝ **********************************************************
  SUBROUTINES USED:
                  ⋆ COEFFLINADJ ⋆
                (GENERATES THE COEFFICIENTMATRIX OF THE
                LINEAR ADJUSTMENT SYSTEM),
                  ⋆ LINEQSOLV ⋆
                (SOLVES THE LINEAR EQUATION AX = B)
```

```
EXAMPLE:
      BLUEF7 ADJUSTCOEF BLUE
0.005882352941 0.0101010101 0.07142857143

∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘
.)
FUNCTION ⋆ ∆NORMAL ⋆

SYNTAX:
NOR ← F ∆NORMAL A

⍝ ⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆
⍝ COMPUTES THE NORMAL TO A FACE F OF A CEPHOID A.
⍝ USES THE LINEAR ADJUSTMENTSYSTEM, THUS COMPUTES
⍝ THE SOLUTION TO THE LINEAR ADJUSTMENTSYSTEM, THEN INVERTS
⍝ THE VECTOR COORDINATEWISE.
⍝ USED AS SUBROUTINES ARE THEREFORE FUNCTIONS
⍝ ⋆ ADJUSTCOEFF ⋆, ⋆ COEFFLINADJ ⋆, ⋆ LMATRIX ⋆ AND
⍝ THEIR SUBROUTINES.
⍝ ⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆

EXAMPLE:

      BLUEF7 ∆NORMAL BLUE
0.14285714285714284 0.125 0.10526315789473684 0.99999999999999984
      0.208955223880597 0.31818181818181816 0.60869565217391312
∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘

.)
FUNCTION ⋆DOBROWS⋆

SYNTAX:
ROWS ← DOBROWS FACE

⍝ ⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆
⍝
⍝ YIELDS THE ROWS OF THE MATRIX ⋆FACE⋆ THAT SHOW
⍝ A DOUBLE ENTRY.
⍝ THESE ARE THE ROWS CORRESPONDING TO ADJUSTMENT
```

```
⍝ INDICES OF THAT FACE
⍝***********************************

°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°
.) FUNCTIONS *BARYCENTER* AND *ALLREFVECS*

SYNTAX:
B ← N BARYCENTER K

⍝ GENERATES THE BARYCETRIC COORDINATES
⍝ OF THE SUM OF K UNIT SIMPLICES
⍝ IN N DIMENSIONS.
......

SYNTAX M ← K ALLREFVECS N
⍝ ***********************************************************
⍝ YIELDS THE REFERENCE VECTORS (R1, ... , RK) FOR DIMENSION N
⍝ USING PROGRAM ⍝ *BARYCENTER*.
⍝ THAT IS, INTEGER VECTORS WITH COORDINATE SUM N+K-1 .
⍝ ***********************************************************

°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°
.) FUNCTIONS *DUALFACE* AND  *DUALFAM*

SYNTAX:
DUAL ← DUALFACE F
FA ← DUALFAM FAM

⍝ **********************************************************
⍝ GENERATE THE DUAL FACE TO A FACE F AND, ACCORDINGLY,
⍝ THE FAMILY OF DUAL FACES TO A FAMILY FAM.
⍝ **********************************************************
°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°
°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°

THE FOLLOWING ARE THE MAIN FUNCTIONS WITHIN THIS WORKSPACE.
THEY ARE DESIGNED TO COMPUTE ALL FACES OF A CEPHOID.

°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°
⍝ °°°°°°°°*********°°°°°°°°°°°°************
```

```
.)
  FUNCTION ⋆∆FC⋆
  SYNTAX:
          FAM ← ∆FC A
⍝ ⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆
⍝ CREATES ALL FACES OF THE CEPHOID REPRESENTED BY THE
⍝ K × N MATRIX A.
⍝ THE OUTPUT, THEREFORE, IS A VECTOR, EACH COMPONENT OF
⍝ WHICH CONTAINS VIA INCLUSION AN N × K MATRIX REPRESENTING
⍝ A FACE.
⍝ ⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆

.)
  FUNCTION ⋆∆FCTIME⋆
  SYNTAX
         FAM ← ∆FCTIME A
⍝ ⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆
⍝ THIS IS FUNCTION ⋆∆FC⋆.  HOWEVER, A TIME MEASURING IS
⍝ INCORPORATED. THE TOTAL TIME FOR TO COMPLETE THE COMPUTATION
⍝ OF ALL FACES OF A CEPHOID IS RECORDED.
⍝ ⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆

⍝ ∘∘∘∘∘∘∘∘⋆⋆⋆⋆⋆⋆⋆⋆⋆∘∘∘∘∘∘∘∘∘∘∘∘⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘
.)
  FUNCTIONS ⋆∆FACK2NBEL⋆ , ⋆∆FACN2KBEL⋆
  ⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆⋆
  ARE THE SAME AS ∆FC AND ∆FCTIME, BUT FOR K=2/N ARBITRARY
  AND N=2/K ARBITRARY.
  HENCE SYNTAX AS IN:
          FAM ← ∆FC A
```

Next, just for curiosity, we present the most important function in this workspace which is "ΔFC" – i.e., the Face Creator.

```
FAM ← ∆FC A;K;N;I;AA;FAM0;REFM;REFM0;R;ZEILE;JANEIN;TIME
⍝ **********************************************************
⍝ CREATES ALL FACES OF THE CEPHOID REPRESENTED BY THE
⍝ K × N MATRIX A.
⍝ THE OUTPUT, THEREFORE, IS A VECTOR, EACH COMPONENT OF
⍝ WHICH CONTAINS VIA INCLUSION AN N × K MATRIX REPRESENTING
⍝ A FACE.
⍝ **********************************************************
⍝ **********************************************************
⍝ 1. THE BASIC DATA: K,N
K←(⍴A)[1] ⋄ N← (⍴A)[2]
⍝ **********************************************************
⍝ 2. THE CASES K=2 UND N=2:
⍝ IF K = 2 IS THE CASE:
→(2<K)/ NGL2
FAM← ∆FACK2NBEL A
→0
⍝ IF N = 2 IS THE CASE:
NGL2:
→(2<N)/ KKLEINN
⍝→(2<N)/ MAIN
```

```
FAM← ∆FACN2KBEL A
→0

⍝ **********************************************************
⍝ °°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°
⍝ WE HAVE TO ASSUME K ≤ N. OTHERWISE NOT ALL FACES CAN BE ACCESSED
⍝ FROM THE BOUNDARY
⍝
⍝ HENCE: 3. THE  RESTRICTION TO K≤N:
⍝            (OTHERWISE WE TAKE THE DUAL FAMILY AND
⍝             MIRROR THE RESULT)

KKLEINN:
→(K≤N)/MAIN

FAM ← DUALFAM ∆FC ⍉ A

→0

⍝ °°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°
⍝ **********************************************************
⍝ 4.  NOW THE INDUCTIVE LOOP:

⍝ °°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°
MAIN:
I←0
FAM←0⍴0
REFM←(0,K)⍴0

⍝ °°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°
UP:
→((⍴A)[2]<I←I+1)/EX
AA←A[;(⍳I-1),I+⍳N-I]
⍝ ................................................
⍝ CREATE THE FAMILY WITH ONE DIMENSION DECREASED BY 1
FAM0 ← ∆FC AA
⍝  ... AND AUGMENT IT:
⍝°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°°
⍝ TO THIS END THE FUNCTION ⋆FCFIRSTSTEP⋆ IS EMPLOYED.
⍝ THE OPERATION REQUIRED ASKS FOR THE FACES AVAILABLE
⍝ TO BE SEEN AS N-1 BOUNDARIES OF N-FACES. THE AUGMENTATION
```

```
⍝ ALSO HAS TO RESPECT THE INDEX I WHICH WAS DELETED ABOVE.
⍝ NOTE THAT I IS THE SAME AS ABOVE, HENCE IS CARRIED OVER
⍝ TO THE PRESENT OPERATION.
⍝
⍝ NOW, THE FUNCTIONS ⋆FACINCREASE⋆,
⍝ ⋆FAMINCREASE INCREASE ALL ENTRIES
⍝ IN ROWS ≥ I BY ONE, THUS CREATES A FAMILY
⍝ THAT IS MISSING INDEX I AND CAN BE USED TO ESTABLISH
⍝ A FAMILY OF THE NEXT HIGHER DIMENSION.

⍝ FUNCTION FCFIRSTSTEP:

FAM0 ←(I FAMINCREASE FAM0) FCFIRSTSTEP A
REFM0 ← REFVECMAT FAM0

⍝ ○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○
⍝ THE REFERENCE MATRIX OBTAINED IS NOW COMPARED WITH THE
⍝ REFERENCE MATRIX FROM THE PREVIOUS INDUCTIVE STEP.
⍝ IF A ROW OF THE NEW MATRIX IS ALREADY PRESENT IN THE OLD ONE,
⍝ THEN BOTH ARE BEING CANCELLED FROM BOTH FAMILIES OF FACES
⍝ THAT CORRESPOND.

R←0
JANEIN←(ρFAM0)ρ1
LINEUP:
→( (ρFAM0) <R←R+1)/CONT
ZEILE ← REFM0[R;]
JANEIN[R] ←0= ZEILE VECINMATRIX REFM
→LINEUP
⍝○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○○
CONT:

⍝ THE NEW REFERENCE VECTORS AND FACES ARE NOW AT HAND
⍝ BOTH ARE BEING CATENATED TO THE DATA OBTAINED IN THE
⍝ PREVIOUS INDUCTIVE STEP.
⍝ THAT IS, REFM0 IS CATENATED TO REFM
⍝ AND FAM0 IS CATENATED TO FAM:


FAM0←JANEIN /FAM0
REFM0 ← JANEIN /[1] REFM0
```

```
[113]  ⍝
[114]  FAM←FAM,FAM0
[115]  REFM←REFM,[1]REFM0
[116]  ⍝
[117]  ⍝ NOW INCREASE I:
[118]
[119]  →UP
[120]  ⍝ ∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘
[121]  EX:
```

## 2 Examples: Large Cephoids

For completeness we repeat some of the examples already discussed in previous chapters. The first one is the “saw” (see Section 2 Chapter 2). As the naive version considered contains zeros and possibly other degeneracies we construct a slightly modified version by adding 100 and perturbing the example.

```
DATA OF CEPHOID ``SAW'':


SAW

 2 3 2
 1 4 0
 5 2 6

 SAW100

 203 303 203
 103 403   3
 503 203 603
```

```
THE FACES OF  SAW100:

    2  ⊛      2  ⊛  1 2 3  ⊛
1 2 3  ⊛      2  ⊛      3  ⊛
    3  ⊛  1 2 3  ⊛      3  ⊛
    1  ⊛    1 2  ⊛    1 3  ⊛
  1 3  ⊛    1 2  ⊛      3  ⊛
  1 2  ⊛      2  ⊛    1 3  ⊛
```

The next one is the notorious windmill (see Section 2, Chapter 2, Figure 2.4 – and Section 6, Chapter 3, Figure 6.2).

```
     THE CEPHOID WINDMILL :

     WINDM

     2 4 2
     3 3 6
     5 1 1

     THE FACES OF WINDMILL

    2  ⊛    2 3  ⊛    1 2  ⊛
    2  ⊛      3  ⊛  1 2 3  ⊛
  2 3  ⊛      3  ⊛    1 3  ⊛
  1 2  ⊛    1 3  ⊛      1  ⊛
1 2 3  ⊛      3  ⊛      1  ⊛
    2  ⊛  1 2 3  ⊛      1  ⊛
```

The following is a simple $3 \times 4$ example – The Cephoid KKK34:

```
THE CEPHOID KKK34:


      KKK34

 701  202 103  904
 505  106 107 1108
 309 1010 611  812


FACES OF THE CEPHOID KKK34:

      WRY KKK34FAC

   1 2 3  ⊛      3 4  ⊛        3  ⊛
 1 2 3 4  ⊛        4  ⊛        3  ⊛
     1 4  ⊛        4  ⊛    2 3 4  ⊛
       2  ⊛  1 2 3 4  ⊛        2  ⊛
   1 2 4  ⊛        4  ⊛      2 3  ⊛
     1 2  ⊛      3 4  ⊛      2 3  ⊛
       1  ⊛        1  ⊛  1 2 3 4  ⊛
       1  ⊛      1 4  ⊛    2 3 4  ⊛
       1  ⊛    1 3 4  ⊛      2 3  ⊛
     1 2  ⊛    1 3 4  ⊛        2  ⊛
```

The Cephoid *Blue* has our special attention. We provide two lists (overlapping in a sense): The first set of faces is obtained by taking two of the involved deGua Simplices and constructing the permutations that create all their faces ( Section 4 Chapter 3). The resulting faces of the Cephoid

"BLUE" are then seen in the light of the "moving index principle". The second list is just the result of applying ΔFC – hence contains all Pareto faces.

```
          BLUEPERMS
 *************************  THE CEPHOID BLUE  *********

             BLUE
 110 130 150 170 190 210 230
  55  33  77  99  11   1  49
  98 112 133  14  67  44  23

      ρ BLUEFACE
28


-------------------------------------------------


      PERMUTATIONS OF  BLUE[1 2 ;] :


6 5 7 2 1 3 4          4     1
6 5 7 2 1 3          3 4     1
6 5 7 2 1          1 3 4     1
6 5 7 2          2 1 3 4     2
6 5 7          7 2 1 3 4     2
6 5          5 7 2 1 3 4     5
6          6 5 7 2 1 3 4     6


+++++++++++++++++++++++++++++++++++++++++++++++++++++++++

      PERMUTATIONS OF  BLUE[1 3 ;] :


4 7 6 5 2 3 1   4    1
4 7 6 5 2 3     4    3 1
```

```
4 7 6 5 2       4    2 3 1
4 7 6 5         4    5 2 3 1
4 7 6           4    6 5 2 3 1
4 7             4    7 6 5 2 3 1
4               4    4 7 6 5 2 3 1


+++++++++++++++++++++++++++++++++++++++++++++++++++++++++

      PERMUTATIONS OF  BLUE[2 3 ;] :

6     4 7 3 1 2 5 6         6
6     4 7 3 1 2 5         5 6
6     4 7 3 1 2         2 5 6
6     4 7 3 1         1 2 5 6
6     4 7 3         3 1 2 5 6
7     4 7         7 3 1 2 5 6
4     4         4 7 3 1 2 5 6



                PARETO FACES OF POLYHEDRON BLUE
****************************************************************
                       *                        *
          6    1    6  *  +++++++++++++++++++++ *
               2       *           4    4    1  *  +++++++++++++++++++++
               3       *           6         2  *           5    4    1
               4       *           7         3  *           6         3
               6       *                     5  *           4         2
               5       *                     6  *           2
               7       *  +++++++++++++++++++++ *           7
 +++++++++++++++++++++ *           6    4    1  *  +++++++++++++++++++++
          7    4    1  *           4         2  *           5    4    1
               7    2  *           5         3  *           6    3    2
                    3  *           7         5  *           2         3
                    7  *  +++++++++++++++++++++ *           7
                    5  *           2    4    1  *  +++++++++++++++++++++
                    6  *           5    3    3  *           2    3    1
 +++++++++++++++++++++ *           3            *           5    4
          6    4    1  *           6            *           1    1
          7    7    2  *           7            *           6
```

```
                   3 * ++++++++++++++++++++ *          7
                   6 *          5    4    1 * ++++++++++++++++++++
                   5 *          7    7    2 *          5    3    2
++++++++++++++++++++ *          6    3    3 *          7    4    1
         5    1    5 * ++++++++++++++++++++ *          6    7
         6    2      *          6    4    1 *               1
              3      *               3    2 * ++++++++++++++++++++
              4      *               7    5 *          6    3    2
              5      *                    3 *               4    5
              7      *                    6 *               1    1
++++++++++++++++++++ * ++++++++++++++++++++ *               7    6
         7    4    1 *          6    4    1 * ++++++++++++++++++++
         5    7    2 *          5    3    2 *          6    3    2
         6         3 *               7    3 *          5    4    1
                   5 *                    5 *               1    5
++++++++++++++++++++ * ++++++++++++++++++++ *               7
         6    1    5 *          2    4    1 * ++++++++++++++++++++
              2    6 *          5         3 *          2    4    1
              3      *          6           *          3
              4      *          4           *          5
              5      *          3           *          6
              7      *          7           *          4
++++++++++++++++++++ * ++++++++++++++++++++ *          1
         4    4    1 *          5    1    2 *          7
                   2 *          2    3      * ++++++++++++++++++++
                   3 *          6    4      *          2    4    1
                   5 *          7    2      *          5    3
                   4 * ++++++++++++++++++++ *          6
                   6 *          5    1    2 *          3
                   7 *          7    3      *          1
++++++++++++++++++++ *          6    4      *          7
         7    4    1 *               7      * ++++++++++++++++++++
         4         2 *               2      *          5    3    2
                   3 * ++++++++++++++++++++ *          6    4    1
                   5 *          6    1    5 *          2    1
                   7 *               3    2 *          7
                   6 *               4    6 * ++++++++++++++
++++++++++++++++++++ *               2      * .
.                    *               7      * .
.                    * ++++++++++++++++++++ * .
.                    *          6    1    2 * .
```

```
.                       *           5    3    5  * .
.                       *                4       * .
.                       *                2       * .
.                       *                7       * .
.                       *  +++++++++++++++++++++ * .
```

Finally, we compute the list of faces of a "large" example. Again, this is just a matter of curiosity – one is not to read all the data but inspect them for possible compositions of interest.

```
      THE CEPHOID LARBLUE:

      LARBLUE
 110 130 150 170 190 210 230
  55  33  77  99  11   1  49
  98 112 133  14  67  44  23
  12  27  55  44  36  95 111
  10  23  49  34  26  83  97
  58  39  86 111  30  19  70

FACES OF LARBLUE:
```

```
5 6 ⊛ 1 3 4 ⊛ 2 ⊛ 6 7 ⊛ 7 ⊛ 2 4 7 ⊛
2 4 5 ⊛ 4 ⊛ 1 2 ⊛ 4 ⊛ 3 4 6 7 ⊛ 4 ⊛
2 4 5 ⊛ 4 ⊛ 1 2 ⊛ 4 6 7 ⊛ 3 7 ⊛ 4 ⊛
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# Chapter 9

# Applications: Optimization

The remainder of this volume is dedicated to various applications for which Cephoids turned out to be relevant. We point out those topics in Applications of Mathematics in which Cephoids appear naturally in context: these are Optimization ("Operations Research"), Mathematical Economics, and Cooperative Game Theory.

The first example is the Standard Rucksack Problem which deals with a simple optimization procedure. Later on, we will turn to Free Trade Theory and applications in the Theory of Cooperative Games.

# 1 The Rucksack Problem: An Interpretation in Linear Programming

Suppose a hiker wanting to ascend a mountain wishes to limit the weight of his rucksack to a unit (of 20 kg). He intends to pack various foods $i = 1, \ldots, n$. The weight per unit of food $i$ is given by $\frac{1}{a_i}$. Now, the hiker wants to obtain maximal nourishment from what he carries and it is known that, for $i \in \boldsymbol{I}$, the nutritive quality of a unit of food $i$ is given by some quantity $n_i$.

Consider any plan $\boldsymbol{x} = (x_1, \ldots x_n) \in \mathbb{R}^n_+$ of the hiker, implying that he takes the quantity $x_i$ of food $i$. Then the weight to be attached to this collection of foods is

$$\sum_{i=1}^{n} \frac{x_i}{a_i}$$

and must not exceed 1. Therefore, the hiker has to solve the Linear Program suggested by

$$\max \left\{ \sum_{i=1}^{n} n_i x_i \,\middle|\, \boldsymbol{x} \in \mathbb{R}^n_+,\ \sum_{i=1}^{n} \frac{x_i}{a_i} \leq 1 \right\} \;=\; \max \left\{ \sum_{i=1}^{n} n_i x_i \,\middle|\, \boldsymbol{x} \in \Pi^{\boldsymbol{a}} \right\} \quad .$$

This kind of a simple Linear Program is generally called a "rucksack problem".

Now it so happens that there is a small elevator available at the mountain area. This device is very sturdy, so the weight to be carried is not a restriction, at least as far as foods are concerned. However, the volume to be transported is limited; for convenience assume that the device carries a unit in volume maximally.

If food $i \in \boldsymbol{I}$ yields a volume of $\frac{1}{b_i}$ per unit, then any plan $\boldsymbol{y} \in \mathbb{R}^n$ of transporting a volume of $y_i$ $(i \in \boldsymbol{I})$ by the elevator results in a total volume of

$$\sum_{i=1}^{n} \frac{y_i}{b_i},$$

hence maximal nourishment is obtaint by solving the Linear Program suggested by

$$\max \left\{ \sum_{i=1}^{n} n_i y_i \,\middle|\, \boldsymbol{y} \in \mathbb{R}^n_+,\ \sum_{i=1}^{n} \frac{y_i}{b_i} \leq 1 \right\} \;=\; \max \left\{ \sum_{i=1}^{n} n_i y_i \,\middle|\, \boldsymbol{y} \in \Pi^{\boldsymbol{b}} \right\} \quad .$$

Finally, a hiker having both, his rucksack and the elevator available is looking for

$$\max\left\{\sum_{i=1}^{n} n_i x_i \,\middle|\, \boldsymbol{x} \in \mathbb{R}_+^n,\ \boldsymbol{x} = \boldsymbol{x}' + \boldsymbol{x}'' \ , \quad \boldsymbol{x}' \in \Pi^{\boldsymbol{a}} \ , \boldsymbol{x}'' \in \Pi^{\boldsymbol{b}} \right\}$$

which is

$$\max\left\{\sum_{i=1}^{n} n_i x_i \,\middle|\, \boldsymbol{x} \in \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}} \right\}.$$

Therefore, let us consider a family $\boldsymbol{a}^{\bullet} = (\boldsymbol{a}^{(1)}, \ldots, \boldsymbol{a}^{(K)}) = \left(\boldsymbol{a}^{(k)}\right)_{k \in \boldsymbol{K}}$ of positive vectors generating the cephoid

$$\Pi \ = \ \Pi^{\boldsymbol{a}^{\bullet}} = \sum_{k=1}^{K} \Pi^{(k)}$$

with $\Pi^{(k)} \ = \ \Pi^{a^{(k)}} (k \in \boldsymbol{K})$, and suppose we want to maximize the linear functional $\boldsymbol{x} \mapsto \boldsymbol{n}\boldsymbol{x}$. Then we can see each $\Pi^{(k)}$ as representing a production process called "line k". All lines produce the same good ("nourishment").

A unit of raw material or production factor $i$ put into activity at line $k$ requires an amount of $\frac{1}{a_i^{(k)}}$ of the capacity of line $k$. The production processes or lines can be operated independently and the results can be added. Thus, maximizing the linear functional defined above amounts to determining

$$\max\left\{\sum_{i=1}^{n} n_i x_i \,\middle|\, \boldsymbol{x} \in \Pi^{\boldsymbol{a}^{\bullet}} = \sum_{k=1}^{K} \Pi^{(k)} \right\} \quad .$$

The maximizers or optimal elements of this Linear Program are to be found in the Pareto faces of $\Pi = \Pi^{\boldsymbol{a}^{\bullet}}$. Clearly, we are now motivated to consider the description of the Pareto faces which has been provided in the previous chapters of this volume.

Suppose now $\boldsymbol{F}$ is a Pareto face of $\Pi$ containing maximizing elements, more precisely, assume it so happens that the normal $\mathfrak{n}^{\star}$ of $\boldsymbol{F}$ is the objective function of the Linear Program. That is, consider the case that $\mathfrak{n}^{\star} = \boldsymbol{n}$ happens to be true. Now, for $\boldsymbol{F}$, let $\boldsymbol{c}^{\star}$ denote the adjustment coefficients which can be computed employing the linear adjustment system. Now there is a global rucksack problem suggested via

$$\max\left\{\mathfrak{n}^{\star}\boldsymbol{x} \,\middle|\, \boldsymbol{x} \in \bigvee_{k \in \boldsymbol{K}} c_k^{\star}\Pi^{(k)} \right\} = \max\left\{\boldsymbol{n}\boldsymbol{x} \,\middle|\, \boldsymbol{x} \in \bigvee_{k \in \boldsymbol{K}} c_k^{\star}\Pi^{(k)} \right\}$$

the optimal solutions of which contain the ones of the original many line problem (suggested by the Cephoid $\Pi$). The deGua Simplex

$$\widehat{\Pi} := \bigvee_{k \in \boldsymbol{K}} c_k^\star \Pi^{(k)}$$

represents the new "global" production line which is obtained via

$$a_i^\star = \max_{k \in \boldsymbol{K}} c_k a_k^{(i)} ,$$

i.e., $\widehat{\Pi} = \Pi^{\boldsymbol{a}^\star}$. Thus, the capacities are defined by

$$\frac{1}{a_i^\star} = \frac{1}{\max_{k \in \boldsymbol{K}} c_k a_k^{(i)}} = \min_{k \in \boldsymbol{K}} \frac{1}{c_k a_k^{(i)}} .$$

This is readily interpreted as follows: we may adjust the capacities of the various production lines appropriately (to admit comparison of productivity !) and then take the minimal capacity in order to obtain the global production process.

The optimal solutions are of the form $c_k^\star \boldsymbol{a}^{(k)i} (i \in \boldsymbol{J}^{(k)})$. Finally, we obtain

$$\begin{aligned}
\max\left\{ \boldsymbol{n}\boldsymbol{x} \,\middle|\, \boldsymbol{x} \in \bigvee_{k \in \boldsymbol{K}} c_k^\star \Pi^{(k)} \right\} &= \max\left\{ \mathfrak{n}^\star \boldsymbol{x} \,\middle|\, \boldsymbol{x} \in \bigvee_{k \in \boldsymbol{K}} c_k^\star \Pi^{(k)} \right\} \\
&= \mathfrak{n}^\star c_k^\star \boldsymbol{a}^{(k)i} \\
&= t^\star \\
&= n_i^\star a_i^{(k)} c_k^\star \\
(1) \qquad &= n_i^\star \bar{a}_k^{(i)} c_k^\star \\
&= n_i^\star \bar{\boldsymbol{a}}^{(i)k} \boldsymbol{c}^\star \\
&= t^\star \\
&= \max\left\{ \boldsymbol{y}\boldsymbol{c}^\star \,\middle|\, \boldsymbol{y} \in \bigvee_{i \in \boldsymbol{I}} n_i^\star \bar{\Pi}^{(i)} \right\} \\
&= \max\left\{ \boldsymbol{y}\boldsymbol{c}^\star \,\middle|\, \boldsymbol{y} \in \bigvee_{i \in \boldsymbol{I}} n_i \bar{\Pi}^{(i)} \right\}
\end{aligned}$$

which is the "duality theorem of Cephoidal Programming".

# Chapter 10

# Applications: Free Trade

Our next example for Cephoids deals with a more than 200 years old concept in Macroeconomics. This theory is considered to be the basis of Free Trade Theory. It is the model of David Ricardo. Ricardo's theory establishes a first version of efficiency gains when Free Trade is admitted. as compared to production in autarky. The author is greatly indebted to Wolfram F. Richter who pointed out to him the relevance of the subject in context with the theory of Cephoids. As a result, the subsequent presentation is a version of [24].

# 1 Ricardian Free Trade

As to the basis for our presentation, we refer to David Ricardos Volume "On the Principles of Political Economy and Taxation", see [23]. There are quite a few modern reprints, we also cite McKenzie [14], Jones [11] or – for a textbook reference – Caves et al. [2]. Ricardo considers a model of two countries (Britain and Portugal), each of them producing two commodities (cloth and wine) but at different production costs (essentially in labor). We exhibit the general structure if one admits an arbitrary number of commodities produced by an arbitrary number of countries – which turns out to be a Cephoid.

We start with a single country producing in autarky. Let $\boldsymbol{I} := \{1, ...n\}$ denote the ***commodities*** and let $L$ denote the country's supply of ***labor***. Then we set up a "Ricardian" model of production and trade as follows.

Let $\hat{b}_i > 0$ denote the ***input coefficient*** i.e., the amount of labor required to produce one unit of commodity $i \in \boldsymbol{I}$. Then $\hat{a}_i := \frac{1}{\hat{b}_i} > 0$ is the ***productivity of labor*** with respect to commodity $i \in \boldsymbol{I}$, i.e., the number of units of commodity $i$ that can be produced with the input of one hour of labor.

We write $x_i$ for a quantity of commodity $i \in \boldsymbol{I}$. Then $\boldsymbol{x} = (x_1, \ldots x_n) \in \mathbb{R}^n_+$ is a ***plan*** according to which the amount $x_i$ of commodity $i$ is being produced. This plan results in an aggregate amount

$$\sum_{i=1}^{n} x_i \hat{b}_i \quad ,$$

of labor reqired to produce the vector $\boldsymbol{x}$. The plan $\boldsymbol{x}$ is ***feasible*** if the aggregate demand does not exceed the total supply of labor $L$. Consequently, the feasible plans are represented by the (deGua) Simplex

$$\left\{ \boldsymbol{x} \in \mathbb{R}^n_+ \;\middle|\; \sum_{i=1}^{n} x_i \hat{b}_i \leq L \right\} .$$

We introduce the notation $a_i := L\hat{a}_i$, hence $b_i := \frac{1}{a_i} = \frac{1}{L\hat{a}_i} = \frac{\hat{b}_i}{L}$ is the *relative* amount of labor neccessary in order to produce a unit of commodity $i$. Accordingly $\boldsymbol{a} = (a_1, \ldots, a_n)$ is the ***capacity vector***.

Then the above ***feasible set*** can as well be rewritten as

$$
(1)\quad \begin{aligned}
&\left\{\boldsymbol{x}\in\mathbb{R}^n_+ \,\middle|\, \sum_{i=1}^n x_i\hat{b}_i \le L\right\} \;=\; \left\{\boldsymbol{x}\in\mathbb{R}^n_+ \,\middle|\, \sum_{i=1}^n \frac{x_i\hat{b}_i}{L} \le 1\right\} \;=\\
&\left\{\boldsymbol{x}\in\mathbb{R}^n_+ \,\middle|\, \sum_{i=1}^n \frac{x_i}{L\hat{a}_i} \le 1\right\} \;=\; \left\{\boldsymbol{x}\in\mathbb{R}^n_+ \,\middle|\, \sum_{i=1}^n \frac{x_i}{a_i} \le 1\right\} \;=:\; \Pi^{\boldsymbol{a}}\,.
\end{aligned}
$$

That is, we obtain the DeGua Simplex generated by the capacity vector $\boldsymbol{a}$.

The ***efficient plans*** are given by the Simplex

$$
\Delta^{\boldsymbol{a}} = \left\{\boldsymbol{x}\in\mathbb{R}^n_+ \,\middle|\, \sum_{i=1}^n \frac{x_i}{a_i} = 1 \right\}
$$

which is the Pareto surface of $\Pi^{\boldsymbol{a}}$.

An efficient plan $\overline{\boldsymbol{x}}$ is said to be supported by a ***price vector*** $\boldsymbol{p} = (p_1,\ldots,p_n)$ if $\boldsymbol{p}\overline{\boldsymbol{x}}$ maximizes th linear funtional $\boldsymbol{x}\to\boldsymbol{p}\boldsymbol{x}$ over $\Pi^{\boldsymbol{a}}$.

The efficient production plan $\boldsymbol{a}^i \;:=\; a_i\boldsymbol{e}^i \in \Delta^{\boldsymbol{a}}$ allots *all* labor available to the production of *one* commodity $i$; these plans represent ***complete specialization*** of the economy. The total amount of labor available is employed to produce just one commodity $i$.

Figure 1.1 represents the familiar DeGua Simplex – now reinterpreted as a feasible set of production plans for 3 commodities. The completely specializing production plans appear as the vertices of the Simplex. Consequently, the length $a_i$ $(i\in\boldsymbol{I})$ of the line segments from the origin to a vertex represent the capacities.

Recall that input coefficients are given by $b_i = \frac{1}{a_i}$. The input coefficients can be seen as the coordinates of a price vector

$$
\mathfrak{n} \;:=\; \left(\frac{1}{a_1},\ldots,\frac{1}{a_n}\right) \;=\; (b_1,\ldots,b_n)
$$

which is *supporting* for *all* efficent production plans of the Simplex $\Delta^{\boldsymbol{a}}$. For short, $\mathfrak{n}$ is supporting $\Delta^{\boldsymbol{a}}$. Geometrically, this vector is the *normal* to $\Delta^{\boldsymbol{a}}$. All price vectors supporting $\Delta^{\boldsymbol{a}}$ are multiples of $\mathfrak{n}$, thus up to a multiple, the supporting prices to $\Delta^{\boldsymbol{a}}$ are uniquely determined.

We now enhance our model by introducing several countries into the production scene, we assume that there are $K$ countries each one producing

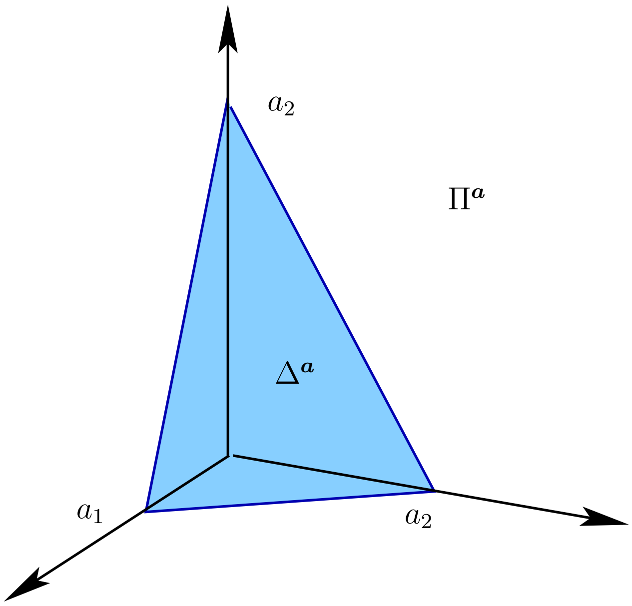


Figure 1.1: Efficient Plans – the DeGua Simplex $\Pi^{\boldsymbol{a}}$

the same commodities $i \in \boldsymbol{I}$. Then $\boldsymbol{K} := \{1, \ldots, K\}$ denotes the list of these countries. Each country $k \in \boldsymbol{K}$ is characterized by a ***capacity vector*** $\boldsymbol{a}^{(k)} = (a_1^{(k)}, \ldots, a_n^{(k)}) \in \mathbb{R}^n$. Accordingly, for each country the feasible production plans are provided by the DeGua Simplex $\Pi^{(k)} := \Pi^{\boldsymbol{a}^{(k)}}$.

A set of feasible production plans, one for each country, is a *production plan schedule*, that is, a list $(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(K)})$. The ***aggregate production*** of the world economy resulting thereby is given by the *sum* $\boldsymbol{x}^{(1)} + \ldots + \boldsymbol{x}^{(K)}$. Thus,

$$(2) \qquad \begin{aligned} \Pi \;=\; \Pi^{\boldsymbol{a}^{\bullet}} &:= \sum_{k=1}^{K} \Pi^{\boldsymbol{a}^{(k)}} \\ &= \left\{ \boldsymbol{x}^{(1)} + \ldots + \boldsymbol{x}^{(K)} \,\middle|\, \boldsymbol{x}^{(1)} \in \Pi^{(1)}, \ldots, \boldsymbol{x}^{(K)} \in \Pi^{(K)} \right\} \end{aligned}$$

is the set of aggregates of production schedules, for short the ***global plans***. Obviously, we obtain a sum of deGua Simplices, that is, the aggregate production schedules constitute a Cephoid.

We recall the sketch presented in Section 1 of Chapter 1. Now we interpret the first two sketches in Figure 1.2 as representing the situation for two countries and two commodities. Then the feasible production sets for each country are represented by the triangles $\Delta^{\boldsymbol{a}^{(1)}}$ and $\Delta^{\boldsymbol{a}^{(2)}}$. The global plans appear as the (algebraic) sum of these triangles in $\mathbb{R}^2_+$, represented by $\Pi$. The Cephoid $\Pi'$ is a sum of 4 triangles.

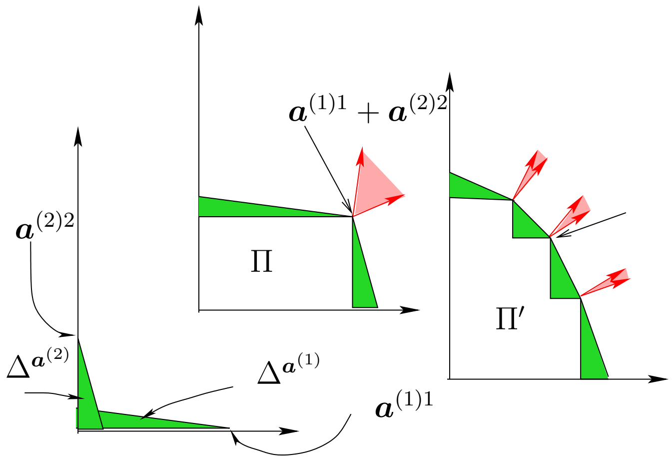


Figure 1.2: Construction of global plans

According to Theorem 2.3 in CHAPTER 1, all polyhedral production plans (i.e., compact, convex comprehensice polyhedra with a piecewise linear Pareto boundary) are being obtained as aggregate production plans of a certain set of countries with suitable capacity vectors.

From an Economical viewpoint, the idea of "comparative advantages" is relevant here: not every aggregation of complete specializations results in a vertex of the cephoid, that is in a Pareto efficient global plan. *The sum of two vertices is a vertex if and only if they admit of a common normal* . That is to say, *a production schedule of complete specializations in all countries results in an efficient global plan if and only if there is a common price vector supporting the plan of each country involved.*

E.g. in Figure 1.2 we observe that $\boldsymbol{a}^{(1)1} + \boldsymbol{a}^{(2)2}$ is Pareto efficient but, say, $\boldsymbol{a}^{(1)2} + \boldsymbol{a}^{(2)1}$ is not. Even more significant, consider the case that, in autarky, both countries are producing at the center of their capacities, i.e, country $k$ chooses

$$\frac{\boldsymbol{a}^{(k)1} + \boldsymbol{a}^{(k)2}}{2} = \frac{(a_1^k, a_2^k)}{2} = \frac{\boldsymbol{a}^{(k)}}{2}$$

Then the global plan resulting if each country sticks to its production schedule would be

$$\frac{\boldsymbol{a}^{(1)}}{2} + \frac{\boldsymbol{a}^{(2)}}{2} = (\frac{a_1^1 + a_1^2}{2}, \frac{a_2^1 + a_2^2}{2}) < (a_1^1, a_2^2) = \boldsymbol{a}^{(1)1} + \boldsymbol{a}^{(2)2} ,$$

as $a_1^2 < a_1^1$ and $a_2^1 < a_2^2$. There are no prices at which countries 1 and 2 can jointly and efficiently produce when each of them chooses the central production plan.

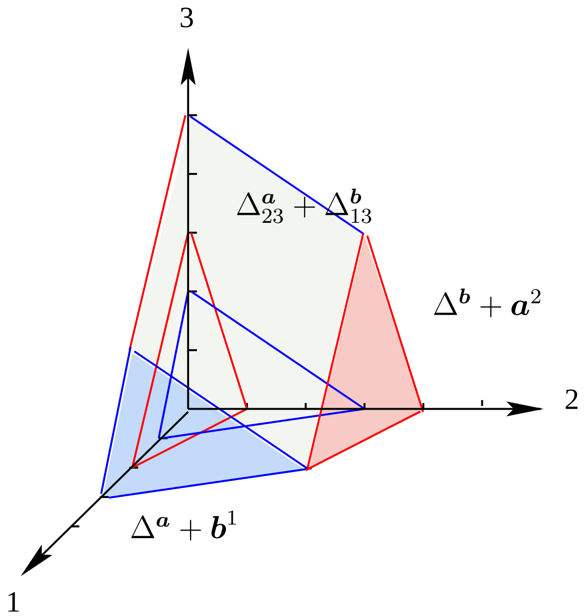


Figure 1.3: Incomplete Specialisation

Similarly Figure 1.3 (again taken from Chapter 1) refers to the production of 3 commodities in 2 countries. The resulting cephoid of global plans $\Pi = \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}$ has a Pareto surface $\partial\Pi$ consising of translates of Simplices ($\Delta^{\boldsymbol{a}} + \boldsymbol{b}^1$ and $\Delta^{\boldsymbol{b}} + \boldsymbol{a}^2$) and the parallelogram or ***rhombus*** $\Delta^{\boldsymbol{a}}_{23} + \Delta^{\boldsymbol{b}}_{13}$.

In the first case, (in order to produce efficiently) one country is completely specialized while for the other one *any* efficient plan is admitted.

In the second the rhombus is a sum of two Subsimplices. Such a Subsimplex, say $\Delta^{\boldsymbol{a}}_{23}$, consist of vectors that are convex combinations of extremals $\boldsymbol{a}^{(2)2}$ and $\boldsymbol{a}^{(2)3}$ hence represent production plans involving commodities 2 and 3 only but not commodity 1. This we interprete naturally as a ***partial*** or ***incomplete specialization*** of the country reflected by $\boldsymbol{a}$ on commodities $2, 3$ while the country reflected by $\boldsymbol{b}$ partially specializes in commodities 1 and 3.

Generally, if $\bar{\boldsymbol{x}} = \sum_{k=1}^{K} \boldsymbol{x}^{(k)}$ is efficient in $\Pi = \sum_{k=1}^{K} \Pi^{\boldsymbol{a}^{(k)}}$, then we use an appropriate notation: we say that country $k$ is ***completely specialized*** if $\boldsymbol{x}^{(k)} = a_i^{(k)} \boldsymbol{e}^i = \boldsymbol{a}^{(k)i}$ holds true for some $i \in \boldsymbol{I}$.

Also, given $\bar{\boldsymbol{x}}$, country $k$ is said to be ***partially specialized*** if there is a nonempty subset of commodities $\boldsymbol{J}^{(k)} \subseteq \boldsymbol{I}$ such that $\boldsymbol{x}^{(k)} \in \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$.

Thus, plans at which economy $k$ is partially specialized w.r.t the same subset

$\boldsymbol{J}^{(k)}$ of commodies used in production constitute the subsimplex $\Delta^{(k)}_{\boldsymbol{J}^{(k)}} \subseteq \Delta^{(k)}$.

For more than two countries and 3 commodities we repeat Figure 2.8 from CHAPTER 1 which illustrates the situation.

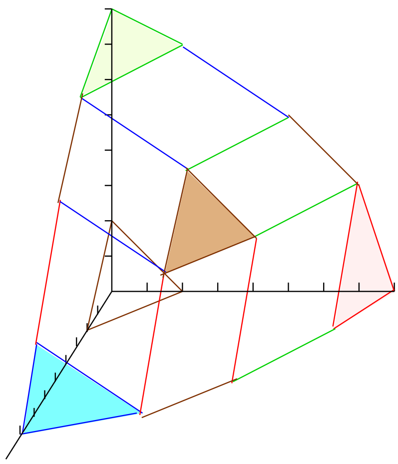

Figure 1.4: Four Countries producing 3 commodities

For $n = 3$ the shape of maximal faces does not change. any Pareto face is the sum of either 3 vertices and a full Simplex (the translates of a Simplex) or else the sum of 2 vertices and two straight lines (the rhombi). The vertices involved do not appear explicitly in the geometrical description. Generally, with $n$ commodities and $K$ economies we obtain various patterns of partial specialization.

Within this context we recall the *reference vector* which describes the great abundance of possibilities: we know from Corollary 1.4 in CHAPTER 6 that any such vector indicates uniquely a Pareto face, hence a possible market equilibrium and hence the corresponding versions of specialization.

Thus an integer vector $\boldsymbol{r} = (r_1, \ldots, r_K)$ satisfying

$$(3) \qquad 1 \leq r_k \leq n \quad (k \in \boldsymbol{K}) \quad \sum_{\kappa=1}^{K} r_k \; = \; K + n - 1$$

demonstrates a possible distribution of production among the countries, where economy 1 specializes in $r_1$ commodities, ..., economy $K$ specializes in $r_K$ commodities. All reference vectors appear as possible partial specializations *when production takes place "in" a Pareto face* or equivalently in the corresponding equilibrium.

This way of organizing the world economy is uniquely determined: *which* $r_1$ of the $n$ commodities are produced by economy 1 ... etc. and which $r_K$ of the commodities are produced by economy $K$ is uniquely specified by reference vector via the bijection established by the above–mentioned Corollary.

Consequently, a ***pattern of specialization*** is a family of subsets $\mathfrak{J} = \{\boldsymbol{J}^{(1)}, \ldots, \boldsymbol{J}^{(K)}\}$ satisfying

$$\bigcup_{k \in \boldsymbol{K}} \boldsymbol{J}^{(k)} \; = \; \boldsymbol{I}$$

resulting from a Pareto face, i.e., representing a partial specialization for each country. The resulting set of global plans is given by

$$\boldsymbol{F} \; = \; \sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \quad . \tag{4}$$

Pareto faces are sets of global plans that admit of a unique normal, i.e., prices are unique (up to a multiple).

On the other hand, we can start out with a positive vector $\mathfrak{p} > \boldsymbol{0}$ to be interpreted as a vector of prices ruling on the world market. Consider an efficient plan $\bar{\boldsymbol{x}} \in \mathbb{R}^n_+$ which is supported by $\mathfrak{p}$. That is we have

$$\mathfrak{p}\bar{\boldsymbol{x}} \; = \; \max\{\mathfrak{p}\boldsymbol{x} \mid \boldsymbol{x} \in \Pi\}\,, \tag{5}$$

Now, if we take $\boldsymbol{x}^k \in \Delta^{(k)}$ $(k \in \boldsymbol{K})$ such that $\bar{\boldsymbol{x}} \; = \; \sum_{\{k \in \boldsymbol{K}\}} \boldsymbol{x}^{(k)}$, then we know that $\mathfrak{p}$ is supporting each $\Pi^{(k)}$ at $\boldsymbol{x}^{(k)}$, i.e.,

$$\mathfrak{p}\boldsymbol{x}^{(k)} \; = \; \max\left\{\mathfrak{p}\boldsymbol{x} \,\middle|\, \boldsymbol{x} \in \Pi^{(k)} \quad (k \in \boldsymbol{K})\right\}.$$

The quantity $p_i a_i^{(k)}$ is regarded as the *value of a unit of labor in country $k$ when labor is used in production of commodity $i$.* Therefore, the quantity

$$w_k(\mathfrak{p}) \; := \; max_{i \in \boldsymbol{I}} p_i a_i^{(k)} \tag{6}$$

is seen as the ***wage rate*** of country $k$ supported by world prices $\boldsymbol{p}$. That is, this is the value of a unit of labor that can be maximally earned in country $k$ when labor is applied to the maximizing commodities.

Returning to the efficient plan $\bar{\boldsymbol{x}}$ supported by $\mathfrak{p}$, note that it is not efficient to produce commodity $i$ if it does not yield the maximal wage, i.e., if $w_k(\mathfrak{p}) > p_i a_i^{(k)}$, that is, we have

$$w_k(p) > p_i a_i^{(k)} \to \overline{x}_i^{(k)} = 0. \tag{7}$$

That is, the maximizer $\bar{\boldsymbol{x}}$ must necessarily put positive weights only on the maximizing coordinates of $\boldsymbol{a}^{(k)}$.

Now, for $k \in \boldsymbol{K}$, let

$$\boldsymbol{J}^{(k)}(\mathfrak{p}) \;=\; \left\{ i \in \boldsymbol{I} \,\middle|\, p_i a_i^{(k)} \;=\; w_k(\mathfrak{p}) \right\} \tag{8}$$

denote the set of commodities that can be produced without a loss at prices $\mathfrak{p}$. Then the efficient vector $\boldsymbol{x}^{(k)}$ introduced above will have positive coordinates at most at coordinates $i \in \boldsymbol{J}^{(k)}(\mathfrak{p})$, that is we have

$$\boldsymbol{x}^{(k)} \in \Delta^{(k)}_{\boldsymbol{J}^{(k)}(\mathfrak{p})} \quad .$$

We consider $\boldsymbol{J}^{(k)}(\mathfrak{p})$ to be country k's profile of specialization and the family

$$\mathfrak{J}(\mathfrak{p}) \;:=\; \{\boldsymbol{J}^{(k)}(\mathfrak{p})\}_{k \in \boldsymbol{K}}$$

in this context is suitably called a ***pattern of specialization supported by*** $\mathfrak{p}$. Combining we formulate

**Remark 1.1.** *Let $\boldsymbol{F}$ be a Pareto face of $\Pi$. Then the reference system of $\boldsymbol{F}$ is the unique pattern of specialization supported by the normal $\mathfrak{p}$ of $\boldsymbol{F}$. This normal represents the prices ruling the world economy when production takes place in an efficient global plan $\overline{x} \in \boldsymbol{F}$. Prices and wages "in equilibrium" are connected via* (6). *Country $k \in \boldsymbol{K}$ will produce only commodities that are efficient to world prices in the sense of* (7)

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In the next step the adjustment system and the generation of prices (under the n.d. assumption) is being reinterpreted in the light of Ricardian Theory. We imagine that world production takes place at some plan in a Pareto face

$$\boldsymbol{F} \;=\; \sum_{k=1}^{K} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$$

of the aggregate production Cephoid $\Pi$. The adjustment set is $\boldsymbol{L}$ and the characteristics are given by

$$\mathbb{L} \;:=\; \left\{ (k,l) \,\middle|\, l \in \boldsymbol{L},\; \boldsymbol{J}^{(k)} \ni l \right\} \quad .$$

Accordingly, we obtain the linear adjustment system of equations in variables $(c_k, \lambda_l)$, $((k,l) \in \mathbb{L})$ given by

$$c_k a_l^{(k)} = \lambda_l \quad ((k,l) \in \mathbb{L}) \;, \tag{9}$$

see SECTION 1 of CHAPTER 3. We know that the solutions of (4) are unique up to a multiple and the quantities

$$a_i^\star := \max_{k\in \boldsymbol{K}} c_k a_i^{(k)} > 0, (i \in \boldsymbol{I}) \; ; \quad a_l^\star := c_k a_l^{(k)} \quad (k,l) \in \mathbb{L} \tag{10}$$

yield the normal (= prices) via $\mathfrak{p}^\star := \mathfrak{n}^\star = \left(\frac{1}{a_1^\star}, \ldots, \frac{1}{a_n^\star}\right)$ up to a positive multiple. Thus, for $(k,l) \in \mathbb{L}$:

$$c_k a_l^{(k)} = \frac{1}{p_l^\star} \quad \text{or} \quad p_l^\star a_l^{(k)} = \frac{1}{c_k} \; = \; w_k \; .$$

That is, the inverse of the adjustment coefficients turn out to be the wage rates in equilibrium. As

$$b_l^{(k)} = \frac{1}{a_l^{(k)}} = \frac{1}{L\widehat{a}_l^{(k)}} = \frac{\widehat{b}_l^{(k)}}{L}$$

we also have

$$p_l^\star = w_k b_l^{(k)} \quad (k,l) \in \mathbb{L} \; . \tag{11}$$

That is, *prices are obtained by relative productivity evaluated with wages in equilibrium* .

Then , according to (10), the prices for commodity $i$ are also

$$p_i^\star = \min_{k\in \boldsymbol{K}} \frac{1}{c_k a_i^{(k)}} \; = \; \min_{k\in \boldsymbol{K}} w_k b^{(k_i)}$$

with (11) prevailing whenever $i = l \in \boldsymbol{L} \cap \boldsymbol{J}^{(k)}$ i.e., $(i,k) \in \mathbb{L}$.

Combining we reformulate the idea of normal and adjustment in terms of a Ricardian equilibrium as follows.

For any pattern of specialization $\mathfrak{J} = \left\{\boldsymbol{J}^{(k)}\right\}_{k\in \boldsymbol{K}}$ we define es previously $K_i \; := \; |\{k \in \boldsymbol{K} \bigm| l \in \boldsymbol{J}^{(k)}\}| - 1$ , such that commodity $i$ is produced by $K_i + 1$ countries simultaneously. Accordingly, let $\boldsymbol{L} \; := \; \{l \in \boldsymbol{I} \mid K_l \geq 1\}$ denote those commodities that are produced in at least two countries. Hence,

$$\boldsymbol{L} \; = \; \{l \in \boldsymbol{I} \mid K_l \geq 1\} \tag{12}$$

**Definition 1.2.** : *Let $\mathfrak{J}^\star$ be a pattern of specialization and let $\mathfrak{p}^\star = (p_i^\star)_{i\in \boldsymbol{I}}$ and $\boldsymbol{w}^\star = (w_k^\star)_{k\in \boldsymbol{K}}$ be a vector of prices and wage rates respectively. $(\mathfrak{J}^\star, \pi^\star, \boldsymbol{w}^\star)$ is called a* ***market equilibrium*** *if the following holds true.*

*1.*

$$p_i^\star \ = \ \min_{k \in \boldsymbol{K}} w_k^\star b^{(k_i)} \ . \tag{13}$$

*Prices result from the least expensive production. The world economy is reflected by a capacity vector*

$$\boldsymbol{a}^\star = (a_k^\star)_{k\in\boldsymbol{K}} \quad \text{with} \quad a_k^\star = \frac{1}{p_k^\star} \quad (k \in \boldsymbol{K}). \tag{14}$$

*2.*

$$p_l^\star = w_k^\star b_l^{(k)} \quad (k,l) \in \mathbb{L} \ . \tag{15}$$

*The wage rate of country $k$ when producing any joint commodity is supported by the prices.*

*3. $\mathfrak{J}$ is supported by $\mathfrak{p}^\star$. For any joint production plan*

$$\bar{\boldsymbol{x}} = \sum_{k\in\boldsymbol{K}} \bar{\boldsymbol{x}}^{(k)}, \quad \bar{\boldsymbol{x}}^{(k)} \in \Delta^{(k)}_{\boldsymbol{J}^{\star(k)}} (k \in \boldsymbol{K}),$$

*it follows that*

$$\mathfrak{p}\bar{\boldsymbol{x}}^{(k)} = \mathfrak{p}\bar{\boldsymbol{x}} \ (k \in \boldsymbol{K}) \tag{16}$$

*as well as*

$$w_k^\star > p_i^\star a_i^{(k)} \quad \text{implies} \quad \overline{x}_i^{(k)} = 0 \quad ((i,k) \in \boldsymbol{I} \times \boldsymbol{K}) \tag{17}$$

*That is, commodities at which the wage rate of country $k$ is not competitive are not being produced in country $k$.*

Now we have an equilibrium whenever production takes place "in" some Pareto face.

**Theorem 1.3.** *Let $\boldsymbol{F}$ be a Pareto face of a cephoid $\Pi$. Then there exists a market equilibrium $(\mathfrak{J}^\star, \mathfrak{p}^\star, \boldsymbol{w}^\star)$ such that $\mathfrak{p}^\star$ is the normal at $\boldsymbol{F}$ and the adjustment coefficients $\boldsymbol{c}^\star$ define the wages. The world economy produces with a capacity $\boldsymbol{a}^\star$ defined via the adjustment coefficients/wages such that countries produce efficiently all commodities which are produced by at least two countries.*

In this case the equilibrium data $\mathfrak{p}^\star, \boldsymbol{w}^\star$ are unique up to multiplication with a positive real number. Or else, as is common in General Equilibrium Theory, one could ask for normalization of price vectors ($\sum_{i\in \boldsymbol{I}} p_i^\star = 1$) and then claim that equilibrium is unique.

Basically, equilibria exist at any Pareto efficient face, not just the maximal ones. In particular, any extremal point of the Pareto surface generates equilibria, the prices result from the normal cone which has dimension $n$. As any extremal point is an element of $n$ maximal faces (not all of them Pareto efficient), we obtain a set of equilibria corresponding to each Pareto face involved exactly as in Theorem 1.3

The somewhat opaque discussion in McKENZIE[14] concerning the question whether world production will take place at some extremal ("specialization of each country to one commodity") or on a maximal face – or rather the "likelihood" of such global plans – is, in our view, begging the question. Which type of "solution" within our model do we have in mind? What is the nature of an equilibrium (to regard countries as price taking agents poses a certain difficulty, it would seem) or of a cooperative solution (countries bargaining about an agreement as to which global plan should be implemented – what about "free trade" in this case ?). In any case, the knowledge of the nature of maximal faces seems to be a necessity before proceeding to any kind of solution concept.

One could think of a solution concept in the sense of Cooperative Game Theory. Though it is doubtful that economists are susceptible to this vague idea we point out that there is a solution concept specially adapted to Cephoids. We return to this concept in CHAPTER 11.

**Example 1.4.** *We return to Figure 1.3 which describes the production of two economies with 3 commodities, i.e., the Cephoid $\Pi = \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}$. The Pareto faces are the* rhombus $\Delta_{23}^{\boldsymbol{a}} + \Delta_{13}^{\boldsymbol{b}}$ *and the two translates of the generating deGua Simplices.*

*The rhombus is the sum of two Subsimplices, economy $\boldsymbol{a}$ ("blue") specializing on commodities $\{2,3\}$ and economy $\boldsymbol{b}$ ("red") specializing on commodities $\{1,3\}$. The only adjustment index is 3, i.e., $\boldsymbol{L} = \{3\}$ holds true.*

*We adjust the third axis of both the Simplices. That is, we seek constants $c_{\boldsymbol{a}}$ and $c_{\boldsymbol{b}}$ such that the vectors $c_{\boldsymbol{a}}\boldsymbol{a}^3$ and $c_{\boldsymbol{b}}\boldsymbol{b}^3$ coincide. See also Figures 1.2 and 1.3 in* CHAPTER *3.*

*Appropriate multiples of the Subsimplices ("specializations") $c_{\boldsymbol{a}}\Delta_{23}^{\boldsymbol{a}}$ and $c_{\boldsymbol{b}}\Delta_{13}^{\boldsymbol{b}}$ generate the triangle $\Delta^\star$ which is the Pareto efficient set of the world economy $\Pi^\star$. This Pareto efficient set is represented by a green triangle in Figure 1.5.*

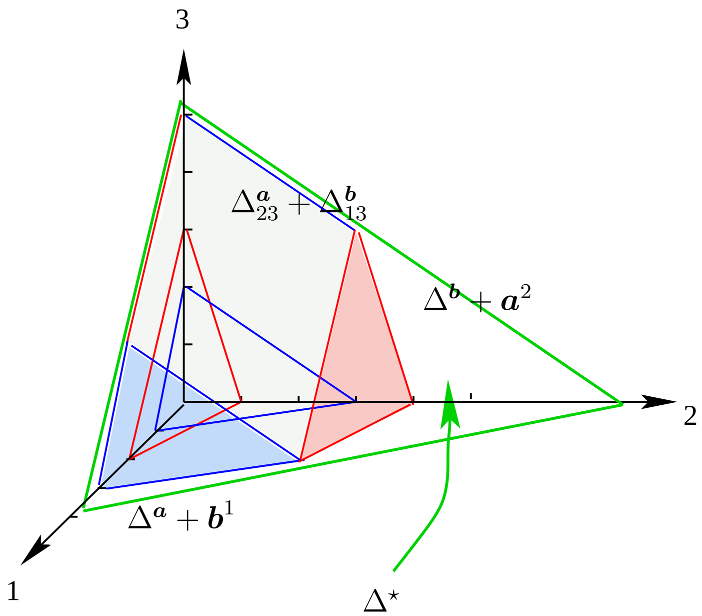


Figure 1.5: The world economy of Figure 1.3

*In Figure 1.5 we have a special arrangement as follows. There is one degree of freedom in choosing the adjustment coefficients. Therefore, in this simple case adjustment can be achieved in particular by choosing*

$$c_{\boldsymbol{a}} \;:=\; \frac{a_3 + b_3}{a_3} \;, \quad c_{\boldsymbol{b}} \;:=\; \frac{a_3 + b_3}{b_3} \;.$$

*Then we have indeed* $c_{\boldsymbol{a}}\boldsymbol{a}^3 = c_b\boldsymbol{b}^3$*. Moreover,* $c_{\boldsymbol{a}}a_3 + c_{\boldsymbol{b}}b_3 = a_3 + b_3$ *indicates that the* $\Pi$ *and* $\Delta^\star$ *coincide at the third coordinate, hence the whole rhombus* $\Delta_{23}^{\boldsymbol{a}} + \Delta_{13}^{\boldsymbol{b}}$ *is located within* $\Delta^\star$*.*

*This means that we have set the wages at*

$$w_{\boldsymbol{a}} = \frac{1}{c_{\boldsymbol{a}}} = \frac{a_3}{a_3 + b_3} = b_{\boldsymbol{a}} \;, \quad w_{\boldsymbol{b}} = \frac{1}{c_{\boldsymbol{b}}} = \frac{b_3}{a_3 + b_3} = b_{\boldsymbol{b}} \;,$$

*in other words, wages are dictated by (relative) productivity. We have*

$$\begin{aligned} \boldsymbol{a}^\star &= (c_{\boldsymbol{a}}a_1, c_{\boldsymbol{b}}b_2, c_{\boldsymbol{a}}a_3) \\ &= (a_1\frac{a_3 + b_3}{a_3} \;, \; b_2\frac{a_3 + b_3}{b_3} \;, \; a_3 + b_3) \\ &= (a_3 + b_3)(\frac{a_1}{a_3}, \frac{b_2}{b_3}, 1) \end{aligned}$$

*and hence prices are given by*

$$\mathfrak{p} \;=\; \frac{1}{a_3+b_3}(\frac{a_3}{a_1},\frac{b_3}{b_2},1) \;=\; \frac{1}{a_1b_2(a_3+b_3)}(a_3b_2,a_1b_3,a_1b_2)\ .$$

∘ ~~~~~~~ ∘

**Remark 1.5.** *Suppose $\boldsymbol{p}$ is a positive vector of prices and $\mathfrak{J}(\boldsymbol{p})$ is a pattern of specializations supported by $\boldsymbol{p}$. Consider the corresponding reference vector $\boldsymbol{r}(\boldsymbol{p}) \in \mathbb{N}^K$ with coordinates $r_k(\boldsymbol{p}) = |\boldsymbol{J}^{(k)}(\boldsymbol{p})|\ \ (k \in \boldsymbol{K})$. In view of nondegeneracy the dimension of the subface of $\Pi$ supported by $\boldsymbol{p}$, i.e., of*

$$\boldsymbol{F}_0(\boldsymbol{p}) \;:=\; \sum_{k\in\boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}(\boldsymbol{p})}\ ,$$

*is given by*

$$d(\boldsymbol{p}) \;:=\; \sum_{k\in\boldsymbol{K}} r_k(\boldsymbol{p}) - K.$$

*In particular, if $\boldsymbol{F}_0$ is a maximal face, then $r(\boldsymbol{p})$ is the corresponding reference vector and satisfies $d(\boldsymbol{P}) = n-1$. And, on the other hand, if $\boldsymbol{F}_0$ is a singleton (hence necessarily an extremal of $\Pi$), then clearly $d(\boldsymbol{P}) = 0$ holds true.*

∘ ~~~~~~~ ∘

Let us call a pattern of specialization $\mathfrak{J}$ *compatible* with a reference vector $\boldsymbol{r} \in \mathbb{R}^K$ if $|\boldsymbol{J}^{(k)}| = r_k\ \ (k \in \boldsymbol{K})$ holds true, that is, if the profile of specialization $\boldsymbol{J}^{(k)}$ assigned by $\mathfrak{J}$ to each country $k$ has cardinality $r_k$. Now let $\boldsymbol{F}$ be the unique maximal face associated with the vector $\boldsymbol{r}$ and let $\overline{\mathfrak{J}}$ be the corresponding pattern of specialization, i.e., the reference system of face $\boldsymbol{F}$. We know that $\overline{\mathfrak{J}}$ collects exactly the maximizing coordinates $p_i a_i^{(k)}$ for each $k$, this is the meaning of formulae (6) and (8). Therefore, for all $k \in \boldsymbol{K}$, we know that

$$p_i a_i^{(k)} \le w_k \quad (i \in \boldsymbol{J}^{(k)}) \tag{18}$$

holds true with a strict inequality for at least one $i$ if $\boldsymbol{J}^{(k)} \neq \overline{\boldsymbol{J}^{(k)}}$. Consequently we have for all $k \in \boldsymbol{K}$

$$\prod_{i\in\boldsymbol{J}^{(k)}} p_i a_i^{(k)} \le w_k^{r_k}\ , \tag{19}$$

again with a strict inequality if $\boldsymbol{J}^{(k)} \neq \overline{\boldsymbol{J}^{(k)}}$ holds true. Finally, we come up with the inequality

$$\prod_{k\in\boldsymbol{K}} \prod_{i\in\boldsymbol{J}^{(k)}} p_i a_i^{(k)} < \prod_{k\in\boldsymbol{K}} w_k^{r_k} \;=\; \prod_{k\in\boldsymbol{K}} \prod_{i\in\overline{\boldsymbol{J}}^{(k)}} p_i a_i^{(k)}\ . \tag{20}$$

This result in economical terms is reformulated in the following

**Corollary 1.6.** *Let* $\Pi$ *be nondegenerate and let* $\boldsymbol{r}$ *be a reference vector. Then a pattern of specialization* $\overline{\mathfrak{J}}$ *is efficient (i.e., associated with* $\boldsymbol{r}$ *or yielding the corresponding maximal face) if and only if it maximizes the associated product of values of labor units among all patterns compatible with* $\boldsymbol{r}$.

For economists it is of interest that JONES [11] has a rudimentary statement of the type of our corollary. However, his result concerns complete specializations (i.e., sums of extremals of the various $\Pi^{(k)}$)) only in which case it is obvious that the product of labor values is not maximal whenever the result of the summation is not an extremal in $\Pi$.

As an interesting detail in history GRAHAM [8] produces a fascinating example of an economy with 10 countries and 10 goods, i.e., a cephoid which is the sum of 10 deGua Simplices in $\mathbb{R}^10$. One is fascinated by his effort to compute a price vector $\boldsymbol{p}$ and an efficient global production $\bar{\boldsymbol{x}} \in \Pi$ for a $10 \times 10$ his example of a world economy.

GRAHAM produces his Example via a table ("TABLEAU OF OPPORTUNITY COST RATES") (see p. 91) from which a set of capacity vectors can be derived by multiplying the first row with 10, the second with 20,... the tenth with 400. The result is (W.F.RICHTER, private communication) the matrix

(21)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 10 | 100 | 80 | 220 | 800 | 250 | 70 | 440 | 510 | 870 |
| 20 | 240 | 240 | 380 | 1080 | 360 | 100 | 580 | 500 | 1920 |
| 30 | 420 | 90 | 450 | 630 | 1500 | 330 | 930 | 900 | 960 |
| 40 | 640 | 240 | 200 | 3840 | 1480 | 160 | 920 | 1440 | 560 |
| 50 | 1400 | 800 | 2400 | 600 | 1550 | 100 | 650 | 4050 | 1450 |
| 80 | 2880 | 400 | 560 | 3600 | 1840 | 960 | 3040 | 2960 | 2480 |
| 120 | 2160 | 480 | 1080 | 7560 | 4080 | 720 | 7200 | 5160 | 4200 |
| 200 | 3400 | 3600 | 5400 | 6600 | 9000 | 400 | 2800 | 10800 | 3400 |
| 300 | 9600 | 2100 | 3900 | 12900 | 3600 | 4800 | 24000 | 19200 | 15600 |
| 400 | 8400 | 8000 | 6800 | 25600 | 15200 | 1200 | 13600 | 10400 | 28800 |

The efficient production plan GRAHAM computes (TABLE $A_{1p}$) is

$$\bar{\boldsymbol{x}} = \sum_{k \in \boldsymbol{K}} \bar{\boldsymbol{x}}^{(k)}$$

with

$$
(22) \qquad
\begin{array}{rcl}
\bar{\boldsymbol{x}}^1 &=& (0,0,0,0,800,0,0,0,0,0) \;=\; 800\boldsymbol{e}^5 \;=\; \boldsymbol{a}^{(1)5} \\
\bar{\boldsymbol{x}}^2 &=& (0,0,0,0,0,0,0,0,0,1920) \;=\; 1920\boldsymbol{e}^{10} \;=\; \boldsymbol{a}^{(2)10} \\
\bar{\boldsymbol{x}}^3 &=& (0,0,0,0,0,1500,0,0,0,0) \;=\; 1500\boldsymbol{e}^6 \;=\; \boldsymbol{a}^{(3)6} \\
\bar{\boldsymbol{x}}^4 &=& (0,0,0,0,3840,0,0,0,0,0) \;=\; 3840\boldsymbol{e}^5 \;=\; \boldsymbol{a}^{(4)5} \\
\bar{\boldsymbol{x}}^5 &=& (0,0,0,2400,0,0,0,0,0,0,) \;=\; 2400\boldsymbol{e}^4 \;=\; \boldsymbol{a}^{(5)4} \\
\bar{\boldsymbol{x}}^6 &=& (0,2880,0,0,0,0,0,0,0,0) \;=\; 2880\boldsymbol{e}^2 \;=\; \boldsymbol{a}^{(6)2} \\
\bar{\boldsymbol{x}}^7 &=& (101\,{}^1/_2,0,0,0,0,0,0,110,0,0) \\
\bar{\boldsymbol{x}}^8 &=& (0,0,0,1204,0,4506\,{}^2/_3,0,0,2984,0) \\
\bar{\boldsymbol{x}}^9 &=& (0,724,0,0,0,0,1802,7900,4224,0) \\
\bar{\boldsymbol{x}}^{10} &=& (48\,{}^2/_3,0,3003\,{}^1/_3,0,4907\,{}^2/_3,0,0,0,0,8892)
\end{array}
$$

The resulting $\bar{\boldsymbol{x}}$ has positive coordinates in subsets of $\boldsymbol{I}$ reflected by a reference vector

$$\boldsymbol{r} = (1, \ldots, 1, 2, 3, 4, 4)$$

As this vector *is* a reference vector for $n = K = 10$ we conclude that $\bar{\boldsymbol{x}}$ necessarily is located within the (relative) interior of the maximal face corresponding to $\boldsymbol{r}$ – that is, Graham indeed computes an efficient pattern of specializations, hence a maximal face of $\Pi$, which he is not aware of. The author remarks that to do this has been a "tedious process of trial and error" – to which one can only applaud. Exhibiting (fast) algorithms for computing all faces of a cephoid is certainly a task justifying the same epitaph.

However, the presentation of the Pareto surface of Grahams example via the list of reference system would explode this volume and tax the readers patience substantially. For once, the number of Pareto faces is given by the function $f$ specified in (8) and (10) of SECTION 2 CHAPTER 5. We compute readily

$$(23) \qquad f(10, 10) = 48620 \; .$$

Now view the Pareto surface of the example "LARBLUE" in SECTION 2 CHAPTER 7. This is $6 \times 7$ Cephoid, hence the number of its faces is

$$(24) \qquad f(6, 7) = 462 \; .$$

Copying this example demonstrates the size of the problem in terms of numerical computation: "LARBLUE" took 8 pages so listing the Pareto faces of Graham's example would take a volume of 842 pages.

# Chapter 11

# Applications: Cooperative Games – Bargaining

The remainder of this volume describes solution concepts in the Theory of Cooperative Games that are based on Cephoids. We start with the most simple version of a Cooperative Game, that is, with "Pure Bargaining Games" or "Bargaining Problems". These are Cooperative Games such that cooperation is either agreed upon in a (the "grand") coalition or else completely fails – in which case the players are thrown back onto a "status quo position" with no gains for anyone.

The Maschler–Perles Bargaining Solution establishes a convincing concept for the solution of Bargaining Problems. It differs notably in esprit and refinement from popular Bargaining Solutions – the Nash Solution and the Kalai–Smorodinsky Solution – concerning the decisive axiom: "superadditivity". To us, this axiom appears to be much more appealing than, say, "Independence of Irrelevant Alternatives" (Nash) or "(one player) Monotonicity" (Kalai–Smorodinsky).

The original work of Maschler and Perles ([13],[22]) establishes the solution for two players; superadditivity turns out to be necessary and sufficient for the unique existence. The result hinges on the fact that every piecewise linear compact convex comprehensive set in $\mathbb{R}^2_+$ is a Cephoid.

For more than 2 players we present a generalization on Cephoidal Bargaining Problems.

# 1 Bargaining

A bargaining problem for $n$ players consists of a "feasible set" (in $\mathbb{R}^n$) and a "status–quo point". Players can "cooperate" and thus agree on some point of the feasible set. The result is then registered as a contract with some law enforcing agency. If they do not agree, then the "status quo point" is registered and executed.

Within this chapter we study essentially bargaining problems for 2 players resulting in 2–dimensional feasible sets. Yet definitions may, in general, be formulated for $n$ persons i.e., feasible sets in $n$ dimensions.

The Maschler–Perles bargaining solution (Maschler–Perles [13], [22], see also [27] for a textbook presentation) is a mapping defined on 2–dimensional bargaining problems respecting anonymity, Pareto efficiency, and affine transformations of utility. Moreover, this mapping is *superadditive* by which property it is uniquely characterized. We elaborate on this concept in view of the Theory of Cephoids.

**Definition 1.1.** *Let* $\underline{\boldsymbol{x}} \in \boldsymbol{U} \subseteq \mathbb{R}^n$. *Assume that* $\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} > \underline{\boldsymbol{x}}\} \neq \emptyset$.

1. $\boldsymbol{U}$ *is called* $\underline{\boldsymbol{x}}$*–**comprehensive** if*

$$\boldsymbol{U} \;=\; \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geq \underline{\boldsymbol{x}}\} \cap \boldsymbol{CmpH}\,\boldsymbol{U}$$

   *holds true.*

2. $\boldsymbol{V} = (\underline{\boldsymbol{x}}, \boldsymbol{U})$ *is called a **bargaining problem** if* $\boldsymbol{U}$ *is compact, convex, and* $\underline{\boldsymbol{x}}$*–comprehensive.*

$\boldsymbol{U}$ is the ***feasible set*** ( of "utility vectors") and $\underline{\boldsymbol{x}}$ is the ***status quo point***. Players $i \in \boldsymbol{I}$ are involved in the bargaining process. They can either reach an agreement (a "contract") regarding some feasible utility vector, then they receive utility $x_i$ $(i \in \boldsymbol{I})$ each. Or else, they may fail to do so in which case they are forced to accept the status quo point, i.e., receive utility $\underline{x}_i$ $(i \in \boldsymbol{I})$ each.

Now let $\mathcal{U}$ denote the family of feasible sets and $\mathcal{V}$ the set of bargaining problems. As a preliminary definition, a ***solution*** is a mapping $\boldsymbol{\varphi}$ that, based on some axiomatic justification, assignes to each bargaining problem $\boldsymbol{V} = (\underline{\boldsymbol{x}}, \boldsymbol{U})$ a Pareto efficient vector $\boldsymbol{\varphi}(\boldsymbol{V})$. We define a solution also w.r.t. a subset $\emptyset \neq \mathcal{V}_0 \subseteq \mathcal{V}$. That is we consider a mapping

$$\boldsymbol{\varphi} \;:=\; \mathcal{V}_0 \to \mathbb{R}^n \tag{1}$$

such that, for all $\boldsymbol{V} = (\underline{\boldsymbol{x}}, \boldsymbol{U}) \in \mathcal{V}_0$ it follows that $\varphi(\underline{\boldsymbol{x}}, \boldsymbol{U}) \in \boldsymbol{U}$ is Pareto efficient in $\boldsymbol{U}$. For short, $\varphi$ is called ***Pareto efficient***.

There are two further requirements that a solution should satisfy, these are scale invariance and anonymity.

"Rescaling utility " is represented by an ***affine transformation of utility*** which is a mapping

$$
\begin{aligned}
L &: \mathbb{R}^n \to \mathbb{R}^n \\
L(\boldsymbol{x}) &= (\alpha_1 x_1, \ldots, \alpha_n x_n) + \boldsymbol{\beta} \qquad (\boldsymbol{x} \in \mathbb{R}^n)
\end{aligned} \tag{2}
$$

with positive $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\boldsymbol{\beta} \in \mathbb{R}^n$. $L$ acts on bargaining problems canonically via

$$L\boldsymbol{V} = L(\underline{\boldsymbol{x}}, \boldsymbol{U}) = (L(\underline{\boldsymbol{x}}), L(\boldsymbol{U})) \qquad \boldsymbol{V} \in \mathcal{V}.$$

Hence a solution is said to be ***scale invariant*** if it commutes with all a.t.u's, i.e., if for any $L$ one has

$$
\begin{aligned}
\varphi(L\boldsymbol{V}) &= L(\varphi(\boldsymbol{V})) \qquad (\boldsymbol{V} \in \mathcal{V}) \\
&\text{or} \\
\varphi \circ L &= L \circ \varphi
\end{aligned} \tag{3}
$$

The definition can be restricted to some $L$–invariant $\mathcal{V}_0 \subseteq \mathcal{V}$, that is, satisfying $L\mathcal{V}_0 = \mathcal{V}_0$

Next, we discuss the notion of anonymity. Intuitively, this concept requires a solution to be independent of the names of the players. More precisely, it is supposed to respect permutations of the axes.

A permutation $\pi : \boldsymbol{I} \to \boldsymbol{I}$ acts on $\mathbb{R}^n$ by permuting the axes, i.e., we define

$$
\begin{aligned}
&\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \\
&\pi(x)_i = x_{\pi^{-1}(i)} \qquad (i \in \boldsymbol{I},\ x \in \mathbb{R}^n).
\end{aligned}
$$

and hence canonically

$$
\begin{aligned}
&\pi : \mathcal{V} \longrightarrow \mathcal{V}, \\
&\pi\boldsymbol{V} = \pi(\underline{x}, \boldsymbol{U}) = (\pi(\underline{x}), \pi(\boldsymbol{U})) \qquad (\boldsymbol{V} \in \mathcal{V}).
\end{aligned}
$$

A solution $\varphi : \mathcal{V}^0 \to \mathbb{R}^n$ is said to be **anonymous** if, for any $\boldsymbol{V} \in \mathcal{V}$ and any permutation $\pi$, we have $\varphi(\pi\boldsymbol{V}) = \pi(\varphi(\boldsymbol{V}))$, or for short, $\varphi \circ \pi = \pi \circ \varphi$

(again with a possible restriction to some $\mathcal{V}_0 \subseteq \mathcal{V}$ which is "anonymous" in an obvious sense).

Scale invariance and anonymity are standard axioms that naturally one imposes on a solution. Combining we come up with the definition as follows.

**Definition 1.2.** *Let $\emptyset \neq \mathcal{V}_0 \subseteq \mathcal{V}$ be scale invariant and anonymous. A* ***bargaining solution*** *for $\mathcal{V}_0$ is a mapping*

$$\boldsymbol{\varphi} \ := \ \mathcal{V}_0 \to \mathbb{R}^n \tag{4}$$

*which is Pareto efficient, scale invariant, and anonymous.*

From scale invariance it follows immediately that for suitable $\underline{\boldsymbol{x}}, \boldsymbol{U}$

$$\boldsymbol{\varphi}(\underline{\boldsymbol{x}}, \boldsymbol{U}) = \underline{\boldsymbol{x}} + \boldsymbol{\varphi}(\boldsymbol{0}, \boldsymbol{U}). \tag{5}$$

As it turns out, most properties of solutions can be formulated for bargaining problems with status quo point $\underline{\boldsymbol{x}} = \boldsymbol{0}$ and then eaasily carried over to the general case via (5).

Therefore we – somewhat sloppily – will frequently write $\boldsymbol{\varphi}(\boldsymbol{U}) := \boldsymbol{\varphi}(\boldsymbol{0}, \boldsymbol{U})$ and consider $\boldsymbol{\varphi}$ as a mapping on some $\mathcal{U}_0 \in \mathcal{U}$.

Now it turns out that one further axiom "characterizes" a bargaining solution and there are several such axioms in the tradition. These axioms are e.g. "I.I.R" for the Nash Bargaining Solution and "one player monotonicity" for the Kalai–Smorodinsky Bargaining Solution (See e.g. [27] for a textbook treatment). In the present treatment, we single out "superadditivity". Thus we discuss the concept due to Maschler–Perles [22], see also [27], chapter VIII, Section 4.

Consider a situation with two bargaining problems $\boldsymbol{V} = (\underline{\boldsymbol{x}}, \boldsymbol{U}), \boldsymbol{V}' = (\underline{\boldsymbol{x}}', \boldsymbol{U}') \in \mathcal{V}$. Formally, we consider the "sum" or "aggregation" of both these problems which is

$$\boldsymbol{V} + \boldsymbol{V}' = (\underline{\boldsymbol{x}} + \underline{\boldsymbol{x}}', \boldsymbol{U} + \boldsymbol{U}') \in \mathcal{V} \ .$$

Then we have

**Definition 1.3.** *A bargaining solution is* ***superadditive*** *if, for all $\boldsymbol{V}, \boldsymbol{V}' \in \mathcal{V}$ we have*

$$\varphi(\boldsymbol{V}) + \varphi(\boldsymbol{V}') \leq \varphi(\boldsymbol{V} + \boldsymbol{V}') \ . \tag{6}$$

First note that "additivity", that is requiring an equation in (6), is too much to ask for: generically, the quantity $\varphi(\boldsymbol{V})+\varphi(\boldsymbol{V}')$ will not be Pareto efficient. On the other hand, by superadditivity *and* scale invariance the inequality (6) at once extends to

$$(7) \qquad \varphi(t\boldsymbol{V}) + \varphi(s\boldsymbol{V}') \leq t\varphi(\boldsymbol{V}) + s\varphi(\boldsymbol{V}') \quad (\boldsymbol{V}, \boldsymbol{V}' \in \mathcal{V}, \quad t, s > 0).$$

.

Now, there are quite a few axiomatic justifications for superadditivity and – most impressively – the axiom turns out to uniquely single out a bargaining solution – in $\mathbb{R}^2$ at least.

Let us divert for an interpretation; the characterizing axiom "superadditivity" results from one of the following stories.

Consider two bargaining problems, say $\boldsymbol{V}^1$ and $\boldsymbol{V}^2$. Imagine that there is a *lottery* represented by a *probability* $\boldsymbol{p} = (p_1, p_2) \geq \boldsymbol{0}$; $p_1 + p_2 = 1$. The lottery will choose the bargaining problem $\kappa \in \{1, 2\}$ with probability $p_\kappa$. The players intend to enter a joint venture which involves the two games and the probability of realizing them. Necessarily they have to agree on the distribution of utility *ex ante*, that is, before the chance move takes place and can be observed.

Now on one hand they can consider an agreement saying "if, for some $k \in \{1, 2\}$, chance results in $\boldsymbol{V}^k$, then $\varphi(\boldsymbol{V}^k)$ will be implemented". This agreement would essentially amount to a distribution of utility "ex post", that is, after the result has been observed. At the contracting instant, i.e., "ex ante", this decision will result in an expected utility of $\mathbb{E}_{\boldsymbol{p}}\boldsymbol{\varphi}(\boldsymbol{U}^\bullet)$.

On the other hand, as players have to contract in advance, they can be seen to face the "expected bargaining problem" $\mathbb{E}_{\boldsymbol{p}}\boldsymbol{V}^\bullet$ which is given via the feasible set

$$\mathbb{E}_{\boldsymbol{p}}(\boldsymbol{U}^\bullet) \ := \ p_1\boldsymbol{U}^1 + p_2\boldsymbol{U}^2.$$

and the expected status quo point

$$\mathbb{E}_{\boldsymbol{p}}(\underline{\boldsymbol{x}}^\bullet) \ := \ p_1\underline{\boldsymbol{x}}^1 + p_2\underline{\boldsymbol{x}}^2.$$

In view of this uncertainty and the requirement to contract "ex ante", the players may choose to agree on the solution $\boldsymbol{\varphi}(\mathbb{E}_{\boldsymbol{p}}\boldsymbol{V}^\bullet)$.

A solution that (for any pair of bargaining problems and any lottery) satisfies

$$(8) \qquad \boldsymbol{\varphi}(\mathbb{E}_{\boldsymbol{p}}\boldsymbol{V}^\bullet) \geq \mathbb{E}_{\boldsymbol{p}}\boldsymbol{\varphi}(\boldsymbol{V}^\bullet)\ .$$

is consistent with contracting "ex ante".

Now, for a bargaining solution (8) is equivalent to (6) and (7). Thus a superadditive solution resolves conflicts that may occur from having to make decisions – agreements – under uncertainty by offering the payers a distribution of utility consistent with the necessary contracting "ex ante". Ideologically, this interpretation is based on the notion of "von Neumann–Morgenstern utility".

There is a second interpretation that does not refer to a probabilistic set–up in the background.

Two players – let us call them **US** and **EU** – are engaged in two "remote" bargaining situations $\boldsymbol{V}^1$ and $\boldsymbol{V}^2$ ( one in Brussels and one in Washington) simultaneously. Initially, these are separate affairs to be carried on by second–ranking officials of the two administrations. One may regard agricultural problems (in Washington): import and export of nutritional commodities, the other one liberalizing the markets for cars and agreeing on joint standards for vehicles under this aspect (in Brussels). Given $\boldsymbol{\varphi}$, the representing officials settle for $\boldsymbol{\varphi}(\boldsymbol{V}^1)$ and $\boldsymbol{\varphi}(\boldsymbol{V}^2)$ respectively and separately.

Later on, ranking officials realize that combining both bargaining projects may be advantageous. A *junctim* evolved and government officials considered giving in w.r. to one contract in favor of receiving concessions in the other one. Both administrations consider concessions in $\boldsymbol{V}^1$ versus gains in $\boldsymbol{V}^2$ and *vice versa*. The utilities and the status quo point available are now $\{\boldsymbol{x}^1 + \boldsymbol{x}^2 | \boldsymbol{x}^1 \in \boldsymbol{U}^1, \boldsymbol{x}^2 \in \boldsymbol{U}^2\} =: \boldsymbol{U}^1 + \boldsymbol{U}^2$ and $\underline{\boldsymbol{x}}^1 + \underline{\boldsymbol{x}}^2$. That is, players now face the bargaining problem $\boldsymbol{V}^1 + \boldsymbol{V}^2$. The solution is then $\boldsymbol{\varphi}(\boldsymbol{V}^1 + \boldsymbol{V}^2)$. Now, whenever the solution concept is superadditive, i.e., $\boldsymbol{\varphi}(\boldsymbol{V}^1 + \boldsymbol{V}^2) \geq \boldsymbol{\varphi}(\boldsymbol{V}^1) + \boldsymbol{\varphi}(\boldsymbol{V}^2)$, then *both players profit* from a *quid quo pro.*

# 2 The Maschler–Perles Solution

The bargaining solution developed by MASCHLER–PERLES [22] the only superadditive solution for 2 players – i.e., in 2 dimensions. For more than 2 dimensions there is no superadditive solution on $\mathcal{V}$, see PERLES [21]. Nevertheless, we shall discuss a generalization of the Maschler–Perles solution to $n$ dimensions in CHAPTER 12.

Our treatment follows the presentation in [28].

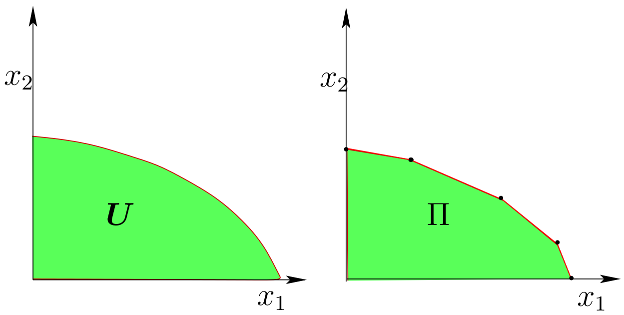


Figure 2.1: Bargaining problems – smooth and polyhedral

A bargaining problem is ***polyhedral*** if the Pareto surface of the feasible set $U$ consists of line segments. For $n = 2$ the polyhedral bargaining problems are exactly the ones with the feasible set $\boldsymbol{U} = \Pi$ being a Cephoid (Theorem 3.1, CHAPTER 5). Accordingly, the original idea of MASCHLER–PERLES is based on the observation that every polyhedral bargaining problem in $\mathbb{R}^2$ is Cephoidal. By continuity with respect to the Hausdorff metric the solution is transferred to bargaining problems with a smooth Pareto curve.

We consider Cephoids $\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ with a further restriction imposed. As it is sufficient to establish the solution on a dense subset with respect to the Hausdorff topology, we restrict ourselves at the moment to families $\left\{\boldsymbol{a}^{(k)}\right\}_{k \in \boldsymbol{K}}$ which are *dyadic* (with the same fixed basis).

A positive vector $\boldsymbol{a}$ is ***dyadic*** (with basis $T$) if there are integers $t_1, t_2$ such that $\boldsymbol{a} = \left(\frac{t_1}{2^T}, \frac{t_2}{2^T}\right)$ holds.

Simultaneously, we want to tentatively relax our condition of nondegeneracy (not essential in two dimensions) as follows. Any DeGua Simplex (triangle) $\Pi^{\boldsymbol{a}}$ can be written as a ("homothetic") sum of two of its copies shrinked by a suitable factor. For example, we have

$$\Pi^{\boldsymbol{a}} \;=\; \frac{1}{2}\Pi^{\boldsymbol{a}} + \frac{1}{2}\Pi^{\boldsymbol{a}} \;=\; \Pi^{\frac{1}{2}\boldsymbol{a}} + \Pi^{\frac{1}{2}\boldsymbol{a}}.$$

By this operation the ***area*** of the triangle $\Pi^{\boldsymbol{a}}$, say, $\pi(\boldsymbol{a}) := \frac{1}{2}a_1a_2$, is divided by 4, i.e.,

$$\pi(\frac{1}{2}\boldsymbol{a}) = \frac{1}{4}\pi(\boldsymbol{a}).$$

Therefore, if we start with a Cephoid resulting from a family of dyadic deGua Simplices, we may assume that all deGua Simplices involved have equal area. The bargaining problems having this property again form a dense subset of the set of all bargaining problems. Similarly, whenever we deal with the sum of two bargaining problems, we may assume that the summands, as well as the sum, are dyadic with the same basis. This is formulated as follows.

**Definition 2.1.** *We call a feasible set* $\boldsymbol{U}$ *(and a resulting bargaining problem* $\boldsymbol{V} = (\underline{\boldsymbol{x}}, \boldsymbol{U})$*)* ***standard dyadic*** *if* $\boldsymbol{U}$ *is a Cephoid obtained by a family* $\left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ *of dyadic vectors, all of them generating equal area, say* $\frac{1}{2}\alpha_T$*, i.e.,*

$$a_1^{(k)}a_2^{(k)} = \frac{t_1^{(k)}t_2^{(k)}}{2^T} = \alpha_T \quad (k \in \boldsymbol{K}). \tag{1}$$

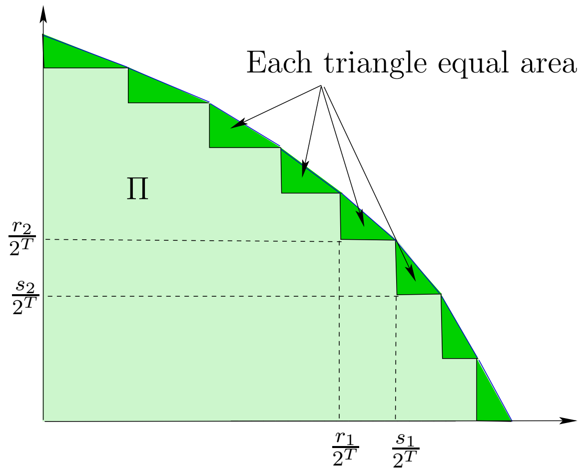


Figure 2.2: A standard dyadic bargaining problem

Observe again that, for the moment, we deviate from our standing assumption of nondegeneracy when adopting the above definition: some of the generating deGua Simplices could be "homothetic". However, the situation in $\mathbb{R}^2_+$ requires little of our refined apparatus and there will be no clash of ideologies.

Frequently we assume that the enumeration of the deGua Simplices is such that the normals are decreasing in $k$, i.e.,

$$\frac{a_2^{(k)}}{a_1^{(k)}} \downarrow_{k\in\boldsymbol{K}} \tag{2}$$

The Maschler–Perles solution for a standard dyadic bargaining problem is then defined inductively as follows:

**Definition 2.2.** *Let*

$$\mathcal{U}^d \;:=\; \{\boldsymbol{U} \mid \boldsymbol{U} \textit{ is standard dyadic } \} \tag{3}$$

*and*

$$\mathcal{V}^d \;:=\; \{\boldsymbol{V} \;=\; (\underline{\boldsymbol{x}}, \boldsymbol{U}) \mid \boldsymbol{U} \textit{ is standard dyadic } \} \;. \tag{4}$$

*The **Maschler–Perles** solution $\boldsymbol{\mu}$ is the mapping*

$$\boldsymbol{\mu} \;:\; \mathcal{U}^d \to \mathbb{R}^2_+$$

*defined recursively as follows:*

1. *For $K = 1$, i.e., $\boldsymbol{U} = \Pi^{\boldsymbol{a}}$, $\boldsymbol{\mu}$ it is the midpoint of the line segment (the Pareto curve), i.e.,*
$$\boldsymbol{\mu}(\boldsymbol{U}) = \frac{\boldsymbol{a}}{2} \;.$$

2. *Let $K = 2$. If both deGua Simplices are homothetic, then apply the Definitioin in the first* item mutatis mutandis. *That is, as $\Pi = \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{a}} = \Pi^{2\boldsymbol{a}}$ we have $\boldsymbol{\mu}(\Pi) \;:=\; \boldsymbol{a}$.*

   *Assume that the two deGua Simplices are not homothetic. Then $\boldsymbol{\mu}$ it is the unique vertex of $\Pi \;=\; \Pi^{\boldsymbol{a}^{(1)}} + \Pi^{\boldsymbol{a}^{(2)}}$. E.g., if $a_1^{(1)} > a_1^{(2)}$ and hence $a_2^{(1)} < a_2^{(2)}$ (as $a_1^{(k)} a_2^{(k)}$ is equal for $k \;=\; 1, 2$), then*

$$\boldsymbol{\mu}(\Pi) \;:=\; \boldsymbol{a}^{(1)1} + \boldsymbol{a}^{(2)2} \;=\; (a_1^{(1)}, a_2^{(2)}) \;. \tag{5}$$

3. *Let $K \geq 3$. Assume that the ordering is chosen according to decreasing slopes, i.e., (2) is satisfied.*

   *Then $\boldsymbol{\mu}$ is defined recursively via*

$$\begin{aligned} \boldsymbol{\mu}(\Pi) \;&=\; \boldsymbol{\mu}\left(\sum_{k \in \boldsymbol{K}} \Pi^{a^{(k)}}\right) \\ &:=\; \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(K)}\right) + \boldsymbol{\mu}\left(\sum_{k \in \boldsymbol{K} - \{1, K\}} \Pi^{a^{(k)}}\right). \end{aligned} \tag{6}$$

4. *Let* $\mathcal{V}^d$ *denote the set of standard dyadic bargaining problems. Then, for* $\boldsymbol{V} = (\underline{\boldsymbol{x}}, \mathcal{V})$, *define*

$$\boldsymbol{\mu}(\boldsymbol{V}) \;=\; \underline{\boldsymbol{x}} + \boldsymbol{\mu}(\mathbf{0}, \Pi) \;:=\; \boldsymbol{\mu}(\Pi) \tag{7}$$

*that is, translate the solution of* $(\mathbf{0}, \mathcal{V})$ *as defined by the first* items *via* $\underline{\boldsymbol{x}}$.

Figure 2.3 illustrates the construction on $\mathcal{U}^d$. One takes the deGua Simplices with the smallest and largest slope separately adding them up; this constitutes bargaining problems $\Upsilon$ and $\Psi$. In view of the arrangement of the slopes, the vertex of $\Upsilon$ admits all normals at the vertices of $\Psi$. Therefore, the sum $\boldsymbol{\mu}(\Upsilon) + \boldsymbol{\mu}(\Psi)$ is Pareto efficient and employed to define the solution of the sum $\Upsilon + \Psi$. Clearly, we construct an *additive* solution in the particular context.

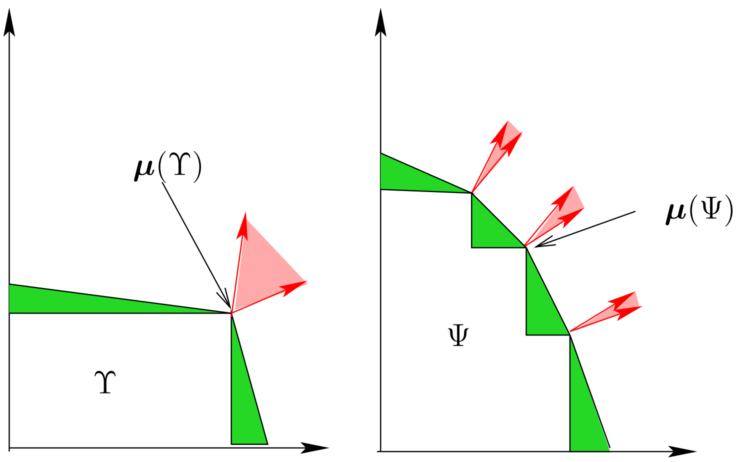


Figure 2.3: Additivity of the solution

The above consideration yields the definition of $\boldsymbol{\mu}$ as well as the additivity within a certain context. Now we prove that $\boldsymbol{\mu}$ generally is indeed superadditive. We wish to provide two simple proofs. We refer the reader to [13], [22] for the original versions which we will discuss to some extent later on. It is sufficient to deal with the mapping on $\mathcal{U}^d$.

**Theorem 2.3.** *Let* $\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ *be a standard dyadic Cephoid in* $\mathbb{R}^2$. *Let*

$$\Pi \;=\; \Upsilon + \Psi \tag{8}$$

*where*

$$\Upsilon = \sum_{k\in\boldsymbol{I}} \Pi^{a^{(k)}} , \quad \Psi = \sum_{k\in\boldsymbol{J}} \Pi^{a^{(k)}} \tag{9}$$

*are both standard dyadic. Then*

$$\boldsymbol{\mu}(\Pi) \geq \boldsymbol{\mu}(\Upsilon) + \boldsymbol{\mu}(\Psi). \tag{10}$$

The first proof hinges on induction, thus it is close to the definition of the solution as discussed above.

**Proof:** If $\Pi$ is the sum of two DeGua Simplices (w.l.g. not homothetic) with equal area, then $\boldsymbol{\mu}(\Pi)$ is the unique vertex on the Pareto surface of $\Pi$ while $\boldsymbol{\mu}(\Upsilon) + \boldsymbol{\mu}(\Psi)$ is a non–Pareto efficient point on the line connecting $\mathbf{0}$ and $\boldsymbol{\mu}(\Pi)$. Hence (10) is obvious.

In order to perform the induction step, consider an arbitrary decomposition such that

$$\Pi = \Upsilon + \Psi, \quad \Upsilon = \sum_{k\in\boldsymbol{I}} \Pi^{a^{(k)}} , \quad \Psi = \sum_{k\in\boldsymbol{J}} \Pi^{a^{(k)}} \tag{11}$$

holds true.

**1st STEP** : Assume that the indices 1 and $K$ are jointly contained in one of the above sets, say $\{1, K\} \subseteq \boldsymbol{I}$. Then, as $\Pi^{(1)} + \Pi^{(K)}$ admits of joint normals at $\boldsymbol{\mu}(\Pi^{(1)} + \Pi^{(K)})$ with all other polyhedra involved, we have

$$\begin{aligned}
\boldsymbol{\mu}(\Pi) &= \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(K)} + \sum_{k\in\boldsymbol{K}-\{1,K\}} \Pi^{a^{(k)}}\right) \\
&= \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(K)}\right) + \boldsymbol{\mu}\left(\sum_{k\in\boldsymbol{K}-\{1,K\}} \Pi^{a^{(k)}}\right) \\
&\qquad \text{(by Definition, see (6) )} \\
&\geq \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(K)}\right) \\
&\quad + \boldsymbol{\mu}\left(\sum_{k\in\boldsymbol{I}-\{1,K\}} \Pi^{a^{(k)}}\right) + \boldsymbol{\mu}\left(\sum_{k\in\boldsymbol{J}} \Pi^{a^{(k)}}\right) \\
&\qquad \text{(by induction hypothesis)} \\
&= \boldsymbol{\mu}\left(\sum_{k\in\boldsymbol{I}} \Pi^{a^{(k)}}\right) + \boldsymbol{\mu}\left(\sum_{k\in\boldsymbol{J}} \Pi^{a^{(k)}}\right) .
\end{aligned} \tag{12}$$

**2$^{\text{nd}}$STEP** : Suppose now, that we have $1 \in \boldsymbol{I}$ and $K \in \boldsymbol{J}$. Let $L$ denote the largest index in $\boldsymbol{I}$, i.e., the one wich induces the largest slope (in absolute value) of a line segment involved in $\Upsilon$. Then we obtain

$$
\begin{aligned}
\boldsymbol{\mu}(\Pi) &= \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(L)} + \sum_{k \in \boldsymbol{I}-\{1,L\}} \Pi^{a^{(k)}} + \sum_{k \in \boldsymbol{J}} \Pi^{a^{(k)}}\right) \\
&\geq \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(L)} + \sum_{k \in \boldsymbol{J}} \Pi^{a^{(k)}}\right) + \boldsymbol{\mu}\left(\sum_{k \in \boldsymbol{I}-\{1,L\}} \Pi^{a^{(k)}}\right) \\
&\qquad \text{(by the } 1^{st} STEP \text{ as} \{1, K\} \subseteq \boldsymbol{J} + \{1, L\}) \\
&\geq \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(L)}\right) + \boldsymbol{\mu}\left(\sum_{k \in \boldsymbol{J}} \Pi^{a^{(k)}}\right) + \boldsymbol{\mu}\left(\sum_{k \in \boldsymbol{I}-\{1,L\}} \Pi^{a^{(k)}}\right) \qquad (13) \\
&\qquad \text{(by induction hypothesis)} \\
&= \boldsymbol{\mu}\left(\sum_{k \in \boldsymbol{J}} \Pi^{a^{(k)}}\right) + \boldsymbol{\mu}\left(\Pi^{(1)} + \Pi^{(L)} \sum_{k \in \boldsymbol{I}-\{1,L\}} \Pi^{a^{(k)}}\right) \\
&\qquad \text{(by Definition applied to } \Upsilon \text{, see (6) )} \\
&= \boldsymbol{\mu}\{\Upsilon\} + \boldsymbol{\mu}\{\Psi\},
\end{aligned}
$$

**q.e.d.**

We provide a second proof which refers to the construction of the solution.

**Proof:** The enumeration is such that the tangent slope decreases with the index $k$. Since the *products* $a_1^{(k)} a_2^{(k)}$ are all equal, it follows that the enumeration satisfies

$$
\begin{aligned}
a_1^{(1)} \geq a_1^{(2)} \ \dots \quad &\geq a_1^{(K)} \ , \\
a_2^{(1)} \leq a_2^{(2)} \ \dots \quad &\leq a_2^{(K)}
\end{aligned} \qquad (14)
$$

W.l.o.g we may assume that K is even (otherwise split every polyhedron homothetically in two). Then we know that

$$
\boldsymbol{\mu}(\Pi) = \left(\sum_{k=1}^{\frac{K}{2}} a_1^{(k)} \ , \ \sum_{k=\frac{K}{2}+1}^{K} a_2^{(k)}\right), \qquad (15)
$$

that is, $\boldsymbol{\mu}(\Pi)$ *collects the* $\frac{K}{2}$ *largest vectors* with respect to each coordinate.

Now with respect to $\Upsilon$ we may as well assume that $|\boldsymbol{I}|$ is even. Thus, there is a decomposition $\boldsymbol{I} = \boldsymbol{I}_1 + \boldsymbol{I}_2$ with $|\boldsymbol{I}_1| = |\boldsymbol{I}_2|$ such that

$$\boldsymbol{\mu}(\Upsilon) = \left( \sum_{k \in \boldsymbol{I}_1} a_1^{(k)} \; , \; \sum_{k \in \boldsymbol{I}_2} a_2^{(k)} \right). \tag{16}$$

The same holds true concerning $\Psi$ with respect to a decomposition $\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2$. Clearly, $|\boldsymbol{I}_1 + \boldsymbol{J}_1| = |\boldsymbol{I}_2 + \boldsymbol{J}_2| = \frac{K}{2}$ and hence

$$\boldsymbol{\mu}_1(\Upsilon) + \boldsymbol{\mu}_1(\Psi) = \sum_{k \in \boldsymbol{I}_1} a_1^{(k)} + \sum_{k \in \boldsymbol{J}_1} a_1^{(k)} = \sum_{k \in \boldsymbol{I}_1 + \boldsymbol{J}_1} a_1^{(k)} \leq \sum_{k=1}^{\frac{K}{2}} a_1^{(k)} \tag{17}$$

as the last sum collects the largest $\frac{K}{2}$ coordinates, it equals $\boldsymbol{\mu}_1(\Pi)$ (see (15)). that is we have

$$\boldsymbol{\mu}_1(\Upsilon) + \boldsymbol{\mu}_1(\Psi) \leq \boldsymbol{\mu}_1(\Pi)$$

Performing the analogous construction for the second coordinate we obtain (10).

**q.e.d.**

This consideration leads immediately to a preliminary uniqueness result.

**Lemma 2.4.** *Let $\boldsymbol{\varphi} : \mathfrak{U}^d \to \mathbb{R}^2_+$ be a mapping satisfying*

1. *$\boldsymbol{\varphi}$ is Pareto efficient*
2. *$\boldsymbol{\varphi}$ chooses the midpoint for $K = 1$ and the unique vertex of $\Pi = \Pi^{\boldsymbol{a}^{(1)}} + \Pi^{\boldsymbol{a}^{(2)}}$ for $K = 2$ and non-homothetic DeGua Simplices.*
3. *$\boldsymbol{\varphi}$ is superadditive.*

   *Then $\boldsymbol{\varphi} = \boldsymbol{\mu}$.*

**Proof:** $\boldsymbol{\varphi}$ coincides with $\boldsymbol{\mu}$ for $K = 1, 2$ by definition.

Moreover, every superadditive solution $\boldsymbol{\varphi}$ is necessarily additive whenever the solutions of the two summands admit of a joint normal.

To see this, consider Figure 2.3. Note that the sum of two Pareto efficient vectors is Pareto efficient if and only if both admit of a joint normal. In Figure 2.3, the corner point of $\Upsilon$ admits of a joint normal with *each* Pareto efficient point of $\Psi$ (some normal cones are indicated). By induction, $\boldsymbol{\varphi}(\Psi) = \boldsymbol{\mu}(\Psi)$ and by definition $\boldsymbol{\varphi}(\Upsilon) = \boldsymbol{\mu}(\Upsilon)$. Therefore

$$\boldsymbol{\varphi}(\boldsymbol{V}) \geq \boldsymbol{\varphi}(\Upsilon) + \boldsymbol{\varphi}(\Psi) \;=\; \boldsymbol{\mu}(\Upsilon) + \boldsymbol{\mu}(\Psi) \;=\; \boldsymbol{\mu}(\boldsymbol{V})$$

But $\boldsymbol{\varphi}$ is Pareto efficient, hence we must have $\boldsymbol{\varphi}(\boldsymbol{V}) = \boldsymbol{\mu}(\boldsymbol{V})$.

**q.e.d.**

# 3 Smooth Pareto Surface: The Surface Integral

The recursive definition of the Maschler–Perles solution provides uniqueness at once. The fact that the solution is superadditive is proved by MASCHLER–PERLES [13], [22] using concepts of "speed" and the "traveling time" for points on the Pareto surface of a Bargaining problem.

To this end, they first extend the solution to smooth bargaining problems. Then they introduce a procedure reflecting the idea of having two points travel on the Pareto curve, simultaneously starting at each player's "bliss point". The speed of the motion is arranged that the product of velocities in directions of both axes is equal at each instant.

The process reflects "concessions" continuously made by the players (Maschler and Perles speak of a "Donkey Cart" moving along the Pareto surface). When both points meet on the Pareto surface, the solution is reached.

This concept is then carried back to polyhedral problems to show superadditivity and hence the coincidence with the concept established for Cephoidal Bargaining Problems. The reader may wish to consult MASCHLER–PERLES [13] or [27] (*CH. VIII, Theorem 4.21, p.588*) for more details.

We will in the present context provide a short descriptioin of the Maschler–Perles solution for bargaining problems with arbitrary (polyhedral od smooth) Pareto surface. For the moment we consider a class $\mathcal{V}_0$ such that the status quo point is $\mathbf{0}$ and (for technical reasons) all normals are positive. Accordingly we write $\boldsymbol{\varphi}(\boldsymbol{U})$ for $\boldsymbol{\varphi}(0, \boldsymbol{U})$ etc. Also we use the notation

(1) $\kappa_1 = \kappa_1(\boldsymbol{U}) = \max\{t \mid (t, 0) \in \boldsymbol{U}\}, \quad \kappa_2 = \kappa_2(\boldsymbol{U}) = \max\{t \mid (0, t) \in \boldsymbol{U}\}$

such that $\boldsymbol{x}^1 = \boldsymbol{x}^1(\boldsymbol{U}) = \kappa_1 \boldsymbol{e}^1$ and $\boldsymbol{x}^2 = \boldsymbol{x}^2(\boldsymbol{U}) = \kappa_2 \boldsymbol{e}^2$ denote the ***bliss points*** for players 1 and 2 respectively.

**Definition 3.1.** *A* ***parametrization*** *of* $\partial \boldsymbol{U}$ *is a bijective mapping*

(2) $$\boldsymbol{x}(\bullet) \; := \; [a, b] \to \partial \boldsymbol{U}$$

*such that*

1. *$[a, b]$ is a nonempty interval, i.e., $a < b$,*
2. *$\boldsymbol{x}(\bullet)$ is differentiable up to countably many points in $[a, b]$.*

**Example 3.2.** *The following examples establish the existence of parametrizations as required by Definition 3.1.*

*Let* $\boldsymbol{U} \in \mathfrak{U}$ *and define a function*

$$C : [0, \kappa_1)] \longrightarrow \mathbb{R}$$

*by*

$$C(t) := \max\{s \mid (t, s) \in U\}. \tag{3}$$

*Then* $C$ *is a concave and decreasing function, hence continuous at all points with the possible exception of* $\kappa_1$. *Moreover,* $C$ *admits of an almost everywhere (Radon-Nikodym) derivative (since it is decreasing and hence absolutely continuous). The derivative exists except for at most countably many points. Therefore, if* $\partial\boldsymbol{U}$ *does not contain a line segment parallel to the* $x_2-$*axis, then we have found a parametrization*

$$\begin{aligned} &\boldsymbol{x}^C : [0, \kappa_1] \longrightarrow \partial\boldsymbol{U} \\ &\boldsymbol{x}^C(t) = (t, C(t)) \qquad (t \in [0, \kappa_1]) \end{aligned} \tag{4}$$

*of* $\partial\boldsymbol{U}$ *which is continuous, monotone decreasing in the second coordinate, and almost surely differentiable.*

*Analogously, we may take the function*

$$\begin{aligned} &D : [0, \kappa_2] \longrightarrow \partial\boldsymbol{U} \\ &D(s) := \max\{t \mid (t, s) \in \boldsymbol{U}\} \end{aligned} \tag{5}$$

*and parametrize* $\partial U$ *by*

$$\begin{aligned} &\boldsymbol{x}^D : [0, \kappa_2] \longrightarrow \mathbb{R}^2 \\ &\boldsymbol{x}^D(t) = (D(t), t) \qquad (t \in [0, \kappa_2]), \end{aligned}$$

*provided there is no line segment parallel to the* $x_1-$*axis.*

∘ ~~~~~~~ ∘

For any parametrization $\boldsymbol{x}(\bullet)$ we denote the derivative

$$\dot{x}_i(t) := \frac{dx_i}{ds}(t) \qquad (i = 1, 2). \tag{6}$$

In particular we observe that

$$\dot{x}_1^C(t) = 1, \qquad \dot{x}_2^C(t) = \frac{dC}{ds}(t) = \dot{C}(t) \qquad (t \in (a, b)\ . \tag{7}$$

**Theorem 3.3.** *Let* $\boldsymbol{V} \in \mathcal{V}_0$.

1. *For any parametrization* $\boldsymbol{x}(\bullet)$

(8)
$$\int_a^b \sqrt{-\dot{x}_1(t)\dot{x}_2(t)}\,dt$$

*is finite and positive.*

2. *Let* $\widehat{\boldsymbol{x}} \in \partial \boldsymbol{U}$, $a < \widehat{x}_1 < \kappa_1$. *Consider the parametrization* $\boldsymbol{x}^C$ *as described in Remark 3.2. Then the integral*

(9)
$$\int_0^{\hat{x}_1} \sqrt{-\dot{x}_1^C(t)\dot{x}_2^C(t)}\,dt \;=\; \int_0^{\hat{x}_1} \sqrt{-\dot{C}(t)}\,dt$$

*is finite and positive.*

3. *Let* $\boldsymbol{x}(\bullet) : [a,b] \to \partial \boldsymbol{U}$ *be a parametrization such that* $x(a) = \boldsymbol{x}^2$ *(the blisspoint of player* 2*), Let* $\widehat{\boldsymbol{x}} \in \partial \boldsymbol{U}$ *and let* $\widehat{t} \in [a,b]$ *be such that* $\boldsymbol{x}(\widehat{t}) = \widehat{\boldsymbol{x}}$. *Then*

(10)
$$\int_a^{\hat{t}} \sqrt{-\dot{x}_1(t)\dot{x}_2(t)}\,dt = \int_0^{\widehat{x}_1} \sqrt{-\dot{x}_1^C(t)\dot{x}_2^C(t)}\,dt \;=\; \int_0^{\hat{x}_1} \sqrt{-\dot{C}(t)}\,dt$$

*and similarly*

(11)
$$\int_{\hat{t}}^{b} \sqrt{-\dot{x}_1(t)\dot{x}_2(t)}\,dt = \int_{\widehat{x}_1}^{\kappa_1} \sqrt{-\dot{x}_1^C(t)\dot{x}_2^C(t)}\,dt \;=\; \int_{\widehat{x}_1}^{\kappa_1} \sqrt{-\dot{C}(t)}\,dt$$

*that is, the integrals are independent of the particular parametrization.*

We do not prove these technical details as they are standard w.r.t. line and surface integrals. (See however [27], *CH. VIII* for a detailed treatment).

Consequently, as the integral under consideration is independent on the parametrization we are justified to use notations like

$$\int_{\partial U} \sqrt{-dx_1\,dx_2} = \int_{\boldsymbol{x}^1}^{\boldsymbol{x}^2} \sqrt{-dx_1\,dx_2} := \int_a^b \sqrt{-\dot{x}_1(t)\dot{x}_2(t)}\,dt$$

or in the case of *item* 3

$$\int_{\boldsymbol{x}^1}^{\widehat{\boldsymbol{x}}} \sqrt{-dx_1\, dx_2} \;:=\; \int_{a}^{\hat{t}} \sqrt{-\dot{x}_1(t)\dot{x}_2(t)} \qquad .$$

**Example 3.4.** *For $\boldsymbol{a} > \boldsymbol{0} \in \mathbb{R}^2_+$ consider in particular a deGua Simplex $\Pi^{\boldsymbol{a}} \in \mathfrak{U}$. Choose the interval $[0, a_1]$ and define $C(t) := a_2 - \frac{a_2}{a_1}t \;\; (0 \le t \le a_1)$.*

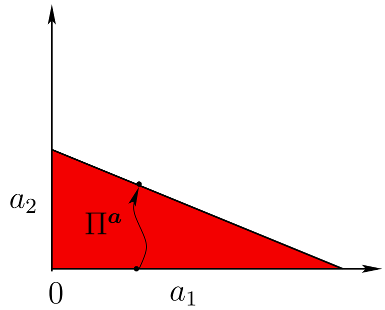


Figure 3.1: Parametrizing a deGua Simplex

*Then*

$$\text{(12)} \qquad \begin{aligned} \boldsymbol{x}^C &: [0, a_1] \longrightarrow \mathbb{R}^2 \\ \boldsymbol{x}^C(t) &= \left(t, a_2 - \frac{a_2}{a_1}t\right) \end{aligned}$$

*parametrizes the Pareto curve $\Delta^{\boldsymbol{a}}$ of $\Pi^{(\boldsymbol{a})}$. Thus we compute the surface integral to be*

$$\int_{\partial U} \sqrt{-dx_1\, dx_2} = \int_0^{a_1} \sqrt{-\dot{x}_1^C \dot{x}_2^C}\, dt = \int_0^{a_1} \sqrt{\frac{a_2}{a_1}}\, dt = \sqrt{a_1 a_2}.$$

*That is, we obtain (up to a factor $\frac{1}{2}$) the squareroot of the area of $\Pi^{\boldsymbol{a}}$.*

*Now let $\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ be a cephoid. By a simple generalization we obtain for the surface integral*

$$\text{(13)} \qquad \int_{\partial U} \sqrt{-dx_1\, dx_2} \;=\; \sum_{k \in \boldsymbol{K}} \sqrt{a_1^{(k)} a_2^{(k)}}\ .$$

*In particular, if $\Pi$ is standard dyadic and $\alpha_T$ the common product resulting from the deGua Simplices involved (Definition 2.1), then of course*

$$\text{(14)} \qquad \int_{\partial U} \sqrt{-dx_1\, dx_2} \;=\; K\sqrt{\alpha_T}.$$

*Observe that in both cases the products $a_1^{(k)}a_2^{(k)}$ have the dimension of an area. However, as we apply the square root, the surface integral has the dimension of a length.*

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**Lemma 3.5.** *Let $\Pi \in \mathfrak{U}^d$ and recall the blisspoints $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ of both the players. Then the Maschler–Perles solution is characterized by the equation*

$$\int_{\boldsymbol{x}^1}^{\boldsymbol{\mu}(\boldsymbol{V})} \sqrt{-dx_1\,dx_2} \;=\; \int_{\boldsymbol{\mu}(\boldsymbol{V})}^{\boldsymbol{x}^2} \sqrt{-dx_1\,dx_2} \qquad . \tag{15}$$

**Proof:** Let $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$; we can assume the $K$ is even and the deGua Simplices involved are ordered according to decreasing slope. Let $\alpha_T$ denote the common product of the deGua Simplices involved as in Definition 2.1. Then according to (15) we have

$$\boldsymbol{\mu}(\boldsymbol{V}) = \boldsymbol{\mu}(\Pi) = \left( \sum_{k=1}^{\frac{K}{2}} a_1^{(k)} \,,\; \sum_{k=\frac{K}{2}+1}^{K} a_2^{(k)} \right) \tag{16}$$

The surface integral collects the surface measure of the various deGua Simplices, i.e., the squareroot of the areas. Therefore we have

$$\begin{aligned} \int_{\boldsymbol{x}^1}^{\boldsymbol{\mu}(\boldsymbol{V})} \sqrt{-dx_1\,dx_2} \;&=\; \sum_{k=1}^{\frac{K}{2}} \sqrt{a_1^{(k)}a_2^{(k)}} \;=\; \frac{K}{2}\sqrt{\alpha_T} \\ &=\; \sum_{k=\frac{K}{2}+1}^{K} \sqrt{a_1^{(k)}a_2^{(k)}} \;=\; \int_{\boldsymbol{\mu}(\boldsymbol{V})}^{\boldsymbol{x}^2} \sqrt{-dx_1\,dx_2}. \end{aligned} \tag{17}$$

**q.e.d.**

Naturally, we make this to be the definition in general.

**Definition 3.6.** *The mapping $\boldsymbol{\mu} : \mathfrak{U} \to \mathbb{R}^n_+$ defined by*

$$\int_{\bar{\boldsymbol{x}}^1}^{\mu} \sqrt{-dx_1\,dx_2} = \int_{\mu}^{\bar{\boldsymbol{x}}^2} \sqrt{-dx_1\,dx_2}. \tag{18}$$

*is the* ***Maschler–Perles solution****.* $\boldsymbol{\mu}$ *is extended to the mapping* $\boldsymbol{\mu} : \mathcal{V} \to \mathbb{R}^n$ *via*

$$\boldsymbol{\mu}(\boldsymbol{V}) \;:=\; \boldsymbol{\mu}(\underline{\boldsymbol{x}}, \boldsymbol{U}) \;:=\; \underline{\boldsymbol{x}} + \boldsymbol{\mu}(\boldsymbol{U}) \;. \tag{19}$$

**Remark 3.7.** *Let* $\boldsymbol{U} \in \mathcal{U}$ *and let* $\boldsymbol{x}(\bullet)$ *be a parametrization of* $\partial \boldsymbol{U}$*. Define*

$$T = T_{\boldsymbol{V}}^{\boldsymbol{x}(\bullet)} \tag{20}$$

*by*

$$\int_a^T \sqrt{-\dot{x}_1 \dot{x}_2}\, dt = \int_T^b \sqrt{-\dot{x}_1 \dot{x}_2}\, dt, \tag{21}$$

*then*

$$\mu(\boldsymbol{V}) := \boldsymbol{x}(T) = \boldsymbol{x}(T_{\boldsymbol{V}}^{\boldsymbol{x}(\bullet)}) \;. \tag{22}$$

*Indeed, the integrals do not depend on the particular parametrization.*

*In particular, we can interpret the "parameter $t$" of a parametrization as "time". Then $\boldsymbol{x}(\bullet)$ describes the motion of a point along $\partial \boldsymbol{U}$ and $\dot{\boldsymbol{x}}(t)$ is the "velocity" or "speed" of the movement of the point. The velocity is a vector pointing in the direction of the motion – a tangency to the curve $\partial \boldsymbol{U}$. The coordinates $\dot{x}_1(t)$ and $\dot{x}_2(t)$ describe the components of the velocity in the directions of the axes. Or, returning to the interpretation of an agreement point of players moving along the Pareto curve, we imagine that they make* concessions or gains *respectively depending on the direction the motion takes. Maschler–Perles like to speak of the "Donkey Cart" (see also Figure 3.2).*

*Roughly, if the motion starts e.g. at player 2's blisspoint $\boldsymbol{x}^2$ and moves from left to right along the Pareto curve, then $-\frac{\dot{x}_2(t)}{\dot{x}_1(t)}$ reflects the* the rate of concessions *player 2 is yielding to player* 1 *during the motion. This kind of process stops when the agreement point reaches the Maschler–Perles solution.*

*Within this picture, $T^{\boldsymbol{x}(\bullet)}$ is the* ***traveling time*** (under $\boldsymbol{x}(\bullet)$*) which the point $\boldsymbol{x}(t)$ requires in order to reach the position of the Maschler–Perles solution when starting* from either one of the blisspoints.

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In order to make this even more lucid, we focus on a particular parametrization which exhibits a distinct view on the traveling time.

**Remark 3.8.** *Let $\boldsymbol{U} \in \mathfrak{U}$ and let $\boldsymbol{x}(\bullet) : [a,b] \to \mathbb{R}^2$ be a parametrization. Consider the strictly monotone function*

$$f : [a,b] \to \mathbb{R}$$

*given by*

$$f(t) := \int_a^t \sqrt{-\dot{x}_1 \dot{x}_2}\, ds \qquad (t \in [a,b]) \tag{23}$$

*and let $\bar{b} := f(b)$ such that*

$$f : [a,b] \to [0,\bar{b}]$$

*is a bijection. Let $h$ be the inverse of $f$, i.e.,*

$$\begin{aligned} &h : [0,\bar{b}] \longrightarrow [a,b] \\ &h(s) = f^{-1}(s) \qquad (s \in [0,\bar{b}])\ . \end{aligned} \tag{24}$$

*Using $'$ for $\frac{d}{ds}$ for the (almost surely defined) derivatives, we obtain*

$$f' = \sqrt{-\dot{x}_1 \dot{x}_2} > 0 \quad ; \quad h'(s) = \frac{1}{f'(f^{-1}(s))} = \frac{1}{(f' \circ h)(s))} \qquad (s \in [0,\bar{b}]). \tag{25}$$

*Consider a further parametrization $\boldsymbol{\xi}(\bullet)$ defined by*

$$\begin{aligned} &\boldsymbol{\xi} : [0,\bar{b}] \longrightarrow \mathbb{R}^2 \\ &\boldsymbol{\xi}(s) = \boldsymbol{x}(h(s)) = (\boldsymbol{x} \circ h)(s) \qquad (s \in [0,\bar{b}])\ . \end{aligned} \tag{26}$$

*then we compute the derivatives in direction of the axis; these are for $i = 1,2$:*

$$\xi_i' = (\dot{x}_i \circ h) \cdot h' = (\dot{x}_i \circ h) \frac{1}{f' \circ h},$$

*With some abbreviation of notation, we write this as*

$$\xi_i' = \frac{\dot{x}_i}{\sqrt{-\dot{x}_1 \dot{x}_2}}\ . \tag{27}$$

*As a consequence we obtain*

$$\xi_1' \cdot \xi_2' = \frac{\dot{x}_1 \dot{x}_2}{\left(\sqrt{-\dot{x}_1 \dot{x}_2}\right)^2} = -1\ . \tag{28}$$

*Formula (28) shows: If $\boldsymbol{\xi}$ is chosen for the parametrization (i.e., a time change applied via (23) and (26)), then the point $\boldsymbol{\xi}(t)$ moves on $\partial \boldsymbol{U}$ with a velocity $\dot{\boldsymbol{\xi}}$ in a way such that the product of the velocities in each coordinate direction is constant and equals $-1$.*

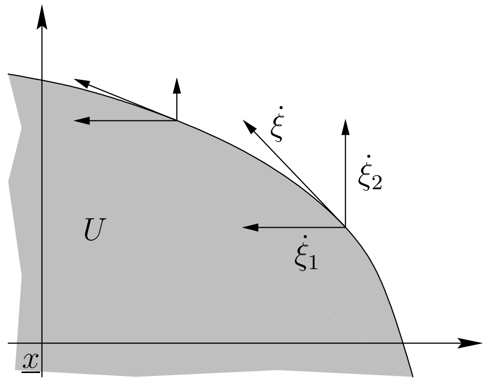


Figure 3.2: The Donkey Cart

*Now in this case the traveling time* $\overline{T} := T_{\boldsymbol{U}}^{\xi(\bullet)}$*, as defined for* $\boldsymbol{\xi}(\bullet)$ *via Remark 3.7 formula (20) or (21) respectively is obtained as follows. We have*

$$(29) \qquad \int_0^{\overline{T}} \sqrt{-\dot{\xi}_1\dot{\xi}_2}\, dt = \int_{\overline{T}}^{\bar{b}} \sqrt{-\dot{\xi}_1\dot{\xi}_2}\, dt \quad \text{i.e.} \quad \int_0^{\overline{T}} 1\, dt = \int_{\overline{T}}^{\bar{b}} 1\, dt\ ,$$

*which results in* $\overline{T} = \bar{b} - \overline{T}$*, i.e.,*

$$(30) \qquad \overline{T} \ = \ T_{\boldsymbol{U}}^{\boldsymbol{\xi}(\bullet)} \ = \ \frac{\bar{b}}{2}.$$

*Thus* $T \ = \ T_{\boldsymbol{U}}^{\xi(\bullet)}$ *is the midpoint of* $[0, \bar{b}]$*.*

*In other words, if two particles move with a speed such that the product of velocities in coordinate directions equals -1 for each of them, and if they start at the different endpoints of* $\partial\boldsymbol{U}$*, then they will meet exactly at half the time needed to traverse* $\partial\boldsymbol{U}$ *and they will meet at* $\mu(\boldsymbol{V})$*.*

*Note that the parametrization* $\boldsymbol{\xi}$ *is almost surely uniquely defined by the requirement* $\dot{\xi}_1\dot{\xi}_2 = -1$*. For, if we return to the parametrization* $\boldsymbol{x}^C(\bullet)$ *as discussed in Example 3.2, then we obtain equations* $\dot{\xi}_1\dot{\xi}_2 = -1$ *and* $\frac{\dot{\xi}_1}{\dot{\xi}_2} = \dot{C}$*. Consequently*

$$\dot{\xi}_2 = \dot{C}\dot{\xi}_1 = -\dot{C}\frac{1}{\dot{\xi}_2} \qquad \dot{\xi}_2 = -\sqrt{\dot{C}}\ .$$

*In view of this observation we may conclude that*

$$(31) \qquad \bar{T}(\boldsymbol{U}) := T_{\boldsymbol{U}}^{\xi(\bullet)}$$

*does not depend on the parametrization. (*$\xi$ *does not significantly change by a reverse of time).*

**Definition 3.9.** *We call* $\boldsymbol{\xi}$ *the* ***standard parametrization*** *and*

$$\bar{T}(\boldsymbol{U}) := T_{\boldsymbol{U}}^{\boldsymbol{\xi}(\bullet)} \tag{32}$$

*the* ***standard traveling time,***

*Recall that the Maschler–Perles solution is*

$$\mu(\boldsymbol{U}) = \boldsymbol{\xi}(\bar{T}(\boldsymbol{U})) \,. \tag{33}$$

*Moreover, as* $\int_{\boldsymbol{x}^1}^{\boldsymbol{\mu}(\boldsymbol{U})} = \sqrt{-dx_1 dx_2}$ *does not depend on the parametrization (Theorem 3.3), we obtain immediately*

$$\int_{\boldsymbol{x}^1}^{\boldsymbol{\mu}(\boldsymbol{U})} \sqrt{-dx_1 dx_2} = \int_0^{\bar{T}(\boldsymbol{U})} \sqrt{-\dot{\xi}_1 \dot{\xi}_2}\, dt = \bar{T}(\boldsymbol{U}) = T_{\boldsymbol{U}}^{\boldsymbol{\xi}(\bullet)} \,. \tag{34}$$

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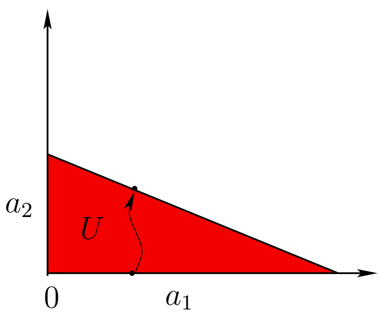



Figure 3.3

**Example 3.10.** *Let* $\boldsymbol{U} \in \mathfrak{U}$ *be a deGua Simplex* $\Pi^{\boldsymbol{a}}$*, that is a triangle as in Figure 3.3. Then the standard parametrization is verified to be*

$$\xi : [0, \sqrt{a_1 a_2}] \to \mathbb{R}^2$$
$$\xi(t) = \left( \sqrt{\frac{a_1}{a_2}} t, a_2 - \sqrt{\frac{a_2}{a_1}} t \right)$$

*(or its time reverse), thus the standard traveling time is the area generated by the deGua Simplex, i.e.,*

$$\bar{T}(\boldsymbol{U}) = \frac{1}{2}\sqrt{a_1 a_2} \,. \tag{35}$$

∘ ~~~~~~~ ∘

**Theorem 3.11.** *1. Let $\Pi \in \mathcal{U}^d$ and let $\alpha_T$ be the common product of the deGua Simplices involved. Then the standard traveling time is*

$$\bar{T}(\boldsymbol{U}) \;=\; T_{\boldsymbol{U}}^{\boldsymbol{\xi}(\bullet)} \;=\; \frac{K}{2}\sqrt{\alpha_T}\ . \tag{36}$$

*2. The standard traveling time is additive on standard dyadic Cephoids, , i.e., if $\Pi, \Pi' \in \mathcal{U}^d$ are both polyhedral, then*

$$\bar{T}(\Pi + \Pi') = \bar{T}(\Pi) + \bar{T}(\Pi')\ . \tag{37}$$

**Proof:** The first statement follows exactly as in Example 3.4. For the second we can assume that T and $\alpha_T$ are common in both dyadic Cephoids. By partitioning the deGua Simplices involved if necessary, we can also assume the $K$ is the same in the representation of both $\Pi$ $\Pi'$.

Then $\Pi + \Pi'$ is the sum of $2K$ deGua Simplices and hence by (36)

$$\bar{T}(\Pi + \Pi') = K\sqrt{T} = \frac{K}{2}\sqrt{T} + \frac{K}{2}\sqrt{T} = \bar{T}(\Pi) + \bar{T}(\Pi')\ . \tag{38}$$

**q.e.d.**

**Theorem 3.12.** $\boldsymbol{\mu}$ *is a bargaining solution on $\mathcal{V}$.*

**Proof:** It suffices to provide a proof for $\boldsymbol{\mu}$ seen as a mapping on $\mathcal{U}$. Pareto efficiency is obvious. We prove scale invariance, it will then be obvious how to do anonymity.

Discussing the situation on $\mathcal{U}$, it suffices to consider (positive) *linear* transformations $L : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$L(x) = (\alpha_1 x_1, \alpha_2 x_2) \quad (x \in \mathbb{R}^2)$$

with $(0 < \alpha \in \mathbb{R}^2)$.

Now, if $\boldsymbol{x}(\bullet)$ parametrizes $\partial \boldsymbol{U}$, then it is not hard to see that $(L \circ \boldsymbol{x})(\bullet)$ parametrizes $\partial L(\boldsymbol{U})$. And if $T = T_{\boldsymbol{V}}^{x(\bullet)}$ is such that

$$\int_a^T \sqrt{-\dot{x}_1\dot{x}_2}\, dt = \int_T^b \sqrt{-\dot{x}_1\dot{x}_2}\, dt\ ,$$

then it follows that

$$
\begin{aligned}
\int_a^T \sqrt{-L_1 \dot{\circ}\, x_1 L_2 \dot{\circ}\, x_2}\, dt &= \int_a^T \sqrt{-\alpha_1 \dot{x}_1 \alpha_2 \dot{x}_2}\, dt \\
&= \sqrt{\alpha_1\alpha_2} \int_a^T \sqrt{-\dot{x}_1 \dot{x}_2}\, dt \\
&= \sqrt{\alpha_1\alpha_2} \int_T^b \sqrt{-\dot{x}_1 \dot{x}_2}\, dt \\
&= \int_T^b \sqrt{-L_1 \dot{\circ}\, x_1 L_2 \dot{\circ}\, x_2}\, dt
\end{aligned} \tag{39}
$$

holds true. Consequently we have $T_{\boldsymbol{LU}}^{L\circ x(\bullet)} = T_{\boldsymbol{U}}^{x(\bullet)}$ and

$$
\mu(L\boldsymbol{U}) = L \circ x(T_{\boldsymbol{LU}}^{L\circ x(\bullet)}) = L \circ x(T_{\boldsymbol{U}}^{x(\bullet)}) = L(\mu(\boldsymbol{U}))\ . \tag{40}
$$

**q.e.d.**

We have established the uniqueness of a superadditive bargaining solution on $\mathcal{U}^d$ (and $\mathcal{V}^d$) preliminarily by Lemma 2.4. We now have to complete this task.

**Theorem 3.13.** $\boldsymbol{\mu}$ *is the unique superadditive bargaining solution on* $\mathcal{V}^d$.

**Proof:** Let $\boldsymbol{\varphi}$ be any other superadditive bargaining solution on $\mathcal{V}^d$.

**1st STEP** :

By scale invariance and anonymity it is clear that, for any deGua Simplex $\Pi^{\boldsymbol{a}}$, it follows that $\boldsymbol{\varphi}(\Pi^{\boldsymbol{a}})$ is the midpoint $\frac{\boldsymbol{a}}{2}$. For, by choosing an appropriate rescaling one can transform $\Pi^{\boldsymbol{a}}$ into $\Pi^{\boldsymbol{e}}$, which is completely symmetric and hence (by anonymity) yields $\boldsymbol{\varphi}(\Pi^{\boldsymbol{e}}) = (\frac{1}{2}, \frac{1}{2})$.

**2nd STEP** :

Next, let $\Pi = \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}$ all ingredients being standard dyadic. The situation is depicted in Figure 3.4. We know that the area of both $\Pi^{\boldsymbol{a}}$ and $\Pi^{\boldsymbol{b}}$ is equal.

Consider the linear transformation of utility $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$
L(x_1, x_2) = (a_2 x_1, b_2 x_2) \quad (\boldsymbol{x} \in \mathbb{R}^2_+)
$$

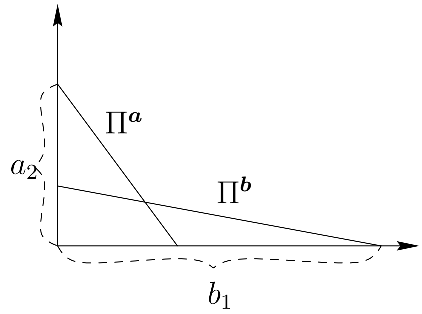



Figure 3.4: Summing two dyadic deGua Simplices

After the transformation the left side of $L(\Pi^{\boldsymbol{a}})$ and the lower side of $L(\Pi^{\boldsymbol{b}})$ have equal length $a_2b_1$. Nevertheless, both deGua Simplices after the transformation do have equal area as before, as the area is simultaneously multiplied by $a_2b_1$. Then necessarily $L(\Pi^{\boldsymbol{a}})$ and $L(\Pi^{\boldsymbol{b}})$ have to be equal up to reflection along the diagonal; the situation is depicted in (Figure 3.5).

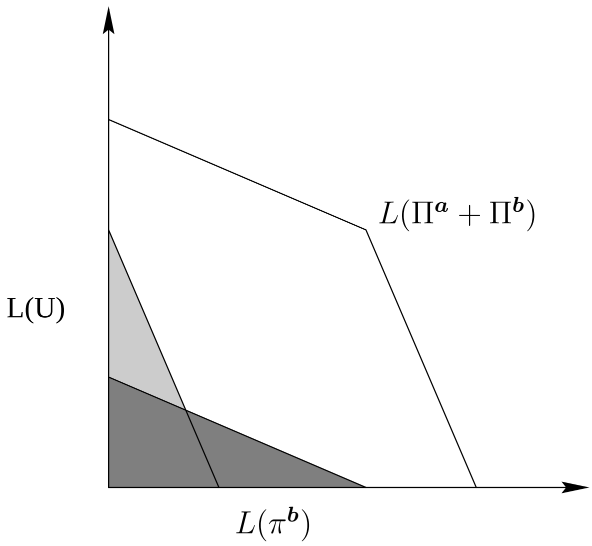



Figure 3.5: After the Transformation

Consequently $L(\Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}) = L(\Pi^{\boldsymbol{a}}) + L(\Pi^{\boldsymbol{b}})$ is symmetric. As $\boldsymbol{\varphi}$ is Pareto efficient and anonymous, we conclude that

$$\boldsymbol{\varphi}(L(\Pi^{\boldsymbol{a}}) + L(\Pi^{\boldsymbol{b}})) = \boldsymbol{\varphi}(L(\Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}})),$$

and hence is the unique edge of $L(\partial\Pi)$. Consequently $\boldsymbol{\varphi}(\Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}})$ is the unique edge of $\partial\Pi$.

**3$^{\text{rd}}$STEP** : We have now proved that $\boldsymbol{\varphi}$ satisfies all conditions of Lemma 2.4, hence $\boldsymbol{\varphi} = \boldsymbol{\mu}$,

**q.e.d.**

In order to expand this result to $\mathcal{V}$ one has to rely on a continuity property as $\boldsymbol{\mu}$ employes the surface integral. One can base the main argument on the Hausdorff topology applied to $\mathcal{U}$ and then , in a natural manner to $\mathcal{V}$.

Within our present context, we will just outline the procedure. The details are to be found with MASCHLER–PERLES [13] or – for a textbook version – in [27]. One starts by extending the Hausdorff topology on compact sets to bargaining solution in an obvious manner. One then shows that $\boldsymbol{\mu}$ is continuous on $\mathcal{U}$.

Clearly, the bargaining problems in $\mathcal{U}^d$ are dense within $\mathcal{U}$. Thus, we are in the position to claim that a unique superadditive *and continuous* bargaining solution exists on $\mathcal{U}$. For completeness, we cite this as a Theorem.

**Theorem 3.14** (PERLES-MASCHLER [13]). *There exists a uniquely defined superadditive and continuous bargaining solution on* $\mathcal{U}$. *This is the Maschler–Perles solution.*

# 4 Cephoids: The Surface Measure

We now return to our topic – Cephoids – and review the Maschler–Perles solution under a slightly changed aspect.

The "donkey card" of Maschler and Perles reflects the intuitive idea of a point traveling along the Pareto surface with a certain speed. This speed is determined by two forces pulling in the direction of the axis (the donkeys pulling in different directions). The product of these velocities can be normalized to be $-1$ (in the standard parametrization).

This process – applied for two such vehicles or points on the Pareto surface – represents the concept of two players yielding to the demands of the opponent continuously until a point of stability is reached: at the Maschler–Perles solution each player has conceded the same amount of "utils".

Now we change our intuition as follows: we consider a *surface measure* defined on the Pareto surface with a density (w.r. to the Lebesgue measure on the Pareto surface) corresponding to the traveling speed as mentioned above.

We start out with a deGua Simplex $\Pi^{\boldsymbol{a}}$. We associate a surface measure $\alpha := \sqrt{a_1 a_2}$ to the Pareto surface $\Delta^{\boldsymbol{a}}$ of $\Pi^{\boldsymbol{a}}$. More generally, to some interval $I \in \Delta^{\boldsymbol{a}}$ with length (Lebesgue measure) $\ell$ we associate the surface measure

$$(1) \qquad \boldsymbol{\iota}_{\Delta}(I) \;=\; \frac{\sqrt{a_1 a_2}}{\sqrt{a_1^2 + a_2^2}} \;=\; \frac{\alpha}{\ell}\boldsymbol{\lambda}(I)\ .$$

such that the total is indeed $\boldsymbol{\iota}_{\Delta}(I) = \alpha$ . This way we generate a measure $\boldsymbol{\iota}_{\Delta}$ on $\Delta^{\boldsymbol{a}}$ with density $\frac{\alpha}{\ell}$; we call this the ***surface measure***.

Next let

$$\widehat{\Pi}^{\boldsymbol{a}} \;:=\; \frac{\alpha}{\sqrt{2}}\,\Pi^{\boldsymbol{e}} \;=\; \Pi^{\frac{\alpha}{\sqrt{2}}\boldsymbol{e}}$$

be an appropriate multiple of the the unit deGua Simplex. The length (Lebesgue measure) of the Pareto surface $\widehat{\Delta}^{\boldsymbol{a}}$ is $\alpha$. Therefore we can establish the mapping

$$\widehat{\boldsymbol{\kappa}} \;:\; \Delta^{\boldsymbol{a}} \to \widehat{\Delta}^{\boldsymbol{a}}$$

by the agreement

$$(2) \qquad \widehat{\boldsymbol{\kappa}}(\boldsymbol{a}^i) = \frac{\alpha}{\sqrt{2}}\boldsymbol{e}^i \quad (i = 1, 2)$$

and

$$(3) \qquad \widehat{\boldsymbol{\kappa}}(t\boldsymbol{a}^1 + (1-t)\boldsymbol{a}^2) \;:=\; t\widehat{\boldsymbol{\kappa}}(\boldsymbol{a}^1) + (1-t)\widehat{\boldsymbol{\kappa}}(\boldsymbol{a}^2) \quad (0 \le t \le 1)\ .$$

The mapping $\widehat{\boldsymbol{\kappa}}$ transports the measure $\boldsymbol{\iota}_\Delta$ from $\Delta^{\boldsymbol{a}}$ to $\widehat{\Delta}^{\boldsymbol{a}}$ this measure is

$$\widehat{\boldsymbol{\kappa}}\boldsymbol{\iota}_\Delta \; := \; \boldsymbol{\iota}_\Delta \circ \widehat{\boldsymbol{\kappa}}^{-1} \; . \tag{4}$$

The result is

$$(\widehat{\boldsymbol{\kappa}}\boldsymbol{\iota}_\Delta)(\widehat{\Delta}^{\boldsymbol{a}}) \; = \; \boldsymbol{\iota}_\Delta(\widehat{\boldsymbol{\kappa}}^{-1}(\widehat{\Delta}^{\boldsymbol{a}})) \; = \; \boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}}) \; = \; \alpha \; , \tag{5}$$

and because of linearity of $\widehat{\boldsymbol{\kappa}}$ (equation (3)) we obtain the Lebesgue measure on $\widehat{\Delta}^{\boldsymbol{a}}$, i.e.,

$$\widehat{\boldsymbol{\kappa}}\boldsymbol{\iota}_\Delta = \boldsymbol{\lambda} \quad \text{on} \quad \widehat{\Delta}^{\boldsymbol{a}} \; . \tag{6}$$

This procedure is now generalized to an arbitrary Cephoid in two dimensions, mapping the Pareto surface on a suitable multiple of the unit Simplex. To this end, consider a Cephoid $\Pi = \sum_{k \in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$. We explicitely assume that the ordering of the line segments follows the slopes as previously. Then, to every line segment $\Delta^{(k)} \; = \; \Delta^{\boldsymbol{a}^{(k)}}$ *and its translate within* $\partial\Pi$ we assign the surface measure $\boldsymbol{\iota}_\Delta(\bullet)$ as above. The surface measure yields

$$\boldsymbol{\iota}_\Delta(\Delta^{(k)}) \; = \; \sqrt{a_1^{(k)} a_2^{(k)}} \; := \; \alpha_k \quad (k \in \boldsymbol{K}) \; . \tag{7}$$

Performing this sumultaneously for all $k$ we obtain $\boldsymbol{\iota}_\Delta(\bullet)$ on $\partial\Pi^{(k)}$ such which has a piecewise constant density with respect to Lebesgue measure (which is $\frac{\alpha_k}{\sqrt{a_1^{(k)} + a_2^{(k)}}}$ on $\Delta^{(k)}$). We call $\boldsymbol{\iota}_\Delta$ the ***surface measure*** on the Pareto surface $\partial\Pi^{\boldsymbol{a}}$ of $\Pi$.

Now we proceed as in the construction of the canonical representation in CHAPTER 2: For $k \in \boldsymbol{K}$ we define a multiple of the unit Simplex

$$\widehat{\Pi}^{(k)} \; := \; \frac{1}{\sqrt{2}} \alpha_k \Pi^{\boldsymbol{e}} \; = \; \Pi^{\frac{\alpha_k}{\sqrt{2}} \boldsymbol{e}} \tag{8}$$

such that the surface has length $\lambda(\widehat{\Delta}^{(k)}) = \alpha_k$. Then we put

$$\widehat{\Pi} \; := \; \sum_{k \in \boldsymbol{K}} \widehat{\Pi}^{(k)} \; = \; (\sum_{k \in \boldsymbol{K}} \frac{1}{\sqrt{2}} \alpha_k) \Pi^{\boldsymbol{e}} \; =: \; \frac{1}{\sqrt{2}} \overline{\alpha} \Pi^{\boldsymbol{e}} \; . \tag{9}$$

the Pareto surface

$$\partial\widehat{\Pi} = \overline{\alpha} \Delta^{\boldsymbol{e}} = \widehat{\Delta}$$

is a multiple of the unit Simplex with length

$$\boldsymbol{\iota}_\Delta(\Delta) = \sum_{k \in \boldsymbol{K}} \frac{1}{\sqrt{2}} \alpha_k \; = \; \frac{1}{\sqrt{2}} \overline{\alpha} \; . \tag{10}$$

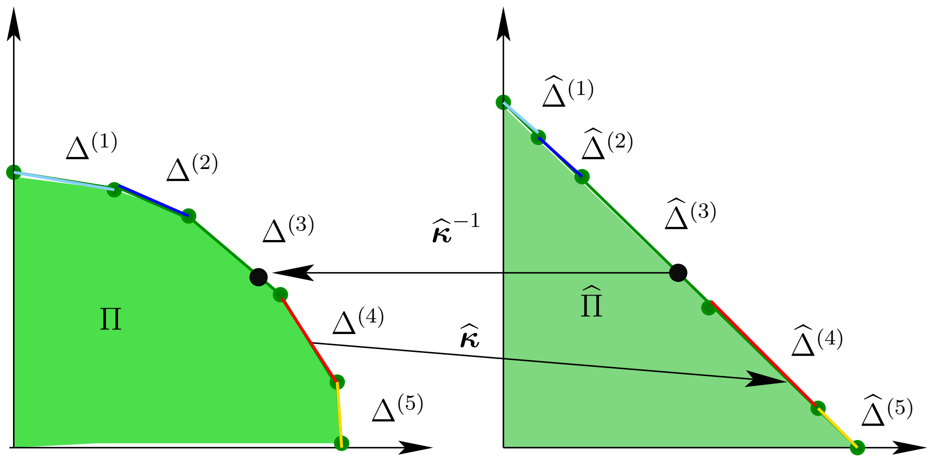



Figure 4.1: Mapping $\partial\Pi$ onto $\widehat{\Delta}$

which is the length of $\partial\Pi$ in terms of the surface measure.

Accordingly, we construct a bijective and piecewise affine mapping $\widehat{\boldsymbol{\kappa}} : \Delta \to \widehat{\Delta}$ exactly in analogy to the canonical mapping $\boldsymbol{\kappa}$ defined in CHAPTER 2. We proceed by mapping the translate of $\Delta^{(k)}$ (as a part of $\partial\Pi$) on the corresponding copy of $\widehat{\Delta}^{(k)}$ such that the order dictated by the slopes $|\frac{a_2^{(k)}}{a_1^{(k)}}|$ is respected. So, if $\Delta^{(1)}$ has the smallest slope $|\frac{a_2^{(1)}}{a_1^{(2)}}|$, then the translate of $\Delta^{(1)}$ is the line segment in the uppermost left corner of $\partial\Pi$ and this is mapped on a line segment of length $\alpha_1$ in the uppermost left corner of $\widehat{\Delta}$ etc. Figure 4.1 indicates the procedure.

Formally, one proceeds as in CHAPTER 1. A "grid" on the Pareto surface

$$\partial\widehat{\Pi} = \frac{1}{\sqrt{2}}\overline{\alpha}\Delta\boldsymbol{e} \tag{11}$$

is provided by the vectors

$$\widehat{\boldsymbol{a}}^{(1)} := \frac{1}{\sqrt{2}}(0, \sum_{l\in\boldsymbol{K}}\alpha_2^l), \quad \widehat{\boldsymbol{a}}^{(k)} := \frac{1}{\sqrt{2}}(\sum_{l\leq(k-1)}\alpha_1^l, \sum_{l\geq k}\alpha_2^l) \quad (k\in\boldsymbol{K})\ . \tag{12}$$

These vectors generate Simplices

$$\widehat{\Delta}^{(k)} = [\widehat{\boldsymbol{a}}^{(k-1)}, \widehat{\boldsymbol{a}}^{(k)}] \quad (k\in\boldsymbol{K}) \tag{13}$$

on $\widehat{\Delta}$ the order of which reflects the ordering of the $\Delta^{(k)}$, i.e., follows deacreasing slope. The mapping $\widehat{\boldsymbol{\kappa}}$ is then extended in the obvious affine way to preserve the lattice structure (the poset).

Combining we formulate

**Theorem 4.1.** *The poset* $\widehat{\mathcal{V}}$ *generated on* $\widehat{\Delta}$ *is isomorphic to the poset* $\mathcal{V}$. *Hence* $(\partial\Pi, \mathcal{V})$ *and* $(\widehat{\Pi}, \widehat{\mathcal{V}})$ *are combinatorically equivalent.*

**Proof:**

Same as in CHAPTER 2 SECTION 1, Theorem 1.3.

**q.e.d.**

**Definition 4.2.** *Let* $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ *be a Cephoid in* $\mathbb{R}^2_+$. *Let*

$$\widehat{\Delta} \ := \ \overline{\alpha}\Delta^{\boldsymbol{e}} \ = \ \frac{1}{\sqrt{2}}\left(\sum_{k\in\boldsymbol{K}} \sqrt{a_1^{(k)} a_2^{(k)}}\right) \Pi^{\boldsymbol{e}} \ . \tag{14}$$

*We call* $\widehat{\Delta}$ *the* ***measure preserving representation*** *of* $\Pi$. *The mapping*

$$\widehat{\boldsymbol{\kappa}} \ := \ \partial\Pi \to \widehat{\Delta} \tag{15}$$

*is the* ***measure preserving mapping***. *The endpoints (blisspoints) of* $\widehat{\Delta}$ *are* $\widehat{\boldsymbol{x}}^1 \ := \ \frac{1}{\sqrt{2}}(\overline{\alpha}, 0)$ *and* $\widehat{\boldsymbol{x}}^2 \ := \ \frac{1}{\sqrt{2}}(0, \overline{\alpha})$; *the center point of* $\widehat{\Delta}$ *is denoted by*

$$\widehat{\boldsymbol{\mu}} \ := \ \frac{1}{2\sqrt{2}}\overline{\alpha}\boldsymbol{e} \ . \tag{16}$$

*Naturally,* $\widehat{\boldsymbol{\mu}} = \boldsymbol{\mu}(\widehat{\Pi})$ *is the Maschler–Perles solution to the bargaining problem* $(\boldsymbol{0}, \widehat{\Pi}$.

**Theorem 4.3.** *Let* $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ *be a Cephoid. Let* $\widehat{\Pi}$ *denote the measure preserving representation and let* $\widehat{\boldsymbol{\kappa}}$ *denote the measure preserving mapping. Then the Maschler–Perles solution of* $(0, \Pi)$ *is the inverse image of the center point, i.e.,*

$$\boldsymbol{\mu}(\Pi) \ = \ \widehat{\boldsymbol{\kappa}}^{-1}(\widehat{\boldsymbol{\mu}}) \ = \ \widehat{\boldsymbol{\kappa}}^{-1}\left(\boldsymbol{\mu}(\widehat{\Pi})\right) \ = \ \widehat{\boldsymbol{\kappa}}^{-1}\left(\frac{1}{2\sqrt{2}}\overline{\alpha}\boldsymbol{e}\right) \ . \tag{17}$$

**Proof:** Consider the function $f = \mathbb{1}_{[\boldsymbol{x}^1, \boldsymbol{\mu}(\Pi)]}$ defined on $\partial\Pi$. $f$ is being transported to $\widehat{\Delta}$ via $\widehat{f} \ := \ f \circ \widehat{\boldsymbol{\kappa}}^{-1}$. Since $\widehat{\boldsymbol{\kappa}}$ is bijective and $\widehat{\boldsymbol{\kappa}}(\boldsymbol{x}^1) \ = \ \widehat{\boldsymbol{x}}^1$, the vector $\boldsymbol{x}^\star \ := \ \widehat{\boldsymbol{\kappa}}(\boldsymbol{\mu}(\Pi))$ yields

$$\widehat{f} = \mathbb{1}_{[\widehat{\boldsymbol{x}}^1, \boldsymbol{x}^\star]} \ .$$

The integral behaves under the transformation of variables initiated by $\widehat{\boldsymbol{\kappa}}$ as follows:

$$\int\limits_{\partial\Pi} f d\boldsymbol{\iota}_{\Delta} \;=\; \int\limits_{\widehat{\Delta}} f \circ \widehat{\boldsymbol{\kappa}}^{-1} d(\widehat{\boldsymbol{\kappa}}\boldsymbol{\iota}_{\Delta}) \;=\; \int\limits_{\widehat{\Delta}} \widehat{f} d(\widehat{\boldsymbol{\kappa}}\boldsymbol{\iota}_{\Delta}) \;=\; \int\limits_{\widehat{\Delta}} \widehat{f} d\boldsymbol{\lambda}\,,$$

which is

$$\begin{aligned}\int\limits_{\boldsymbol{x}^1}^{\boldsymbol{\mu}(\Pi)} d\boldsymbol{\iota}_{\Delta} \;&=\; \int\limits_{\partial\Pi} \mathbb{1}_{[\boldsymbol{x}^1,\boldsymbol{\mu}(\Pi)]} d\boldsymbol{\iota}_{\Delta} \;=\; \int\limits_{\partial\Pi} f d\boldsymbol{\iota}_{\Delta} \\ &=\; \int\limits_{\widehat{\Delta}} \widehat{f} d\boldsymbol{\lambda} \;=\; \int\limits_{\widehat{\Delta}} \mathbb{1}_{[\widehat{\boldsymbol{x}}^1,\boldsymbol{x}^\star]} d\boldsymbol{\lambda} \;=\; \int\limits_{\widehat{\boldsymbol{x}}^1}^{\boldsymbol{x}^\star} d\boldsymbol{\lambda} \;=\; \boldsymbol{\lambda}([\widehat{\boldsymbol{x}}^1, \boldsymbol{x}^\star])\,.\end{aligned} \tag{18}$$

Now we perform the same operation for the function $\mathbb{1}_{[\boldsymbol{\mu}(\Pi),\boldsymbol{x}^2]}$. Then analogously

$$\int\limits_{\boldsymbol{\mu}(\Pi)}^{\boldsymbol{x}^2} d\boldsymbol{\iota}_{\Delta} \;=\; \int\limits_{\boldsymbol{x}^\star}^{\boldsymbol{x}^2} d\boldsymbol{\lambda} \;=\; \boldsymbol{\lambda}([\boldsymbol{x}^\star, \boldsymbol{x}^2])\,. \tag{19}$$

However, the integrals in (18) and (19) are equal according to Definition 3.6 as they are

$$\int\limits_{\boldsymbol{x}^1}^{\boldsymbol{\mu}(\Pi)} d\boldsymbol{\iota}_{\Delta} \;=\; \int\limits_{\boldsymbol{x}^1}^{\boldsymbol{\mu}(\Pi)} \sqrt{-dx_1 dx_2} \quad \text{and} \quad \int\limits_{\boldsymbol{\mu}(\Pi)}^{\boldsymbol{x}^2} d\boldsymbol{\iota}_{\Delta} \;=\; \int\limits_{\boldsymbol{\mu}(\Pi)}^{\boldsymbol{x}^2} \sqrt{-dx_1 dx_2}$$

see also (34).

Therefore,

$$\boldsymbol{\lambda}([\widehat{\boldsymbol{x}}^1, \boldsymbol{x}^\star] \;=\; \boldsymbol{\lambda}([\boldsymbol{x}^\star, \boldsymbol{x}^2])$$

which shows that $\boldsymbol{x}^\star \;=\; \widehat{\boldsymbol{\mu}}$ is the centerpoint of $\widehat{\Delta}$. Thus

$$\widehat{\boldsymbol{\kappa}}\,(\boldsymbol{\mu}(\Pi)) \;=\; \boldsymbol{x}^\star \;=\; \widehat{\boldsymbol{\mu}}\,,$$

**q.e.d.**

Figure 4.2 illustrates the procedure. The length of each segment in $\widehat{\Delta}$ equals the square root of the area of the corresponding triangle on $\partial\Pi$.

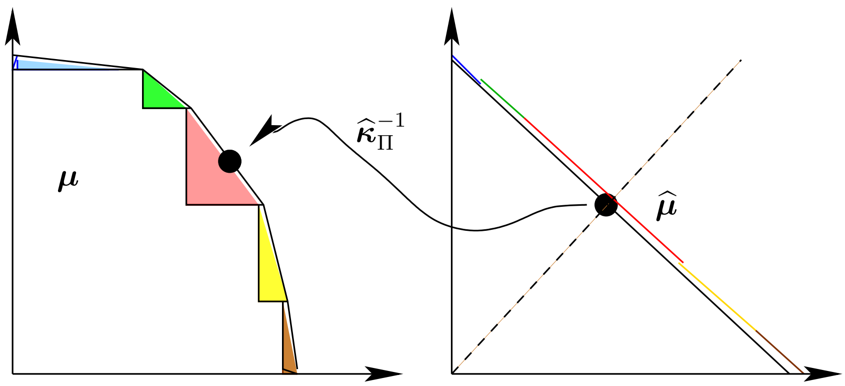



Figure 4.2: $\boldsymbol{\mu}$ as the inverse image of the center point.

# Chapter 12

# The Surface Measure and the $\mu\pi$ Solution

Within this chapter, we present a generalization of the Maschler–Perles solution to $n$ dimensions. As a general superadditive solution does not exist (Perles [21]), we have to restrict our discussion to some narrower domain in some way or other. Naturally, we restrict our interest to Cephoids.

Hence, we follow the path indicated in Section 4 of Chapter 11. Thus, we focus on Cephoids as feasible sets of bargaining problems for $n$ players and elaborate on the concept of the surface measure. To this end, for any Cephoid we adjust the (higher dimensional) "volume" of a Pareto face such that the result has the correct dimension and size of a surface to be assigned to that Pareto face. The measure preserving mapping is then arranged to map the complete Pareto surface to a suitable multiple of the unit Simplex endowed with the Surface Measure. The solution envisioned is then the inverse of the center point under the measure preserving mapping.

# 1 Preliminary Example: Three dimensions

As an introductory exercise consider the "sum of two" deGua simplices as indicated in Figure 2.4. This image appeared very early in Chapter 1 of our presentation, see Figure 2.4 of that Chapter. The figure represents a Cephoid

$$\Pi \;=\; \Delta^{\boldsymbol{a}} + \Delta^{\boldsymbol{b}} \;,$$

the surface of which shows the two translates of the deGua Simplices involved and a rhombus

$$\Lambda^{\boldsymbol{ab}} \;=\; \Lambda^{\boldsymbol{ab}}_{23\;13} \;=\; \Delta^{\boldsymbol{a}}_{23} + \Delta^{\boldsymbol{b}}_{13}$$

This rhombus is the sum of two Subsimplices of $\Delta^{\boldsymbol{a}}$ and $\Delta^{\boldsymbol{b}}$.

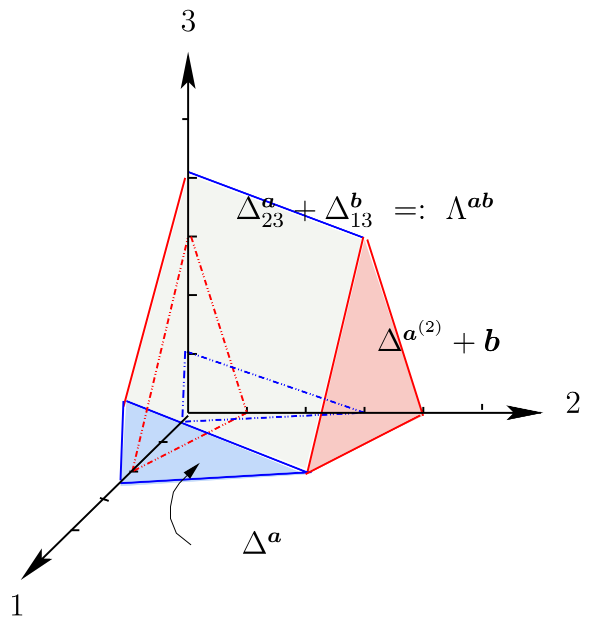



Figure 1.1: The sum of two deGua Simplices

For the Maschler–Perles solution in two dimensions the surface measure results from the *area* of a deGua Simplex. Analogously, we want to construct a surface measure for the above Cephoid in 3 dimensions via the volume generated by a deGua simplex. However, as we have a new type of a Pareto face – the rhombus $\Lambda^{\boldsymbol{ab}}$ appears – there has to be a surface measure defined on this rhombus as well. Moreover, we need to have a certain compatibility of the definitions on the various Pareto faces.

First we focus on $\Delta^{\boldsymbol{a}}$ with $\boldsymbol{a} = (a_1, a_2, a_3) > 0$. The volume of the deGua Simplex $\Pi^{\boldsymbol{a}}$ as computed in $\mathbb{R}^3$ is $V(\Pi^{\boldsymbol{a}}) = \frac{a_1 a_2 a_3}{6}$.

We assign an area

$$\boldsymbol{\iota}_{\Delta}(\Delta^{\boldsymbol{a}}) \;=\; \sqrt[3]{(a_1 a_2 a_3)^2} \tag{1}$$

to $\Delta^{\boldsymbol{a}}$. $\boldsymbol{\iota}_{\Delta}$ will eventually be called the ***surface measure***.

The same area is associated to any translate of $\Delta^{\boldsymbol{a}}$. Then for $\boldsymbol{d} \in \mathbb{R}^3_+$ and $\varepsilon > 0$

$$\boldsymbol{\iota}_{\Delta}(\boldsymbol{d} + \Delta^{\varepsilon \boldsymbol{a}}) = \varepsilon^2 \boldsymbol{\iota}_{\Delta}(\Delta^{\boldsymbol{a}}). \tag{2}$$

This indicates that $\boldsymbol{\iota}_{\Delta}$ behaves indeed like an area although derived from a volume.

A slight extension of this definition generates a $\sigma$–additive set function on $\Delta^{\boldsymbol{a}}$. For, decompose $\Delta^{\boldsymbol{a}}$ canonically into 4 similar Simplices as indicated by Figure 1.2. Define the central triangle – which is not the surface of a DeGua Simplex – to be

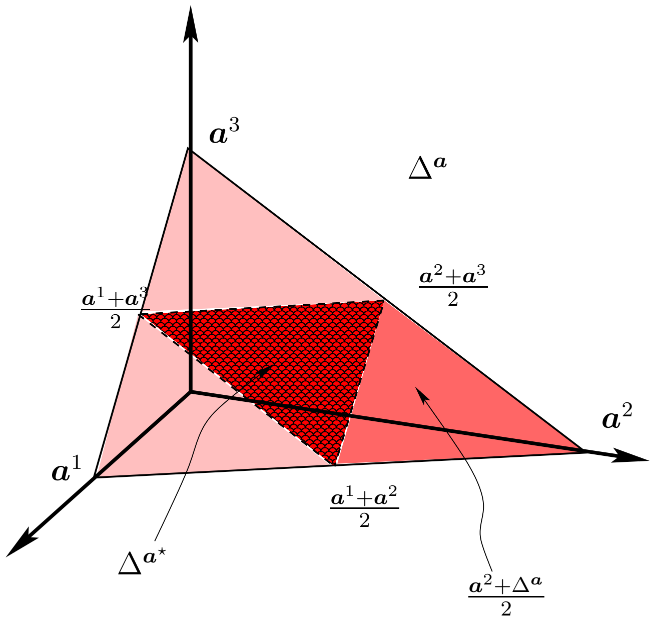



Figure 1.2: Canonical Decomposition of a Simplex $\Delta^{\boldsymbol{a}}$

$$\Delta^{a*} \;:=\; \operatorname{convH}\left(\left\{\frac{\boldsymbol{a}^1+\boldsymbol{a}^2}{2}, \frac{\boldsymbol{a}^1+\boldsymbol{a}^2}{2}, \frac{\boldsymbol{a}^2+\boldsymbol{a}^3}{2}, \right\}\right) \;. \tag{3}$$

Then

$$\text{(4)} \qquad \Delta^{\boldsymbol{a}} \;=\; \Delta^{a*} \cup \bigcup_{i=1}^{3} \left( \frac{\boldsymbol{a}^i + \Delta^{\boldsymbol{a}}}{2} \right) \; .$$

Each of the 4 triangles involved has measure $\boldsymbol{\iota}_\Delta(\frac{1}{2}\boldsymbol{d} + \frac{1}{2}\Delta^{\boldsymbol{a}})$ and because of (2) we have

$$\text{(5)} \qquad \boldsymbol{\iota}_\Delta(\frac{1}{2}\boldsymbol{d} + \frac{1}{2}\Delta^{\boldsymbol{a}}) = \frac{1}{4}\boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}}).$$

Continue this kind of decomposition of a Simplex into 4 subsimplices to obtain arbitrarily fine decompositions. This way we obtain an additive set function on the field generated by these Simplices on $\Delta^{\boldsymbol{a}}$. By the usual extension theorems we then we obtain the surface measure $\boldsymbol{\iota}_\Delta$ on the Pareto surface $\boldsymbol{d} + \Delta^{\boldsymbol{a}}$ of every translate $\boldsymbol{d} + \Pi^{\boldsymbol{a}}$ of some deGua Simplex $\Pi^{\boldsymbol{a}}$ (the $\sigma$–algebra is generated by the relative topology). This rather clumsy con-

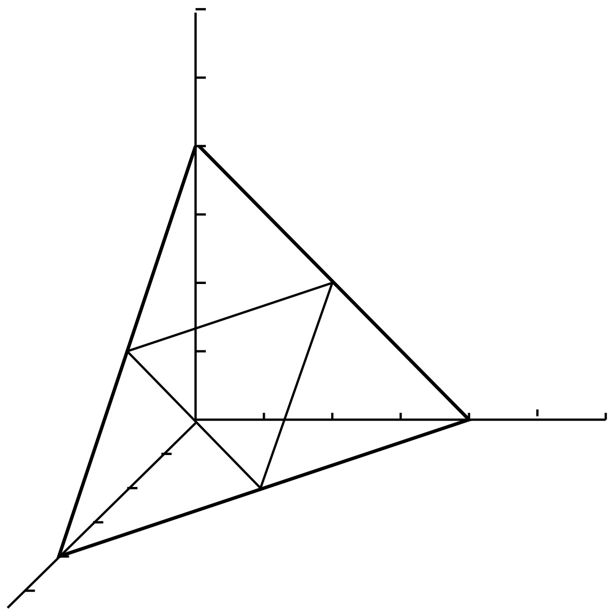


Figure 1.3: The Measure Preserving Image

struction will be omitted in what follows: assigning a measure to a Pareto face of a Cephoid induces a measure on this surface in a natural way.

Next we consider the construction of a mapping $\widehat{\boldsymbol{\kappa}}$ which throws $\Delta^{\boldsymbol{a}}$ (or a translate) onto the multiple $\widehat{\Delta}^{\boldsymbol{a}} = \boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}})\Delta^{\boldsymbol{e}}$ of the unit Simplex. $\widehat{\boldsymbol{\kappa}}$ will be called the "***measure preserving mapping***". The definition is obvious –

one has to identify the vertices and construct a linear mapping canonically. The image is depicted in Figure 1.3 and the surface measure $\boldsymbol{\iota}_{\Delta}$ is transferred into the Lebesgue measure on $\widehat{\Delta}^{\boldsymbol{a}}$.

Now proceed with an analogous procedure for $\Lambda^{\boldsymbol{ab}}$. We define a measure on the surface $\Lambda^{\boldsymbol{ab}}$ in a "compatible" way to the one on the surface of the Simplices. That is, the relations *in size and ordering* have to be preserved by the mapping $\widehat{\boldsymbol{\kappa}}$.

There is a marked difference to the two dimensional case (CHAPTER 11) in which a polyhedral Cephoid is just defined by line segments and hence *ex ante* is a Cephoid. The rhombus is the first new type of a Pareto face that appears in three dimensions. (The next new type is the *block* in four dimensions).

We assign to $\Lambda^{\boldsymbol{ab}}_{23\ 13}$ a measure that depends on the volumes of the two deGua Simplices involved and allows for a consistent bijection onto a suitable multiple of $\Delta^{\boldsymbol{e}}$. To this end, consider Figure 1.4.

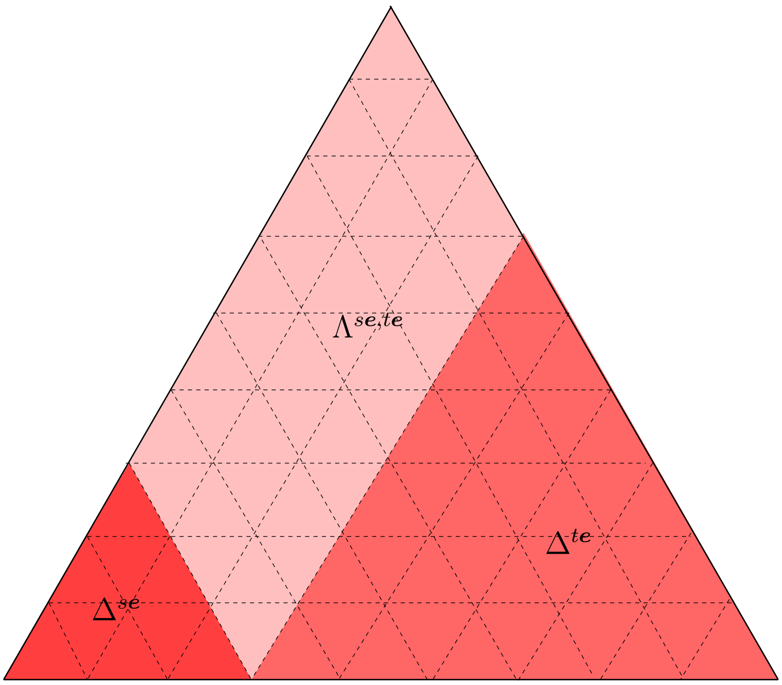



Figure 1.4: Decomposition of $\Delta^{(s+t)\boldsymbol{e}}$

Denote the area (Lebesgue measure) of the unit Simplex by $\boldsymbol{\lambda}(\Delta^{\boldsymbol{e}}) =: \lambda$ $(= \frac{1}{2}\sqrt{3})$. Then the area of a multiple $\Delta^{s\boldsymbol{e}}$ is $\boldsymbol{\lambda}(\Delta^{s\boldsymbol{e}}) = s^2\lambda$. Therefore the area

of the rhombus is
(6)
$$\boldsymbol{\lambda}(\Lambda^{se,te}) = \boldsymbol{\lambda}(\Delta^{(s+t)e} - \boldsymbol{\lambda}(\Delta^{se}) - \boldsymbol{\lambda}(\Delta^{te}) = \left((s+t)^2 - s^2 - t^2\right)\lambda = 2st\lambda \ .$$

E.g., in Figure 1.4, we have $s = 3$, $t = 6$ and hence $D^{se}$ contains 9 units, $\Delta^{te}$ contains 36 units, and $\Lambda^{se,te}$ containes 18 units.

By (1) the surface measure is normalized to $\boldsymbol{\iota}_\Delta(\Delta^{e}) = 1$. Then (6) translates into

(7)
$$\boldsymbol{\iota}_\Delta(\Lambda^{se,te}) = \left((s+t)^2 - s^2 - t^2\right) = 2st \ .$$

Figure 1.4 interpretes (7) as well – one just has to change the normalisation. Generally, let $s, t$ be determined by

$$\boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}}) \ = \ \sqrt[3]{(a_1a_2a_3)^2} \ =: \ s^2 \ , \ \boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{b}}) \ = \ \sqrt[3]{(b_1b_2b_3)^2} \ =: \ t^2$$

Then, by (7) we have for the area of the rhombus

$$2st \ = \ 2\sqrt[3]{a_1a_2a_3}\sqrt[3]{b_1b_2b_3} \ = \ 2\sqrt[3]{(a_1a_2a_3)(b_1b_2b_3)} \ .$$

Therefore we now *define* the measure of the rhombus to be

$$\boldsymbol{\iota}_\Delta(\Lambda^{\boldsymbol{ab}}) \ := \ 2\sqrt[3]{(a_1a_2a_3)(b_1b_2b_3)}$$

For a general Cephoid in 3 dimensions this is formulated as follows.

**Definition 1.1.** *Let $n = 3$ and let $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{(k)}$ be a Cephoid. Write*

$$\boldsymbol{P}^{(k)} \ := \ a_1^{(k)}a_2^{(k)}a_3^{(k)}$$

*Then the* ***surface measure*** *$\boldsymbol{\iota}_\Delta(\bullet)$ is defined by*

(8)
$$\boldsymbol{\iota}_\Delta(\Delta^{(k)}) \ = \ \sqrt[3]{\left(\boldsymbol{P}^{(k)}\right)^2}$$

*for the translates of deGua Simplices and*

(9)
$$\boldsymbol{\iota}_\Delta(\Lambda^{(kk')}) \ := \ = \ 2\sqrt[3]{\left(\boldsymbol{P}^{(k)}\right)\left(\boldsymbol{P}^{(k')}\right)} \ .$$

*for the translates of the rhombi.*

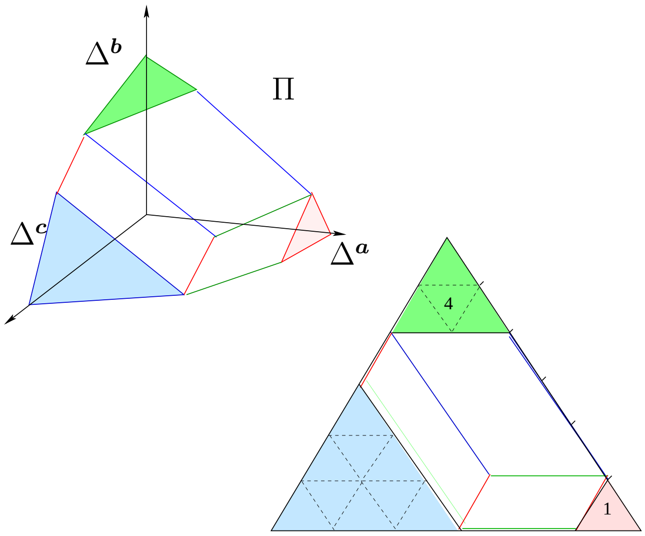



Figure 1.5: The Surface Measure of deGua Simplices and Rhombi

An example showing a sum of 3 deGua Simplices is provided by Figure 1.5. Let $\Delta^{\boldsymbol{a}}$ be red, $\Delta^{\boldsymbol{b}}$ green, and $\Delta^{\boldsymbol{c}}$ blue. Assume the surface measure to have been arranged in a way that

$$\boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{b}}) = 4\boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}}) \ , \boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{c}}) = 9\boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}}) \ ,$$

which is suggested by the number of triangles contained in each of the image Simplices. So we kind of compute in $\boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}})$–units.

Generally, for some integer $p$ and $\boldsymbol{\iota}_\Delta(\Delta^{\bullet}) = p\ \boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}})$ we obtain

$$\boldsymbol{P}^{\bullet} = p^{\frac{3}{2}}\boldsymbol{P}^{\boldsymbol{a}}$$

and hence

$$\boldsymbol{\iota}_\Delta(\Lambda^{\boldsymbol{a}\bullet}) \ = \ 2\sqrt[3]{(\boldsymbol{P}^{\boldsymbol{a}})p^{\frac{3}{2}}\boldsymbol{P}^{\boldsymbol{a}})} \ = \ 2\sqrt{p}\sqrt[3]{(\boldsymbol{P}^{\boldsymbol{a}})^2} \ = \ 2\sqrt{p}\boldsymbol{\iota}_\Delta(\Pi^{\boldsymbol{a}}) \ .$$

Therefore in Figure 1.5 the red–green rhombus receives

$$\boldsymbol{\iota}_\Delta(\Lambda^{\boldsymbol{ab}}) = 4\boldsymbol{\iota}_\Delta(\Pi^{\boldsymbol{a}}) \ ,$$

and the red–blue rhombus receives

$$\boldsymbol{\iota}_\Delta(\Lambda^{\boldsymbol{ac}}) = 6\boldsymbol{\iota}_\Delta(\Pi^{\boldsymbol{a}}) \ .$$

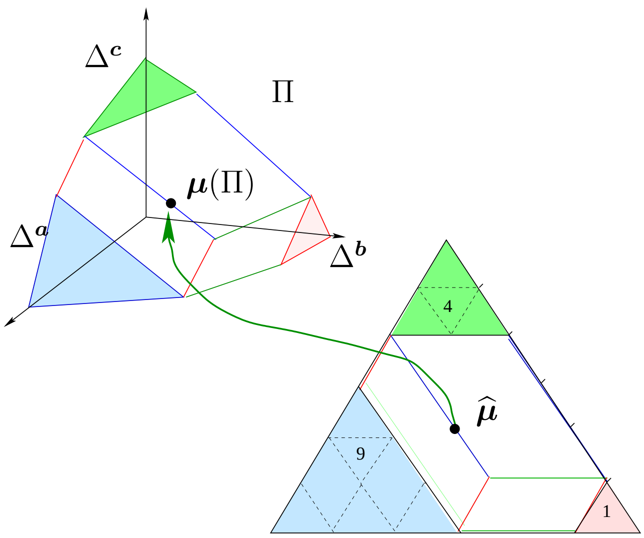
For the the green–blue rhombus we compute

$$\boldsymbol{\iota}_\Delta(\Lambda^{\boldsymbol{bc}}) = 2\sqrt[3]{(\boldsymbol{P}^{\boldsymbol{b}})^2(\boldsymbol{P}^{\boldsymbol{c}})^2} = 2\sqrt[3]{4^{\frac{3}{2}}(\boldsymbol{P}^{\boldsymbol{a}})^2 9^{\frac{3}{2}}(\boldsymbol{P}^{\boldsymbol{a}})^2} = 12\boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}})$$

which one can nicely reenact in Figure 1.5 by counting the number of triangles within the various Simplices and rhombi.

Based on these preliminary results we describe the program of this chapter to be presented in the following sections.

1. Assign a surface measure to the Pareto surface of a cephoid.
2. Construct a bijective mapping $\widehat{\boldsymbol{\kappa}}$ ("***the measure preserving mapping***") from the Pareto surface of a Cephoid $\Pi$ onto a multiple $\widehat{\Pi}$ of the unit Simplex ("the ***measure preserving representation***") such that




(a)

Figure 1.6: Constructing the Bargaining Solution

(b) the poset of Pareto faces is preserved,

(c) the surface measure is carried into Lebesgue measure,

3. Define and justify a bargaining solution as the inverse image $\widehat{\boldsymbol{\kappa}}^{-1}(\boldsymbol{\mu})$ of the center point $\widehat{\boldsymbol{\mu}}$ of the measure preserving representation.

# 2 The Surface Measure

For the general construction of the surface measure, we make use of the *volume* in order to define a measure on the *surface* of a Cephoid. We start with a deGua Simplex. Let $\boldsymbol{a} = (a_1, \ldots, a_n) > 0$ be a positive vector and let $\Pi^{\boldsymbol{a}}$ be the deGua Simplex associated, the surface is the Simplex $\Delta^{\boldsymbol{a}}$.

For $\boldsymbol{a} \in \mathbb{R}^n_+$ and any $\boldsymbol{J} \subseteq \boldsymbol{I}$ we write

$$\mathbf{P}^{\boldsymbol{a}}_{\boldsymbol{J}} := \prod_{i \in \boldsymbol{J}} a_i \ . \tag{1}$$

For the corresponding deGua Simplex $\Pi^{\boldsymbol{a}}$ the ***adjustment factor*** is

$$\boldsymbol{\tau}_{\Pi^{\boldsymbol{a}}} := \boldsymbol{\tau}_{\boldsymbol{a}} := \sqrt[n]{(\mathbf{P}^{\boldsymbol{a}}_{\boldsymbol{I}})}. \tag{2}$$

In particular, the unit deGua Simplex $\Pi^{\boldsymbol{e}}$ receives the adjustment factor

$$\boldsymbol{\tau}_{\Pi^{\boldsymbol{e}}} := 1 \ . \tag{3}$$

This notion is extended to Cephoids by additivity, that is, for a Cephoid $\Pi = \Pi^{\boldsymbol{a}^\bullet}$ we define the ***adjustment factor*** to be

$$\boldsymbol{\tau}_{\Pi} := \sum_{k \in \boldsymbol{K}} \boldsymbol{\tau}_{\Pi^{(k)}} \ . \tag{4}$$

Now we turn to the ***surface measure*** of a Pareto face. We start with a deGua Simplex.

**Definition 2.1.** *1. For positive $\boldsymbol{a} \in \mathbb{R}^n_+$ the **surface measure** assigned to $\Delta^{\boldsymbol{a}}$ is*

$$\boldsymbol{\iota}_{\Delta}(\Delta^{\boldsymbol{a}}) := \sqrt[n]{(\mathbf{P}^{\boldsymbol{a}}_{\boldsymbol{I}})^{n-1}}. = \boldsymbol{\tau}^{n-1}_{\boldsymbol{a}}. \tag{5}$$

*The same measure is established for any translate of $\Delta^{\boldsymbol{a}}$.*

*2. In particular, the Simplex $\Delta^{\boldsymbol{e}}$ (the surface of the unit deGua Simplex $\Pi^{\boldsymbol{e}}$) receives surface measure 1 according to (3):*

$$\boldsymbol{\iota}_{\Delta}(\Delta^{\boldsymbol{e}}) = 1 \ . \tag{6}$$

Now we have to establish $\boldsymbol{\iota}_\Delta$ "compatibly" on faces of a Cephoid. To this end, let $\Pi = \sum_{k\in \boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ be a Cephoid and let $\boldsymbol{F}$ be a Pareto face with reference system $\mathfrak{J} = \left\{\boldsymbol{J}^{(k)}\right\}_{k\in\boldsymbol{K}}$, such that

$$(7) \qquad \boldsymbol{F} \;=\; \sum_{k\in\boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$$

holds true. The numbers $j_k \;:=\; |\boldsymbol{J}^{(k)}| \;\; (k \in \boldsymbol{K})$ satisfy

$$(8) \qquad (j_1 - 1) + \ldots + (j_K - 1) = n - 1 \;, \quad j_1 + \ldots + j_K = n + K - 1 \;,$$

meaning that the dimensions of the sub-simplices involved in the construction of $\boldsymbol{F}$ add up to the dimension of $\boldsymbol{F}$. (see the Reference Theorem (1.2) in Chapter 3).

Let $c_{\mathfrak{J}}$ denote the quotient of the volume of $\Delta^{\boldsymbol{e}}_{\boldsymbol{J}^{(1)}} + \ldots + \Delta^{\boldsymbol{e}}_{\boldsymbol{J}^{(K)}}$ and the volume of $\Delta^{\boldsymbol{e}}$, we write

$$(9) \qquad c_{\mathfrak{J}} \;=\; c_{j_1,\ldots,j_K} \;:=\; \frac{\boldsymbol{\lambda}(\Delta^{\boldsymbol{e}}_{\boldsymbol{J}^{(1)}} + \ldots + \Delta^{\boldsymbol{e}}_{\boldsymbol{J}^{(K)}})}{\boldsymbol{\lambda}(\Delta^{\boldsymbol{e}})},$$

where $\boldsymbol{\lambda}$ denotes the Lebesgue measure.

**Example 2.2.** *Generally, for arbitrary $n$, one has $c_{1,\ldots,n,\ldots,1} = 1$. For $n = 3$ obviously $c_{13} = 1$. Also, two triangles will fit into a rhombus, see e.g. Figures 1.3 or 1.4, hence $c_{22} = 2$.*

*For $n = 4$ one has $c_{114} = c_{141} = c_{411} = 1$. Observe that the numbers $c_{\mathfrak{J}}$ can at once be interpreted by inspecting the canonical representation. Compare e.g. Figures 6.4 and 6.5 in Chapter 3. In four dimensions (three dimensions for the canonical representation), three tetrahedra fill a cylinder (the third deGua adds a translation), implying $c_{123} = c_{213} = ... = 3$. A block containes exactly 6 tetrahedra (two cylinders fill a block) – hence $c_{222} = 6$, etc.*

◦ ~~~~~~~ ◦

Also, in passing we observe

**Lemma 2.3.** *The coefficient $c_{j_1,\ldots,j_K}$ is the volume of the convex body*

$$(10) \qquad \begin{aligned} &\boldsymbol{CovH}\{0, \boldsymbol{e}^1, \ldots, \boldsymbol{e}^{j_1-1}\} \times \boldsymbol{CovH}\{0, \boldsymbol{e}^{j_1}, \ldots, \boldsymbol{e}^{j_1+j_2-1}\} \\ &\times \boldsymbol{CovH}\{0, \boldsymbol{e}^{j_1+j_2}, \ldots, \boldsymbol{e}^{j_1+j_2+j_3-1}\} \times \ldots \\ &\ldots \times \boldsymbol{CovH}\{0, \boldsymbol{e}^{j_1+j_2+\ldots+j_{K-1}}, \ldots, \boldsymbol{e}^{j_1+\ldots+j_K-1}\}. \end{aligned}$$

**Proof:**

This follows from the fact that the Subsimplices involved are located in orthogonal subspaces.

**q.e.d.**

Having obtained the above defined "normalizing coefficients" we can now proceed by defining a surface measure on any face of a Cephoid.

**Definition 2.4.** *Let* $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ *be Cephoid and let*

$$\boldsymbol{F} \;=\; \sum_{k\in\boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \tag{11}$$

*be a be a Pareto face with reference system* $\mathfrak{J} = \left\{\boldsymbol{J}^{(k)}\right\}_{k\in\boldsymbol{K}}$. *Then the* ***surface measure*** *of* $\boldsymbol{F}$ *is given by*

$$\boldsymbol{\iota}_\Delta(\boldsymbol{F}) = c_{\mathfrak{J}} \sqrt[n]{\left[\mathbf{P}^{(1)}_{\boldsymbol{I}}\right]^{j_1-1} \cdot \ldots \cdot \left[\mathbf{P}^{(K)}_{\boldsymbol{I}}\right]^{j_K-1}} \tag{12}$$

*with* $\mathbf{P}^{(k)}_{\boldsymbol{I}} \;:=\; \mathbf{P}^{\boldsymbol{a}^{(k)}}_{\boldsymbol{I}} \quad (k \in \boldsymbol{K})$.

*The* ***surface measure*** *of* $\partial\Pi$ *is obtained as the collection of all copies on the various Pareto faces.*

We list some properties of the surface measure indicating that it exhibits the "appropriate behavior".

**Lemma 2.5.**

1. *For* $\boldsymbol{t} = (t_1, \ldots, t_K) > 0$ *and* $\boldsymbol{t}\boldsymbol{a}^{(\bullet)} = (t_k\boldsymbol{a}^{(k)})_{k\in\boldsymbol{K}}$ *let* $\boldsymbol{tF}$ *denote the face corresponding to a face* $\boldsymbol{F}$. *Then*

$$\boldsymbol{\iota}_\Delta(\boldsymbol{tF}) \;=\; t_1^{j_1-1} \cdot \ldots \cdot t_K^{j_K-1} \boldsymbol{\iota}_\Delta(\boldsymbol{F}). \tag{13}$$

2. *In particular, for* $\boldsymbol{t} = (\varepsilon, \ldots, \varepsilon)$, *we obtain from* (8)

$$\boldsymbol{\iota}_\Delta(\varepsilon\boldsymbol{F}) \;=\; \varepsilon^{n-1}\boldsymbol{\iota}_\Delta(\boldsymbol{F}). \tag{14}$$

*Equations* (13) *and* (14) *show that* $\boldsymbol{\iota}_\Delta(\bullet)$ *behaves like the Lebesgue measure of the surface up to normalization.*

3. *If for some family* $\left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ *all the volumes of the DeGua Simplices involved are equal, i.e.,*

$$\mathbf{P}_{\boldsymbol{I}}^{\boldsymbol{a}^{(1)}} = \ldots = \mathbf{P}_{\boldsymbol{I}}^{\boldsymbol{a}^{(K)}} ,$$

*then it follows that a face* $\boldsymbol{F}$ *represented by* (2.4) *satisfies*

$$\boldsymbol{\iota}_{\Delta}(\boldsymbol{F}) \;=\; c_{\mathfrak{J}}\boldsymbol{\iota}_{\Delta}(\Delta^{\boldsymbol{a}^{(1)}}). \tag{15}$$

**Proof:** The first two items are obtained by obvious computations with volumes and surface areas involving the definition (12). The last item is a consequence of the convention established by (9).

**q.e.d.**

Next we establish a mapping $\widehat{\boldsymbol{\kappa}} = \widehat{\boldsymbol{\kappa}}_{\Pi}$ which carries the surface $\partial\Pi$ of a Cephoid $\Pi$ bijectively onto a suitable multiple $\widehat{\Delta}$ of the unit Simplex endowed with a Cephoidal structure. $\widehat{\boldsymbol{\kappa}}$ preserves the poset and carries the surface measure into the Lebesgue measure. The procedure is the same as for the canonical representation in Chapter 2. However, the relative size, i.e., the areas, volumes, measures of the various Pareto faces are normalized differently. Within the canonical representation of Chapter 2 the image of a deGua Simplex is normalized to the unit. Presently, the image of a deGua Simplex is adjusted according to the surface measure.

The following data will be used.

**Definition 2.6.** *Let* $\left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ *be a family of of positive vectors and let* $\Pi = \sum_{k\in\boldsymbol{K}}\Pi^{\boldsymbol{a}^{(k)}}$ *be the Cephoid generated. For* $k \in \boldsymbol{K}$ *let*

$$\widehat{\boldsymbol{a}}^{(k)} \;:=\; \boldsymbol{\tau}_{\boldsymbol{a}^{(k)}}\boldsymbol{e} \quad , \quad \widehat{\Delta}^{(k)} := \Delta^{(\widehat{\boldsymbol{a}}^{(k)})} , \quad \widehat{\Pi}^{(k)} := \Pi^{(\widehat{\boldsymbol{a}}^{(k)})} \tag{16}$$

*such that*

$$\boldsymbol{\iota}_{\Delta}(\widehat{\Delta}^{(k)}) = \boldsymbol{\iota}_{\Delta}(\Delta^{(k)}) \quad (k \in \boldsymbol{K}) \tag{17}$$

*is satisfied. Define*

$$\widehat{\Delta} \;:=\; \sum_{k=1}^{K}\widehat{\Delta}^{(k)} \;=\; \boldsymbol{\tau}_{\Pi}\Delta^{\boldsymbol{e}} \quad , \quad \widehat{\Pi} \;:=\; \sum_{k=1}^{K}\widehat{\Pi}^{(k)} \;=\; \boldsymbol{\tau}_{\Pi}\Pi^{\boldsymbol{e}} \;. \tag{18}$$

*such that*

$$\boldsymbol{\lambda}(\widehat{\Delta}^{(k)}) = \boldsymbol{\tau}_{\boldsymbol{a}^{(k)}}\boldsymbol{\lambda}(\Delta^{\boldsymbol{e}}) \;\; (k \in \boldsymbol{K}) \;\; , \;\; \boldsymbol{\lambda}(\widehat{\Delta}) = \boldsymbol{\tau}_{\Pi}\boldsymbol{\lambda}(\Delta^{\boldsymbol{e}}) \tag{19}$$

*holds true.*

Next we embed the Simplices $\widehat{\Delta}^{(k)}$ suitably into $\widehat{\Delta}$ via a mapping $\widehat{\boldsymbol{\kappa}} = \widehat{\boldsymbol{\kappa}}_{\Pi} : \partial\Pi^{\boldsymbol{a}} \rightarrow \widehat{\Delta}$ in a way such that the complete poset $\mathcal{V}$ of $\partial\Pi^{\boldsymbol{a}}$ is bijectively mapped onto the corresponding poset $\widehat{\mathcal{V}}$ of $\widehat{\Delta}$. In particular, each $\Delta^{(k)}$ is mapped onto an image $\widehat{\Delta}^{(k)}$ with a size given by the surface measure. The normalization with the Lebesgue measure involves the unit Simplex – this is the meaning of (19). Alternatively, one focuses on a normalization w.r.t. the surface measure – which is the second part of Formula (19).

We arrange the mapping $\widehat{\boldsymbol{\kappa}}$ on $\partial\Pi$ by mapping the extremals of $\partial\Pi$ bijectively onto the corresponding vectors of $\widehat{\Delta}$. This exactly done in analogy to the procedure in CHAPTER 2, SECTION 1.

By nondegeneracy every vertex of $\partial\Pi$ is a unique sum of vertices of the $\Delta^{\boldsymbol{a}^{(k)}}$ involved. More precisely, for every vertex $\boldsymbol{u}$ of $\partial\Pi$, there is a unique mapping $\mathbf{i}_{\bullet}$ such that $\boldsymbol{u}$ can be written via

$$\begin{aligned} \mathbf{i}_{\bullet} \ &: \quad \boldsymbol{K} \rightarrow \boldsymbol{I} \\ \boldsymbol{u} = \boldsymbol{a}^{\mathbf{i}_{\bullet}} \ &:= \ \sum_{k \in \boldsymbol{K}} \boldsymbol{a}^{(k)\mathbf{i}_k} \ . \end{aligned} \tag{20}$$

Thus we obtain

**Definition 2.7.** *1. Let* $\Pi$ *be a Cephoid and let*

$$\widehat{\Delta} \ , \ \widehat{\Pi}$$

*be given by* (18). *Then* $\widehat{\Delta}$ *is the* ***measure preserving representation*** *of* $\Delta^{\boldsymbol{a}}$.

2. *Let* $\boldsymbol{u}$ *be a vertex of* $\partial\Pi$ *and let* $\mathbf{i}_{\bullet}$ *be the corresponding mapping as described by* (20). *Then*

$$\widehat{\boldsymbol{\kappa}}(\boldsymbol{u}) \ = \ \widehat{\boldsymbol{\kappa}}_{\Pi}(\boldsymbol{u}) \ := \ \sum_{k \in \boldsymbol{K}} \widehat{\boldsymbol{a}}^{(k)\mathbf{i}_k} \ =: \ \widehat{\boldsymbol{u}} \in \widehat{\Delta} \tag{21}$$

*is called the* ***measure preserving representation*** *of* $\boldsymbol{u}$ *on* $\widehat{\Delta}$ .

3. *Let* $\boldsymbol{F}$ *be a Pareto face of* $\Pi$ *and let* $\boldsymbol{u}^1, \ldots, \boldsymbol{u}^L$ *be its extremal points. Then the convex hull of the images, i.e.,*

$$\widehat{\boldsymbol{\kappa}}(\boldsymbol{F}) \ = \ \widehat{\boldsymbol{\kappa}}_{\Pi}(\boldsymbol{F}) \ := \ \boldsymbol{CovH}\{\widehat{\boldsymbol{\kappa}}(\boldsymbol{u}^1), \ldots, \widehat{\boldsymbol{\kappa}}(\boldsymbol{u}^L)\}, \ =: \ \widehat{\boldsymbol{F}} \tag{22}$$

*is called the* ***measure preserving representation*** *of* $\boldsymbol{F}$ *on* $\widehat{\Delta}$.

4. *In particular, for $\kappa \in \boldsymbol{K}$, let*

$$\boldsymbol{\Delta}^{\{\kappa\}} = \sum_{k \in \boldsymbol{K}\setminus\{\kappa\}} \boldsymbol{a}^{(k)i_k} + \Delta^{(\kappa)} \subseteq \partial\Pi \tag{23}$$

   *be the translate of $\Delta^{(k)}$ located on $\partial P$ as in the Translation Theorem 4.9 of* CHAPTER *3. Then the measure preserving representation of $\boldsymbol{\Delta}^{\{\kappa\}}$ is*

$$\widehat{\boldsymbol{\kappa}}(\boldsymbol{\Delta}^{\{\kappa\}}) = \widehat{\boldsymbol{\kappa}}_\Pi(\boldsymbol{\Delta}^{\{\kappa\}}) = \widehat{\boldsymbol{\Delta}}^{\{\kappa\}} := \sum_{k \in \boldsymbol{K}\setminus\{\kappa\}} \widehat{\boldsymbol{a}}^{(k)i_k} + \widehat{\Delta}^{(\kappa)} \subseteq \widehat{\Delta}\ . \tag{24}$$

5. *Let $\mathcal{V}$ be the poset of faces of $\partial\Pi$ and let*

$$\widehat{\boldsymbol{\kappa}}(\mathcal{V}) := \{\widehat{\boldsymbol{\kappa}}(\boldsymbol{F}) \,\big|\, \boldsymbol{F} \in \mathcal{V}\} =: \widehat{\mathcal{V}} \tag{25}$$

   *denote the collection of images of faces under the mapping $\widehat{\boldsymbol{\kappa}}$. Then $\widehat{\mathcal{V}}$ is the **measure preserving representation** of $\mathcal{V}$ on $\widehat{\Delta}$.*

6. *We use the term $\boldsymbol{\mu\pi}$ to indicate "measure preserving" as a homage to Maschler and Perles. Thus, $\widehat{\boldsymbol{\kappa}}$ is the "$\boldsymbol{\mu\pi}$ mapping" and the term "$\boldsymbol{\mu\pi}$ representation" is somewhat loosely applied to the triple $(\widehat{\Delta}, \widehat{\mathcal{V}}, \widehat{\boldsymbol{\kappa}})$ as well as to one or two of the ingredients.*

**Theorem 2.8.** *$\widehat{\mathcal{V}}$ is a poset isomorphic to $\mathcal{V}$. Hence $(\Delta, \mathcal{V})$ and $(\widehat{\Delta}, \widehat{\mathcal{V}})$ are combinatorial equivalent. The mapping $\widehat{\boldsymbol{\kappa}} = \widehat{\boldsymbol{\kappa}}_\Pi$ constitutes a piecewise linear isomorphism between $\Delta$ and $\widehat{\Delta}$. This isomorphism transfers the surface measure on $\partial\Pi$ into the Lebesgue measure on $\widehat{\Delta}$(up to normalization via the unit Simplex).*

The proof is the same as in CHAPTER 2, SECTION 1 The mapping $\widehat{\boldsymbol{\kappa}}$ is bijective between the vertices of $\partial\Pi$ and the appropriate subset of grid vectors as described in equations (20) and (21). The (lattice theoretical) minimum of two faces (whenever it exists) is obtained by taking the intersection of the corresponding two sets of extremal points. Similarly, if the maximum of two faces exists, then it is obtained via the union of the sets of extremal points.

Compare the two approaches: the *canonical* representation (treated in CHAPTER 3) is the isomorphism of the Pareto surface $\partial\Pi$ onto the $n-1$–dimensional Simplex $K\Delta^{\boldsymbol{e}}$. The figures in SECTION 2 provide examples.

On the other hand, the *measure preserving* representation is a structure on the Simplex $\widehat{\Delta} = \boldsymbol{\tau}_\Pi\Delta^{\boldsymbol{e}}$. The isomorphism $\widehat{\boldsymbol{\kappa}} = \widehat{\boldsymbol{\kappa}}_\Pi$ assigns a certain size (area, volume, ...) to each of the Simplices, rhombi, ... on the surface. This is done consistently and arranges the images in the same way as the situation on the Pareto surface $\partial\Pi$ dictates.

For examples see the figures in SECTION 1 of the present chapter. Returning to Figure 1.4 and the discussion in the context we observe the adjustment of the size of the translates of the deGua Simplices either in terms of Lebesgue measure or in terms of the surface measure.

The example of Figure 1.4 demonstrates the two essential features of the measure preserving representation: the relative location of a translate $\widehat{\boldsymbol{\kappa}}(\boldsymbol{\Delta}^{\{\kappa\}})$ in $\widehat{\boldsymbol{\Delta}}$ is dictated by the poset – i.e., by the relative location in Figure 2.4. The size of such a translate – expressed either in terms of Lebesgue measure or in terms of the surface measure is given by Formula (19). Based on these data, the size of other Pareto faces is determined – see the discussion to Figure 1.4. This is essentially the content of Formula (12).

# 3 The $\boldsymbol{\mu\pi}$ Bargaining Solution – Axioms and Interpretations

Within this section we describe – and justify – a generalization of the MASCHLER–PERLES solution ([13]). A counterexample provided by PERLES([21]) shows that we cannot expect a superadditive solution on the full class of bargaining solutions. Yet, in a further section, we exhibit the possibility to construct a subclass on which a superadditive solution exists. Later on, (CHAPTER 13) we will point out that a modified version of superadditivity successfully characterizes the solution concept: this modification is *Conditional Additivity.*

We begin with the definition generalizing the two–dimensional version.

**Definition 3.1.** *Let* $\Pi$ *be a Cephoid and let* $(\widehat{\Delta}, \widehat{\nabla}, \widehat{\boldsymbol{\kappa}})$ *be the measure preserving representation (with* $\widehat{\Delta} = \Delta^{\boldsymbol{\tau}_{\Pi}\Delta^{\boldsymbol{e}}}$, $\widehat{\boldsymbol{\kappa}} = \widehat{\boldsymbol{\kappa}}_{\Pi}$ *). Let*

$$(1) \qquad \widehat{\boldsymbol{\mu}} \ := \ \frac{\boldsymbol{\tau}_{\Pi}}{n}\boldsymbol{e} \ =: \ \boldsymbol{\mu}(\Pi^{\boldsymbol{\tau}_{\Pi}\boldsymbol{e}})$$

*denote the barycenter (i.e., the "midpoint" of the extremal elements) of* $\widehat{\Delta} = \Delta^{\boldsymbol{\tau}_{\Pi}\boldsymbol{e}}$*. Then*

$$(2) \qquad \boldsymbol{\mu}(\Pi) \ := \ \widehat{\boldsymbol{\kappa}}_{\Pi}^{-1}(\widehat{\boldsymbol{\mu}})$$

*is the* $\boldsymbol{\mu\pi}$ ***solution*** *of* $\Pi$*.*

We discuss a first interpretation extensively. To this end we revisit the interpretation of bargaining problems with a Cephoidal feasible set and interpret it's $\boldsymbol{\mu\pi}$ representation. Based on these exhibitions we will motivate the $\boldsymbol{\mu\pi}$ solution.

**Remark 3.2. Interpreting the Solution –**
**. Bargaining in a Cephoidal Setup**

We will distinguish the three PO sets involved with a Cephoid: apart from the PO set of the Pareto surface of the Cephoid we have also the Canonical Representation and the Measure Preserving Representation. The Canonical Representation will serve as the commodity space, the Cephoid is the utility space and the Measure Preserving (or $\boldsymbol{\mu\pi}$) representation appears as an adjusted utility space to allow universal comparison of utility.

Imagine a commodity being available for distribution among the players via cooperation. Each player has a utility function that depends on one ("his") commodity only (represented by his corresponding axis), this leads to the Cephoid located within the utility space. The measure preserving image is interpreted as to indicate an adjustment of utilities respecting transfer costs within various countries.

Let us make this precise.

First of all, consider bargaining in a deGua Simplex as follows. Assume that $n$ players can allocate a unit of one commodity by agreement about the distribution. Each player has a linear utility function, say $u^i(t) = a_i t \ \ (t \in [0,1])$ for $i \in \boldsymbol{I}$ and some positive $a_i$. The feasible allocations of the commodity are represented by $\Delta^{\boldsymbol{e}} = \{\widehat{\boldsymbol{x}} \in \mathbb{R}^n_+ \mid \sum_{i\in \boldsymbol{I}} \widehat{x}_i = 1\}$. The Pareto surface of the resulting bargaining problem ("in utility space") is

$$\begin{aligned}\Delta^{\boldsymbol{a}} &= \left\{(u^1(\widehat{x}_1),\ldots,u^n(\widehat{x}_n)) \,\middle|\, \widehat{\boldsymbol{x}} \in \Delta^{\boldsymbol{e}}\right\} \\ &= \{(a_1\widehat{x}_1,\ldots,a_n\widehat{x}_n) \mid \widehat{\boldsymbol{x}} \in \Delta^{\boldsymbol{e}}\} \\ &= \boldsymbol{CovH}(\{\boldsymbol{a}^1,\ldots,\boldsymbol{a}^n\})\end{aligned} \tag{3}$$

Thus, technically, the bargaining problem is given by (the status quo point $\mathbf{0}$ and) the feasible set $\Pi^{\boldsymbol{a}}$. We regard this situation – where the deGua Simplex represents just a set of linear utilities – as a "primitive" bargaining problem.

Now assume that our players are involved in several ("primitive") bargaining situations of this type in various countries, remote and with varying infrastructure. Similarly to the "primitive case", we imagine that players bargain about the distribution of one unit of a commodity in each country $k \in \boldsymbol{K}$.

While bargaining takes place via the internet and feasible solutions can be agreed upon immediately, the actual transfer of the commodity may be difficult as the case may be. For example, the transfer of money to a certain player may be costly and involve red tape when different countries or currency domains (the Dollar, the Euro) are involved. Thus, each player (depending on his own location) has a different linear utility function, say $u^{(k)i}(t) = a_i^{(k)} t \ \ (t \in [0,1])$ referring to his utility of obtaining a unit of money in country $k$. Thus, $\boldsymbol{a}^{(k)} = (a_1^{(k)},\ldots,a_n^{(k)})$ represents the utility functions of the players regarding assignments in country $k$.

Bargaining takes place simultaneously with respect to the commodities in each country. Players may consider giving in by an $\varepsilon$ with respect to the bargaining problem in country $k$ for obtaining a $\delta$ in country $l$. If players agree on a final distribution $\widehat{\boldsymbol{x}}^{(k)}$ in county $k$, then according to (3) the result in utilities is

$$(a_1^{(k)}\widehat{x}_1^{(k)},\ldots,a_n^{(k)}\widehat{x}_n^{(k)}) \in \Delta^{\boldsymbol{a}^{(k)}} \ . \tag{4}$$

Players simply add their utility received in each country ("intrapersonal comparison of utility"). Hence, the resulting ("global") bargaining problem is represented exactly by the Cephoid

$$\Pi \;=\; \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}.$$

We expect that players agree to bargain only about Pareto efficient distribution of the commodity. A Pareto efficient result of the bargaining process, i.e.,

$$\widehat{\boldsymbol{x}} \;=\; \sum_{k\in\boldsymbol{K}} \widehat{\boldsymbol{x}}^{(k)} \quad , \quad \widehat{\boldsymbol{x}}^{(k)} \in \Delta^{\boldsymbol{e}} \;\; (k \in \boldsymbol{K})$$

reflects a set of distributions $\widehat{\boldsymbol{x}}^{(k)}$ of the commodity in each country $k \in \boldsymbol{K}$. The utilities according to (4) are

$$\bar{\boldsymbol{x}}^{(k)} \;=\; (\overline{x}_1^{(k)},\ldots,\overline{x}_n^{(k)}) = (a_1^{(k)}\widehat{x}_1^{(k)},\ldots,a_n^{(k)}\widehat{x}_n^{(k)}) \in \Delta^{\boldsymbol{a}^{(k)}} \;. \tag{5}$$

The sum

$$\bar{\boldsymbol{x}} \;=\; \sum_{k\in\boldsymbol{K}} \bar{\boldsymbol{x}}^{(k)} \tag{6}$$

is supposed to be Pareto efficient, hence located in some Pareto face $\boldsymbol{F}$ of $\Pi$. Hence, according to the Reference Theorem 1.2, we know that with a suitable reference set $\mathfrak{J}$,

$$\boldsymbol{F} \;=\; \sum_{k\in\boldsymbol{K}} \Delta^{(k)}_{\boldsymbol{J}^{(k)}}, \text{ and } \bar{\boldsymbol{x}}^{(k)} \in \Delta^{(k)}_{\boldsymbol{J}^{(k)}} \tag{7}$$

holds true.

We impose the PO–set of $\Pi$ on $K\Delta^{\boldsymbol{e}} = \Delta^{K\boldsymbol{e}}$ according to the Canonical Representation Theorem. Then, the feasible allocations are given by the Simplex

$$K\Delta^{\boldsymbol{e}} \;=\; \Delta^{K\boldsymbol{e}} \;.$$

and a commodity distribution $\widehat{\boldsymbol{x}}^{(k)}$ in country $k$ leads to a utility distribution $\bar{\boldsymbol{x}}^{(k)}$ via (3). Therefore, the commodity distribution available in country $k \in \boldsymbol{K}$ is marked by the relative location of the corresponding utilities in $\Pi$, that is, the location is supplied by the situation in $\Pi$ via the isomorphic PO–sets. For short, the location of commodities available for bargaining in a country is reflected by the canonical representation.

For an example we consider the Cephoid *Odot* introduced in Chapter 2 Figure 2.8. This figure is repeated in Figure 3.1. There are three players bargaining

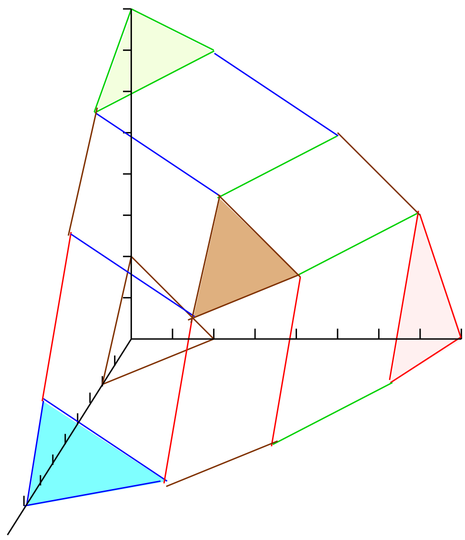


Figure 3.1: The Cephoid Odot

in four countries. The possible allocations of (4 units of) commodity are given by the Simplex $4\Delta^{\boldsymbol{e}} = \Delta^{4\boldsymbol{e}}$, this is represented in Figure 3.2.

In Figure 3.2 there are 4 units of the commodity to be allocated, the allocations are given by $4\Delta^{\boldsymbol{e}}$.Depending on the particular Pareto face of $\Pi$, a unit of commodity is transferred at different rates. This defines a decomposition of the allocation space into various regions corresponding to the maximal faces in the utility space.

Therefore, the relevance of a Pareto efficient distribution of commodity $\widehat{\boldsymbol{x}} = \sum_{k \in \boldsymbol{K}} \widehat{\boldsymbol{x}}^{(k)}$ is dictated by the location of the corresponding utiltiy vector in the *canonical* representation!

Each player has a *concave, piecewise linear utility function*, the slopes vary in each of the regions of the fourfold unit Simplex $4\Delta^{\boldsymbol{e}}$ as indicated in Figure 3.2.

Transferring the allocations from the commodity space $K\Delta^{\boldsymbol{e}}$ into utility space yields the Cephoid *Odot* in Figure 3.1. Triangles and rhombi reflect the (locally constant) rates of utility transfer resulting from the shape of the utility functions evaluated at Pareto efficient commodity allocations. The precise slope of the utilities in each region is obtained from the normal of the corresponding face (simplex, rhombus).

Thus, in a "primitive" bargaining problem $\Pi^{(\boldsymbol{a}^{(k)})}$ the transfer of utils between players takes place at fixed transfer rates on the Pareto surface (determined

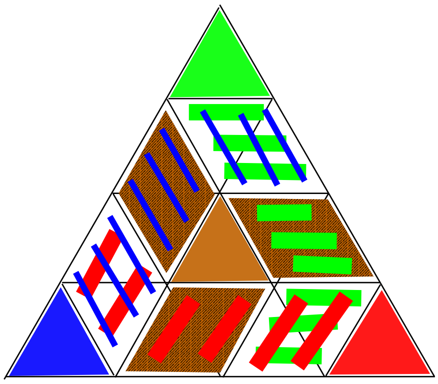


Figure 3.2: Canonical Representation of Odot

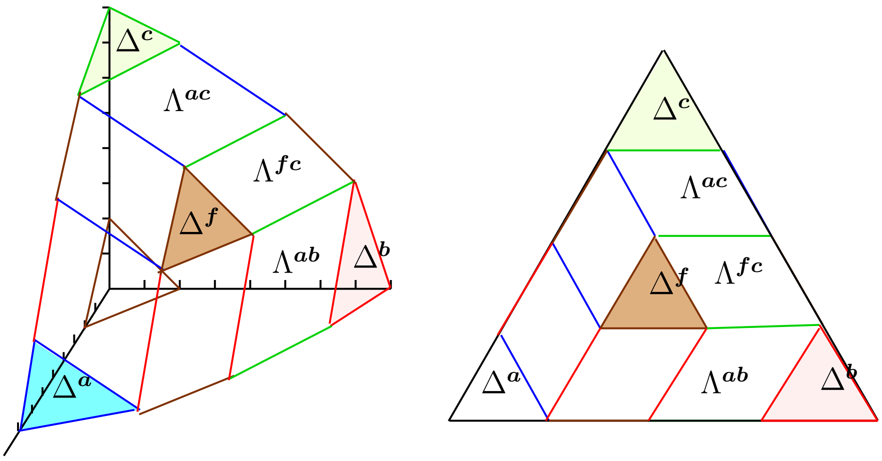



Figure 3.3: Assigning Rhombi to pairs of Simplices

by the normal of $\Delta^{(\boldsymbol{a}^{(k)})}$). For the "global" bargaining problem which involves bargaining in different countries, this is no longer true. We observe new effects as players can give and take in different countries $k$ and $l$. These effects correspond to the (maximal) faces of the sum of the primitive bargaining problems, i.e., the Cephoid $\Pi$.

Next, we enhance our interpretation by a further step as follows. We introduce a new utility – let us call it "adjusted utility". It reflects a players transfer rate for yielding an $\varepsilon$ in country $k$ vs. obtaining a $\delta$ in country $l$, hence induces a consistent measurement of utility over the various countries. The allocation of " adjusted utility" simultaneously takes care of all transfer effects in a new allocation space – this is going to be $\widehat{\alpha}\Delta^{\boldsymbol{e}}$.

In utility space (i.e., in $\Pi$) consider a Pareto efficient vector of the form

$$\bar{\boldsymbol{x}} = \bar{\boldsymbol{x}}^{(l)} + \bar{\boldsymbol{x}}^{(k)} \in \Lambda^{kl} \;=\; \Delta^{(l)}_{\boldsymbol{J}^{(l)}} + \Delta^{(k)}_{\boldsymbol{J}^{(k)}}$$

which reflects utilities of commodities traded in countries $l$ and $k$. That is, the rhombus represents an area of $\{k, l\}$–exchange. Along the boundary lines of said rhombus $\{k, l\}$ and the translate of Simplices (copies of primitive situations) the measurement of utility should be consistent – i.e., the length of the boundary segments of the rhombus should be consistently determined by the length measurement in the primitive situations. As a rhombus has two linear boundary segments (determined by two Simplices), this implies that the area should be consistently defined by the area in the generating Simplices.

Can one arrange for an adjusted utility space consistently? Can this be done via a consistent bijective mapping of the adjusted utility allocations?

To rephrase it slightly, there is *intra*–personal comparison of utility among the players w.r.t. different countries – as they add their utilities. There is also *inter*–personal transfer of utility at various rates in the various countries. The allocation space of the original commodity does not reflect this. Yet exchanging a unit has different effects when restricted to the various countries. Now, we can indeed define a new utility space in which the areas of the various regions of utility transfer are arranged consistently with the transfer rates of utility.

Indeed, Theorem 2.8 shows that the construction of such an adjusted utility space is possible. We can arrange for the *measure preserving* representation, which reflects a decomposition of the Simplex $\Delta^{\widehat{\alpha}\boldsymbol{e}} = \widehat{\alpha}\Delta^{\boldsymbol{e}}$ of the "adjusted utility space" corresponding to the Pareto faces of $\partial\Pi$. Figure 3.4 illustrates a possible shape of the Pareto surface and the adjusted utility space of the three player bargaining problem involving four countries (deGua Simplices) as induced by the Cephoid $\Pi$ of Figure 3.1.

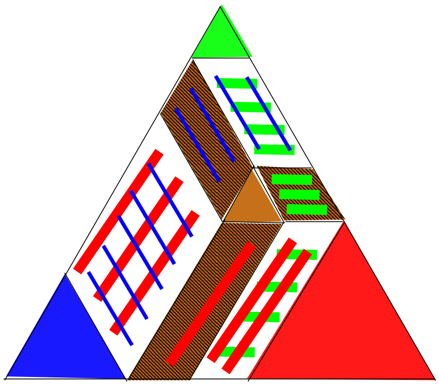


Figure 3.4: Measure Preserving Representation of Odot

As the adjusted utility space reflects a universally transferable utility, the midpoint of this new allocation Simplex is the solution suggested, generalizing the Maschler–Perles solution. Hence, in terms of adjusted utiltiy, players receive the same assignment: their utility is correctly and universally adjusted, so each of them gets the same amount.

With the appropriate version of "adjusted commodity", the new utility space reflects all necessary information regarding the exchange effects as explained above. Therefore, equal allocation of this commodity results in the solution in utility space.

It remains to justify the size (measurement) of the polyhedra in the adjusted utility space. The generating deGua Simplices determine a Cephoid (and its PO set) completely. Analogously the size of the images $\alpha_k \Delta^{\boldsymbol{e}}$ in $K\Delta^{\boldsymbol{e}}$ determine the shape of the measure preserving representation completely, as the relative locations are prescribed by the PO set which is dictated by the one of $\Pi$.

Therefore, it suffices to justify the size of $\widehat{\Delta}^{(k)}$ or, more precisely the term

$$\boldsymbol{\iota}_{\Delta}(\widehat{\Delta}^{(k)}) = \alpha_k \boldsymbol{\iota}_{\Delta}(\Delta^{\boldsymbol{e}}) = \alpha_k \ , \tag{8}$$

that is, it all boils down to interpreting the value chosen in Definition 2.1. We argue, that this is dictated by the rationale of the Maschler–Perles solution.

According to Definition 4.2 of Chapter 11 we know that the length of a line segment $\Delta^{\boldsymbol{a}}$ with $\boldsymbol{a} = (a_1, a_2)$ in the two dimensional Cephoid should be $\sqrt{a_1 a_2}$. That is, the measure preservng representation for $n = 2$ is dictated by the Maschler–Perles solution which is based and axiomaticly determined by superadditivity.

Indeed, for 2 dimensions (i.e., bargaining problems with 2 players) the concessions players are made when departing from their bliss points during the bargaining process are measured in terms of the surface measure.

Consider a Cephoid (a sum of 5 prisms, hence representing 5 states) as indicated by Figure 3.5; the right hand side shows the corresponding canonical representation.

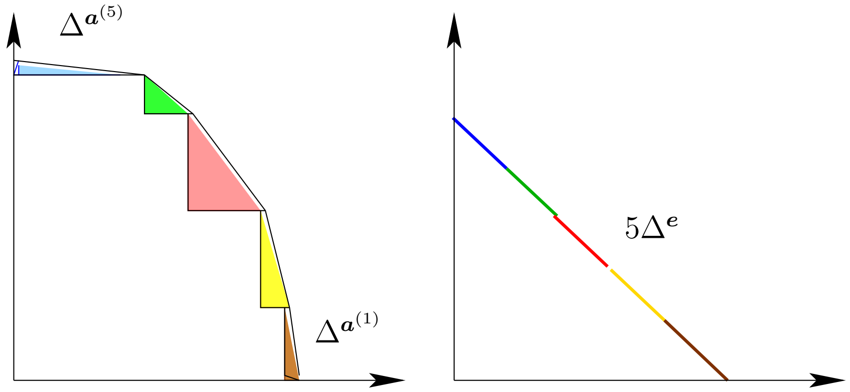



Figure 3.5: A two dimensional Cephoidal bargaining problem

Due to the superadditivity axiom, the Maschler–Perles solution evaluates concessions of the players along maximal faces (i.e., line segments) according to the corresponding *area* of the triangles (deGua Simplices). We are led to assign a new length measurement (the surface measure) to a maximal face. E.g., if the feasible set in Figure 3.5 is the Cephoid $\Pi = \sum_{k=1}^{5} \Pi^{\boldsymbol{a}^{(k)}}$ then the surface measure of (translate of) the line segment (simplex) $\Delta^{a^{(k)}}$ is

$$\boldsymbol{\tau}_k \;:=\; \boldsymbol{\iota}_{\Delta}(\Delta^{\boldsymbol{a}^{(k)}}) \;:=\; \sqrt{a_1^{(k)} a_2^{(k)}}$$

(see Figure 3.6).

The concession of player 1 when he moves from his bliss point $\boldsymbol{x}^1$ to $\boldsymbol{x}^2$ along $\Delta^{\boldsymbol{a}^{(1)}}$ is considered to be equal to the concession of player 2 to move from $\boldsymbol{y}^1$ to $\boldsymbol{y}^2$ along $\Delta^{\boldsymbol{a}^{(5)}}$ if and only if $\boldsymbol{\tau}_1 = \boldsymbol{\tau}_5$ holds true. Eventually, this procedure results in a distribution of utility at the Maschler–Perles solution at which both players have made equal overall concessions.

In order to construct the solution, the total sum $\boldsymbol{\tau} \;:=\; \sum_{k=1}^{5} \boldsymbol{\tau}_k$ determines the size of a new Simplex $\boldsymbol{\tau}\Delta^{\boldsymbol{e}}$. Each line segment $\Delta^{\boldsymbol{a}^{(k)}}$ is bijectively mapped

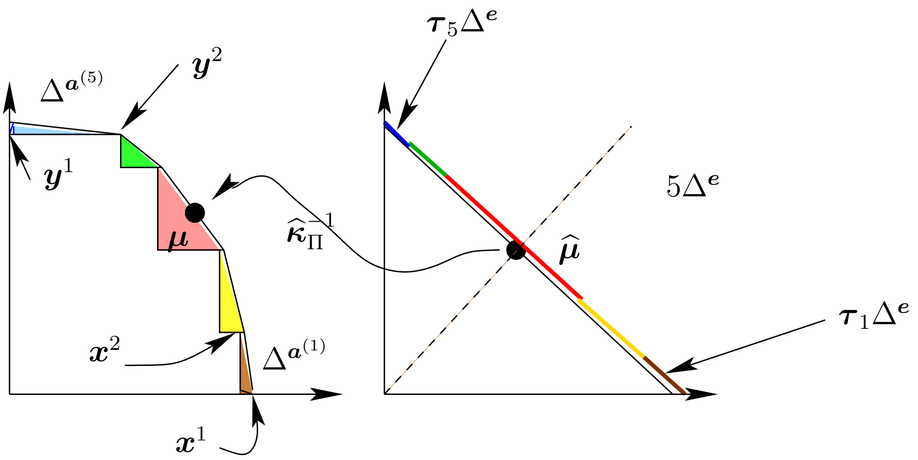



Figure 3.6: The M–P solution as the inverse image of the center point

onto a copy in $\boldsymbol{\tau}\Delta^{\boldsymbol{e}}$, the size of this copy is the surface measure $\boldsymbol{\tau}_k$ of the line segment. This way a bijective mapping $\widehat{\boldsymbol{\kappa}}_\Pi$ of the Pareto surface $\partial\Pi$ onto a multiple of the unit Simplex (the space of adjusted commodity) appears. With respect to this representation, concessions of players along the Pareto surface a measured by Lebesgue measure. Hence the midpoint $\widehat{\boldsymbol{\mu}}$ of $\boldsymbol{\tau}\Delta^{\boldsymbol{e}}$ generates the Maschler–Perles solution $\boldsymbol{\mu}$ as the inverse image in $\partial\Pi$, i.e., $\boldsymbol{\mu} = \widehat{\boldsymbol{\kappa}}_\Pi^{-1}(\widehat{\boldsymbol{\mu}})$.

The construction is completely determined by the axiomatic of the Maschler Perles solution. Superadditivity of the solution dictates the evaluation of concessions via the area ("volume") of the prisms involved.

While we cannot create a superadditive solution for $n = 3$, we hold that the area assigned to a deGua Simplex $\boldsymbol{a} = (a_1, a_2, a_3)$ in $\mathbb{R}^3$ as a first approach should be of the order of the product of the line segments which constitute the boundary, hence

$$\sqrt{a_1 a_2}\ \sqrt{a_1 a_3}\ \sqrt{a_2 a_3}$$

However, this quantity has the dimension of a volume and not of a surface. Hence this quantitiy should be adjusted to

$$\{\sqrt{a_1 a_2}\ \sqrt{a_1 a_3}\ \sqrt{a_2 a_3}\}^{\frac{2}{3}}$$

which is

$$\sqrt[3]{(a_1 a_2 a_3)^2}\ .$$

In addition, some normalization should be chosen which, naturally, is the one of Definition 2.1, for this leads to a surface measure 1 for $\Delta^{\boldsymbol{e}}$ as stated in that Definition.

Finally, as the measurements in the measure preserving representation allow for a universal comparison of utility, we hold that the barycenter or midpoint of this new allocation, space corresponds to the solution in utility space. In two dimensions, this results in the unique superadditive solution for *all* polyhedral bargaining problems, that is the Maschler–Perles solution. In 3 dimensions, we would have to choose the barycenter/midpoint of e.g. Figure 3.4. In general, this leads to accepting Definition 3.1.

For an additional justification, we will demonstrate within the following section that superadditivity can be observed in suitably restricted classes of Cephoids, although of course not in general.

Perles [21] proved that a superadditive bargaining solution does not exist for more than 2 players (i.e., bargaining problems in 3 and more dimensions). This is the most hindering drawback to a generalization of the solution. Calvo–Gutierrez (see [6]) presented an extension to $n$–person games. They generalized a procedure to compute the solution. An axiomatic justification is missing and they offer no examples. Of course, their approach as well cannot yield a superadditive solution.

One has to ponder about our approach for a while. For various reasons the Maschler–Perles solution has never been very popular compared with, say, the Nash solution. This is not the place to speculate about the fact, however, it is obvious that the axiomatic as well as the computational aspects constitute a barrier for Economists. From the viewpoint of this author, superadditvity is a much more convincing axiom than, say (with all due respect) the axiom "Independence of Irrelevant alternatives" (Nash) or "One Player Monotonicity" (Kalai–Smorodinsky).

Our results in SECTION 4 will establish superadditivity for certain "well behaved" bargaining problems. Yet the solution exists for all Cephoids, i.e., for bargaining problems resulting from a sum of deGua Simplices.

In CHAPTER 13 we will come up with a modified version of Superadditivity called "Conditional Additivity" – this version supplies a further axiomatic justification of our solution concept.

# 4 Symmetry Considerations

For the following development we introduce certain requirements of symmetry that allow to imitate the arguments used in the 2–dimensional case regarding supperadditivity. Compare the discussion centering around Standard Dyadic Cephoids in $\mathbb{R}^2_+$ in Section 2.

Analogously, we introduce a restricting requirement concerning a family of positive vectors $\left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ and the resulting Cephoid $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$ as follows: we assume that all deGua Simplices involved have *equal volume.*

Similarly to Section 2 we argue that this assumption is not as severe as it may seem on first sight. For, a deGua Simplex generated by a rational vector can be replaced by a homothetic sum of small multiples of itself. This way, any family $\boldsymbol{a}^\bullet$ with volumes being multiples of the same small number qualifies. Of course, we loose nondegeneracy by this procedure – but *weak* nondegeneracy (see Definition 1.1 of Chapter 2) is preserved.

In Section 2 we have used a version of this requirement and it was justified by continuity with respect to the Hausdorff metric. Presently we do not know anything about denseness of Cephoids within the set of convex bodies for general $n \in \mathbb{N}$. But for Cephoids resulting from families $\left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ with rational vectors, we argue that the restriction to equal volumes is feasible.

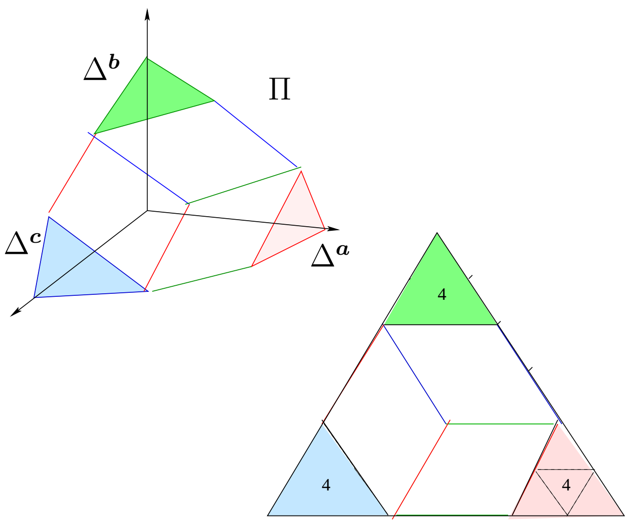



Figure 4.1: Surface Measure with equal volumes of deGua Simplices

Additionally, we require that the total number $K$ of deGua Simplices involved in a family of vectors is a multiple of the dimension $n$.

This can be achieved in a similar way by replacing each deGua Simplex by a sum of $n$ homothetic $\frac{1}{n}$–copies of itself.

Figure 4.1 provides a simple example of a Cephoid with the properties discussed.

Combining we come up with the following definition.

**Definition 4.1.** *A family* $\boldsymbol{a}^{\bullet}$ *of positive vectors as well as the Cephoid* $\Pi$ *generated are called* ***standard*** *if the following conditions are satisfied.*

1. $\boldsymbol{a}^{\bullet}$ *is weakly nondegenerate (see Definition 1.1).*
2. *The* $K$ *deGua Simplices involved have equal volume.*
3. $n$ *is a divisor of* $K$*.*

We treat the extension of our theory to weakly n.d. Cephoids and in particular to Standard Cephoids rather sloppily. For example, we do not explicitly construct a measure preserving representation for weakly nondegenerate families $\boldsymbol{a}^{\bullet}$. Indeed, points on the surfaces $\Delta$ and $\widehat{\Delta}$ can be identified consistently despite the fact that there are non–unique representations of some extremals given by vectors of the type exhibited in (20) of SECTION 2. Actually, weak degeneracy is essentially a vehicle for providing a simplified access to examples and lemmata – the concept is not really needed for the development of the theory. One could always restrict the discussion to the n.d. situation – in which case the set of examples as provided in the following would just be a bit smaller but not essentially so.

**Lemma 4.2.** *Let*

$$\Pi \;=\; \sum_{k=1}^{K} \Pi^{a^{(k)}}$$

*be a standard Cephoid. Then* $\boldsymbol{\mu}(\Pi)$ *has in each coordinate* $\frac{K}{n}$ *summands, i.e., there is a partition* $\boldsymbol{K} = \boldsymbol{K}_1 \cup \ldots \cup \boldsymbol{K}_n$ *with* $|\boldsymbol{K}_1| = \ldots = |\boldsymbol{K}_n| = \frac{K}{n}$ *such that*

$$\boldsymbol{\mu}(\Pi) = \left( \sum_{k\in\boldsymbol{K}_1} a_1^{(k)} \;,\; \ldots,\; \sum_{k\in\boldsymbol{K}_n} a_n^{(k)} \right). \tag{1}$$

**Proof:** Consider the measure preserving representation on $\widehat{\Delta} = \Delta^{\widehat{\alpha}\boldsymbol{e}}$ (see Definition 2.7). As the volumes of all deGua Simplices involved are equal, there is a positive number $\alpha_0$ satisfying $\sqrt[n-1]{\alpha_k} = \alpha_0 \quad (k \in \boldsymbol{K})$.

Thus

$$\widehat{\boldsymbol{a}}^{(k)} = \alpha_0 \boldsymbol{e} \quad (k \in \boldsymbol{K}). \tag{2}$$

Therefore the barycenter of $\widehat{\Delta}$ is given by

$$\begin{aligned} \boldsymbol{\mu}(\widehat{\Pi}) = \quad & \frac{\boldsymbol{e}}{n}\,\widehat{\alpha} & = \quad & \frac{\boldsymbol{e}}{n}\sum_{k\in\boldsymbol{K}} \sqrt[n-1]{\alpha_k} \\ = \quad & \frac{\boldsymbol{e}}{n} K\alpha_0 & = \quad & (K_0\alpha_0, \ldots, K_0\alpha_0), \end{aligned} \tag{3}$$

where $K_0 \;:=\; \frac{K}{n}$ is an integer.

Now consider the pre–image $\boldsymbol{\mu}(\Pi) = \kappa^{-1}(\boldsymbol{\mu}(\widehat{\Pi}))$. In view of Definition 2.7 (see also formula (20)), there is mapping $\boldsymbol{i}_\bullet : \boldsymbol{K} \to \boldsymbol{I}$ such that

$$\boldsymbol{\mu}(\Pi) \;=\; \sum_{k\in\boldsymbol{K}} \boldsymbol{a}^{(k)\mathbf{i}_k} \;\;,\;\; \boldsymbol{\mu}(\widehat{\Pi}) \;=\; \sum_{k\in\boldsymbol{K}} \widehat{\boldsymbol{a}}^{(k)\mathbf{i}_k} \;=\; (K_0, \ldots, K_0)\alpha_0. \tag{4}$$

In view of 2 the sets $\boldsymbol{K}_i \;:=\; \{k \,|\, \boldsymbol{i}_k = i\}$ necessarily satisfy

$$|\boldsymbol{K}_1| \;=\; \ldots \;=\; |\boldsymbol{K}_n| \;=\; K_0 \;. \tag{5}$$

**q.e.d.**

**Definition 4.3.** *Let $a^\bullet$ be a standard family of positive vectors an let*

$$\Pi \;=\; \sum_{k=1}^{K} \Pi^{a^{(k)}}$$

*be the Cephoid generated. We call $\boldsymbol{a}^\bullet$ as well as $\Pi$ **well behaved** if there is a partition of $\boldsymbol{K}$, say*

$$\boldsymbol{K} = \bigcup_{i\in\boldsymbol{I}} \overline{\boldsymbol{K}}_i \tag{6}$$

*such that*

1. $\left|\,\overline{\boldsymbol{K}}_i\,\right| = \left|\;\overline{\boldsymbol{K}}_j\;\right| \quad (i,j \in \boldsymbol{I})$
2. $a_i^{(k)} \geq a_i^{(l)} \quad (k \in \overline{\boldsymbol{K}}_i, l \notin \overline{\boldsymbol{K}}_i)$.

**Remark 4.4.** *Recall the scenario of our interpretation in* SECTION *3. A bargaining problem is "well behaved" if, when each player specifies the countries in which he receives maximal marginal utilities, this constitutes a (possible) partition. Thus, players have disjoint "most preferred countries" (not quite stringent: equalities are admitted in Definition 4.3 – explaining the above "possible"). In this situation, the bargaining solution will turn out to behave superadditively. That is, whenever one decomposes the bargaining problem into the sum of two smaller ones, then there is no incentive for the players to insist on separate bargaining. Everybody is better off when the "global" bargaining problem is taken into account.*

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**Lemma 4.5.** *If $\boldsymbol{a}^{\bullet}$ is well behaved, then*

$$\boldsymbol{\mu}(\Pi) = \left( \sum_{k \in \overline{\boldsymbol{K}}_1} a_1^k \ , \ \dots \ , \ \sum_{k \in \overline{\boldsymbol{K}}_n} a_n^k \right), \tag{7}$$

*that is, $\boldsymbol{\mu}(\Pi)$ collects the $\frac{K}{n}$ largest vectors with respect to each coordinate.*

**Proof:** By Lemma 4.2 we know that the solution satisfies

$$\boldsymbol{\mu}(\Pi) = \left( \sum_{k \in \boldsymbol{K}_1} a_1^{(k)} \ , \ \dots , \sum_{k \in \boldsymbol{K}_3} a_n^{(k)} \right) \tag{8}$$

with $|\boldsymbol{K}_1| = \dots = |\boldsymbol{K}_n| = \frac{K}{n}$. Now suppose that some $k_1 \in \bar{\boldsymbol{K}}_1$ is not contained in $K_1$, i.e., the summand $a_1^{(k_1)}$ does not appear in the first sum in (8). Then it is contained in some other set $\boldsymbol{K}_i$, assume for simplicity that this is $\boldsymbol{K}_2$. Necessarily, there is $k_2 \in \bar{\boldsymbol{K}}_2$ that is not contained in $\boldsymbol{K}_2$. Again assume for simplicity, that $k_2 \in \boldsymbol{K}_3$ holds true and find $k_3$ such that $k_3 \in \bar{\boldsymbol{K}}_3$, $k_3 \notin \boldsymbol{K}_3$. Proceeding this way, we must close the circle after finitely many steps, again let us assume that this is after $n$ steps. Thus we have found $k_n \in \bar{\boldsymbol{K}}_n$, $k_n \notin \boldsymbol{K}_n$ such that $k_n \in \boldsymbol{K}_1$ is the case. Now we exchange the indices cyclically, i.e., consider the vector

$$\overline{\boldsymbol{x}} := \left( \sum_{k \in (\boldsymbol{K}_1 \backslash \{k_n\}) \cup \{k_1\}} a_1^{(k)} \ , \sum_{k \in (\boldsymbol{K}_2 \backslash \{k_1\}) \cup \{k_2\}} a_2^{(k)} \ , \dots , \sum_{k \in (\boldsymbol{K}_n \backslash \{k_{n-1}\}) \cup \{k_n\}} a_n^{(k)} \right).$$

Because of Definition 4.3, we have increased the coordinates in each position. But the vector $\overline{\boldsymbol{x}}$ is a sum of vertices of the Simplices involved, hence it cannot Pareto dominate the vector $\boldsymbol{\mu}(\Pi)$ which is located on $\partial\Pi$.

**q.e.d.**

We are now in the position to prove a version of superadditivity.

**Theorem 4.6.** *The mapping $\boldsymbol{\mu}$ behaves superadditively along decompositions of a well behaved Cephoid. That is, if $\Pi$ is a well behaved Cephoid and $\Pi = \Upsilon + \Psi$, then ($\Upsilon, \Psi$ are Cephoids and)*

$$\boldsymbol{\mu}(\Pi) \geq \boldsymbol{\mu}(\Upsilon) + \boldsymbol{\mu}(\Psi). \tag{9}$$

**Proof:**

**1$^{\mathbf{st}}$STEP** : First of all, consider the case that both $\Upsilon$ and $\Psi$ are sums of those deGua Simplices that generate $\Pi$. That is, assume

$$\Upsilon = \sum_{k \in \boldsymbol{I}} \Pi^{\boldsymbol{a}^{(k)}} \quad , \quad \Psi = \sum_{k \in \boldsymbol{J}} \Pi^{\boldsymbol{a}^{(k)}}$$

with suitable disjoint index sets $\boldsymbol{I}, \boldsymbol{J}$ satisfying $\boldsymbol{I} \cup \boldsymbol{J} = \boldsymbol{K}$. In each family, the deGua Simplices have equal volume. Possibly $n$ is not a divisor of $|\boldsymbol{I}|$ or $|\boldsymbol{J}|$. If so, we replace each deGua Simplex by a sum of $n$ homothetic $\frac{1}{n}$–copies of itself. This does not change the order property of $\Pi$ and preserves weak nondegeneracy. Hence we can at once assume w.l.o.g that $\Upsilon$ and $\Psi$ are standard.

According to Lemma 4.2 we know that

$$\begin{aligned} \boldsymbol{\mu}(\Upsilon) &= \left( \sum_{k \in \boldsymbol{I}_1} a_1^{(k)} \ , \ \ldots \ , \ \sum_{k \in \boldsymbol{I}_n} a_n^{(k)} \right), \\ \boldsymbol{\mu}(\Psi) &= \left( \sum_{k \in \boldsymbol{J}_1} a_1^{(k)} \ , \ \ldots \ , \ \sum_{k \in \boldsymbol{J}_n} a_n^{(k)} \right), \end{aligned} \tag{10}$$

with

$$\begin{aligned} |\boldsymbol{I}_1| = \ldots = |\boldsymbol{I}_n|, \\ |\boldsymbol{J}_1| = \ldots = |\boldsymbol{J}_n|. \end{aligned}$$

Obviously, we have

$$|\boldsymbol{I}_1 + \boldsymbol{J}_1| = \ldots = |\boldsymbol{I}_n + \boldsymbol{J}_n|,$$

and as the sum of all $n$ terms is $K$, each of them has to be $\frac{K}{n}$. Now consider the first coordinate of the solutions. We obtain

$$\boldsymbol{\mu}_1(\Upsilon) + \boldsymbol{\mu}_1(\Psi) = \sum_{k \in \boldsymbol{I}_1 \cup \boldsymbol{J}_1} a_1^{(k)} \leq \sum_{k \in \boldsymbol{K}_1} a_1^{(k)} = \boldsymbol{\mu}_1(\Pi), \tag{11}$$

as the $1^{st}$ coordinate of the $\boldsymbol{a}^{(k)}$ is maximal in $\boldsymbol{K}_1$.

**2$^{\text{nd}}$STEP** : Next, assume that there are $n$–adic numbers $t_k = \frac{r_k}{2^{Tn}}$ $(k \in \boldsymbol{K})$ such that

$$\Upsilon = \sum_{k \in \boldsymbol{K}} \Pi^{t_k \boldsymbol{a}^{(k)}} \quad , \quad \Psi = \sum_{k \in \boldsymbol{K}} \Pi^{(1-t_k)\boldsymbol{a}^{(k)}}. \tag{12}$$

It is no loss of generality to assume that all these numbers have a common basis $2^{Tn}$. Therefore, as in our introductory remark, we can decompose every deGua Simplex in each of the families into small homothetic multiples of each other until all deGua Simplices involved have equal volume and each deGua Simplex $\Pi^{t_k \boldsymbol{a}^{(k)}}$ is a sum of such deGua Simplices with equal volume. We may then apply the result of the first step to prove superadditivity in the above sense.

**3$^{\text{rd}}$STEP** : Now suppose that the decomposition is arbitrary.

By a well–known criterion (see PALLASCHKE–URBANSKI ([20]), Theorem 8.3.3 or SCHNEIDER([30]), Theorem 3.2.8) the two polyhedra $\Upsilon$ and $\Psi$ have to satisfy equation (12) possibly with non $n$–adic real numbers $t_k$, $0 \leq t_k \leq 1$ $(k \in \boldsymbol{K})$. But the $n$–adic numbers are dense and it is not hard to see that, whenever $\Upsilon$ and $\Psi$ are approximated using decompositions of $\Pi$ that are $n$–adic in the sense of the $2^{nd}STEP$, then the solution behaves continuously. Superadditivity follows, therefore, from the result of the $2^{nd}STEP$. **q.e.d.**

# 5 Examples

Finally, we present an example that demonstrates the merits and demerits of the $\boldsymbol{\mu\pi}$ solution concept.

**Example 5.1.** *The symmetric example "The Circle" represented already in Figure 2.1 in* Chapter 2 *is repeated in Figure 5.1. The canonical representation given in Figure 5.2.*

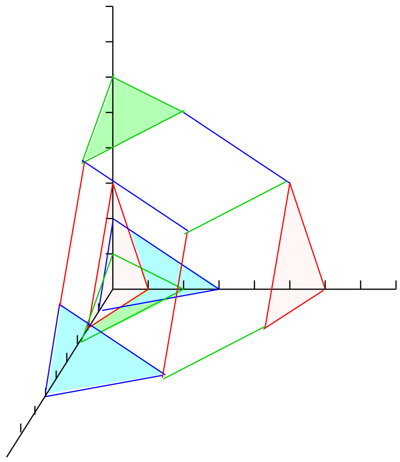

Figure 5.1: " The Circle"

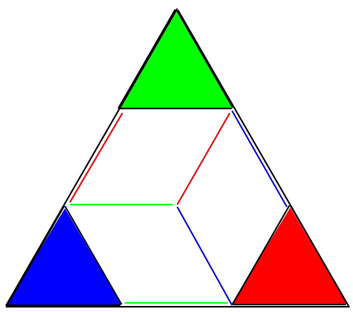

Figure 5.2: Canonical Representation of The Circle

*The Circle, i.e., the Cephoid* $\Pi^{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}}$ *is generated by the family of positive vectors* $\boldsymbol{a} = (1,3,2)$, $\boldsymbol{b} = (2,1,3)$, $\boldsymbol{c} = (3,2,1)$.

*Without further investigation, we know by symmetry that any bargaining solution* $\boldsymbol{\varphi}$ *yields the midpoint of the Pareto surface, i.e.,*

(1) $$\boldsymbol{\varphi}(\Pi) = \boldsymbol{a}^2 + \boldsymbol{b}^3 + \boldsymbol{c}^1.$$

*Formally, the canonical representation and the measure preserving representation are equal, the center point of the latter is* $\frac{1}{3}(3\boldsymbol{e}) = \boldsymbol{e}$ *and the inverse under* $\widehat{\boldsymbol{\kappa}}$ *is*

$$\boldsymbol{\mu}(\Pi) = \widehat{\boldsymbol{\kappa}}^{-1}(\boldsymbol{e}) = \boldsymbol{a}^2 + \boldsymbol{b}^3 + \boldsymbol{c}^1 \ . \tag{2}$$

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**Example 5.2.** *We consider some variants of this example given by*

$$\Pi = \Pi^{l_a \boldsymbol{a}} + \Pi^{l_b \boldsymbol{b}} + \Pi^{l_c \boldsymbol{c}}$$

*with constants* $l_a, l_b, l_c$ *that are nonnegative integers.*

*To begin with, let* $\boldsymbol{a}^{\bullet}$ *be given via*

$$\begin{array}{rcccl} \boldsymbol{a}^{(1)} = \ldots = \boldsymbol{a}^{(4)} & = & \boldsymbol{a} & = & (1,3,2)\ ; \\ \boldsymbol{a}^{(5)} = \boldsymbol{a}^{(6)} = \ \boldsymbol{a}\ ;\ \boldsymbol{a}^{(7)} = \boldsymbol{a}^{(8)} & = & \boldsymbol{b} & = & (2,1,3)\ ; \\ \boldsymbol{a}^{(9)} = \ldots = \boldsymbol{a}^{(12)} & = & \boldsymbol{c} & = & (3,2,1)\ . \end{array} \tag{3}$$

*Now put*

$$\bar{\boldsymbol{K}}_1 \ := \ \{9, \ldots, 12\},\ \bar{\boldsymbol{K}}_2 \ := \ \{1, \ldots, 4\},\ \bar{\boldsymbol{K}}_3 \ := \ \{5, \ldots, 8\}. \tag{4}$$

*Then* $\boldsymbol{a}^{\bullet}$ *is well behaved in the sense of Definition 4.3 as*

$$\begin{array}{rclcl} c_1 & \geq & a_1, b_1 & & \\ a_2 & \geq & b_2, c_2 & & \\ b_3 & \geq & a_3 & \geq & c_3. \end{array}$$

*Therefore we can apply Theorem 4.6 which shows that* $\boldsymbol{\mu}$ *behaves superadditively along any decomposition of*

$$\Pi \ = \ \sum_{k=1}^{K} \Pi^{a^{(k)}} \ = \ \Pi^{6\boldsymbol{a}} + \Pi^{2b} + \Pi^{4c} \ .$$

*Indeed, the proof of Theorem 4.6 can immediately be specified; we observe that* $\boldsymbol{\mu}(\Pi)$ *collects the largest quantities in each coordinate. As it turns out, we obtain*

$$\begin{array}{rl} \boldsymbol{\mu}(\Pi) = & 4\boldsymbol{c}^1 + 4\boldsymbol{a}^2 + 2\boldsymbol{b}^3 + 2\boldsymbol{a}^3 \\ = & (4 \times 3, 4 \times 3, 2 \times 3 + 2 \times 2) \\ = & (12, 12, 10)\ . \end{array} \tag{5}$$

*Obviously, the procedure works for any triple* $(l_a, l_b, l_c)$ *satisfying*

$$l_b \leq l_c \leq l_a \ , \quad l_a + l_b = 2l_c. \tag{6}$$

*Moreover, we may exchange the roles of* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ *in a cyclic order. Then we obtain similar statements whenever*

$$\begin{gathered} l_a \leq l_b \leq l_c \ , \quad l_a + l_c = 2l_b \\ \textit{or} \\ l_c \leq l_a \leq l_b \ , \quad l_c + l_b = 2l_a \ . \end{gathered} \tag{7}$$

*The smallest term is permitted to be* $0$*, e.g.,* $l_c = 0, \ l_b = 2l_a$ *is feasible. This yields the sum of two deGua Simplices similar to Figure 1.1, but the translate of* $\Pi^{2\boldsymbol{b}}$ *has twice the area of the one of* $\Pi^{\boldsymbol{a}}$*.*

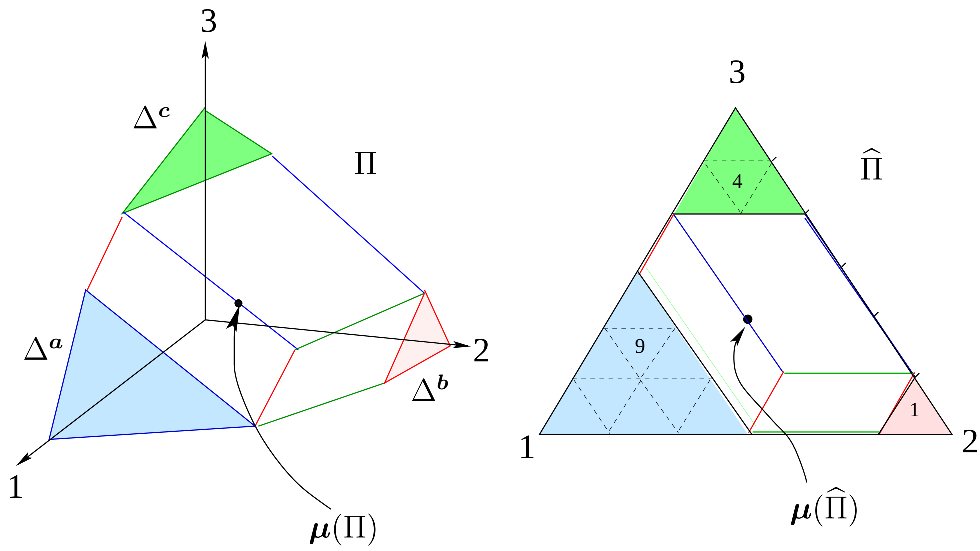


Figure 5.3: The $\boldsymbol{\mu\pi}$ solution for $(l_a, l_b, l_c) = (3, 1, 2)$

*To clarify the situation, Figure 5.3 depicts the case* $(l_a, l_b, l_c) = (3, 1, 2)$ *which is structurally the same as the one treated above with* $(l_a, l_b, l_c) = (6, 2, 4)$*. We observe that*

$$\boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}}) = \boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{b}}) = \boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{c}}) = \sqrt[3]{6^2} \ := \ \alpha$$

*and it is convenient to compute quantities in terms of* $\alpha$*. Hence, the Simplex of adjusted commodity is* $6\alpha\Delta^{\boldsymbol{e}}$ *which is the union of* $36$ *Simplices of area* $\alpha$ *(the right–hand side in Figure 5.3).*

*The translate of* $\Delta^{\boldsymbol{b}}$ *is represented by a Simplex with area of one unit* $\alpha$*,* $\Delta^{\boldsymbol{c}}$ *is reflected by a Simplex with* $4$ *units, and* $\Delta^{\boldsymbol{a}}$ *receives* $9$ *units. It is seen that the barycenter* $\boldsymbol{\mu}(\widehat{\Pi}) = \alpha(2, 2, 2)$ *corresponds to*

$$\begin{aligned} \boldsymbol{\mu}(\Pi) = & \ 2\boldsymbol{c}^1 + 2\boldsymbol{a}^2 + \boldsymbol{b}^3 + \boldsymbol{a}^3 \\ = & \ (6,6,5) \ . \end{aligned} \tag{8}$$

*The Nash solution for this example is* $\boldsymbol{\nu}(\Pi) = (6, \frac{27}{4}, \frac{9}{2})$*. This point (maximizing the coordinate product) is located on the edge connecting* $\boldsymbol{\mu}(\Pi) = (6,6,5)$ *and the vertex* $\boldsymbol{\kappa} = (6,9,3)$*, more precisely,*

$$\boldsymbol{\nu} = \frac{\boldsymbol{\kappa}}{4} + \frac{3\boldsymbol{\mu}}{4}.$$

*Thus, the superadditive solution gives slightly more to player* 3 *and slightly less to player* 2 *compared to the Nash solution and, in addition, treats players* 1 *and* 2 *equally.*

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# Chapter 13

# Conditional Additivity

We present a further axiomatic justification of the (generalized Maschler–Perles) $\boldsymbol{\mu\pi}$ solution for $n$ dimensions. Again we focus on Cephoids as feasible sets and refer to the surface measure.

The essential conceptual change is provided by the idea of *Conditional Additivity*. This is a suitable modification of the Superadditivity concept delt with so far.

The idea rests on AUMANN'S exposition in [1]. Aumann provides an axiomatization of SHAPLEY'S NTU–value (SHAPLEY [32]) . See also HART [10] and DE CLIPPEL[3]).

In our present context, we apply it first to Bargaining Problems, thus obtaining an axiomatic description of the $\boldsymbol{\mu\pi}$ Solution. Later on ( CHAPTER 14) we will also characterize a version of the Shapley Value for NTU–Games. However, other than the above mentioned authors who deal with correspondences (set valued mappings), we always consider solutions/values to be (point valued) functions.

# 1 Revisiting the $\boldsymbol{\mu\pi}$ Solution

Let us revisit the solution concept introduced in CHAPTER 12 and exhibit a decisive property: it's conditional additivity. The framework is provided by a (non–degenerate) family $\left\{\boldsymbol{a}^{(k)}\right\}_{k\in\boldsymbol{K}}$ and the generated Cephoid $\Pi = \sum_{k\in\boldsymbol{K}} \Pi^{\boldsymbol{a}^{(k)}}$. The mapping

$$(1) \qquad \widehat{\boldsymbol{\kappa}}_{\Pi} \;:\; \partial\Pi \to \boldsymbol{\tau}_{\Pi}\Delta^{e} = \widehat{\Delta},$$

constitutes a piecewise linear isomorphism which preserves the poset. The (Simplex $\widehat{\Delta}$ together with) the mapping $\widehat{\boldsymbol{\kappa}}_{\Pi}$ is the ***measure preserving representation*** of $\partial\Pi$. A Cephoid $\Pi$ stands for the bargaining problem $(\mathbf{0}, \Pi)$, hence the solution concept can be seen as a function on Cephoids. As previously we write $\widehat{\boldsymbol{\mu}} := \frac{1}{n}(1, \ldots, 1) = \frac{1}{n}\boldsymbol{e} \in \mathbb{R}^n$. Let us denote the set of (nondegenerate) Cephoids in $\mathbb{R}^n$ by $\mathfrak{C}^n$.

Then the $\boldsymbol{\mu\pi}$ solution for Cephoidal bargaining problems is the mapping $\boldsymbol{\mu} \;:\; \mathfrak{C}^n \to \mathbb{R}^n$ given by

1. $\boldsymbol{\mu}_{t\Delta^e} \;:=\; t\widehat{\boldsymbol{\mu}} \;=\; \frac{t}{n}(1, \ldots, 1) \quad (t > 0)$
2. $\boldsymbol{\mu}_{\Pi} \;:=\; \widehat{\boldsymbol{\kappa}}_{\Pi}^{-1}(\boldsymbol{\mu}_{\boldsymbol{\tau}_{\Pi}\Delta^e}) \;=\; \widehat{\boldsymbol{\kappa}}_{\Pi}^{-1}(\boldsymbol{\tau}_{\Pi}\widehat{\boldsymbol{\mu}})$

For $n = 2$, this is the Maschler–Perles superadditive solution. Hence $\boldsymbol{\mu}$ is a generalized Maschler–Perles solution. Now we provide a different approach to the concept of superadditivity which we cannot expect to be valid for any bargaining solution for more than two players. The appropriate change is provided by the idea of ***conditional additivity*** which eventually will also result in a further axiomatization of the concept.

**Definition 1.1.** *[see* AUMANN *[1]] A mapping $\boldsymbol{\varphi} : \mathfrak{C}^n \to \mathbb{R}^n$ is **conditionally additive** if, for any two Cephoids $\Pi$ and $\Pi'$ such that $\boldsymbol{\varphi}(\Pi) + \boldsymbol{\varphi}(\Pi')$ is Pareto efficient in $\Pi + \Pi'$, it follows that*

$$(2) \qquad \boldsymbol{\varphi}(\Pi) + \boldsymbol{\varphi}(\Pi') = \boldsymbol{\varphi}(\Pi + \Pi')$$

*holds true. Equivalently one requires that for any family of Cephoids $\Pi^{\bullet} \;=\; \{\Pi^q\}_{q\in\boldsymbol{Q}}$ and any lottery $\boldsymbol{p} = \{p_q\}_{q\in\boldsymbol{Q}}$ (see* SECTION *1 of* CHAPTER *11) it follows that*

$$(3) \qquad \boldsymbol{\varphi}(\mathbb{E}_{\boldsymbol{p}}\Pi^{\bullet}) \;=\; \mathbb{E}_{\boldsymbol{p}}\boldsymbol{\varphi}(\Pi^{\bullet})\ .$$

*holds true whenever* $\mathbb{E}_{\boldsymbol{p}}\boldsymbol{\varphi}(\Pi^{\bullet})$ is Pareto efficient *in* $\mathbb{E}_{\boldsymbol{p}}\Pi^{\bullet}$.

Analogous definitions will be used for related concepts like the measure preserving mapping $\widehat{\boldsymbol{\kappa}}_{\bullet}$ etc. For *two players* conditional additivity is equivalent to superadditivity in order to characterize the Maschler–Perles solution . This follows easily from the construction presented in CHAPTER 12.

First of all, we deal with related properties of the basic ingredients.

**Theorem 1.2.** *1. The adjustment factor $\boldsymbol{\tau}_{\bullet}$ is additive, i.e., for any two Cephoids $\Pi$ and $\Pi'$*

$$\boldsymbol{\tau}_{\Pi+\Pi'} = \boldsymbol{\tau}_{\Pi} + \boldsymbol{\tau}_{\Pi'} \tag{4}$$

*holds true.*

*2. The measure preserving representation is conditionally additive. That is, for any two Cephoids $\Pi$ and $\Pi'$ and any $\boldsymbol{x} \in \partial\Pi$, $\boldsymbol{x}' \in \partial\Pi'$ satisfying $\boldsymbol{x} + \boldsymbol{x}' \in \partial(\Pi + \Pi')$, it follows that*

$$\widehat{\boldsymbol{\kappa}}_{\Pi}(\boldsymbol{x}) + \widehat{\boldsymbol{\kappa}}_{\Pi'}(\boldsymbol{x}') = \widehat{\boldsymbol{\kappa}}_{\Pi+\Pi'}(\boldsymbol{x} + \boldsymbol{x}') \tag{5}$$

*holds true.*

**Proof:**

**1stSTEP** :

The first statement is obvious from the definition, i.e., by (4) in SECTION 2 of CHAPTER 12.

**2ndSTEP** :

Regarding the second statement, we start out with two extremal points $\boldsymbol{u} \in \partial\Pi$ and $\boldsymbol{u}' \in \partial\Pi'$. Consider the corresponding mappings as given by (20), and Definition 2.7 in SECTION 2 of CHAPTER 12, say $\mathbf{i}_{\bullet} : \boldsymbol{K} \to \boldsymbol{I}$ and $\mathbf{i}'_{\bullet} : \boldsymbol{K}' \to \boldsymbol{I}$. We assume that the sum $\boldsymbol{u}+\boldsymbol{u}'$ is Pareto efficient, hence extremal in $\partial(\Pi+\Pi')$. We write

$$\boldsymbol{u} + \boldsymbol{u}' = \sum_{k\in\boldsymbol{K}} \boldsymbol{a}^{(k)\mathbf{i}_k} + \sum_{k'\in\boldsymbol{K}'} \boldsymbol{a}'^{(k')\mathbf{i}'_{k'}} = \sum_{l\in\boldsymbol{K}\cup\boldsymbol{K}'} \bar{\boldsymbol{a}}^{l\bar{\mathbf{i}}_l} \tag{6}$$

with canonically defined quantities

$$\bar{\boldsymbol{a}}^{l} = \boldsymbol{a}^{l} \ \ (l \in \boldsymbol{K}), \ \ \bar{\boldsymbol{a}}^{l} = \boldsymbol{a}'^{l} \ \ (l \in \boldsymbol{K}'). \tag{7}$$

and

$$\bar{\mathbf{i}}_l = \bar{\mathbf{i}}_l \ \ (l \in \boldsymbol{K}), \ \ \bar{\mathbf{i}}_l = \mathbf{i}'_l \ \ (l \in \boldsymbol{K}'). \tag{8}$$

Consequently

$$(9) \qquad \widehat{\boldsymbol{\kappa}}_{\Pi}(\boldsymbol{u}) + \widehat{\boldsymbol{\kappa}}_{\Pi'}(\boldsymbol{u}') = \sum_{k \in \boldsymbol{K}} \widehat{\boldsymbol{a}}^{(k)\mathbf{i}_k} + \sum_{k' \in \boldsymbol{K}'} \widehat{\boldsymbol{a}}'^{(k')\mathbf{i}'_{k'}} = \sum_{l \in \boldsymbol{K} \cup \boldsymbol{K}'} \widehat{\widehat{\boldsymbol{a}}}^{l \bar{\boldsymbol{i}}_l}$$

and since the mapping $\bar{\mathbf{i}}_{\bullet}$ corresponding to $\boldsymbol{u} + \boldsymbol{u}'$ is uniquely defined, the right hand side in (9) has to be $\widehat{\boldsymbol{\kappa}}_{\Pi+\Pi'}(\boldsymbol{u} + \boldsymbol{u}')$.

**3rdSTEP :**

Suppose now that $\boldsymbol{x}$ and $\boldsymbol{x}'$ sum up to a Pareto efficient point, hence admit of a joint normal. Pick extremal points of $\partial\Pi$ and $\partial\Pi'$ in the tangent hyperplane generated by that normal for $\Pi$ and $\Pi'$ respectively. Then we have convex representations, say

$$(10) \qquad \boldsymbol{x} = \sum_{\rho} \alpha_\rho \bar{\boldsymbol{x}}^\rho \ , \quad \boldsymbol{x}' = \sum_{\sigma} \alpha'_\sigma \bar{\boldsymbol{x}}'^\sigma .$$

with positive coefficients adding up to 1. All extremal points admit of the same normal, hence our result from the $2^{nd}STEP$ holds for the sum of any two of them taken from the different Cephoids. Also, the subfaces generated by the normal add up to a subface of the sum and all mappings behave affinely linear on these subfaces. In view of

$$(11) \qquad \widehat{\boldsymbol{\kappa}}_{\Pi}(\boldsymbol{x}) = \sum_{\rho} \alpha_\rho \widehat{\boldsymbol{\kappa}}_{\Pi}(\bar{\boldsymbol{x}}^\rho) = \sum_{\rho\ \sigma} \alpha_\rho \alpha'_\sigma \widehat{\boldsymbol{\kappa}}_{\Pi}(\bar{\boldsymbol{x}}^\rho) \ ,$$

we obtain

$$(12) \qquad \begin{aligned} \widehat{\boldsymbol{\kappa}}_{\Pi}(\boldsymbol{x}) + \widehat{\boldsymbol{\kappa}}_{\Pi'}(\boldsymbol{x}') &= \sum_{\rho\ \sigma} \alpha_\rho \alpha'_\sigma \left( \widehat{\boldsymbol{\kappa}}_{\Pi}(\bar{\boldsymbol{x}}^\rho) + \widehat{\boldsymbol{\kappa}}_{\Pi'}(\bar{\boldsymbol{x}}'^\sigma) \right) \\ &= \sum_{\rho\ \sigma} \alpha_\rho \alpha'_\sigma \left( \widehat{\boldsymbol{\kappa}}_{\Pi+\Pi'}(\bar{\boldsymbol{x}}^\rho + \bar{\boldsymbol{x}}'^\sigma) \right) \\ &= \widehat{\boldsymbol{\kappa}}_{\Pi+\Pi'} \left( \sum_{\rho\ \sigma} \alpha_\rho \alpha'_\sigma (\bar{\boldsymbol{x}}^\rho + \bar{\boldsymbol{x}}'^\sigma) \right) \\ &= \widehat{\boldsymbol{\kappa}}_{\Pi+\Pi'} \left( \sum_{\rho} \alpha_\rho \bar{\boldsymbol{x}}^\rho + \sum_{\sigma} \alpha'_\sigma \bar{\boldsymbol{x}}'^\sigma \right) \\ &= \widehat{\boldsymbol{\kappa}}_{\Pi+\Pi'} \left( \bar{\boldsymbol{x}} + \bar{\boldsymbol{x}}' \right) \ , \end{aligned}$$

**q.e.d.**

As a consequence we can immediately conclude:

**Theorem 1.3.** $\boldsymbol{\mu}$ *is conditionally additive.*

**Proof:** This is an immediate consequence of Theorem 1.2 as the midpoints of multiples of of the unit Simplices behave additively. Formally we have

$$
\begin{aligned}
\boldsymbol{\mu}(\Pi) + \boldsymbol{\mu}(\Pi') &= \widehat{\boldsymbol{\kappa}}_{\Pi}^{-1}\left(\widehat{\boldsymbol{\mu}}(\Delta^{\boldsymbol{\tau}(\Pi))\boldsymbol{e}}\right) + \widehat{\boldsymbol{\kappa}}_{\Pi'}^{-1}\left(\widehat{\boldsymbol{\mu}}(\Delta^{\boldsymbol{\tau}(\Pi'))\boldsymbol{e}}\right) \\
&= \widehat{\boldsymbol{\kappa}}_{\Pi}^{-1}\left(\boldsymbol{\tau}(\Pi)\frac{\boldsymbol{e}}{n}\right) + \widehat{\boldsymbol{\kappa}}_{\Pi'}^{-1}\left(\boldsymbol{\tau}(\Pi')\frac{\boldsymbol{e}}{n}\right) \\
&= \widehat{\boldsymbol{\kappa}}_{\Pi+\Pi'}^{-1}\left((\boldsymbol{\tau}(\Pi) + \boldsymbol{\tau}(\Pi'))\frac{\boldsymbol{e}}{n}\right) \\
&= \widehat{\boldsymbol{\kappa}}_{\Pi+\Pi'}^{-1}\left((\boldsymbol{\tau}(\Pi + \Pi'))\frac{\boldsymbol{e}}{n}\right) \\
&= \boldsymbol{\mu}(\Pi + \Pi')
\end{aligned}
\tag{13}
$$

**q.e.d.**

For completeness, we show that $\boldsymbol{\mu}$ is a bargaining solution, performing the necessary routine operations for anonymity and scale covariance.

**Lemma 1.4.** $\boldsymbol{\mu}$ *is anonymous.*

**Proof:**

**1$^{\text{st}}$STEP** :

A permutation $\pi : \boldsymbol{I} \to \boldsymbol{I}$ constitutes a linear mapping on $\mathbb{R}^n$ via $(\pi(\boldsymbol{x}))_i := x_{\pi^{-1}(i)}(\boldsymbol{x} \in \mathbb{R}^n,\ i \in \boldsymbol{I}))$. For Subsimplices this implies $\pi(\Delta_{\boldsymbol{J}}^{\boldsymbol{a}}) = \Delta_{\pi^{-1}(\boldsymbol{J})}^{\boldsymbol{a}}$ whenever $\boldsymbol{a}$ is a positive vector and $\boldsymbol{J} \subseteq \boldsymbol{I}$. It follows at once that a Pareto face

$$
\boldsymbol{F} = \Delta_{\boldsymbol{J}^{(1)}}^{(1)} + \ldots + \Delta_{\boldsymbol{J}^{(K)}}^{(K)}
\tag{14}
$$

of a Cephoid $\Pi$ induces a maximal face

$$
\pi(\boldsymbol{F}) := \Delta_{\pi^{-1}(\boldsymbol{J}^{(1)})}^{(1)} + \ldots + \Delta_{\pi^{-1}(\boldsymbol{J}^{(K)})}^{(K)}
\tag{15}
$$

of the permuted Cephoid $\pi(\Pi)$. This defines the reference system $\pi\mathfrak{J}$. Consider the surface measure of the face $\boldsymbol{F}$ as presented in Definition 2.4 of CHAPTER 12. We obtain for the permuted version

$$
\boldsymbol{\iota}_{\Delta}(\pi(\boldsymbol{F})) = c_{(\pi\mathfrak{J})} \sqrt[n]{\left[\boldsymbol{P}_{\boldsymbol{I}}^{\pi(\boldsymbol{a}^{(1)})}\right]^{\overline{j}_1 - 1} \cdot \ldots \cdot \left[\boldsymbol{P}_{\boldsymbol{I}}^{\pi(\boldsymbol{a}^{(K)})}\right]^{\overline{j}_K - 1}}
\tag{16}
$$

Here the exponents $\overline{j}_k$ are written for the size $|\pi^{-1}(J^{(k)})|$ of the permuted index sets which, for each $k$, equals $j_k$. The volume of a deGua Simplex does not change under a permutation, so the term under the square root is invariant. Finally, the coefficient $c_{(\pi\mathcal{J})}$ attached to the permuted reference system equals $c_{\mathcal{J}}$ as it depends on the size of the reference sets only. Hence the surface measure is invariant under permutations, i.e.,

$$\boldsymbol{\iota}_\Delta(\pi(\boldsymbol{F})) \;=\; \boldsymbol{\iota}_\Delta(\boldsymbol{F}). \tag{17}$$

Since the Pareto faces are being permuted, so are the extremal points of $\partial\Pi$ and as the surface measure is invariant, we conclude that the complete P.E. structure as well as the mapping $\widehat{\boldsymbol{\kappa}}_{\bullet}$ complies with the permutation.

Formally, for any Cephoid $\Pi$ and any $\boldsymbol{x} \in \Pi$

$$\widehat{\boldsymbol{\kappa}}_{\pi(\Pi)}(\pi(\boldsymbol{x})) \;=\; \pi(\widehat{\boldsymbol{\kappa}}_{\Pi}(\boldsymbol{x})) \tag{18}$$

or

$$\widehat{\boldsymbol{\kappa}}_{\pi(\Pi)} = \pi \circ \widehat{\boldsymbol{\kappa}}_{\Pi} \circ \pi^{-1}. \tag{19}$$

Also,

$$\boldsymbol{\tau}_{\pi(\Pi)} \;=\; \boldsymbol{\tau}_{\Pi} \tag{20}$$

is obvious, i.e., the adjustment factor is invariant under permutations.

**2ndSTEP** : Symmetry of the solution follows now at once; we have

$$\begin{aligned} \boldsymbol{\mu}_{\pi(\Pi)} \;=\; \widehat{\boldsymbol{\kappa}}^{-1}_{\pi(\Pi)}\left(\boldsymbol{\tau}_{\pi(\Pi)}\widehat{\boldsymbol{\mu}}\right) \;&=\; \pi \circ \widehat{\boldsymbol{\kappa}}^{-1}_{\Pi} \circ \pi^{-1}\left(\boldsymbol{\tau}_{\Pi}\widehat{\boldsymbol{\mu}}\right) \\ &=\; \pi \circ \widehat{\boldsymbol{\kappa}}^{-1}_{\Pi}\left(\boldsymbol{\tau}_{\Pi}\widehat{\boldsymbol{\mu}}\right) = \pi \circ \boldsymbol{\mu}_{\Pi} \end{aligned} \tag{21}$$

**3rdSTEP** :

Now covariance with a.t.u. is verified similarly. Consider a linear mapping

$$L \;:\; \mathbb{R}^n \to \mathbb{R}^n\,, \qquad L(\boldsymbol{x}) \;=\; (\alpha_1 x_1, \ldots, \alpha_n x_n) \;\; (\boldsymbol{x} \in \mathbb{R}^n)$$

for positive $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$.

First, observe that

$$\boldsymbol{\tau}_{L(\Pi)} \;=\; \sum_{k\in\boldsymbol{K}} \sqrt[n]{\left(\prod_{i\in\boldsymbol{I}} \alpha_i a_i^{(k)}\right)} \;=\; \sqrt[n]{\prod_{i\in\boldsymbol{I}} \alpha_i} \sum_{k\in\boldsymbol{K}} \sqrt[n]{\left(\prod_{i\in\boldsymbol{I}} a_i^{(k)}\right)} \;=\; \boldsymbol{\tau}_{\boldsymbol{\alpha}}\boldsymbol{\tau}_{\Pi}\;. \tag{22}$$

Next, define the translation $T : \mathbb{R}^n \to \mathbb{R}^n$ via

$$T(\boldsymbol{x}) \;=\; \boldsymbol{\tau}_{\boldsymbol{\alpha}}\boldsymbol{x} \quad (\boldsymbol{x} \in \mathbb{R}^n) \tag{23}$$

such that in particular

$$T \;:\; \Delta^{\boldsymbol{\tau}_{\Pi}\boldsymbol{e}} \to \Delta^{\boldsymbol{\tau}_{L(\Pi)}\boldsymbol{e}} \tag{24}$$

holds true. Now, for some Cephoid $\Pi$, let

$$\boldsymbol{u} \;:= \sum_{k \in \boldsymbol{K}} \boldsymbol{a}^{(k)\mathbf{i}_k} \tag{25}$$

be a vertex of $\partial\Pi$ as previously. Then

$$L(\boldsymbol{u}) \;:= \sum_{k \in \boldsymbol{K}} L(\boldsymbol{a}^{(k)\mathbf{i}_k}) \;=\; \sum_{k \in \boldsymbol{K}} L(\boldsymbol{a}^{(k)})^{\mathbf{i}_k} \tag{26}$$

is the corresponding vertex of $L(\Pi)$. If $\widehat{\boldsymbol{u}}$ is the image of $\boldsymbol{u}$ under $\widehat{\boldsymbol{\kappa}}_{\Pi}$, then

$$\begin{aligned} \widehat{\boldsymbol{\kappa}}_{L(\Pi)}(L(\boldsymbol{u})) \;&=\; \sum_{k \in \boldsymbol{K}} (T(\widehat{\boldsymbol{a}}^{(k)})^{\mathbf{i}_k}) \;=\; T\left(\sum_{k \in \boldsymbol{K}} (\widehat{\boldsymbol{a}}^{(k)})^{\mathbf{i}_k}\right) \\ &=\; T(\widehat{\boldsymbol{u}}) \;=\; T(\widehat{\boldsymbol{\kappa}}_{\Pi}(\boldsymbol{u})). \end{aligned} \tag{27}$$

Because of the linearity of $\widehat{\boldsymbol{\kappa}}_{\bullet}$ on the faces, we have

$$\widehat{\boldsymbol{\kappa}}_{L(\Pi)}(L(\boldsymbol{x})) = T(\widehat{\boldsymbol{\kappa}}_{\Pi}(\boldsymbol{x})) \tag{28}$$

for all $\boldsymbol{x}$ in some face $\boldsymbol{F}$ and then for all $\boldsymbol{x} \in \partial\Pi$. This is now reformulated to

$$\widehat{\boldsymbol{\kappa}}_{L(\Pi)}(L(\kappa_{\Pi}^{-1}(\bullet))) = T(\bullet)$$

or

$$L(\widehat{\boldsymbol{\kappa}}_{\Pi}^{-1}(\bullet)) = \widehat{\boldsymbol{\kappa}}_{L(\Pi)}^{-1}(T(\bullet)) \;. \tag{29}$$

**4^th^STEP** :

Finally, the behavior of $\boldsymbol{\mu}$ under a.t.u. is demonstrated by

$$\begin{aligned} \boldsymbol{\mu}(L(\Pi)) \;&=\; \widehat{\boldsymbol{\kappa}}_{L(\Pi)}^{-1}\left(\boldsymbol{\tau}_{L(\Pi)}\widehat{\boldsymbol{\mu}}\right) \\ &=\; \widehat{\boldsymbol{\kappa}}_{L(\Pi)}^{-1}\left(\boldsymbol{\tau}_{\boldsymbol{\alpha}}\boldsymbol{\tau}_{\Pi}\widehat{\boldsymbol{\mu}})\right) && \text{(by (22))} \\ &=\; \widehat{\boldsymbol{\kappa}}_{L(\Pi)}^{-1}\left(T(\boldsymbol{\tau}_{\Pi}\widehat{\boldsymbol{\mu}})\right) && \text{(by definition of } T\text{)} \\ &=\; L\left(\widehat{\boldsymbol{\kappa}}_{\Pi}^{-1}\left(\boldsymbol{\tau}_{\Pi}\widehat{\boldsymbol{\mu}}\right)\right) && \text{(by (29))} \\ &=\; L(\boldsymbol{\mu}_{\Pi}), \end{aligned} \tag{30}$$

**q.e.d.**

Thus, the $\boldsymbol{\mu\pi}$ solution is indeed a ***conditionally additive*** bargaining solution.

# 2 Axiomatic Characterization of the $\boldsymbol{\mu\pi}$ Solution

With due modifications conditional additivity uniquely defines the $\boldsymbol{\mu\pi}$ Solution. Here we introduce the relevant concepts and axioms.

**Definition 2.1.** *An **adjustment** is a pair of mappings* $(\boldsymbol{\gamma_\bullet}, \boldsymbol{\sigma})$ *with the following properties:*

1. $\boldsymbol{\sigma} : \mathfrak{C} \to \mathbb{R}$ *(the **scaling factor**) is a positively homogeneous mapping.*

2. $\boldsymbol{\gamma_\bullet}$ *(the **transfer mapping**) assigns to every Cephoid* $\Pi$ *a bijective mapping*
$$\boldsymbol{\gamma}_\Pi : \partial\Pi \to \boldsymbol{\sigma}_\Pi \Delta^{\boldsymbol{e}}.$$
$\boldsymbol{\gamma_\bullet}$ *is positively homogeneous ,i.e., satisfies*
$$\boldsymbol{\gamma}_{t\Pi} = t\boldsymbol{\gamma}_\Pi : \partial\Pi \to \boldsymbol{\sigma}_{t\Pi}\Delta^{\boldsymbol{e}} = t\boldsymbol{\sigma}_\Pi\Delta^{\boldsymbol{e}} \quad (t > 0). \tag{1}$$

3. *For* $\mathbf{0} < \boldsymbol{a} \in \mathbb{R}^n$*, the mapping*
$$\boldsymbol{\gamma_a} \; := \; \boldsymbol{\gamma}_{\Pi^{\boldsymbol{a}}} : \Delta^{\boldsymbol{a}} \to \boldsymbol{\sigma}^{\boldsymbol{a}}\Delta^{\boldsymbol{e}} \tag{2}$$
*is the canonical affine identification of Simplices, i.e., the mapping*
$$\sum_{i\in \boldsymbol{I}} \beta_i \boldsymbol{a}^i \to \boldsymbol{\sigma_a}(\beta_1, \ldots, \beta_n) \;\; (\boldsymbol{\beta} > 0, \boldsymbol{e\beta} = 1).$$

For deGua Simplices we write $\boldsymbol{\gamma_a}, \boldsymbol{\sigma_a}$ instead of $\boldsymbol{\gamma}_{\Pi^{\boldsymbol{a}}}, \;\; \boldsymbol{\sigma}_{\Pi^{\boldsymbol{a}}}$.

**Remark 2.2.** *A transfer mapping* $\boldsymbol{\gamma_\bullet}$ *may be **conditionally additive** in the sense of Definition 1.1 and as also used for* $\widehat{\boldsymbol{\kappa}}_\bullet$ *e.g. in Theorem 1.2. Note that this* implies *that* $\boldsymbol{\gamma_\bullet}$ *is **piecewise convex**, i.e., for any set of extremal points* $\boldsymbol{b}^1, \ldots, \boldsymbol{b}^L$ *of* $\partial\Pi$ *and convex coefficients* $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_L)$ $(\boldsymbol{\beta} > 0, \boldsymbol{e\beta} = 1)$ *with* $\sum_{l=1}^L \beta_l \boldsymbol{b}^l \in \partial\Pi$*, we have*
$$\boldsymbol{\gamma}_\Pi \left( \sum_{l=1}^L \beta_l \boldsymbol{b}^l \right) \;=\; \sum_{l=1}^L \beta_l \boldsymbol{\gamma}_\Pi \left( \boldsymbol{b}^l \right) \tag{3}$$

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**Definition 2.3.** *A scaling factor* $\boldsymbol{\sigma}$ *is said to be* ***consistent****, if, there are real valued functions* $g$ *and* $f$ *such that for any* $i, j \in \boldsymbol{I}$

$$\boldsymbol{\sigma}_{\boldsymbol{a}} \;=\; g\left(\boldsymbol{\sigma}_{(a_i,a_j)}\right) f\left(\boldsymbol{\sigma}_{(a_k)_{k\neq i,j}}\right) \;. \tag{4}$$

This means that the assessment of concessions by $n$ players is consistent with the one by any two players. If $(a_k)_{k\neq i,j}$ is fixed, then concessions between two players are evaluated according to the scaling factor for two persons. The next lemma says that $n$ players should consistently evaluate a Simplex in terms of the coordinate product. Together with positive homogeneity, this amounts to choosing $\boldsymbol{\tau}$.

**Lemma 2.4.** *Let* $\boldsymbol{\sigma}$ *be a consistent scaling factor. Assume that for* $n = 2$ $\boldsymbol{\sigma}$ *is a function of the product. Then, up to some postive constant,* $\boldsymbol{\sigma} = \boldsymbol{\tau}$ *holds true.*

**Proof:** Let $\mathbf{0} < \boldsymbol{a} \in \mathbb{R}^n$. Choose any $i, j \in \boldsymbol{I}$. Then, for fixed values of $a_k \;\; (k \neq i,j)$, $\boldsymbol{\sigma}_{\boldsymbol{a}}$ is a function of the product $a_i a_j$. This is true for any arbitrary choice of $\{i,j\}$. We show that the function $\boldsymbol{\sigma}_{\boldsymbol{a}}$ is exponential in the coordinate product, say

$$\boldsymbol{\sigma}_{\boldsymbol{a}} = (a_1 \cdot \ldots \cdot a_n)^r$$

Indeed, for fixed $a_4, \ldots, a_n$ write $h^3(a_3) = f_{12}(a_3, a_4, \ldots, a_n)$ etc. such that

$$\boldsymbol{\sigma}_{\boldsymbol{a}} = g(a_1a_2)h^3(a_3) = g(a_1a_3)h^2(a_2) = g(a_2a_3)h^1(a_1).$$

Then

$$\begin{aligned} \frac{\boldsymbol{\sigma}_{\boldsymbol{a}}}{h^3(a_3)h^2(a_2)h^1(a_1)} &= \frac{g(a_1a_2)}{h^2(a_2)h^1(a_1)} &&= \frac{g(a_1a_3)}{h^3(a_3)h^1(a_1)} \\ &= \frac{g(a_3a_2)}{h^2(a_2)h^3(a_3)} &&= const. \end{aligned}$$

Hence

$$\boldsymbol{\sigma}_{\boldsymbol{a}} = const \prod_{123} h^i(a_i)$$

and

$$g(a_1a_2) = h^1(a_1)h^2(a_2)$$

Then

$$g(t) = h^1(t)h^2(1) = h^2(t)h^1(1)$$

thus

$$\frac{h^1(t)}{h^1(1)} = \frac{h^2(t)}{h^2(1)} := h(t)$$

with $h(1) = 1$. Consequently

$$g(t) = h(t)\alpha, \quad h(a_1 a_2)\alpha = g(a_1 a_2) = h(a_1)h(a_2)\alpha$$

meaning

$$h(a_1 a_2) = h(a_1)h(a_2).$$

Hence $h$ is exponential and so is $g$. Now

$$\begin{aligned} \boldsymbol{\sigma}_{\boldsymbol{a}} &= g(a_1 a_2) f(a_3, ...a_n) = (a_1)^r (a_2)^r f(a_3, ...a_n) \\ &= (a_1)^r (a_3)^r f(a_2, a_4...a_n) = (a_2)^r (a_3)^r f(a_1, a_4...a_n), \end{aligned}$$

thus

$$\frac{\boldsymbol{\sigma}_{\boldsymbol{a}}}{(a_1)^r (a_2)^r \dots (a_n)^r} = \frac{f(a_3, ...a_n)}{(a_3)^r \dots (a_n)^r} = \frac{f(a_1, a_4...a_n)}{(a_1)^r (a_4)^r \dots (a_n)^r} = \dots = const$$

and

$$\boldsymbol{\sigma}_{\boldsymbol{a}} = const\,(a_1 \dots a_n)^r$$

Because of $\boldsymbol{\sigma}_{t\boldsymbol{e}} = t\boldsymbol{\sigma}_{\boldsymbol{e}}$ it follows that $r = \frac{1}{n}$. Ignoring a constant if necessary, we come up with

$$\boldsymbol{\sigma}_{\boldsymbol{a}} = \sqrt[n]{a_1 \cdot \ldots \cdot a_n} = \sqrt[n-1]{\boldsymbol{\iota}_\Delta(\Delta^{\boldsymbol{a}})} = \boldsymbol{\tau}^{\boldsymbol{a}}. \tag{5}$$

**q.e.d.**

**Definition 2.5.** *Let $\boldsymbol{\eta}$ be a bargaining solution on Cephoids, and let $(\boldsymbol{\gamma}_\bullet, \boldsymbol{\sigma})$ be an adjustment. We say that $(\boldsymbol{\eta}, \boldsymbol{\gamma}_\bullet, \boldsymbol{\sigma})$ satisfies the **adjusted value axioms** if the following holds true.*

1. *$\boldsymbol{\eta}$ is conditionally additive.*
2. *$\boldsymbol{\gamma}_\bullet$ is conditionally additive.*
3. *$\boldsymbol{\sigma}$ is additive and consistent.*
4. *The solution concept respects the adjustment. That is,*

$$\boldsymbol{\gamma}_\Pi(\boldsymbol{\eta}(\Pi)) = \boldsymbol{\eta}(\boldsymbol{\gamma}_\Pi(\Pi)) = \boldsymbol{\eta}(\boldsymbol{\sigma}_\Pi \Delta^{\boldsymbol{e}}). \tag{6}$$

**Theorem 2.6.** *If $(\boldsymbol{\eta}, \boldsymbol{\gamma}_\bullet, \boldsymbol{\sigma})$ satisfies the adjusted value axioms, then*

1. *$\boldsymbol{\eta}$ is the $\boldsymbol{\mu\pi}$ solution, i.e., generalized superadditive solution $\boldsymbol{\mu}$,*

*Up to some positive common constant,*

2. *$\boldsymbol{\gamma}_{\bullet}$ is the measure preserving mapping $\widehat{\boldsymbol{\kappa}}_{\bullet}$, and*
3. *$\boldsymbol{\sigma}$ is the assessment function $\boldsymbol{\tau}$.*

**Proof:**

For arbitrary $n$, every bargaining solution yields the center point whenever the bargaining problem is a Simplex. Therefore, with respect to the last axiom, equation (6) can be rewritten

$$\boldsymbol{\gamma}_{\Pi}(\boldsymbol{\eta}(\Pi)) = \widehat{\boldsymbol{\mu}}(\boldsymbol{\gamma}_{\Pi}(\Pi)) = \widehat{\boldsymbol{\mu}}(\boldsymbol{\sigma}_{\Pi}\Delta^{\boldsymbol{e}}) = \boldsymbol{\sigma}_{\Pi}(\frac{1}{n},\ldots,\frac{1}{n}) \ . \tag{7}$$

**1st STEP** :

For $n = 2$ all polyhedral bargaining problems are Cephoids. There is one and only one solution which is conditionally additive on polyhedral bargaining problems, this is the Maschler–Perles solution $\boldsymbol{\mu}$, see CHAPTER 11. Hence we have $\boldsymbol{\eta} = \boldsymbol{\mu}$.

**2nd STEP** :

We prove that, for $n = 2$, $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ coincide up to a constant. Let $\mathbb{R}^2 \ni \boldsymbol{a}, \boldsymbol{b} > \boldsymbol{0}$ be positive vectors and let $\Delta^{\boldsymbol{a}}$ and $\Delta^{\boldsymbol{b}}$ be the corresponding Simplices (line segments) in $\mathbb{R}^2$ (we assume non-degeneracy). Assume that $a_1a_2 \geq b_1b_2$. Also, choose $\alpha \leq 1$ such that $\alpha a_1 a_2 = b_1 b_2$. Furthermore, let $\Pi := \Pi^{\boldsymbol{a}} + \Pi^{\boldsymbol{b}}$ and $\Pi^{\alpha} = \Pi^{\alpha\boldsymbol{a}} + \Pi^{\boldsymbol{b}}$. Then $\Pi^{\alpha}$ is symmetric up to an affine transformation, so $\boldsymbol{\eta}(\Pi^{\alpha}) = \boldsymbol{\mu}(\Pi^{\alpha})$ is the unique vertex. Hence, $\boldsymbol{\gamma}_{\Pi^{\alpha}}$ maps the two line segments of $\partial\Pi^{\alpha}$ bijectively linear onto the two line segments of $2\boldsymbol{\sigma}_{\Pi^{\boldsymbol{a}}}\Delta^{\boldsymbol{e}}$ that are generated by the midpoint $(\boldsymbol{\sigma}_{\Pi^{\boldsymbol{a}}}, \boldsymbol{\sigma}_{\Pi^{\boldsymbol{a}}})$. We conclude that $\boldsymbol{\gamma}_{\Pi^{\boldsymbol{a}}} = \widehat{\boldsymbol{\kappa}}_{\Pi^{\boldsymbol{a}}}$ and $\boldsymbol{\gamma}_{(1-\alpha)\Pi^{\boldsymbol{a}}} = \widehat{\boldsymbol{\kappa}}_{(1-\alpha)\Pi^{\boldsymbol{a}}}$ holds true. Now any $\boldsymbol{x}$ on the “left side” of $\partial\Pi^{\alpha}$ and any $\boldsymbol{x}'$ on $(1-\alpha)\Delta^{\boldsymbol{a}}$ add up to a Pareto efficient sum $\boldsymbol{x}+\boldsymbol{x}'$. By conditional additivity we have

$$\begin{aligned} \boldsymbol{\gamma}_{\Pi}(\boldsymbol{x}+\boldsymbol{x}') &= \boldsymbol{\gamma}_{\Pi^{\alpha}}(\boldsymbol{x}) + \boldsymbol{\gamma}_{(1-\alpha)\Pi^{\boldsymbol{a}}}(\boldsymbol{x}') \\ &= \widehat{\boldsymbol{\kappa}}_{\Pi^{\alpha}}(\boldsymbol{x}) + \widehat{\boldsymbol{\kappa}}_{(1-\alpha)\Pi^{\boldsymbol{a}}}(\boldsymbol{x}') \ = \ \widehat{\boldsymbol{\kappa}}_{\Pi}(\boldsymbol{x}+\boldsymbol{x}'). \end{aligned} \tag{8}$$

That is, $\boldsymbol{\gamma}_{\Pi}$ and $\widehat{\boldsymbol{\kappa}}_{\Pi}$ coincide on the “left side” of $\partial\Pi$. In particular $\boldsymbol{\mu}$ behaves additively, i.e., $\boldsymbol{\mu}_{\Pi} = \boldsymbol{\mu}_{\Pi^{\alpha}} + \boldsymbol{\mu}_{(1-\alpha)\Pi^{\boldsymbol{a}}}$. Consequently, the midpoint of $\widetilde{\Delta}$ (cf.

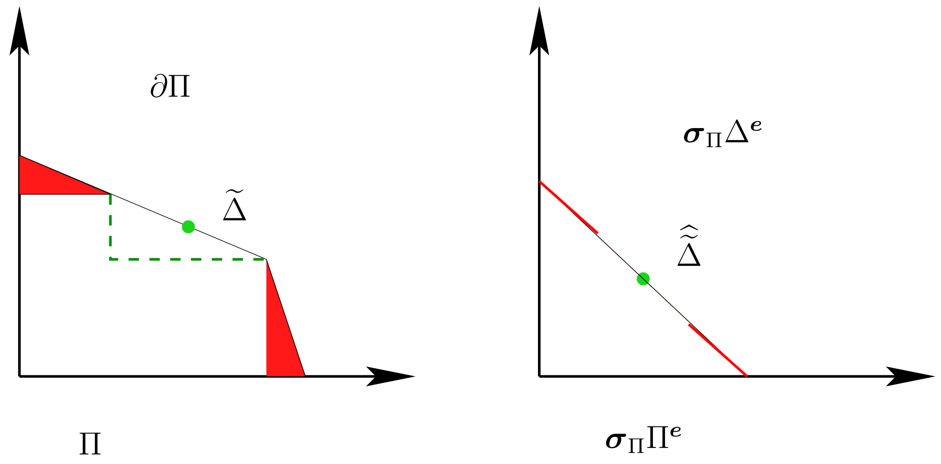


Figure 2.1: A sum of two prisms and the $\boldsymbol{\gamma}$–image

Figure 2.1) is mapped onto the midpoint of $\boldsymbol{\sigma}_{\Pi}\Delta^{\boldsymbol{e}}$. Hence, all line segments on $\partial\Pi$ are mapped onto the corresponding line segments on $\boldsymbol{\sigma}_{\Pi}\Delta^{\boldsymbol{e}}$ in the same ratio of length as is the case with the mapping $\widehat{\boldsymbol{\kappa}}_{\Pi}$. We conclude that

$$(9) \qquad \boldsymbol{\gamma}_{\Pi} \;=\; \frac{\boldsymbol{\sigma}_{\Pi}}{\boldsymbol{\tau}_{\Pi}}\widehat{\boldsymbol{\kappa}}_{\Pi} \;=:\; r_{\Pi}\widehat{\boldsymbol{\kappa}}_{\Pi}$$

holds indeed.

We claim that the ratio $r_{\Pi}$ does not depend on $\Pi$. Indeed, change $\boldsymbol{b}$ to $\boldsymbol{b}'$ in the above argument such that the product $\alpha a_1 a_2 = b_1 b_2 = b_1' b_2'$ is the same. Then the length of the line segments involving $\Delta^{\boldsymbol{a}}$ and $\alpha\Delta^{\boldsymbol{a}}$ does not change. As the ratios are again the ones indicated above, the total length of the image $\boldsymbol{\sigma}_{\Pi}\Delta^{\boldsymbol{e}}$ does not change.

The above procedure is naturally extended to a sum of $K$ prisms in $\mathbb{R}^2$ (see CHAPTER 11). Hence, for some positive $r$, we have $\boldsymbol{\gamma}_{\bullet} \;=\; r\boldsymbol{\kappa}_{\bullet}\,, \quad \boldsymbol{\sigma}_{\bullet} \;=\; r\boldsymbol{\tau}_{\bullet}$. We assume that the constant is 1, hence

$$(10) \qquad \boldsymbol{\sigma}_{\Pi^{\boldsymbol{a}}} \;=\; \boldsymbol{\tau}_{\Pi^{\boldsymbol{a}}} \;=\; \boldsymbol{\iota}_{\Delta}(\partial\Pi^{\boldsymbol{a}}) = \sqrt{a_1 a_2}.$$

which exactly describes the Maschler–Perles Solution.

**3$^{\text{rd}}$STEP** : Now we turn to bargaining problems in $\mathbb{R}^n$. First of all, we determine the nature of $\boldsymbol{\sigma}_{\Pi}$. By the previous step, $\boldsymbol{\sigma}$ equals $\boldsymbol{\tau}$ on two dimensional Simplices. By Lemma 2.4 it follows that $\boldsymbol{\sigma}$ equals $\boldsymbol{\tau}$ on all $n$–dimensional deGua simplices. As both functions are additive, they coincide necessarily on Cephoids.

**4th STEP** : Let $\Pi$ be a Cephoid and let $\boldsymbol{u}$ be a vertex of $\partial\Pi$. By non-degeneracy $\boldsymbol{u}$ is a unique sum of vertices of the Simplices $\Delta^{(k)}$ say

$$\boldsymbol{u} = \boldsymbol{a}^{\mathbf{i}_\bullet} = \sum_{k \in \boldsymbol{K}} \boldsymbol{a}^{(k)\mathbf{i}_k} \ . \tag{11}$$

with suitable $\mathbf{i}_\bullet \ : \ \boldsymbol{K} \to \boldsymbol{I}$ (see (20) and Definition 2.7 in SECTION 2 of CHAPTER 12). By *item* 3 we know that

$$\boldsymbol{\gamma}_{\boldsymbol{a}^{(k)}}\left(\boldsymbol{a}^{(k)\mathbf{i}_k}\right) = \boldsymbol{\sigma}_{\boldsymbol{a}^{(k)}} \boldsymbol{e}^{\mathbf{i}_k} \quad (k \in \boldsymbol{K}).$$

As $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ coincide on deGua Simplices, we use conditional additivity in order to conclude that

$$\begin{aligned}
\boldsymbol{\gamma}_\Pi(\boldsymbol{u}) &= \boldsymbol{\gamma}_\Pi\left(\sum_{k\in\boldsymbol{K}} \boldsymbol{a}^{(k)\mathbf{i}_k}\right) &&= \sum_{k\in\boldsymbol{K}} \boldsymbol{\gamma}_{\boldsymbol{a}^{(k)}}\left(\boldsymbol{a}^{(k)\mathbf{i}_k}\right) \\
&= \sum_{k\in\boldsymbol{K}} \boldsymbol{\sigma}_{\boldsymbol{a}^{(k)}} \boldsymbol{e}^{\mathbf{i}_k} &&= \sum_{k\in\boldsymbol{K}} \boldsymbol{\tau}^{\boldsymbol{a}^{(k)}} \boldsymbol{e}^{\mathbf{i}_k} \\
&= \sum_{k\in\boldsymbol{K}} \widehat{\boldsymbol{\kappa}}_\Pi(\boldsymbol{a}^{(k)})\left(\boldsymbol{a}^{(k)\mathbf{i}_k}\right) &&= \widehat{\boldsymbol{\kappa}}_\Pi\left(\sum_{k\in\boldsymbol{K}} \boldsymbol{a}^{(k)\mathbf{i}_k}\right) \\
&= \widehat{\boldsymbol{\kappa}}_\Pi(\boldsymbol{u}).
\end{aligned} \tag{12}$$

Thus, $\boldsymbol{\gamma}_\Pi$ and $\widehat{\boldsymbol{\kappa}}_\Pi$ coincide on the extremal points of $\partial\Pi$. By piecewise convexity, they coincide necessarily on all of $\partial\Pi$. Consequently, $\boldsymbol{\eta}(\Pi) = \boldsymbol{\mu}(\Pi)$ holds true by (7) and by the $3^{rd}STEP$.

**q.e.d.**

The adjusted value axioms uniquely determine the generalized conditionally additive $\boldsymbol{\mu\pi}$ bargaining solution $\boldsymbol{\mu}$.

# Chapter 14

# NTU Games: The Shapley Value

Non–Transferable–Utility Games constitute a generalization of bargaining problems as well as of "TU Games".

A TU Game as introduced by von Neumann–Morgenstern [33] assigns a real value to any coalition of players taken from a (finite) player set, this reflects a (monetary or utilitarian) worth to such a coalition. It is thought that this worth can be secured by the coalition and serves as an argument in the bargaining process leading to a final distribution of wealth resulting from the "game".

More generally then, an NTU game ensures a set of payoffs or monetary wealth assignments to the members of each coalition. It imitates the structure of a bargaining situation for each coalition and it imitates the situation of a TU–Games inasmuch as it considers coalitions in all variety. Commonly the function representing the assignment of wealth distributions to coalitions is denoted as the "coalitional function".

In analogy to the solution concepts for bargaining solutions as well as for TU games, Cooperative Game Theory is concerned with exhibiting "solutions" for NTU games, that is assignments of wealth that reflect bargaining power, fairness considerations, efficiency, and invariance properties via some set of axioms or at least via certain plausibility considerations. In the context of Cooperative Game Theory (that is TU games and NTU games under some unifying aspect) one calls such a solution concept a "value" – referring the reader to the work of Shapley [31] in Cooperative Game Theory or Debreu [5] in General Equilibrium Theory.

Thus the present chapter will deal with a version of the "Shapley value". We will attempt to define and axiomatize a concept that is based on the appropriate axiomatic considerations; specifically on the concept of "conditional superadditivity".

# 1 NTU Games

We recall some basic concepts of Cooperative NTU Game Theory.

**Definition 1.1.** *A (cooperative)* ***NTU game*** *is a triple*$(\boldsymbol{I}, \underline{\underline{\mathbf{P}}}, \boldsymbol{V})$. $\boldsymbol{I} = \{1,\ldots,n\}$ *is the* ***set of players***, $\underline{\underline{\mathbf{P}}} = \{S | S \subseteq \boldsymbol{I}\} = \mathcal{P}(\boldsymbol{I})$ *is the* ***system of coalitions***, *and* $\boldsymbol{V} := \underline{\underline{\mathbf{P}}} \rightarrow \mathcal{P}(\mathbb{R}^n)$ *is the* ***coalitional function***. $\boldsymbol{V}$ *assigns to any coalition a nonempty, compact, convex set of "utility vectors"* $\boldsymbol{V}(S) \subseteq \mathbb{R}^n_S$ *such that there exists* $\underline{\boldsymbol{x}} \in \mathbb{R}^n$ *satisfying*

1. $$\boldsymbol{V}(\{i\}) = \{\underline{x}_i\} \quad i \in \boldsymbol{I} \ ,$$

2. $$\boldsymbol{V}(S) = \left\{ \boldsymbol{x} \in \mathbb{R}^n_S \,\middle|\, \boldsymbol{x} \geq \underline{\boldsymbol{x}}_{\,|\,S} \right\} \cap \boldsymbol{CmpH}\, \boldsymbol{V}(S) \neq \emptyset \quad (S \in \underline{\underline{\mathbf{P}}}) \ .$$

$\boldsymbol{V}(S)$ is thought to be "feasible for the members of S", that is, a set of utility vectors coalition $S$ can ensure to its members (see e.g. the extensive treatment in [26], Definition 1.3, SECTION 1, CHAPTER 4 and also SECTION 5). $\underline{\boldsymbol{x}}$ reflects the *status quo position*, i.e., player $i \in \boldsymbol{I}$ obtains $\underline{x}_i$ whenever cooperation fails in every coalition $S$ with at least 2 players containing him. We write $\underline{\boldsymbol{x}}(\boldsymbol{V})$ if reference to a particular game should be necessary.

Sloppily, as $\boldsymbol{I}$ and $\underline{\underline{\mathbf{P}}}$ remain fixed, we frequently refer to $\boldsymbol{V}$ as to "the game". Also, via the linear transformation $\boldsymbol{x} \rightarrow \boldsymbol{x} - \underline{\boldsymbol{x}}$ we can frequently assume that $\underline{\boldsymbol{x}} = \mathbf{0}$ holds true.

A bargaining problem $\boldsymbol{V} = (\underline{\boldsymbol{x}}, \boldsymbol{U})$ (Chapter 12, Definition 1.1) can be seen as an NTU game. Put

$$\boldsymbol{V}(\boldsymbol{I}) = \boldsymbol{U} \text{ and } \boldsymbol{V}(S) = \left\{ \underline{\boldsymbol{x}}_{\,|\,S} \right\} \quad (S \in \underline{\underline{\mathbf{P}}} \setminus \{\boldsymbol{I}\}) \ .$$

A ***value*** is a Pareto efficient, symmetric mapping $\boldsymbol{\varphi}$ from a class of games into $\mathbb{R}^n$ that respects affine transformations of utility (a.t.u.). $\boldsymbol{\varphi}(\boldsymbol{V})$ reflects the distribution of utility considered to "solve" the game. We write $\boldsymbol{\varphi}_U$ to emphasize that we are dealing with bargaining problems. In this context we use also the term ***solution***.

As in SECTION 1 of CHAPTER 11 a ***lottery*** is a probability distribution with finite carrier defined over a family of games. Given such a family, say, $\boldsymbol{V}^\bullet = \{\boldsymbol{V}^q\}_{q\in\boldsymbol{Q}}$ , a lottery can be represented by a nonnegative vector

$$\boldsymbol{p} = (p_q)_{q\in\boldsymbol{Q}}, \quad \sum_{q\in\boldsymbol{Q}} p_q = 1 . \tag{1}$$

Accordingly, we apply the standard operations for subsets of $\mathbb{R}^n$ in order to define the action of a lottery on games. E.g. the expected game is

$$\mathbb{E}_{\boldsymbol{p}}\boldsymbol{V}^\bullet(S) := \sum_{q\in\boldsymbol{Q}} p_q \boldsymbol{V}^q(S) \quad (S \in \underline{\underline{\mathbf{P}}}) . \tag{2}$$

Thus, the mapping $\mathbb{E}_{\boldsymbol{p}}\boldsymbol{V}^\bullet := \underline{\underline{\mathbf{P}}} \to \mathcal{P}(\mathbb{R}^n)$ is a game in the sense of Definition 1.1. in particular, the status quo position will be affected by a lottery, we find

$$\mathbb{E}_{\boldsymbol{p}}\underline{\boldsymbol{x}}(\boldsymbol{V}^\bullet) = \sum_{q\in\boldsymbol{Q}} p_q \underline{\boldsymbol{x}}(\boldsymbol{V}^q) . \tag{3}$$

The interaction of lotteries and values is reflected by the axiomatic treatment of solution concepts. SHAPLEY [31] characterizes the value (for TU–games) – among other axioms – by additivity, which (given the concept to be positively homogeneous) is equivalent to "risk neutrality", i.e., $\boldsymbol{\varphi}(\mathbb{E}(\boldsymbol{V}^\bullet)) = \mathbb{E}\boldsymbol{\varphi}(\boldsymbol{V}^\bullet)$. There is no discussion of this concept in SHAPLEY'S generalization of the value to NTU games [32].

The MASCHLER–PERLES solution ([22],[13]) as presented in CHAPTER 12 rests on the concept of superadditivity. That is

$$\boldsymbol{\varphi}(\boldsymbol{V} + \boldsymbol{V}') \geq \boldsymbol{\varphi}(\boldsymbol{V}) + \boldsymbol{\varphi}(\boldsymbol{V}')$$

for pairs of bargaining problems or, equivalently,

$$\boldsymbol{\varphi}(\mathbb{E}_{\boldsymbol{p}}\boldsymbol{V}^\bullet) \geq \mathbb{E}_{\boldsymbol{p}}\boldsymbol{\varphi}(\boldsymbol{V}^\bullet) .$$

for a family $\{\boldsymbol{V}^q\}_{q\in\boldsymbol{Q}}$ of bargaining problems. Superadditivity is interpreted to consistently favor contracting *ex ante*, thereby increasing expected utility ([13]), see also CHAPTER 12 SECTION 2.

The Maschler Perles solution works for two players only. PERLES[21] showed that for more than two players, a superadditive solution for bargaining problems does not exist. The $\boldsymbol{\mu\pi}$–Solution presented in CHAPTER 12 generalizes the Maschler–Perles procedure and exhibits a class of games for which superadditivity prevails. (As for further approaches see also CALVO–GUTIÉRREZ [6]).

We will now discuss AUMANN'S[1] concept of *conditional superadditivity* in our context. Aumann provides an axiomatization of SHAPLEY'S [32] NTU value . In his version as well as in SHAPLEY'S and others (see HART [10], DE CLIPPEL[3]), authors consider correspondences, that is a "value" is a set–valued function. In our context, as a value is a (point valued) function, we choose the following

**Definition 1.2.** *A value* $\boldsymbol{\varphi}$ *is* ***conditionally additive*** *if, for any two games* $\boldsymbol{V}$ *and* $\boldsymbol{W}$ *such that* $\boldsymbol{\varphi}(\boldsymbol{V}) + \boldsymbol{\varphi}(\boldsymbol{W})$ *is Pareto efficient in* $\boldsymbol{V}(\boldsymbol{I}) + \boldsymbol{W}(\boldsymbol{I})$*, it follows that*

$$\boldsymbol{\varphi}(\boldsymbol{V}) + \boldsymbol{\varphi}(\boldsymbol{W}) = \boldsymbol{\varphi}(\boldsymbol{V} + \boldsymbol{W}) \tag{4}$$

*holds true. Equivalently, one requires that for any family of games* $\boldsymbol{V}^{\bullet} = \{\boldsymbol{V}^q\}_{q\in\boldsymbol{Q}}$ *it follows that*

$$\boldsymbol{\varphi}(\mathbb{E}_{\boldsymbol{p}}\boldsymbol{V}^{\bullet}) \;=\; \mathbb{E}_{\boldsymbol{p}}\boldsymbol{\varphi}(\boldsymbol{V}^{\bullet}) \;. \tag{5}$$

*holds true whenever* $\mathbb{E}_{\boldsymbol{p}}\boldsymbol{\varphi}(\boldsymbol{V}^{\bullet})$ is Pareto efficient *in* $\mathbb{E}_{\boldsymbol{p}}\boldsymbol{V}^{\bullet}$.

For *two players* conditional additivity is equivalent to superadditivity in order to characterize the Maschler–Perles solution . This follows easily from the construction presented in CHAPTER 12.

AUMANN'S concept is based on games with smooth surfaces of each $\boldsymbol{V}(S)$ while MASCHLER and PERLES start out from a polyhedral setup. More recently, DE CLIPPEL ET AL. elaborate on the problem imposed by choosing the domain of definition for the axiomatic treatment of a value. It is obvious that in 2 dimensions conditional additivity and the IIA axiom characterizing the Nash solution are *not* compatible.

Within this chapter, we discuss a value concept $\boldsymbol{\chi}$ for NTU games based on the surface measure and respecting conditional superadditivity. We provide an axiomatization of this value. Other than, say, in SHAPLEY'S [32] NTU version, the value we exhibit is characterized without any version of IIA, its existence does not rely on a fixed point theorem, and it can be computed straightforwardly – if one is willing to adopt the idea of the surface measure as "computable".

On the other hand, we will restrict ourselves to Cephoids as the basic ingredients for feasible sets. Accordingly, we will assume that the status quo position is normalized $\underline{\boldsymbol{x}} = \mathbf{0}$. This may be considered as a drawback one has to weigh against the advantages.

For reference, we denote the set of (nondegenerate) Cephoids in $\mathbb{R}^n$ by $\mathfrak{C}^n$.

**Definition 1.3.** *An NTU game* $\boldsymbol{V}$ *is* ***cephoidal*** *if*

1.

(6) $$\underline{\boldsymbol{x}} \;=\; \underline{\boldsymbol{x}}(\boldsymbol{V}) \;=\; \mathbf{0}\;.$$

2. *For any* $S \in \underline{\underline{\mathbf{P}}}$ *there exists a (n.d.) family of positive vectors*

(7) $$\left\{\boldsymbol{a}^{S,(k)}\right\}_{k \in \boldsymbol{K}_S} \subseteq \mathbb{R}^n_{S+}$$

*such that*

(8) $$\boldsymbol{V}(S) \;=\; \sum_{k \in \boldsymbol{K}_S} \Pi^{\boldsymbol{a}^{S,(k)}} \;=:\; \sum_{k \in \boldsymbol{K}_S} \Pi^{S,(k)} \subseteq \mathfrak{C}^{|S|}$$

*holds.*

We denote the set of Cephoidal NTU games by $\mathfrak{V}^n$.

Thus, we restrict the discussion to Cephoids as feasible sets in $\mathbb{R}^n_{S+}$. Naturally, the term affine transformation of utility ("a.t.u.") is restricted to positive dilatations of the axes.

# 2 The TU game

Now we introduce the TU game derived from an NTU game. To this end, let $\boldsymbol{V}$ be a Cephoidal NTU game. Our approach suggests that players evaluate concessions and gains in accordance with the coordinate product. It seems plausible that a "side payment game" derived from an NTU situation has to be calibrated accordingly.

The foremost candidate is suggested by the surface measure and the adjustment factor.

Recall the Definitions (1) and (2) in Chapter 12 Section 2 for the adjustment factor of a deGua Simplex and of a Cephoid. Suitably adapted to a deGua Simplex $\Pi^{\boldsymbol{a}^S} \subseteq \mathbb{R}^n_{S+}$ and a Cephoid in $\mathbb{R}^n_{S+}$, say

$$\Pi^S \;=\; \sum_{k\in\boldsymbol{K}_S} \Pi^{\boldsymbol{a}^{S,(k)}} \in \mathfrak{C}^S$$

these definitions amount to adjustment factors

$$\text{(1)} \qquad \boldsymbol{\tau}_{\Pi^{\boldsymbol{a}^S}} \;:=\; \sqrt[s]{\prod_{i\in S} a_i^S}\;, \quad \boldsymbol{\tau}_{\Pi^S} = \sum_{k\in\boldsymbol{K}_S} \boldsymbol{\tau}_{\Pi^{\boldsymbol{a}^{S,(k)}}}\;.$$

with $s \;:=\; |S|$. Accordingly, we consider the "worth" of coalition $S$ implied by $\boldsymbol{V}$ to be given follows.

The standard definition of a TU game is repeated, see e.g. [27]. We will only consider nonnegative TU games in our context. There is then a natural way to embed such TU games into Cephoidal (!) NTU games.

**Definition 2.1.** *1. A (nonnegative)* ***TU game*** *is a mapping*

$$\text{(2)} \qquad \boldsymbol{v} : \underline{\underline{\mathbf{P}}} \to \mathbb{R}_+\;, \quad \boldsymbol{v}(\emptyset) = 0\;.$$

*The set of TU games is denoted by $\mathbb{V}^n_+$.*

*2. Let $\boldsymbol{V}$ be a Cephoidal NTU game. The* ***TU game induced*** *by $\boldsymbol{V}$ is*

$$\text{(3)} \qquad \widehat{\boldsymbol{v}} \;=\; \widehat{\boldsymbol{v}}^{\boldsymbol{V}} \;:\; \underline{\underline{\mathbf{P}}} \to \mathbb{R}$$

$$\text{(4)} \qquad \widehat{\boldsymbol{v}}(S) \;=\; \widehat{\boldsymbol{v}}^{\boldsymbol{V}}(S) \;=\; \boldsymbol{\tau}_{\boldsymbol{V}(S)} \quad (S \in \underline{\underline{\mathbf{P}}}).$$

The following first example underlines that we have chosen the canonically induced version.

**Example 2.2.** *A* ***hyperplane NTU game*** *illustrates the relevance of our version. Let* $\boldsymbol{v} : \underline{\underline{\mathbf{P}}} \to \mathbb{R}_+$ *be a nonnegative TU game. Also, for* $S \in \underline{\underline{\mathbf{P}}}$*, let* $\boldsymbol{a}^S \in \mathbb{R}^n_{S+}$ *be a positive vector. Define* $\boldsymbol{V} = \boldsymbol{V}^{\boldsymbol{v}}_{\boldsymbol{a}}$ *by*

$$\boldsymbol{V}(S) = \boldsymbol{V}^{\boldsymbol{v}}_{\boldsymbol{a}}(S) = \boldsymbol{v}(S)\Pi^{\boldsymbol{a}^S} \quad (S \in \underline{\underline{\mathbf{P}}}).$$

*That is,* $\boldsymbol{V}(S)$ *is a deGua Simplex, essentially determined by a hyperplane (which intuitively reflects the rate of exchanging utility in coalition* $S$*). Then we have (using* $s := |S|$*)*

$$\begin{aligned} \boldsymbol{\tau}_{\boldsymbol{V}(S)} &= \sqrt[s]{\prod_{i\in S}\left(\boldsymbol{v}(S)a_i^S\right)} &&= \boldsymbol{v}(S)\sqrt[s]{\prod_{i\in S} a_i^S} \\ &= \boldsymbol{v}(S)\boldsymbol{\tau}_{\Pi^{\boldsymbol{a}^S}} &&= \boldsymbol{v}(S)\boldsymbol{\tau}_{\boldsymbol{a}^S} \end{aligned}$$

*That is, for* $S \in \underline{\underline{\mathbf{P}}}$*:*

$$\widehat{\boldsymbol{v}}(S) = \boldsymbol{v}(S)\boldsymbol{\tau}_{\boldsymbol{a}^S} \ . \tag{5}$$

*The worth of coalition* $S \in \underline{\underline{\mathbf{P}}}$ *is adjusted or rescaled utilizing the adjustment factor. Clearly,* $\widehat{\boldsymbol{v}}$ *and* $\boldsymbol{v}$ *coincide whenever* $\boldsymbol{a}^S = \boldsymbol{e}_S$ *is the restriction of the unit vector* $\boldsymbol{e}$ *and hence* $\boldsymbol{V}^{\boldsymbol{v}}_{\boldsymbol{e}}$ *is the standard* embedding *of the side payment game* $\boldsymbol{v}$ *into the NTU framework.*

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Next, we analyze the behavior of the TU game mapping under the transformations which underly the axiomatic treatment, i.e., we prepare the ground for invariance properties. To this end, we refer to the presentation regarding bargaining problems as initiated in SECTION 1 of CHAPTER 13.

**Lemma 2.3.** *The TU game mapping* $\boldsymbol{V} \to \widehat{\boldsymbol{v}}^{\boldsymbol{V}}$ *is additive. That is, for* $\boldsymbol{V}, \boldsymbol{W} \in \mathfrak{V}^n$

$$\widehat{\boldsymbol{v}}^{\boldsymbol{V}} + \widehat{\boldsymbol{v}}^{\boldsymbol{W}} = \widehat{\boldsymbol{v}}^{\boldsymbol{V}+\boldsymbol{W}} \ . \tag{6}$$

**Proof:** The proof follows immediately from the additivity of the adjustment factor, i.e., from Theorem 1.2 of SECTION 1 CHAPTER 13. **q.e.d.**

Next, we wish to assess the behavior of side payment games under affine positive transformations. For positive $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ let

$$L : \mathbb{R}^n \to \mathbb{R}^n, \quad L(\boldsymbol{x}) := (\alpha_1 x_1, \ldots, \alpha_n x_n) \quad (\boldsymbol{x} \in \mathbb{R}^n)$$

denote such a transformation, then (22) of SECTION 1 CHAPTER 13 implies

$$\boldsymbol{\tau}_{L(\boldsymbol{V}(S))} = \boldsymbol{\tau}_{\boldsymbol{\alpha}_S}\boldsymbol{\tau}_{\boldsymbol{V}(S)} \ . \tag{7}$$

Accordingly, we have to define the action of an affine transformation of utility on Cephoidal NTU games. The appropriate version of the transformed game is given as follows.

**Definition 2.4.** *Let* $\boldsymbol{V} \in \mathfrak{V}^n$ *and let* $L$ *be an a.t.u.. Define the transformed game* $\widehat{L}\boldsymbol{V}$ *by*

$$(\widehat{L}\boldsymbol{V})(S) \ := \ \frac{\boldsymbol{\tau}_{\boldsymbol{\alpha}}}{\boldsymbol{\tau}_{\boldsymbol{\alpha}_S}} L(\boldsymbol{V}(S)) \quad (S \in \underline{\underline{\mathbf{P}}}). \tag{8}$$

To justify this, observe that we eventually strive to explain values suggesting a cooperative agreement within the grand coalition. In particular, the Shapley Value is based on the assumption that agreement eventually takes place within the grand coalition.

Now transformation of utility also refers to the grand coalition. When coalitions $S$ take $\widehat{\boldsymbol{v}}$ into account, they should respect the different measurements when rescaling according to an affine transformation of utility is performed. Thus we come up with

**Lemma 2.5.** *Let* $L$ *be an a.t.u.. Then the TU game mapping* $\boldsymbol{V} \to \widehat{\boldsymbol{v}}^{\boldsymbol{V}}$ *satisfies*

$$\widehat{\boldsymbol{v}}^{\widehat{L}\boldsymbol{V}}(S) \ = \ \boldsymbol{\tau}_{\boldsymbol{\alpha}} \widehat{\boldsymbol{v}}^{\boldsymbol{V}}(S) \tag{9}$$

*for all* $\boldsymbol{V} \in \mathfrak{V}^n$ *and* $S \in \underline{\underline{\mathbf{P}}}$. *Thus, an affine transformation is reflected in the corresponding TU game by a rescaling via the factor*

$$\boldsymbol{\tau}_{\boldsymbol{\alpha}} \ := \ \sqrt[n]{\prod_{i \in \boldsymbol{I}} \alpha_i} \ . \tag{10}$$

The **Proof** is obvious.

**Remark 2.6.** *Clearly, anonymity causes no problems within our development. Naturally, one defines the action of a permutation* $\pi \ := \ \boldsymbol{I} \to \boldsymbol{I}$ *on NTU games by*

$$\pi \boldsymbol{V}(S) \ := \ \boldsymbol{V}(\pi^{-1}(S)) \quad \{S \in \underline{\underline{\mathbf{P}}}\} \ . \tag{11}$$

*Analogously, for TU games. Then clearly*

$$\boldsymbol{v}^{\pi \boldsymbol{V}} \ = \ \pi \boldsymbol{v}^{\boldsymbol{V}} \tag{12}$$

*describes the compatibility of the TU mapping with renaming the players.*

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# 3 Conditional Additivity: The MPS Value

Within this section, we describe a version of the NTU Shapley value. The concept is based on Shapley's seminal paper [31] (see also [32]) but also refers to the Maschler–Perles solution ([13]). Thus, it is appropriate to say that we define a Maschler–Perles–Shapley value. Of course, we deal with Cephoidal NTU games and generalize the $\boldsymbol{\mu\pi}$ solution developed in Chapters 12 and 13. Hence the term $\boldsymbol{\mu\pi\sigma}$ value would possibly be more characterizing ...

Recall the formula for the Shapley value, given some TU game $\boldsymbol{v}$. The value assigns to player $i \in \boldsymbol{I}$ a worth

$$
(1) \quad \begin{aligned} \Phi_i(\boldsymbol{v}) &:= \sum_{S \in \underline{\underline{\mathbf{P}}}} \frac{(n-s)!(s-1)!}{n!}(v(S) - v(S \setminus \{i\}) \\ &:= \frac{1}{n!}\sum_{\pi \in \Pi}(v(S^{\pi}_{\pi(i)}) - v(S^{\pi}_{\pi(i)} \setminus \{i\})) \quad (i \in \boldsymbol{I}) \ . \end{aligned}
$$

The notation involves $s = |S|$, $\Pi$ is the set of permutations $\pi : \boldsymbol{I} \to \boldsymbol{I}$; also $S^{\pi}_{k} := \{j \in \boldsymbol{I} | \pi(i) \leq k\}$ such that $S^{\pi}_{\pi(i)}$ denotes the predecessors of $i$ with respect to the ordering induced by $\pi$. The concept is characterized by an axiomatic foundation relying on anonymity, Pareto efficiency, additivity, and a dummy or null–player property. See Shapley [31], also [25] for a textbook treatment.

Let $\boldsymbol{V}$ be a Cephoidal NTU game. The TU game derived is $\widehat{\boldsymbol{v}} = \widehat{\boldsymbol{v}}^{\boldsymbol{V}}$ (see (4) of Section 2). In view of Pareto efficiency the Shapley value $\Phi(\widehat{\boldsymbol{v}})$ satisfies

$$
(2) \quad (\Phi(\widehat{\boldsymbol{v}}))(\boldsymbol{I}) \ = \ \sum_{i \in \boldsymbol{I}} \Phi_i(\widehat{\boldsymbol{v}}) \ = \ \widehat{\boldsymbol{v}}(\boldsymbol{I}) = \boldsymbol{\tau}_{\boldsymbol{V}(\boldsymbol{I})}
$$

that is

$$
(3) \quad \Phi(\widehat{\boldsymbol{v}}) \in \Delta^{\boldsymbol{\tau}_{\boldsymbol{V}(\boldsymbol{I})}\boldsymbol{e}} \ = \ \boldsymbol{\tau}_{\boldsymbol{V}(\boldsymbol{I})}\Delta^{\boldsymbol{e}} \ .
$$

Recalling Definition 2.7 of the measure preserving mappig $\widehat{\boldsymbol{\kappa}}$ (Chapter 12 Section 2), we observe that $\Phi(\widehat{\boldsymbol{v}})$ is located in the range of $\widehat{\boldsymbol{\kappa}}_{(\boldsymbol{V}(\boldsymbol{I}))}$, i.e., we have

$$
(4) \quad \widehat{\boldsymbol{\kappa}}_{\boldsymbol{V}(\boldsymbol{I}))} \ : \ \partial\boldsymbol{V}(\boldsymbol{I}) \to \boldsymbol{\tau}_{\boldsymbol{V}(\boldsymbol{I})}\Delta^{\boldsymbol{e}} \ = \ \widehat{\boldsymbol{v}}^{\boldsymbol{V}}(\boldsymbol{I})\Delta^{\boldsymbol{e}} \ .
$$

This permits us to formulate the following definition.

**Definition 3.1.** *The* ***MPS value*** *(* ***conditionally additive value, c.a. value,*** $\boldsymbol{\mu\pi\sigma}$ ***value****) is the mapping* $\boldsymbol{\chi} : \mathfrak{V}^n \to \mathbb{T}^n$ *defined on Cephoidal games by*

$$\boldsymbol{\chi}(\boldsymbol{V}) := \widehat{\boldsymbol{\kappa}}^{-1}_{(\boldsymbol{V}(\boldsymbol{I}))}(\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{V}^n})) \quad (\boldsymbol{V} \in \mathfrak{V}) . \tag{5}$$

We can then start with the introduction of the axioms which are implicitely formulated by the following list of properties of the MPS value.

**Theorem 3.2.** *The MPS value*

1. *is Pareto efficient,*
2. *is symmetric,*
3. *respects a.t.u.*
4. *is conditionally additive.*

**Proof: 1stSTEP** :

As we are concerned with functions on (Cephoidal) NTU games, we introduce a suitable notation for the measure preserving mapping $\widehat{\boldsymbol{\kappa}}$, we write

$$\widehat{\boldsymbol{\kappa}}^{\boldsymbol{V}} := \widehat{\boldsymbol{\kappa}}_{\boldsymbol{V}(\boldsymbol{I}))} : \partial\boldsymbol{V}(\boldsymbol{I}) \to \boldsymbol{\tau}_{\boldsymbol{V}(\boldsymbol{I})}\Delta^e = \widehat{\boldsymbol{v}}^{\boldsymbol{V}}(\boldsymbol{I})\Delta^e \quad (\boldsymbol{V} \in \mathfrak{V}) . \tag{6}$$

Then the defining equation (5) reads

$$\boldsymbol{\chi}(\boldsymbol{V}) := (\boldsymbol{\kappa}^{\boldsymbol{V}})^{-1}(\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{V}^n})) \quad (\boldsymbol{V} \in \mathfrak{V}) .$$

**2ndSTEP** :

Pareto efficiency is obvious from the definition. As for anonymity the action of a permutation $\pi$ on a game must be well defined. For a TU game $\boldsymbol{v}$ we have

$$\pi\boldsymbol{v}(S) := \boldsymbol{v} \circ \pi^{-1}(S) \quad (S \in \underline{\underline{\mathbf{P}}}) ,$$

while for NTU games, the appropriate definition is

$$\pi\boldsymbol{V}(S) := \pi \circ \boldsymbol{V} \circ \pi^{-1}(S) \quad (S \in \underline{\underline{\mathbf{P}}}) .$$

Now, in view of (20) SECTION 1 CHAPTER 13 we have for $S \in \underline{\underline{\mathbf{P}}}$

$$\begin{array}{lrcl} (7) & \widehat{\boldsymbol{v}}^{\pi \boldsymbol{V}}(S) & = & \boldsymbol{\tau}_{(\pi \boldsymbol{V})(S)} \quad = \boldsymbol{\tau}_{\pi(\boldsymbol{V}(\pi^{-1}(S)))} \\ & & = & \boldsymbol{\tau}_{\boldsymbol{V}(\pi^{-1}(S))} \quad = \boldsymbol{\tau}_{\pi^{-1}(\pi(\boldsymbol{V}(\pi^{-1}(S))))} \\ & & = & \boldsymbol{\tau}_{(\pi^{-1}\boldsymbol{V})(S)} \end{array}$$

that is,

$$(8) \qquad \widehat{\boldsymbol{v}}^{\pi \boldsymbol{V}} = \pi \widehat{\boldsymbol{v}}^{\boldsymbol{V}}$$

proving anonymity of the function $\widehat{\boldsymbol{v}}^{\bullet}$. Analogously, we prove the anonymity of $\widehat{\boldsymbol{\kappa}}^{\bullet}$. By (19) Section 1 Chapter 13 we have

$$\begin{array}{lrcll} (9) & \widehat{\boldsymbol{\kappa}}^{\pi \boldsymbol{V}} & = & \widehat{\boldsymbol{\kappa}}_{(\pi \boldsymbol{V})(\boldsymbol{I})} & = \widehat{\boldsymbol{\kappa}}_{(\pi(\boldsymbol{V}(\pi^{-1}(\boldsymbol{I}))))} \\ & & = & \pi \circ \widehat{\boldsymbol{\kappa}}_{\boldsymbol{V}(\boldsymbol{I})} \circ \pi^{-1} & = \pi \circ \widehat{\boldsymbol{\kappa}}^{\boldsymbol{V}} \circ \pi^{-1}. \end{array}$$

That is, $\widehat{\boldsymbol{\kappa}}$ behaves eaxtly as an NTU game $\boldsymbol{V}$ w.r.t. the application of permutations. Combining we obtain

$$\begin{array}{lrcl} (10) & \boldsymbol{\chi}(\pi \boldsymbol{V}) & = & \left(\widehat{\boldsymbol{\kappa}}^{\pi \boldsymbol{V}}\right)^{-1}\left(\Phi(\widehat{\boldsymbol{v}}^{\pi \boldsymbol{V}})\right) \\ & & = & \left(\pi \circ \widehat{\boldsymbol{\kappa}}^{\boldsymbol{V}} \circ \pi\right)^{-1}\left(\Phi(\pi \widehat{\boldsymbol{v}}^{\boldsymbol{V}})\right) \\ & & = & \pi \circ \left(\widehat{\boldsymbol{\kappa}}^{\boldsymbol{V}}\right)^{-1} \circ \pi^{-1}\left(\Phi(\pi \widehat{\boldsymbol{v}}^{\boldsymbol{V}})\right) \\ & & = & \pi \circ \left(\widehat{\boldsymbol{\kappa}}^{\boldsymbol{V}}\right)^{-1} \circ \pi^{-1}\pi\left(\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{V}})\right) \\ & & = & \pi \circ \left(\widehat{\boldsymbol{\kappa}}^{\boldsymbol{V}}\right)^{-1} \circ \left(\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{V}})\right) \\ & & = & \pi(\boldsymbol{\chi}(\boldsymbol{V})). \end{array}$$

**3$^{\text{rd}}$STEP** :

Next, invariance with a.t.u is verified. To this end, let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) > \mathbf{0}$ and let

$$\begin{array}{rcl} L & : & \mathbb{R}^n \to \mathbb{R}^n \\ L(\boldsymbol{x}) & = & (\alpha_1 x_1, \ldots, \alpha_n x_n) \quad (\boldsymbol{x} \in \mathbb{R}^n) \end{array}$$

be the corresponding positive linear mapping. The action of affine transformation on an NTU game is given by the mapping $\widehat{L}$ (see (8) in Section 2 ) Also, recall the translation $T$ defined in (23) that satisfies

$$(11) \qquad \boldsymbol{\tau}_{L(\Pi)} = \boldsymbol{\tau}_{\boldsymbol{\alpha}} \boldsymbol{\tau}_{\Pi} = T(\boldsymbol{\tau}_{\Pi})$$

(see (22)). Also, Lemma 2.5 reads

$$\widehat{\boldsymbol{v}}^{\widehat{L}\boldsymbol{V}} = \boldsymbol{\tau}_{\boldsymbol{\alpha}} \widehat{\boldsymbol{v}}^{\boldsymbol{V}}. \tag{12}$$

The relation of the mappings $\boldsymbol{\kappa}_{\Pi}$ and $\boldsymbol{\kappa}_{L(\Pi)}$ is explained by (28) of SECTION 1 CHAPTER 13 which reads

$$\widehat{\boldsymbol{\kappa}}_{L(\Pi)}(L(\boldsymbol{x})) = T(\widehat{\boldsymbol{\kappa}}_{\Pi}(\boldsymbol{x})) \ .$$

Therefore the applications of $\boldsymbol{\chi}$ and $L$ commute as

$$\begin{aligned}
\boldsymbol{\chi}(\widehat{L}\boldsymbol{V}) &= \widehat{\boldsymbol{\kappa}}^{-1}_{(\widehat{\boldsymbol{L}}V)(\boldsymbol{I})}\left(\Phi(\widehat{\boldsymbol{v}}^{\widehat{L}\boldsymbol{V}})\right) && \text{(by definition of } \boldsymbol{\chi}) \\
&= \widehat{\boldsymbol{\kappa}}^{-1}_{(\widehat{L}V)(\boldsymbol{I})}\left(\Phi(\boldsymbol{\tau}_{\boldsymbol{\alpha}}\widehat{\boldsymbol{v}}^{\boldsymbol{V}})\right) && \text{(by (9) of Lemma 2.5)} \\
&= \widehat{\boldsymbol{\kappa}}^{-1}_{L(\boldsymbol{V}(\boldsymbol{I}))}\left(\Phi(\boldsymbol{\tau}_{\boldsymbol{\alpha}}\widehat{\boldsymbol{v}}^{\boldsymbol{V}})\right) && \text{as } L(\boldsymbol{V}(\boldsymbol{I})) = (\widehat{L}\boldsymbol{V})(\boldsymbol{I}) \\
&= \widehat{\boldsymbol{\kappa}}^{-1}_{L(\boldsymbol{V}(\boldsymbol{I}))}\left(\boldsymbol{\tau}_{\boldsymbol{\alpha}}\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{V}})\right) && \text{(as } \Phi \text{ is linear)} \\
&= \widehat{\boldsymbol{\kappa}}^{-1}_{L(\boldsymbol{V}(\boldsymbol{I}))}\left(T(\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{V}}))\right) && \text{(by definition of T)} \\
&= L\left(\widehat{\boldsymbol{\kappa}}^{-1}_{\boldsymbol{V}(\boldsymbol{I})}\left(\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{V}})\right)\right) && \text{(by (29))} \\
&= L(\boldsymbol{\chi}(\boldsymbol{V})) \ .
\end{aligned} \tag{13}$$

**4$^{\text{th}}$STEP** : Finally, the proof for conditional additivity runs quite analogously to the one of Lemma 1.3. If, for two games $\boldsymbol{V}$ and $\boldsymbol{W}$ the values $\boldsymbol{\chi}(\boldsymbol{V})$ and $\boldsymbol{\chi}(\boldsymbol{W})$ yield a Pareto efficient sum, then they are located within faces that admit of a joint normal and $\widehat{\boldsymbol{\kappa}}$ behaves additively. Consequently

(14)

$$\begin{aligned}
& \boldsymbol{\chi}(\boldsymbol{V}) + \boldsymbol{\chi}(\boldsymbol{W}) \\
= \ & \widehat{\boldsymbol{\kappa}}^{-1}_{(\boldsymbol{V}(\boldsymbol{I}))}(\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{V}})) + \widehat{\boldsymbol{\kappa}}^{-1}_{(\boldsymbol{W}(\boldsymbol{I}))}(\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{W}})) \ \text{ by definition,} \\
= \ & \widehat{\boldsymbol{\kappa}}^{-1}_{(\boldsymbol{V}(\boldsymbol{I}))+(\boldsymbol{W}(\boldsymbol{I}))}\left(\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{V}}) + \Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{W}})\right) \ \text{ by (5 ) in Theorem 1.2 of CH 13 , SEC 1 ,} \\
= \ & \widehat{\boldsymbol{\kappa}}^{-1}_{(\boldsymbol{V}(\boldsymbol{I}))+(\boldsymbol{W}(\boldsymbol{I}))}\left(\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{V}} + \widehat{\boldsymbol{v}}^{\boldsymbol{W}})\right) \ \text{, as the Shapley value is additive,} \\
= \ & \widehat{\boldsymbol{\kappa}}^{-1}_{(\boldsymbol{V}(\boldsymbol{I}))+(\boldsymbol{W}(\boldsymbol{I}))}\left(\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{V}+\boldsymbol{W}})\right) \ \text{ by Lemma (2.3),} \\
= \ & \boldsymbol{\chi}(\boldsymbol{V} + \boldsymbol{W}),
\end{aligned}$$

**q.e.d.**

# 4 Axioms for the Conditionally Additive Value

Finally, it turns out that the axioms define the value uniquely, hence we have a ***characterization*** of the MPS value. As we have to deal with the concept of an abstract value, we start with formal definitions.

Recall Definition 2.1 of SECTION 2 CHAPTER 13. Accordingly, an ***adjustment*** is a pair $(\boldsymbol{\gamma}_{\bullet}, \boldsymbol{\sigma})$ consisting of a scaling factor and a transfer mapping. An adjustment reflects a generalization of the pair $(\widehat{\boldsymbol{\kappa}}, \boldsymbol{\tau})$ that is given by the measure preserving mapping and the adjustment factor. Clearly adjustments induce mappings on games, we introduce

$$(1) \qquad \gamma^{\boldsymbol{V}} := \gamma_{\boldsymbol{V}(\boldsymbol{I})}\ , \quad \boldsymbol{\sigma}^{\boldsymbol{V}} := \boldsymbol{\sigma}_{\boldsymbol{V}(\boldsymbol{I})}$$

and

$$(2) \qquad \mathring{\boldsymbol{v}}(S) = \mathring{\boldsymbol{v}}^{\boldsymbol{V}}(S) = \mathring{\boldsymbol{v}}^{\boldsymbol{\sigma},\boldsymbol{V}}(S) := \boldsymbol{\sigma}_{\boldsymbol{V}(S)} \quad (S \in \underline{\underline{\mathbf{P}}})\ .$$

These defintions are quite in accordance with those given for $(\widehat{\boldsymbol{\kappa}}, \boldsymbol{\tau})$, that is, with $\widehat{\boldsymbol{\kappa}}^{\boldsymbol{V}}$, $\boldsymbol{\tau}^{\boldsymbol{V}}$, and $\widehat{\boldsymbol{v}}$; see SECTION 2.

**Definition 4.1.** *1. A **value** is a mapping $\boldsymbol{\psi} : \mathfrak{V}^n \to \mathbb{R}^n$ which is Pareto efficient, symmetric, and a.t.u. covariant.*

*2. A value $\boldsymbol{\psi}$ obeys the **null player axiom** w.r.t an adjustment $(\boldsymbol{\gamma}_{\bullet}, \boldsymbol{\sigma})$ if any null player of $\mathring{\boldsymbol{v}}^{\boldsymbol{\sigma},\boldsymbol{V}}$ receives $\boldsymbol{\psi}_i(\boldsymbol{V}) = 0$.*

Clearly $\boldsymbol{\chi}$ is a value that obeys the null player axioms w.r.t. $(\widehat{\boldsymbol{\kappa}}, \boldsymbol{\tau})$. We then formulate the Axiomatic of the MPS value.

**Definition 4.2.** *We say that $(\boldsymbol{\psi}, \boldsymbol{\gamma}_{\bullet}, \boldsymbol{\sigma})$ satisfies the **adjusted value axioms** if the following holds true.*

1. *$\boldsymbol{\psi}$ is conditionally additive.*
2. *$\boldsymbol{\psi}$ obeys the null player axiom w.r.t. $(\boldsymbol{\gamma}_{\bullet}, \boldsymbol{\sigma})$*
3. *$\boldsymbol{\gamma}_{\bullet}$ is conditionally additive.*
4. *$\boldsymbol{\sigma}$ is additive and consistent (Definition 2.3 of SECTION 2 CHAPTER 13).*
5. *The solution concept respects the adjustment. That is,*

$$(3) \qquad \gamma^{\boldsymbol{V}}(\boldsymbol{\psi}(\boldsymbol{V})) = \boldsymbol{\psi}(\mathring{\boldsymbol{v}}^{\boldsymbol{\sigma},\boldsymbol{V}}(\bullet)\Delta^e_{\bullet}) \quad (\boldsymbol{V} \in \mathfrak{V}^n).$$

In view of SECTION 3 and as the null player axiom is satisfied, it follows that that the tripel $(\boldsymbol{\chi}, \widehat{\boldsymbol{\kappa}}, \boldsymbol{\tau})$ satisfies these axioms. The task is to show that it does so uniquely.

**Theorem 4.3.** *If $(\boldsymbol{\psi}, \boldsymbol{\gamma}_{\bullet}, \boldsymbol{\sigma})$ satisfies the adjusted value axioms, then $\boldsymbol{\psi}$ is the MPS value $\boldsymbol{\chi}$, and (up to some positive common constant)*

1. *$\boldsymbol{\gamma}_{\bullet}$ is the measure preserving mapping $\widehat{\boldsymbol{\kappa}}_{\bullet}$, and*
2. *$\boldsymbol{\sigma}$ is the adjustment factor $\boldsymbol{\tau}$.*

**Proof:**

**1stSTEP** :

For bargaining problems, we know that the MPS value $\boldsymbol{\chi}$ equals the generalized Maschler–Perles solution $\boldsymbol{\mu}$. Now, the axiomatic of Definition 4.2 is the one presented in Definition 2.5 when restricted to bargaining problems. It follows from Theorem 2.6 SECTION 2 CHAPTER 13 that $(\boldsymbol{\gamma}, \boldsymbol{\sigma}) = (\widehat{\boldsymbol{\kappa}}, \boldsymbol{\tau})$ holds true. Thus, statements *1* and *2* are immediately verified..

**2ndSTEP** :

It follows that the derived side payment game is

$$(4) \qquad \overset{\circ}{\boldsymbol{v}}^{\boldsymbol{V}} \;=\; \overset{\circ}{\boldsymbol{v}}^{\boldsymbol{\sigma},\boldsymbol{V}} = \widehat{\boldsymbol{v}}^{\boldsymbol{\tau},\boldsymbol{V}} = \widehat{\boldsymbol{v}}^{\boldsymbol{V}} \quad (\boldsymbol{V} \in \mathfrak{V}).$$

We claim that $\boldsymbol{\psi}$ has to coincide with the Shapley value (more precisely: with $\boldsymbol{\chi}$) on hyperplane games.

Indeed, consider the function $\boldsymbol{\vartheta}$ on TU–games defined by

$$\boldsymbol{\vartheta}(\boldsymbol{v}) \;:=\; \boldsymbol{\psi}(\boldsymbol{v}(\bullet)\Delta^{\boldsymbol{e}}_{\bullet}) \;=\; \boldsymbol{\psi}(\boldsymbol{V}^{\boldsymbol{v}}_{\boldsymbol{e}}) \;.$$

As $\boldsymbol{\psi}$ is conditionally additive it follows that $\boldsymbol{\vartheta}$ is additive. Also, by (4) and Example 2.2 we have $\overset{\circ}{\boldsymbol{v}} = \widehat{\boldsymbol{v}} = \boldsymbol{v}$, hence null players of $\overset{\circ}{\boldsymbol{v}}$ and $\boldsymbol{v}$ coincide. Consequently, $\boldsymbol{\vartheta}$ satisfies the axioms of the Shapley value.

**3rdSTEP** :

In particular, the fifth axiom (formula (3) in Definition 4.2) can be replaced by

$$(5) \qquad \boldsymbol{\gamma}^{\boldsymbol{V}}(\boldsymbol{\psi}(\boldsymbol{V})) = \Phi(\overset{\circ}{\boldsymbol{v}}^{\boldsymbol{\sigma},\boldsymbol{V}}) = \Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{\tau},\boldsymbol{V}}) \;\; (\boldsymbol{V} \in \mathfrak{V}^n).$$

As $\boldsymbol{\gamma}^{\boldsymbol{V}} = \widehat{\boldsymbol{\kappa}}^{\boldsymbol{V}}$, this implies

$$(6) \qquad \boldsymbol{\psi}(\boldsymbol{V}) \;=\; (\widehat{\boldsymbol{\kappa}}^{\boldsymbol{V}})^{-1}\left(\Phi(\widehat{\boldsymbol{v}}^{\boldsymbol{\tau},\boldsymbol{V}})\right) \;=\; \boldsymbol{\chi}(\boldsymbol{V}) \;\; (\boldsymbol{V} \in \mathfrak{V}^n).$$

**q.e.d.**

**Remark 4.4.** *Our solution or value respectively is point valued and does not require a fixed point theorem. With this respect it differs from other NTU–Shapley values given previously. See e.g.* Aumann *[1], who axiomatizes the Shapley's transfer value and* Hart *[10], who axiomatizes Harsanyi's [9] NTU–value. See also* de Clippel-Peters-Zank *[3] who discuss the dependence on regularity conditions and add axiomatizations for several values including the Consistent NTU–Shapley value of* Maschler–Owen *[12].*

*The present value yields results differing from the above mentioned concepts. To demonstrate this, we refer to Example 5.1 in [18]. The example shows a bargaining problem for which the Nash solution and our solution differ. However, a bargaining problem is a particular 3–person NTU–game, hence the value $\boldsymbol{\chi}$ is different from all values that (by definition or via the axiomatic) coincide with the Nash solution for bargaining problems. This is so for the Shapley NTU value (although e.g. Aumann's axiomatization works for the smooth case only) and Harsanyi's NTU value.*

*As for the Maschler–Owen consistent value, we turn to the example provided by these authors in Section 6, p 403 of [12]. Their 3–person NTU game is essentially given by $\boldsymbol{V}(\{1,2\}) = \{\boldsymbol{x} \in \mathbb{R}^2_+ \mid 2x_1 + 3x_2 \leq 180\}$ and $\boldsymbol{V}(\{1,2,3\}) = \{\boldsymbol{x} \in \mathbb{R}^3_+ \mid x_1 + x_2 + x_3 \leq 120\}$, all other coalitions receive $0$ at most. Maschler–Owen come up with the value $(55, 50, 15)$.*

*In our present framework, the TU–game $\widehat{\boldsymbol{v}}$ is given by $\widehat{\boldsymbol{v}}(\{1,2\}) = \sqrt{60 \times 90} = 30\sqrt{6} \sim 73.485$ and $\widehat{\boldsymbol{v}}(\{1,2,3\}) = 120$. The Shapley value of this sidepayment game is $\sim (52.25, 52.25, 15.5)$ and this is the value $\boldsymbol{\chi}(\boldsymbol{V})$ as well. So our solution treats the third player almost in the same way, but considers the first two players to be symmetric. It can be seen that the Shapley NTU–value assigns more to the third player but treats players 1 and 2 symmetrically as well.*

*Our solution concept is "constructive". The maximal faces of a cephoid can be determined by a recursive procedure (see [19]), thus the scaling factor and the surface measure are attainable by computational methods. We do not know as yet whether these problems are "NP–hard".*

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**Remark 4.5.** *For $n = 2$ every convex polyhedral bargaining set is a cephoid. For $n \geq 2$ the variety of convex polyhedra is much greater; this topic is discussed extensively within the framework of Convex Geometry. In particular, the decomposition of polyhedra into (nonhomothetic) summands is an issue. This constitutes* *by no means a simple problem, see* SCHNEIDER*[30]. E.g., Corollary 3.2.13 on p. 152 of [30] implies that a convex polyhedron in* $\mathbb{R}^3$*, all two–dimensional faces of which are triangles, is "indecomposable", i.e., does not allow for (nonhomothetic) summands. In particular, it cannot be written as a sum of deGua Simplices, thus it is not a Cephoid. The simplest example that comes to mind is the convex hull of the vectors* $\mathbf{0}, \boldsymbol{e}^1, \boldsymbol{e}^2, \boldsymbol{e}^3$, and $(1, 1, 1)$ *which reflects a small tetrahedron affixed to the Pareto surface of the unit simplex. The "typical" cephoid in* 3 *dimensions must have faces that are triangles* and *rhombi.*

*It is an open question whether with a suitable topology on Pareto surfaces, cephoids are dense within a large class of convex, compact, and comprehensive ("bargaining") sets. In two dimensions this is well known assuming that no line segments parallel to an axis appears in the Pareto surface. In* $n$ *dimensions a more restrictive condition may be necessary. Quite likely, the surface measure can be extended to certain smooth Pareto surfaces, a plausible candidate would be obtained by integrating the "volume element" over the Pareto surface, vaguely*

$$\boldsymbol{\iota}_{\Delta}(\partial U) \ := \ \int\limits_{\partial U} \sqrt[n]{(dx_1 \cdot \ldots \cdot dx_n)^{n-1}} \ ,$$

*However, a precise meaning has to be assigned to the "volume" element. Moreover, the continuity properties of the MPS value are to be studied carefully. This task exceeds the scope of our present framework.*

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# Bibliography


[1] R. J. Aumann, *An axiomatization of the non–transferable utility value*, **Econometrica 53** (1985), 599 – 612.

[2] R. E. Caves, J. A. Frankel, and R. W. Jones, *World trade and payments: an Introduction. 6th Edition*, The Harper Collins series in Economics, vol. 168, Harper Collins College Publishers, New York, NY, 1992.

[3] G. de Clippel, H. Peters, and H. Zank, *Axiomatizing the Harsanyi solution, the symmetric egalitarian solution and the consistent solution for NTU–games*, **International Journal of Game Theory 33** (2004), 145–158.

[4] J. P. de Gua de Malves, *Article on : The n-dimensional Pythagorean Theorem*, in: Histoire de l'Academie des Sciences pour l'ann'ee 1783, cf. Guggenheimer, Mathematical Reviews 1136610, 1783.

[5] G. Debreu, *Theory of value: An axiomatic analysis of economic equilibrium*, Wiley, New York, 1959.

[6] Calvo E. and E. Gutiérrez, *Extension of the Perles–Maschler solution to n-person bargaining games*, **International Journal of Game Theory 23** (1994), 325–346.

[7] G. Ewald, *Combinatorial convexity and algebraic geometry*, Graduate Texts in Mathematics, vol. 168, Springer Verlag, New York, Berlin, Heidelberg, 1996.

[8] F. D. Graham, *The Theory of International Values*, Princeton University Press, Princeton, NY, 1948.

[9] J.C. Harsanyi, *A bargaining model for the cooperative n-person game*, Contributions to the Theory of Games IV (A.W.Tucker and R.D. Luce, eds.), Princeton University Press, Princeton, 1959, pp. 325 – 355.

[10] S. Hart, *An axiomatization of Harsanyi's non-transferable utility solution*, **Econometrica** (1985), 1295 – 1313.

[11] W. R. Jones, *Triangles and trade: Lionel McKenzie as a trade theorist*, Preprint, University of Rochester, 2004.

[12] M. Maschler and G. Owen, *The consistent Shapley value for hyperplane games*, **International Journal of Game Theory 18** (1989), 389–407.

[13] M. Maschler and M. A. Perles, *The present status of the superadditive solution*, Essays in Game Theory and Mathematical Economics, Bibliographisches Institut, Mannheim, 1981.

[14] L. McKenzie, *Specialization and efficiency in world production*, **Review of Economic Studies 21** (1954), 165–180.

[15] D. Pallaschke and J. Rosenmüller, *Cephoids: Minkowski sums of prisms*, Working Paper 360, Institute of Mathematical Economics, University of Bielefeld, 2004.

[16] ———, *A superadditive solution for cephoidal bargaining problems*, Working Paper 361, Institute of Mathematical Economics, University of Bielefeld, 2004.

[17] D. Pallaschke and J. Rosenmüller, *Computing the Minkowski sum of prisms*, **Journal of Global Optimization 35** (2006), 321–341.

[18] ———, *A superadditive solution for cephoidal bargaining problems*, **International Journal of Game Theory 35** (2007), 569–590.

[19] ———, *Cephoids: Minkowski sums of deGua simplexes*, **Journal of Optimization 59(4)** (2010), 515–540.

[20] D. Pallaschke and R. Urbański, *Pairs of compact convex sets — fractional arithmetic with convex sets*, Mathematics and its Applications, vol. 548, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2002.

[21] M. A. Perles, *Non–existence of superadditive solutions for 3–person games*, **International Journal of Game Theory 11** (1982), 151–161.

[22] M. A. Perles and M. Maschler, *The superadditive solution for the Nash bargaining game*, **International Journal of Game Theory 10** (1981), 163–193.

[23] D. Ricardo, *On the Principles of Political Economy and Taxation*, John Murray, Albemarle-Street, London, 1817.

[24] W.F. Richter and J. Rosenmüller, *Efficient specialization in Ricardian Production*, **German Economic Review 13** (2012), 117–241.

[25] J. Rosenmüller, *Kooperative Spiele und Märkte*, **Lecture Notes in Operations Research and Mathematical Systems 53** (1971), 152.

[26] ______, *Some topics in cooperative game theory*, **Modern Applied Mathematics** (1981), 557 – 592.

[27] ______, *Game theory: Stochastics, information, strategies and cooperation*, Theory and Decision Library, C, vol. 25, Kluwer Academic Publishers Boston, Dordrecht, London, 2000.

[28] ______, *The Maschler-Perles solution: Two simple proofs for superadditivity*, **Economics Bulletin** (2004), 1–8.

[29] J. Rosenmüller and D. Pallaschke, *Cephoids: Duality and reference vectors*, **Pacific Journal of of Optimization 8** (2012), no. 1, 157 – 184.

[30] R. Schneider, *Convex bodies: The Brunn–Minkowski theory*, Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, UK, 1993.

[31] L. S. Shapley, *A value for n–person games*, **Annals of Math. Studies**.

[32] L.S. Shapley, *Utility comparisons and the theory of games*, La Decision: Aggregation et Dynamique des Ordres de Preference (Paris), Editions du Centre National de la Recherche Scientifique, 1969, pp. 251 – 263.

[33] J. von Neumann and O. Morgenstern, *Theory of games and economic behavior*, Princeton University Press, Princeton, 1944,1947,1953, *This basic volume containes the foundation of Game Theory* .

[34] G.M. Ziegler, *Lectures on polytopes*, Graduate Texts in Mathematics, vol. 152, Springer Verlag, Heidelberg, Berlin, New York, 1995.

# Index