Center for
Mathematical Economics
Working Papers

552

February 2016

Convex vNM–Stable Sets for a Semi Orthogonal Game

Part V:

All Games have vNM–Stable Sets

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http://www.imw.uni-bielefeld.de/wp/
ISSN: 0931-6558

Abstract

Within this paper we conclude the treatise of vNM–Stable Sets for (cooperative) linear production games with a continuum of players. The paper resumes a series of presentations of this topic, for Part I,II,III,IV, see [1], [2], [3] [4]

The framework has been outlined previously. The coalitional function is generated by $r+1$ "production factors" (non atomic measures). r factors are given by orthogonal probabilities ("cornered" production factors) establishing the core of the game. Factor $r+1$ (the "central" production factor) is represented by a nonantomic measure with carrier "across the corners" of the market. I.e., this factor is available in excess and the representing measure is no element of the core of the game.

Generalizing our set-up, we assume now that the "central" production factor is represented by an arbitrary measure not necessarily of step function character. Then the existence theorem is achieved by an approximation procedure.

Again it turns out that there is a (not necessarily unique) imputation outside of the core which, together with the core generates the vNM–Stable Set as the convex hull. Significantly, this additional imputation can be seen as a truncation of the "central" distribution, i.e., the $r+1^{st}$ production factor. Hence, there is a remarkable similarity *mutatis mutandis* regarding the Characterization Theorem that holds true for the "purely orthogonal case" ([5],[6]). This justifies to use the term "Standard vNM–Stable Set" in the presence of a central production factor.

1 Continuous vs. Uniform Models

We continue the discussion of convex vNM–Stable sets for a Semi Orthogonal Game ([1], [2], [3], [4]). Our notation is adapted accordingly. Thus, we consider a (cooperative) ***game*** with a continuum of players, i.e., a triple $(\boldsymbol{I}, \underline{\underline{\mathbf{F}}}, \boldsymbol{v})$ where $\boldsymbol{I} := [0, r)$ reflects the ***players***, $\underline{\underline{\mathbf{F}}}$ is the $\sigma-$field of (Borel) measurable sets (the ***coalitions***), and $\boldsymbol{v}$ (the ***coalitional function***) is a mapping $\boldsymbol{v} : \underline{\underline{\mathbf{F}}} \to \mathbb{R}_+$ which is absolutely continuous w.r.t. Lebesgue measure $\boldsymbol{\lambda}$. We focus on "linear production games", that is, $\boldsymbol{v}$ is described by finitely many measures $\boldsymbol{\lambda}^\rho, (\rho \in \{0, 1, \ldots, r\})$ via

$$\boldsymbol{v}(S) := \min\{\boldsymbol{\lambda}^\rho(S) \mid \rho \in \{0, 1, \ldots, r\}\} \quad (S \in \underline{\underline{\mathbf{F}}}), \tag{1.1}$$

for short,

$$\boldsymbol{v} = \bigwedge\{\boldsymbol{\lambda}^0, \boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r\} = \bigwedge_{\rho \in \mathbf{R}_0} \boldsymbol{\lambda}^\rho , \tag{1.2}$$

(with $\mathbf{R} = \{1, \ldots, r\}$ and $\mathbf{R}_0 = \mathbf{R} \cup \{0\}$). The measures $\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r$ are orthogonal copies of Lebesgue measure on the carriers $\boldsymbol{C}^\rho = (\rho - 1, \rho]$ $(\rho = 1, \ldots, r)$. The "central measure" $\boldsymbol{\lambda}^0$ is absolutely continuous w.r.t. Lebesgue measure such that $\boldsymbol{\lambda}^0(\boldsymbol{I}) > 1$.

The measures generating $\boldsymbol{v}$ are seen as ***production factors*** or ***commodities*** (as $\boldsymbol{v}$ can be interpreted as a production game or a market game). Hence $\{\boldsymbol{\lambda}^\rho\}_{\rho \in \mathbf{R}}$ represent the "cornered" production factors/commodities while $\boldsymbol{\lambda}^0$ represents a "central" production factor/commodity.

We write

$$l^0_\rho := \text{ess inf}_{\boldsymbol{C}^\rho} \dot{\boldsymbol{\lambda}}^0 \quad (\rho \in \mathbf{R}), \quad l^\star_\sigma := \sum_{\rho \in \mathbf{R} \setminus \{\sigma\}} l^0_\rho \tag{1.3}$$

and require

$$l^0_\rho > 0 \quad (\rho \in \mathbf{R}) , \quad \sum_{\rho \in \mathbf{R}} l^0_\rho < 1 , \tag{1.4}$$

which implies

$$l^0_\sigma < 1 - l^\star_\sigma < 1 \quad (\sigma \in \mathbf{R}) . \tag{1.5}$$

We claim without proof that these conditions eventually can be disposed of. For , if the second condition in (1.4) is violated, then the core is the (unique) vNM–Stable Set of the game. The first one can be removed by some kind of limiting procedure as will be presented in SECTION 4 and we do not want to overburden this text.

The density $\dot{\boldsymbol{\lambda}}^0$ is not necessarily piece-wise constant as in our previous set–ups. Somewhat sloppily we will, therefore, refer to the present setup as to a

continuous model. The previous model in which the central measure has a step function density shall – in accordance with *Definition 2.3.* of *Part IV* – be referred to as the ***uniform model***.

With any continuous model we will derive uniform models with central measure $\boldsymbol{\lambda}^{(t)}$ derived from $\boldsymbol{\lambda}^0$ via the conditional expectations of $\boldsymbol{\lambda}^0$ w.r.t some finite (Borel) field (see SECTION 2 of [4]).

More precisely, given some central measure $\boldsymbol{\lambda}^0$, we consider for $t \in \mathbb{N}$ a *uniform* model as follows. Let $\mathbf{T} = \mathbf{T}^{(t)} := \{1, \ldots, rt\})$ and let

$$(1.6) \qquad {}^{(t)}\underline{\underline{\mathbf{D}}} = \left\{ {}^{(t)}\boldsymbol{D}^{\tau} \right\}_{\tau \in \mathbf{T}}$$

be a family of subsets of $\boldsymbol{I}$ such that

$$(1.7) \qquad \boldsymbol{C}^{\rho} = \bigcup_{\tau = (\rho-1)t+1}^{\rho t} {}^{(t)}\boldsymbol{D}^{\tau}$$

is partitioned into t pieces of equal Lebesgue measure,

$$(1.8) \qquad \lambda^t_{\tau} = \boldsymbol{\lambda}({}^{(t)}\boldsymbol{D}^{\tau}) = \frac{1}{t} \quad (\tau \in \mathbf{T} = \mathbf{T}^{(t)}) \ .$$

Let $\underline{\underline{\mathbf{F}}}^{(t)}$ denote the σ–algebra generated by the atoms ("blocks") of ${}^{(t)}\underline{\underline{\mathbf{D}}}$. The description of a uniform model is completed by introducing a central measure $\boldsymbol{\lambda}^{(t)}$ with piece-wise constant density, adjusted to the family ${}^{(t)}\underline{\underline{\mathbf{D}}}$, or rather measurable w.r.t. $\underline{\underline{\mathbf{F}}}^{(t)}$. We choose the conditional measure

$$(1.9) \qquad \boldsymbol{\lambda}^{(t)} := P\left\{ \boldsymbol{\lambda}^0 \,\middle|\, \underline{\underline{\mathbf{F}}}^{(t)} \right\}(\bullet) \ ,$$

the density of which is conditional expectation of $\dot{\boldsymbol{\lambda}}^0$ w.r.t. $\underline{\underline{\mathbf{F}}}^{(t)}$, i.e.,

$$(1.10) \qquad \dot{\boldsymbol{\lambda}}^{(t)}(\star) = E\left\{ \dot{\boldsymbol{\lambda}}^0 \,\middle|\, \underline{\underline{\mathbf{F}}}^{(t)} \right\}(\star) = \sum_{\tau \in \mathbf{T}} \frac{\boldsymbol{\lambda}^0({}^{(t)}D^{\tau})}{\boldsymbol{\lambda}({}^{(t)}D^{\tau})} \mathbb{1}_{{}^{(t)}D^{\tau}}(\star) \ ,$$

where $\mathbb{1}_{\bullet}$ denotes the indicator function of a coalition. Now $\{\boldsymbol{\lambda}^{\rho}\}_{\rho \in \mathbf{R}}$ and $\boldsymbol{\lambda}^{(t)}$ generate a coalitional function

$$(1.11) \qquad \boldsymbol{v}^{(t)} := \boldsymbol{\lambda}^{(t)} \wedge \bigwedge_{\rho \in \mathbf{R}} \boldsymbol{\lambda}^{\rho}$$

and hence a game $(\boldsymbol{I}, \underline{\underline{\mathbf{F}}}, \boldsymbol{v}^{(t)})$. At this stage, there is a considerable freedom in choosing these partitions, essentially we want the partitioning sets to have equal measure.

By definition, $\boldsymbol{\lambda}^{(t)}$ coincides with $\boldsymbol{\lambda}^0$ on $\underline{\underline{\mathbf{F}}}^{(t)}$. The measures $\boldsymbol{\lambda}^{\rho}$ $(\rho \in \mathbf{R})$ do not change essentially when conditioned to $\underline{\underline{\mathbf{F}}}^{(t)}$; consequently in particular

$$(1.12) \qquad \boldsymbol{v}^{(t)}(S) = \boldsymbol{v}(S) \quad (S \in \underline{\underline{\mathbf{F}}}^{(t)}) \ .$$

This way, we have connected a continuous model with families of uniform models ${}^{(t)}\underline{\underline{\mathbf{D}}}$ as established in *Definition 2.3.* of *Part IV.*

Given some uniform model recall the definition of ***relevant vectors*** (see SECTION 3 of [1]) $\boldsymbol{a}^{\odot}$, $\boldsymbol{a}^{\oplus}$, and $\boldsymbol{a}^{\ominus}$. We now augment the notation writing

$$ {}^{(t)}\boldsymbol{a}^{\odot}, \quad {}^{(t)}\boldsymbol{a}^{\oplus}, \quad \text{and} \quad {}^{(t)}\boldsymbol{a}^{\ominus}. $$

We will then speak of "vectors relevant w.r.t ${}^{(t)}\underline{\underline{\mathbf{D}}}$ ". Also, it may be necessary to distinguish between dominance regarding $\boldsymbol{v}$ and regarding $\boldsymbol{v}^{(t)}$. We use the well established notations $\boldsymbol{\xi}\,\mathrm{dom}_S^{\boldsymbol{v}}\,\boldsymbol{\eta}$ and $\boldsymbol{\xi}\,\mathrm{dom}_S^{\boldsymbol{v}^{(t)}}\,\boldsymbol{\eta}$.

We start out with a useful detail:

Lemma 1.1. *Let $t \in \mathbb{N}$ and let ${}^{(t)}\underline{\underline{\mathbf{D}}}$ constitute a uniform model. Let $\boldsymbol{\xi}, \boldsymbol{\eta}$ be imputations and let $\boldsymbol{\xi}^{(t)} := P\left\{\boldsymbol{\xi} \,\middle|\, \underline{\underline{\mathbf{F}}}^{(t)}\right\}(\bullet)$. Let S be a coalition. If*

1. *$S \in \underline{\underline{\mathbf{F}}}^{(t)}$,*
2. *$\{\boldsymbol{\xi} > \boldsymbol{\eta}\} \in \underline{\underline{\mathbf{F}}}^{(t)}$,*

then the relations

$$ \boldsymbol{\xi}\,\mathrm{dom}_S^{\boldsymbol{v}}\,\boldsymbol{\eta} \quad , \quad \boldsymbol{\xi}\,\mathrm{dom}_S^{\boldsymbol{v}^{(t)}}\,\boldsymbol{\eta} \quad \textit{and} \quad \boldsymbol{\xi}^{(t)}\,\mathrm{dom}_S^{\boldsymbol{v}^{(t)}}\,\boldsymbol{\eta} $$

are equivalent.

Proof: $\boldsymbol{\xi}$ and $\boldsymbol{\xi}^{(t)}$ coincide on $\underline{\underline{\mathbf{F}}}^{(t)}$, hence by (1.12) we know that

$$ \boldsymbol{\xi}(S) \le \boldsymbol{v}(S) \quad \text{if and only if} \quad \boldsymbol{\xi}^{(t)}(S) \le \boldsymbol{v}^{(t)}(S)\ . \tag{1.13} $$

Denote $T_> := \{\overset{\bullet}{\boldsymbol{\xi}} > \overset{\bullet}{\boldsymbol{\eta}}\}$ and $T_\le := \{\overset{\bullet}{\boldsymbol{\xi}} \le \overset{\bullet}{\boldsymbol{\eta}}\}$. Then, as $T_> \in \underline{\underline{\mathbf{F}}}^{(t)}$, any atom ${}^{(t)}\boldsymbol{D}^{\tau}$ of $\underline{\underline{\mathbf{F}}}^{(t)}$ is either contained in $T_>$ or disjoint, i.e.,

$$ {}^{(t)}\boldsymbol{D}^{\tau} \subseteq T_> \text{ or } {}^{(t)}\boldsymbol{D}^{\tau} \subseteq T_\le \tag{1.14} $$

Assume now that $\boldsymbol{\xi}^{(t)} > \boldsymbol{\eta}$ holds true " on S ". Let ${}^{(t)}\boldsymbol{D}^{\tau}$ be an atom of S, i.e. ${}^{(t)}\boldsymbol{D}^{\tau} \subseteq S$. Then necessarily $\boldsymbol{\lambda}\left({}^{(t)}\boldsymbol{D}^{\tau} \cap \{\boldsymbol{\xi} > \boldsymbol{\eta}\}\right) > 0$. From (1.14) it follows that ${}^{(t)}\boldsymbol{D}^{\tau} \subseteq \{\boldsymbol{\xi} > \boldsymbol{\eta}\}$ and hence necessarily

$$ \boldsymbol{\xi} > \boldsymbol{\eta} \quad \text{on} \quad {}^{(t)}\boldsymbol{D}^{\tau} $$

Therefore, $\boldsymbol{\xi} > \boldsymbol{\eta}$ " on S " holds true, that is,

$$ \boldsymbol{\xi}^{(t)} > \boldsymbol{\eta} \ \text{ on } \ S \quad \text{implies} \quad \boldsymbol{\xi} > \boldsymbol{\eta} \ \text{ on } \ S. \tag{1.15} $$

On the other hand, if $\boldsymbol{\xi} > \boldsymbol{\eta}$ " on S " is the case, then nessarily $\boldsymbol{\xi} > \boldsymbol{\eta}$ on every atom ${}^{(t)}\boldsymbol{D}^{\tau}$ of S and hence $\boldsymbol{\xi}^{(t)} > \boldsymbol{\eta}$ on every such atom, which implies $\boldsymbol{\xi}^{(t)} > \boldsymbol{\eta}$ "on S ". Thus

$$ \boldsymbol{\xi} > \boldsymbol{\eta} \ \text{ on } \ S \quad \text{implies} \quad \boldsymbol{\xi}^{(t)} > \boldsymbol{\eta} \ \text{ on } \ S. \tag{1.16} $$

Now (1.13), (1.16), (1.15) as well as (1.12) prove the Lemma.

q.e.d.

Write $\vec{\boldsymbol{\lambda}}$ to denote the vectorvalued measure $(\boldsymbol{\lambda}(\bullet\cap C^1),\ldots,\boldsymbol{\lambda}(\bullet\cap C^r))$. Then, given some measurability, we can improve upon the notion of "ε–relevant coalitions" as follows.

Theorem 1.2 (The ε-free Inheritance Theorem). *Let $t\in\mathbb{N}$ and let ${}^{(t)}\underline{\underline{\mathbf{D}}}$ constitute a uniform model. Let $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ be imputations and let $S\in\underline{\underline{\mathbf{F}}}^{(t)}$ be a coalition such that $\boldsymbol{\xi}\,\mathrm{dom}_S^{\boldsymbol{v}^{(t)}}\,\boldsymbol{\eta}$. Assume that,*

1. *$\boldsymbol{\xi}$ is $\underline{\underline{\mathbf{F}}}^{(t)}$–measurable,*
2. *$T_> \;:=\; \{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\eta}}\} \;\in\; \underline{\underline{\mathbf{F}}}^{(t)}$.*

Then there exists a vector $\boldsymbol{a}^{(t)}$ relevant for $\boldsymbol{v}^{(t)}$ and a coalition $T^{\frac{1}{t}\boldsymbol{a}^{(t)}}$ such that the following holds true.

$$\vec{\boldsymbol{\lambda}}(T^{\frac{1}{t}\boldsymbol{a}^{(t)}}) = \frac{1}{t}\boldsymbol{a}^{(t)}\ , \tag{1.17}$$

and

$$\boldsymbol{\xi}\,\mathrm{dom}^{\boldsymbol{v}^{(t)}}_{T^{\frac{1}{t}\boldsymbol{a}^{(t)}}}\,\boldsymbol{\eta} \tag{1.18}$$

holds true.

In other words, one can dispose of the ε and its quantifiers as described in the Inheritance Theorem (Part I, Theorem 3.3, [1]).

Proof: 1stSTEP : Apply the Inheritance Theorem 3.3 of [1] to $\underline{\underline{\mathbf{F}}}^{(t)}$ and $\boldsymbol{v}^{(t)}$. Thereby find a relevant vector $\boldsymbol{a}^{(t)}$ and $\varepsilon_0>0$ such that, for any $\varepsilon<\varepsilon_0$, there is $T^{\varepsilon} \;=\; T^{\varepsilon\boldsymbol{a}^{(t)}} \subseteq S$ satisfying $\boldsymbol{\xi}\,\mathrm{dom}^{\boldsymbol{v}^{(t)}}_{T^{\varepsilon\boldsymbol{a}^{(t)}}}\,\boldsymbol{\eta}$. Clearly, $\varepsilon\le\frac{1}{t}$ as ${}^{(t)}\underline{\underline{\mathbf{D}}}$ constitutes the basic reference model.

2ndSTEP :

Presently, we focus on some relevant vector ${}^{(t)}\boldsymbol{a}^{\ominus}$ of the third type. It will be obvious how to proceed for the (simpler to treat) versions $\boldsymbol{a}^{\odot}$ and $\boldsymbol{a}^{\oplus}$ of the first and second type according to Theorem 3.5 of *Part I*.

Also we assume that the critical coordinate for the construction of ${}^{(t)}\boldsymbol{a}^{\ominus}$ is r. That is, ${}^{(t)}\boldsymbol{a}^{\ominus}$ is of the shape

$${}^{(t)}\boldsymbol{a}^{\ominus} = (1,\ldots,1,\ldots,1,\ldots\ldots,\alpha_r,\beta_r) \tag{1.19}$$

with

$$\begin{aligned}\alpha_r \;&=\; {}^{(t)}a^{\ominus}_{\widehat{\tau}_r} = \frac{(h_{\overline{\tau}_1}+\ldots+h_{\overline{\tau}_{r-1}}+h_{\overline{\overline{\tau}}_r})-1}{h_{\overline{\overline{\tau}}_r}-h_{\overline{\tau}_r}}\ ,\\ \beta_r \;&=\; {}^{(t)}a^{\ominus}_{\overline{\overline{\tau}}_r} = \frac{1-(h_{\overline{\tau}_1}+\ldots+h_{\overline{\tau}_r})}{h_{\overline{\overline{\tau}}_r}-h_{\overline{\tau}_r}}\ ;\end{aligned} \tag{1.20}$$

according to the definition of relevant vectors (Theorem 3.5. of *Part I*). Within the present framework, the the quantities in (1.20) carry an additional index, say ${}^{(t)}h_\tau$. Note that $\alpha_r + \beta_r = 1$. Let

$$\widehat{\boldsymbol{\tau}} \;=\; (\widehat{\tau}_1, \ldots, \widehat{\tau}_r, \overline{\overline{\tau}}_r)$$

denote the sequence corresponding to the positive coordinates of ${}^{(t)}\boldsymbol{a}^{\ominus}$. For $\rho \in \mathbf{R} \setminus \{r\}$, consider the coalition

$$T^{\frac{1}{t}\boldsymbol{a}^{(t)}} \cap \boldsymbol{C}^\rho \;=\; {}^{(t)}\boldsymbol{D}^{\widehat{\tau}_\rho} \quad (\rho \in \mathbf{R} \setminus \{r\}) \;. \tag{1.21}$$

As $T_> \in \underline{\underline{\mathbf{F}}}^{(t)}$, any atom/block ${}^{(t)}\boldsymbol{D}^\tau$ of $\underline{\underline{\mathbf{F}}}^{(t)}$ is either contained in $T_>$ or disjoint, i.e.,

$${}^{(t)}\boldsymbol{D}^\tau \subseteq T_> \text{ or } {}^{(t)}\boldsymbol{D}^\tau \subseteq T_\leq \tag{1.22}$$

From $T^{\varepsilon\boldsymbol{a}^{(t)}} \cap \boldsymbol{C}^\rho \subseteq {}^{(t)}\boldsymbol{D}^{\widehat{\tau}_\rho}$ it follows that $\boldsymbol{\lambda}({}^{(t)}\boldsymbol{D}^{\widehat{\tau}} \cap \{\boldsymbol{\xi} > \boldsymbol{\eta}\}) > 0$. Then, according to (1.22), ${}^{(t)}\boldsymbol{D}^{\widehat{\tau}_\rho} \subseteq \{\boldsymbol{\xi} > \boldsymbol{\eta}\}$. Consequently,

$$T^{\frac{1}{t}\boldsymbol{a}^{(t)}} \cap \boldsymbol{C}^\rho \;=\; {}^{(t)}\boldsymbol{D}^{\widehat{\tau}_\rho} \subseteq \{\boldsymbol{\xi} > \boldsymbol{\eta}\} \tag{1.23}$$

3rdSTEP : Next, for $\rho = r$ and $\tau = \widehat{\tau}_r, \overline{\overline{\tau}}_r$ the argument has to be adapted. Again, $T^{\varepsilon\boldsymbol{a}^{(t)}} \cap \boldsymbol{C}^r \subseteq {}^{(t)}\boldsymbol{D}^{\widehat{\tau}_\rho} \cup {}^{(t)}\boldsymbol{D}^{\overline{\overline{\tau}}_r}$ and $\boldsymbol{\lambda}({}^{(t)}\boldsymbol{D}^\tau \cap \{\boldsymbol{\xi} > \boldsymbol{\eta}\}) > 0$ implies

$${}^{(t)}\boldsymbol{D}^\tau \subseteq \{\boldsymbol{\xi} > \boldsymbol{\eta}\} \quad (\tau = \widehat{\tau}_r, \overline{\overline{\tau}}_r).$$

However, in this case we cannot choose the full atom in order to construct $T^{\frac{1}{t}\boldsymbol{a}^{(t)}}$. Instead, we choose an *arbitrary* subset $T^{\frac{1}{t}\alpha_r}$ of ${}^{(t)}\boldsymbol{D}^{\widehat{\tau}}$ with measure $\frac{1}{t}\alpha_r$ and an *arbitrary* subset $T^{\frac{1}{t}\beta_r}$ of ${}^{(t)}\boldsymbol{D}^{\overline{\overline{\tau}}}$ with measure $\frac{1}{t}\beta_r$ and define $T^{\frac{1}{t}\boldsymbol{a}^{(t)}}$ via

$$T^{\frac{1}{t}\boldsymbol{a}^{(t)}} \cap \boldsymbol{C}^\rho \;:=\; T^{\frac{1}{t}\alpha_r} \cup T^{\frac{1}{t}\beta_r} \;.$$

Then of course

$${}^{(t)}\boldsymbol{D}^\tau \cap T^{\frac{1}{t}\boldsymbol{a}^{(t)}} \;\subseteq\; {}^{(t)}\boldsymbol{D}^\tau \;\subseteq \{\boldsymbol{\xi} > \boldsymbol{\eta}\} \quad (\tau = \widehat{\tau}_r, \overline{\overline{\tau}}_r) \;, \tag{1.24}$$

4thSTEP :

Now,

$$\boldsymbol{\lambda}(T^{\frac{1}{t}\boldsymbol{a}^{(t)}}) \;=\; \frac{1}{t}(\alpha_r + \beta_r) \;=\; \frac{1}{t\varepsilon}(\alpha_r + \beta_r)\boldsymbol{\lambda}(T^{\varepsilon\boldsymbol{a}^{(t)}})$$

and

$$\boldsymbol{\lambda}^0(T^{\frac{1}{t}\boldsymbol{a}^{(t)}}) \;=\; \frac{1}{t}(\alpha_r + \beta_r) \;=\; \frac{1}{t}\left(1 - \sum_{\rho \in \mathbf{R}\setminus\{r\}} h^{(t)}_{\widehat{\tau}_\rho}\right) \;=\; \frac{1}{t\varepsilon}\boldsymbol{\lambda}^0(T^{\varepsilon\boldsymbol{a}^{(t)}})$$

and it follows that

$$\overrightarrow{\boldsymbol{\lambda}}(T^{\frac{1}{t}\boldsymbol{a}^{(t)}}) \;=\; \frac{1}{t\varepsilon}\overrightarrow{\boldsymbol{\lambda}}(T^{\varepsilon\boldsymbol{a}^{(t)}}) \quad , \quad \boldsymbol{\lambda}^0(T^{\frac{1}{t}\boldsymbol{a}^{(t)}}) \;=\; \frac{1}{t\varepsilon}\boldsymbol{\lambda}^0(T^{\varepsilon\boldsymbol{a}^{(t)}}) \tag{1.25}$$

and hence

$$\boldsymbol{v}(T^{\frac{1}{t}\boldsymbol{a}^{(t)}}) \;=\; \frac{1}{t\varepsilon}\boldsymbol{v}(T^{\varepsilon\boldsymbol{a}^{(t)}}) \;=\; \frac{1}{t}\,. \tag{1.26}$$

As $\boldsymbol{\xi}$ is measurable w.r. $\underline{\underline{\mathbf{F}}}^{(t)}$ (i.e., has constant density on the atoms), we have $\boldsymbol{\xi}(T^{\frac{1}{t}\boldsymbol{a}^{(t)}}) = \frac{1}{t\varepsilon}\boldsymbol{\xi}(T^{\varepsilon\boldsymbol{a}^{(t)}})$. Hence,

$$\boldsymbol{\xi}(T^{\frac{1}{t}\boldsymbol{a}^{(t)}}) = \frac{1}{t\varepsilon}\boldsymbol{\xi}(T^{\varepsilon\boldsymbol{a}^{(t)}}) \leq \frac{1}{t\varepsilon}\boldsymbol{v}^{(t)}(T^{\varepsilon\boldsymbol{a}^{(t)}}) = \boldsymbol{v}^{(t)}(T^{\frac{1}{t}\boldsymbol{a}^{(t)}}). \tag{1.27}$$

On the other hand by (1.23) and (1.24) ,

$$\boldsymbol{\xi} > \boldsymbol{\eta} \text{ on } T^{\frac{1}{t}\boldsymbol{a}^{(t)}}. \tag{1.28}$$

Combining (1.27) and (1.28), we have

$$\boldsymbol{\xi}\,\mathrm{dom}_{\frac{1}{t}T^{\boldsymbol{a}^{(t)}}}\,\boldsymbol{\eta},$$

q.e.d.

Remark 1.3. We emphasize that within the constructions in the $2^{nd}STEP$ and the $3^{rd}STEP$ the choice of

$$T^{\frac{1}{t}\boldsymbol{a}^{(t)}} \cap \boldsymbol{C}^r \;=\; T^{\frac{1}{t}\alpha_r} \cup T^{\frac{1}{t}\beta_r} \subseteq \{\boldsymbol{\xi} > \boldsymbol{\eta}\} \tag{1.29}$$

is arbitrary up to generating the correct measures α_r and β_r. This leads directly to formulating the next Theorem.

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Theorem 1.4. *Let $t \in \mathbb{N}$ and let ${}^{(t)}\underline{\underline{\mathbf{D}}}$ constitute a uniform model. Let $\boldsymbol{\xi}$ be an imputation and let $\boldsymbol{\xi}^{(t)} \;=\; \mathbb{P}\left\{\boldsymbol{\xi} \,\middle|\, \underline{\underline{\mathbf{F}}}^{(t)}\right\}$. Let $\boldsymbol{\eta}$ be an imputation and $S \in \underline{\underline{\mathbf{F}}}^{(t)}$ be a coalition such that $\boldsymbol{\xi}^{(t)}\,\mathrm{dom}_S^{\boldsymbol{v}^{(t)}}\,\boldsymbol{\eta}$. Assume that $T_> \;:=\; \{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\eta}}\}$ is $\underline{\underline{\mathbf{F}}}^{(t)}$–measurable . Then there exists a vector $\boldsymbol{a}^{(t)}$ relevant for ${}^{(t)}\underline{\underline{\mathbf{D}}}$ and a coalition $T^{\frac{1}{t}\boldsymbol{a}^{\ominus(t)}}$ such that*

$$\vec{\boldsymbol{\lambda}}(T^{\frac{1}{t}\boldsymbol{a}^{(t)}}) = \frac{1}{t}\boldsymbol{a}^{(t)}\;, \tag{1.30}$$

and

$$\boldsymbol{\xi}\,\mathrm{dom}^{\boldsymbol{v}}_{T^{\frac{1}{t}\boldsymbol{a}^{(t)}}}\,\boldsymbol{\eta} \tag{1.31}$$

holds true.

Proof:

1st STEP :

Run through the steps of the proof of the previous theorem. For all the atoms $\boldsymbol{v}$ and $\boldsymbol{v}^0$ coincide. For $\boldsymbol{C}^r \cap T^{\frac{1}{t}\boldsymbol{a}^{(t)}}$ both $\boldsymbol{v}$ and $\boldsymbol{v}^0$ coincide because of

(1.25) - an this still allows for a free choice of this set up to generating the correct measures. All one has to do is in the $4^{th} STEP$ to arrange for $T^{\frac{1}{t}\alpha_r}$ and $T^{\frac{1}{t}\beta_r}$ in a way that

$$\boldsymbol{\xi}(T^{\frac{1}{t}\alpha_r}) = \boldsymbol{\xi}^{(t)}(T^{\frac{1}{t}\alpha_r}) \quad \text{and} \quad \boldsymbol{\xi}(T^{\frac{1}{t}\beta_r}) = \boldsymbol{\xi}^{(t)}(T^{\frac{1}{t}\beta_r}) \tag{1.32}$$

is satisfied. However, $\boldsymbol{\xi}^{(t)}$ is the conditional probability of $\boldsymbol{\xi}$. Therefore, by taking equal chunks out of $\left\{\boldsymbol{\xi} > \boldsymbol{\xi}^{(t)}\right\} \cap {}^{(t)}\boldsymbol{D}^{\widehat{\tau}_\rho}$ and $\left\{\boldsymbol{\xi} < \boldsymbol{\xi}^{(t)}\right\} \cap {}^{(t)}\boldsymbol{D}^{\widehat{\tau}_\rho}$ if necessary, we can achieve the desired form of $T^{\frac{1}{t}\alpha_r}$ and similarly for $T^{\frac{1}{t}\beta_r}$.

2ndSTEP :

We are going to make this more precise (sometimes omitting an index $^{(t)}$ for clarity).

Indeed, as $\frac{1}{t}h^{(t)}_{(\widehat{\tau}_\rho)}$ is the conditional probability of $\boldsymbol{\lambda}^0$ over $\boldsymbol{D}^{\widehat{\tau}_\rho}$ it follows that

$$\int\limits_{\left\{\dot{\boldsymbol{\lambda}}^0 > h^{(t)}_{(\widehat{\tau}_\rho)}\right\}} \dot{\boldsymbol{\lambda}}^0 - h^{(t)}_{(\widehat{\tau}_\rho)} \;=\; \int\limits_{\left\{\dot{\boldsymbol{\lambda}}^0 < h^{(t)}_{(\widehat{\tau}_\rho)}\right\}} h^{(t)}_{(\widehat{\tau}_\rho)} - \dot{\boldsymbol{\lambda}}^0 \,. \tag{1.33}$$

Now, for $\varepsilon > 0$ we can choose

$$F^\varepsilon_> \subseteq \left\{\dot{\boldsymbol{\lambda}}^0 > h^{(t)}_{(\widehat{\tau}_\rho)}\right\} \quad , \quad F^\varepsilon_= \subseteq \left\{\dot{\boldsymbol{\lambda}}^0 = h^{(t)}_{(\widehat{\tau}_\rho)}\right\} \quad , \quad F^\varepsilon_< \subseteq \left\{\dot{\boldsymbol{\lambda}}^0 < h^{(t)}_{(\widehat{\tau}_\rho)}\right\}$$

such that

$$\boldsymbol{\lambda}\left(F^\varepsilon_> \cup F^\varepsilon_= \cup F^\varepsilon_<\right) = \varepsilon \tag{1.34}$$

and

$$\int\limits_{F^\varepsilon_>} \left(\dot{\boldsymbol{\lambda}}^0 - h^{(t)}_{(\widehat{\tau}_\rho)}\right) d\boldsymbol{\lambda} \;=\; \int\limits_{F^\varepsilon_<} \left(h^{(t)}_{(\widehat{\tau}_\rho)} - \dot{\boldsymbol{\lambda}}^0\right) d\boldsymbol{\lambda} \,. \tag{1.35}$$

via Ljapounoffs Theorem. Then we have

(1.36)

$$\begin{aligned}
&\int\limits_{F^\varepsilon_>} \dot{\boldsymbol{\lambda}}^0 d\boldsymbol{\lambda} + \int\limits_{F^\varepsilon_=} \dot{\boldsymbol{\lambda}}^0 d\boldsymbol{\lambda} + \int\limits_{F^\varepsilon_<} \dot{\boldsymbol{\lambda}}^0 d\boldsymbol{\lambda} \\
&= \int\limits_{F^\varepsilon_> \cup F^\varepsilon_= \cup F^\varepsilon_<} h^{(t)}_{(\widehat{\tau}_\rho)} d\boldsymbol{\lambda} \\
&\quad + \int\limits_{F^\varepsilon_>} \left(\dot{\boldsymbol{\lambda}}^0 - h^{(t)}_{(\widehat{\tau}_\rho)}\right) d\boldsymbol{\lambda} + \int\limits_{F^\varepsilon_=} \left(\dot{\boldsymbol{\lambda}}^0 d\boldsymbol{\lambda} - h^{(t)}_{(\widehat{\tau}_\rho)}\right) d\boldsymbol{\lambda} + \int\limits_{F^\varepsilon_<} \left(\dot{\boldsymbol{\lambda}}^0 d\boldsymbol{\lambda} - h^{(t)}_{(\widehat{\tau}_\rho)}\right) d\boldsymbol{\lambda} \\
&= \varepsilon h^{(t)}_{(\widehat{\tau}_\rho)} d\boldsymbol{\lambda} + \int\limits_{F^\varepsilon_>} \left(\dot{\boldsymbol{\lambda}}^0 - h^{(t)}_{(\widehat{\tau}_\rho)}\right) d\boldsymbol{\lambda} - \int\limits_{F^\varepsilon_<} \left(\dot{\boldsymbol{\lambda}}^0 d\boldsymbol{\lambda} - h^{(t)}_{(\widehat{\tau}_\rho)}\right) d\boldsymbol{\lambda} \\
&= \varepsilon h^{(t)}_{(\widehat{\tau}_\rho)} d\boldsymbol{\lambda} \,.
\end{aligned}$$

the last equation following from (1.35). That is, for $\varepsilon = \frac{\alpha_r}{t}$ we obtain

$$\boldsymbol{\lambda}^0(F^\varepsilon_> \cup F^\varepsilon_= \cup F^\varepsilon_<) \;=\; \int\limits_{F^\varepsilon_> \cup F^\varepsilon_= \cup F^\varepsilon_<} \dot{\boldsymbol{\lambda}}^0 d\boldsymbol{\lambda} \;=\; \frac{\alpha_r h^{(t)}_{(\widehat{\tau}_\rho)}}{t}\;. \tag{1.37}$$

So that, when we choose $T^{\frac{1}{t}\alpha_r} \;:=\; F^\varepsilon_> \cup F^\varepsilon_= \cup F^\varepsilon_<$, then

$$\boldsymbol{\lambda}^0(T^{\frac{1}{t}\alpha_r}) = \frac{\alpha_r h^{(t)}_{(\widehat{\tau}_\rho)}}{t} = \frac{\overline{x}^{(t)}_{\widehat{\tau}_\rho} h^{(t)}_{(\widehat{\tau}_\rho)}}{t} = \boldsymbol{\lambda}^{(t)}(T^{\frac{1}{t}\alpha_r})\;. \tag{1.38}$$

That is

$$\boldsymbol{\lambda}^0(\boldsymbol{D}^{\widehat{\tau}_\rho} \cap T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^\ominus}) \;=\; \boldsymbol{\lambda}(\boldsymbol{D}^{\widehat{\tau}_r}) h^{(t)}_{(\widehat{\tau}_r)} \;=\; \frac{\alpha_r}{t} h^{(t)}_{(\widehat{\tau}_r)} \;=\; \frac{1}{t}\bar{\boldsymbol{x}}^{(t)}_{\widehat{\tau}_r}{}^{(t)}\boldsymbol{a}^\ominus_{\widehat{\tau}_r} \;=\; \frac{1}{t}{}^{(t)}\boldsymbol{a}^\ominus_{\widehat{\tau}_r} h^{(t)}_{(\widehat{\tau}_r)}$$

and

$$\begin{aligned} \bar{\boldsymbol{\xi}}^{(t)}({}^{(t)}\boldsymbol{D}^{\widehat{\tau}_r} \cap T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^\ominus}) \;&=\; \bar{\boldsymbol{\xi}}^{(t)}({}^{(t)}\boldsymbol{D}^{\widehat{\tau}_r} \cap T^{\frac{1}{t}\alpha_r}) \\ &=\; \alpha_r \boldsymbol{\lambda}^{(t)}(\boldsymbol{D}^{\widehat{\tau}_\rho}) = \alpha_r \boldsymbol{\lambda}^0({}^{(t)}\boldsymbol{D}^{\widehat{\tau}_r}) \\ &=\; \frac{1}{t}\bar{\boldsymbol{x}}^{(t)}_{\widehat{\tau}_r}{}^{(t)}\boldsymbol{a}^\ominus_{\widehat{\tau}_r} \;=\; \frac{1}{t}{}^{(t)}\boldsymbol{a}^\ominus_{\widehat{\tau}_r} h^{(t)}_{(\widehat{\tau}_r)} \\ &=\; \frac{\alpha_r}{t} h^{(t)}_{(\widehat{\tau}_r)} \end{aligned} \tag{1.39}$$

i.e.,

$$\bar{\boldsymbol{\xi}}(T^{\frac{1}{t}\alpha_r}) \;=\; \bar{\boldsymbol{\xi}}^{(t)}(T^{\frac{1}{t}\alpha_r}) \;=\; \frac{\alpha_r}{t} h^{(t)}_{(\widehat{\tau}_r)}\;.$$

3rdSTEP :

For $T^{\frac{1}{t}\beta_r}$ the argument is rather similar.

Finally for $\overline{\overline{\tau}}_r$ we have ${}^{(t)}\boldsymbol{D}^{\overline{\overline{\tau}}_r} \subseteq \overset{\wedge}{\boldsymbol{E}}$, hence

$$\dot{\bar{\boldsymbol{\xi}}} \;=\; 1 - l^\star_r \;=\; 1 - h^{(t)\star}_r = \boldsymbol{x}^{(t)}_{\overline{\overline{\tau}}_r} \quad \text{on} \quad {}^{(t)}\boldsymbol{D}^{\overline{\overline{\tau}}_r}.$$

Therefore, no matter the choice of $T^{\frac{1}{t}\beta_r}$, we have

$$\bar{\boldsymbol{\xi}}(T^{\frac{1}{t}\beta_r}) \;=\; \bar{\boldsymbol{\xi}}^{(t)}(T^{\frac{1}{t}\beta_r}) \;=\; \beta_r \frac{1 - l^\star_r}{t} \;=\; \beta_r \frac{1 - h^{(t)\star}_r}{t} \;=\; \beta_r \frac{1}{t}\bar{\boldsymbol{x}}^{(t)}_{\overline{\overline{\tau}}_r}.$$

Now, exactly as in the $2^{nd}STEP$ we can *choose* $T^{\frac{1}{t}\beta_r} \subseteq {}^{(t)}\boldsymbol{D}^{\overline{\overline{\tau}}_r}$ via Ljapounoffs Theorem such that

$$\boldsymbol{\lambda}^0(T^{\frac{1}{t}\beta_r}) \;=\; \beta_r \boldsymbol{\lambda}^0(\boldsymbol{D}^{\overline{\overline{\tau}}_r})$$

is true. Then again

$$\boldsymbol{\lambda}^0(\boldsymbol{D}^{\overline{\overline{\tau}}_r} \cap T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^\ominus}) \;=\; \frac{\beta_r}{t} h^{(t)}_{(\overline{\overline{\tau}}_r)} \;=\; \frac{1}{t}{}^{(t)}\boldsymbol{a}^\ominus_{\overline{\overline{\tau}}_r} h^{(t)}_{(\overline{\overline{\tau}}_r)} \;=\; \frac{1}{t}\bar{\boldsymbol{x}}^{(t)}_{\overline{\overline{\tau}}_r}{}^{(t)}\boldsymbol{a}^\ominus_{\overline{\overline{\tau}}_r} \tag{1.40}$$

This way we obtain (1.32).

q.e.d.

Now let S be a set with rational measure $\boldsymbol{\lambda}(S) = \frac{p}{t}$. Then we can construct a uniform model ${}^{(t)}\underline{\underline{\mathbf{D}}}$ such that $S \in \underline{\underline{\mathbf{F}}}^{(t)}$ just by cutting S into q pieces each of them having measure $\frac{1}{t}$ and choosing the decomposition of $\boldsymbol{I}\backslash S$ suitably. This suggests that we can apply the previous Theorem with somewhat relaxed conditions as follows.

Corollary 1.5 (The rational Inheritance Theorem). *Let $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ be imputations and let S be a coalition such that $\boldsymbol{\xi}\,\mathrm{dom}_S^{\boldsymbol{v}}\,\boldsymbol{\eta}$. Assume that,*

1. *$\boldsymbol{\lambda}(S) = \frac{p}{t}$ is rational,*
2. *$T_> := \{\overset{\bullet}{\boldsymbol{\xi}} > \overset{\bullet}{\boldsymbol{\eta}}\}$ has rational measure $\boldsymbol{\lambda}(T_>) = \frac{q}{t}$ (w.l.o.g. with the same t).*

Let ${}^{(t)}\underline{\underline{\mathbf{D}}}$ be a uniform model such that $S \in \underline{\underline{\mathbf{F}}}^{(t)}$ and $T_> \in \underline{\underline{\mathbf{F}}}^{(t)}$ and let $\boldsymbol{\xi}^{(t)} := P\left\{\boldsymbol{\xi} \,\middle|\, \underline{\underline{\mathbf{F}}}^{(t)}\right\}(\bullet)$. Then the relations

$$\boldsymbol{\xi}\,\mathrm{dom}_S^{\boldsymbol{v}}\,\boldsymbol{\eta}, \quad \boldsymbol{\xi}\,\mathrm{dom}_S^{\boldsymbol{v}^{(t)}}\,\boldsymbol{\eta}, \qquad \textit{and} \quad \boldsymbol{\xi}^{(t)}\,\mathrm{dom}_S^{\boldsymbol{v}^{(t)}}\,\boldsymbol{\eta} \tag{1.41}$$

hold true simulataneously. Moreover, there exists a vector $\boldsymbol{a}^{(t)}$ relevant for $\boldsymbol{v}^{(t)}$ such that $T^{\frac{1}{t}\boldsymbol{a}^{(t)}} \subseteq S$ and

$$\boldsymbol{\xi}\,\mathrm{dom}^{\boldsymbol{v}}_{T^{\frac{1}{t}\boldsymbol{a}^{(t)}}}\,\boldsymbol{\eta}, \quad \boldsymbol{\xi}\,\mathrm{dom}^{\boldsymbol{v}^{(t)}}_{T^{\frac{1}{t}\boldsymbol{a}^{(t)}}}\,\boldsymbol{\eta}, \boldsymbol{\xi}^{(t)}\,\mathrm{dom}^{\boldsymbol{v}^{(t)}}_{T^{\frac{1}{t}\boldsymbol{a}^{(t)}}}\,\boldsymbol{\eta}\ . \tag{1.42}$$

Proof: The first claim follows from Lemma 1.1 and the second from Theorem 1.2,

q.e.d.

Naturally the next step is to establish that a rational measure for the coalition involved in a dominance relation can be enforced. In the following Lemma domination refers to $\boldsymbol{v}$.

Lemma 1.6. *Let $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ be imputations, S a coalition and let $\boldsymbol{\xi}\,\mathrm{dom}_S\,\boldsymbol{\eta}$. Then there exists a coalition $T \subseteq S$ such that $\boldsymbol{\lambda}(T^\rho) = \boldsymbol{\lambda}(T \cap \boldsymbol{C}^\rho)$ is rational for every $\rho \in \mathbf{R}$ and*

$$\boldsymbol{\xi}\,\mathrm{dom}_T\,\boldsymbol{\eta}\ .$$

Proof:

1st STEP : First of all assume that $\boldsymbol{\xi}(S) < \boldsymbol{v}(S)$ is the case. For each $\rho \in \mathbf{R}$ choose a sequence of rational numbers r_n^ρ such that $r_n^\rho \uparrow_n \boldsymbol{\lambda}(S^\rho)$ $(n \to \infty)$ and by Ljapounoffs Theorem an increasing sequence of coalitions $S_n^\rho \subseteq S_{n+1}^\rho \subseteq S^\rho$ such that $\boldsymbol{\lambda}(S_n^\rho) = r_n^\rho$ and $S_n^\rho \uparrow S^\rho$ $(n \in \mathbb{N})$, hence $S_n \uparrow S$ $(n \in \mathbb{N})$. Then

$$\boldsymbol{\xi}(S_n) \uparrow \boldsymbol{\xi}(S) \ \text{ and } \ \boldsymbol{\lambda}^\rho(S_n) \uparrow \boldsymbol{\lambda}^\rho(S) \ \text{ for all } \ \rho \in \mathbf{R}_0 \quad (n \in \mathbb{N}),$$

hence $\boldsymbol{v}(S_n) \uparrow \boldsymbol{v}(S)$ $(n \in \mathbb{N})$. Consequently, for sufficiently large $n \in \mathbb{N}$ we have $\boldsymbol{\xi}(S_n) < \boldsymbol{v}(S_n)$ and, as $S_n \subseteq S$, also $\boldsymbol{\xi} > \boldsymbol{\eta}$ on S_n. That is indeed,

$$\boldsymbol{\xi} \operatorname{dom}_{S_n} \boldsymbol{\eta}$$

for sufficiently large $n \in \mathbb{N}$.

2ndSTEP :

Now we consider the case

$$\boldsymbol{\xi}(S) = \boldsymbol{v}(S) \ . \tag{1.43}$$

Denote $s_\rho := \boldsymbol{\lambda}(S^\rho)(\rho \in \mathbf{R})$. If, for some $T \subseteq S$ we have $\boldsymbol{\xi}(T) < \boldsymbol{v}(T)$, then we may proceed as in the first step.

Then, consider the case that

$$\text{For all} \quad T \subseteq S \text{ with } \boldsymbol{\lambda}(T) < \boldsymbol{\lambda}(S), \text{ we have } \boldsymbol{\xi}(T) = \boldsymbol{v}(T). \tag{1.44}$$

Then we construct for all $0 \leq t \leq 1$ a subcoalition $S^{t,\rho} \subseteq S^\rho$ such that $\boldsymbol{\lambda}(S^{t,\rho}) = ts_\rho$ holds true. Then, for rational r_ρ such that $t := \frac{r_\rho}{s_\rho} < 1$ for all $\rho \in \mathbf{R}$, we observe that

$$\boldsymbol{\lambda}(S^{t,\rho}) = ts_\rho = \frac{r_\rho s_\rho}{s_\rho} \ = \ r_\rho \ ;$$

is rational and yields $\boldsymbol{\lambda}(S^t) = \boldsymbol{v}(S^t)$ (by (1.44)). As $S^t \subseteq S$ we know that $\boldsymbol{\xi} > \boldsymbol{\eta}$ on S^t, that is $\boldsymbol{\xi} \operatorname{dom}_{S^t} \boldsymbol{\eta}$ and we are done.

3rdSTEP :

Therefore, let us now assume that $\boldsymbol{\xi}(S) = \boldsymbol{v}(S)$ and that for all coalitions $T \subseteq S$ with $\boldsymbol{\lambda}(T) < \boldsymbol{\lambda}(S)$, we have $\boldsymbol{\xi}(T) > \boldsymbol{v}(T)$.

Let

$$\mathbf{R}_\star := \{\rho \in \mathbf{R} \,|\, \boldsymbol{\xi}(S) = \boldsymbol{\lambda}^\rho(S)\} \ = \ \{\rho \in \mathbf{R} \,|\, \boldsymbol{\lambda}^\rho(S) \ = \ \boldsymbol{v}(S)\} \neq \emptyset \tag{1.45}$$

such that

$$\boldsymbol{\xi}(S) < \boldsymbol{\lambda}^\rho(S) \ \ (\rho \in \mathbf{R} \setminus \mathbf{R}_\star) \ . \tag{1.46}$$

Next define

(1.47)

$$\begin{aligned} \mathbf{R}_\bullet \ := \ & \{\rho \in \mathbf{R} \,|\, \text{there exists } \varepsilon_0 > 0 \text{ such that for all } 0 < \varepsilon < \varepsilon_0 \\ & \text{and all } T \subseteq S \text{ with } \boldsymbol{\lambda}(S) - \varepsilon < \boldsymbol{\lambda}(T) < \boldsymbol{\lambda}(S) \text{ it follows that} \\ & \boldsymbol{\xi}(T) > \boldsymbol{\lambda}^\rho(T) \ \} \neq \emptyset \ . \end{aligned}$$

Clearly

$$\mathbf{R}_\bullet \subseteq \mathbf{R}_\star \ . \tag{1.48}$$

Now choose $\varepsilon_\bullet$ jointly for all $\rho \in \mathbf{R}_\bullet$, i.e., in a way such that

$$\begin{aligned} &\text{for all } 0 < \varepsilon < \varepsilon_\bullet \text{ and all } T \subseteq S \text{ with } \boldsymbol{\lambda}(S) - \varepsilon < \boldsymbol{\lambda}(T) < \boldsymbol{\lambda}(S), \\ &\text{it follows that} \\ &\boldsymbol{\xi}(T) > \boldsymbol{\lambda}^\rho(T) \quad (\rho \in \mathbf{R}_\bullet) \text{ and } \boldsymbol{\xi}(T) = \boldsymbol{\lambda}^\rho(T) \quad (\rho \in \mathbf{R}_\star \setminus \mathbf{R}_\bullet) \end{aligned} \tag{1.49}$$

Consider now the vector–valued measure $\boldsymbol{\mu} := (\boldsymbol{\xi}, \boldsymbol{\lambda}, \boldsymbol{\lambda}^0, \ldots, \boldsymbol{\lambda}^r)$. For some $\varepsilon < \varepsilon_\bullet$ and $T \subseteq S$ satisfying $\boldsymbol{\lambda}(S) - \varepsilon < \boldsymbol{\lambda}(T) < \boldsymbol{\lambda}(S)$ we have

$$\begin{aligned} \boldsymbol{\xi}(S \setminus T) &= \boldsymbol{\xi}(S) - \boldsymbol{\xi}(T) = \boldsymbol{\lambda}^\rho(S) - \boldsymbol{\xi}(T) \\ &< \boldsymbol{\lambda}^\rho(S) - \boldsymbol{\lambda}^\rho(T) = \boldsymbol{\lambda}^\rho(S \setminus T) \quad (\rho \in \mathbf{R}_\bullet) \end{aligned} \tag{1.50}$$

and

$$\begin{aligned} \boldsymbol{\xi}(S \setminus T) &= \boldsymbol{\xi}(S) - \boldsymbol{\xi}(T) = \boldsymbol{\lambda}^\rho(S) - \boldsymbol{\xi}(T) \\ &= \boldsymbol{\lambda}^\rho(S) - \boldsymbol{\lambda}^\rho(T) = \boldsymbol{\lambda}^\rho(S \setminus T) \quad (\rho \in \mathbf{R}_\star \setminus \mathbf{R}_\bullet) \ . \end{aligned} \tag{1.51}$$

Next, in view of (1.46), we can decrease $\varepsilon_\bullet$ if necessary in order to make sure that for all $T \subseteq S$ with $\boldsymbol{\lambda}(S) - \varepsilon < \boldsymbol{\lambda}(T) < \boldsymbol{\lambda}(S)$, we have

$$\boldsymbol{\xi}(T) < \boldsymbol{\lambda}^\rho(T) \quad (\rho \in \mathbf{R} \setminus \mathbf{R}_\star) \ . \tag{1.52}$$

Now we choose rational numbers $q, \{q_\rho\}_{\rho \in \mathbf{R}}$ such that

$$0 < q < s \quad , \quad s - q < \varepsilon_\bullet \quad , \tag{1.53}$$

as well as

$$0 < q_\rho < s_\rho \quad , \quad \sum_{\rho \in \mathbf{R}} q_\rho = q \quad , \tag{1.54}$$

holds true. Thereafter choose r such that

$$0 < r < 1, \quad s - rs < \varepsilon_\bullet \tag{1.55}$$

is satisfied. Finally put

$$r_\rho := r \frac{\left(1 - \frac{q_\rho}{s_\rho}\right)}{\left(1 - \frac{q}{s}\right)} \ . \tag{1.56}$$

Now choose $T \subseteq S$ such that $\boldsymbol{\lambda}(T) = rs$ and accordingly T satisfies (1.50) and (1.51).

Consider the straight line connecting $\boldsymbol{\mu}(S \setminus T)$ and $\boldsymbol{\mu}(S)$. By Ljapounoffs Theorem we find for every $0 \le t \le 1$ some coalition S^t, $S \setminus T \subseteq S^t \subseteq S$ with values

$$\boldsymbol{\mu}(S^t) = t\boldsymbol{\mu}(S \setminus T) + (1 - t)\boldsymbol{\mu}(S)$$

on this line.

S^t obviously satisfies

$$\begin{aligned} \boldsymbol{\xi}(S^t) &< \boldsymbol{\lambda}^\rho(S^t) \quad (\rho \in \mathbf{R}_\bullet) \\ \boldsymbol{\xi}(S^t) &= \boldsymbol{\lambda}^\rho(S^t) \quad (\rho \in \mathbf{R}_\star \setminus \mathbf{R}_\bullet) \ . \end{aligned} \tag{1.57}$$

Here the first line is based on (1.45) and (1.51), while the second line is based on (1.45) and (1.50).

In particular choose t satisfying

$$t = \frac{1}{r}\left(1 - \frac{q}{s}\right) \ = \ \left(1 - \frac{q_\rho}{s_\rho}\right) \quad \rho \in \mathbf{R} \ ; \tag{1.58}$$

this is possible by (1.56).

Then

$$\begin{aligned} \boldsymbol{\lambda}(S^t) &= t\boldsymbol{\lambda}(S \setminus T) + (1-t)\boldsymbol{\lambda}(S) \\ &= t(s - rs) + (1-t)s \\ &= s - trs \ = \ s - s\left(1 - \frac{q}{s}\right) \\ &= q \end{aligned} \tag{1.59}$$

and

$$\begin{aligned} \boldsymbol{\lambda}(S^{t\rho}) &= \boldsymbol{\lambda}^\rho(S^t) \\ &= t\boldsymbol{\lambda}^\rho(S \setminus T) + (1-t)\boldsymbol{\lambda}^\rho(S) \\ &= t(s_\rho - r_r s_\rho) + (1-t)s_\rho \\ &= s_\rho - trs_\rho \ = \ s_\rho - s_\rho\left(1 - \frac{q_\rho}{s_\rho}\right) \\ &= q_\rho \end{aligned} \tag{1.60}$$

are rational numbers. As $s - q < \varepsilon_\bullet$, we know that S^t satisfies (1.52), i.e.,

$$\boldsymbol{\xi}(S^t) \ < \ \boldsymbol{\lambda}^\rho(S^t) \quad (\rho \in \mathbf{R} \setminus \mathbf{R}_\star) \ . \tag{1.61}$$

Combining (1.57) and (1.61) we see that $\boldsymbol{\xi}(S^t) \leq \boldsymbol{v}(S^t)$ and as $S^t \subseteq S$ we have $\boldsymbol{\xi} > \boldsymbol{\eta}$ on S and hence

$$\boldsymbol{\xi} \operatorname{dom}_{S^t} \boldsymbol{\eta} \ , \tag{1.62}$$

q.e.d.

The following lemma collects the above results.

Lemma 1.7. *Let $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ be imputations and let T be a coalition such that $\boldsymbol{\xi} \operatorname{dom}^{\boldsymbol{v}}_T \boldsymbol{\eta}$. Then there exists a coalition S and an imputation $\boldsymbol{\vartheta}$ such that*

1. $S \subseteq T$,

2. $\boldsymbol{\vartheta} = \boldsymbol{\eta}$ *on* S ,

3. $\{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\vartheta}}\} \subseteq \{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\eta}}\}$,

4. $\boldsymbol{\lambda}(S \cap \boldsymbol{C}^\rho)$ *is rational for all* $\rho \in \mathbf{R}$,

5. $\boldsymbol{\lambda}(\{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\vartheta}}\}) \cap \boldsymbol{C}^\rho$ *is rational for all* $\rho \in \mathbf{R}$,

6. $\boldsymbol{\xi}\, \mathrm{dom}_S\, \boldsymbol{\vartheta}$.

Proof: According to Lemma 1.6 we find $S \subseteq T$ such that that $\boldsymbol{\lambda}(S) \cap \boldsymbol{C}^\rho$ is rational for all $\rho \in \mathbf{R}$ and $\boldsymbol{\xi}\, \mathrm{dom}_S\, \boldsymbol{\eta}$. As $S \subseteq \{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\eta}}\}$ we are done if it so happens that $S = \{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\eta}}\}$ is true. Assume, therefore, that $\{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\eta}}\} \setminus S$ has positive measure. Then, as $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are imputations, it follows that $\{\dot{\boldsymbol{\xi}} \leq \dot{\boldsymbol{\eta}}\} \subseteq \boldsymbol{I} \setminus S$ has positive measure.

Denote $\boldsymbol{T}_> := \{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\eta}}\}$ and $\boldsymbol{T}_\leq := \{\dot{\boldsymbol{\xi}} \leq \dot{\boldsymbol{\eta}}\}$. Choose $R^+ \subseteq \boldsymbol{T}_>$ and $R^- \subseteq \boldsymbol{T}_\leq$ of sufficiently small but positive measure such that $\boldsymbol{\lambda}((\boldsymbol{T}_> \setminus R^+) \cap \boldsymbol{C}^\rho)$ is rational for all $\rho \in \mathbf{R}$ and

$$\int_{R^+} (\dot{\boldsymbol{\xi}} - \dot{\boldsymbol{\eta}}) d\boldsymbol{\lambda} = \int_{R^-} (\dot{\boldsymbol{\eta}} - \dot{\boldsymbol{\xi}}) d\boldsymbol{\lambda} \tag{1.63}$$

holds true. Then

$$\boldsymbol{\vartheta} := \boldsymbol{\eta} + (\boldsymbol{\xi} - \boldsymbol{\eta}) \mathbb{1}_{R^+} - (\boldsymbol{\eta} - \boldsymbol{\xi}) \mathbb{1}_{R^-} \tag{1.64}$$

is an imputation. Moreover $\boldsymbol{\vartheta} = \boldsymbol{\xi} + (\boldsymbol{\eta} - \boldsymbol{\xi}) = \boldsymbol{\eta} > \boldsymbol{\xi}$ on R^+. Similarly, $\boldsymbol{\vartheta} = \boldsymbol{\eta} + (\boldsymbol{\xi} - \boldsymbol{\eta}) = \boldsymbol{\xi}$ on R^-.

Thus

$$\{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\vartheta}}\} = \boldsymbol{T}_> \setminus R_+ \subseteq \{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\eta}}\} \tag{1.65}$$

and $\{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\vartheta}}\} \cap \boldsymbol{C}^\rho$ has rational measure $\boldsymbol{\lambda}(\{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\vartheta}}\} \cap \boldsymbol{C}^\rho) = \boldsymbol{\lambda}((\boldsymbol{T}_> \setminus R^+) \cap \boldsymbol{C}^\rho)$ for all $\rho \in \mathbf{R}$. Finally, as $\boldsymbol{\vartheta} = \boldsymbol{\eta}$ holds true on S, we have clearly $\boldsymbol{\xi}\, \mathrm{dom}_S\, \boldsymbol{\vartheta}$ as well.

q.e.d.

Corollary 1.8. *Let* $\boldsymbol{\xi}$ *and* $\boldsymbol{\eta}$ *be imputations and let* S *be a coalition such that* $\boldsymbol{\xi}\, \mathrm{dom}_S^v\, \boldsymbol{\eta}$. *Then there exists* $t_0 \in \mathbb{N}$ *such that for every multiple* $t = rt_0$ *of* t_0 *there is a uniform model constituted by* ${}^{(t)}\underline{\underline{\mathbf{D}}}$ *as well as a coalition* R *and an imputation* $\boldsymbol{\vartheta}$ *such that the following is satisfied:*

1. $R \subseteq S$,

2. $\boldsymbol{\vartheta} = \boldsymbol{\eta}$ *on* R ,

3. $\{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\vartheta}}\} \subseteq \{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\eta}}\}$,

4. $R \in \underline{\underline{\mathbf{F}}}^{(t)}$,

5. $\{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\vartheta}}\} \in \underline{\underline{\mathbf{F}}}^{(t)}$,

6. $\boldsymbol{\xi} \operatorname{dom}_R \boldsymbol{\vartheta}$.

Proof: Follows immediately from Lemma 1.7: choose R and $\boldsymbol{\vartheta}$ accordingly and let t_0 be an integer such that $\boldsymbol{\lambda}(R \cap \boldsymbol{C}^\rho)$ and $\boldsymbol{\lambda}(\{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\vartheta}}\}) \cap \boldsymbol{C}^\rho$ are multiples of $\frac{1}{t_0}$ for all $\rho \in \mathbf{R}$. Then decompose each $\boldsymbol{C}^\rho$ $(\rho \in \mathbf{R})$ into atoms ${}^tD^\tau$ in a way that $R \cap \boldsymbol{C}^\rho$ and $\{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\vartheta}}\} \cap \boldsymbol{C}^\rho$ are respected.

q.e.d.

Remark 1.9. The choice of a model via ${}^{(t)}\underline{\underline{\mathbf{D}}}$ in the context of the previous results still allows for certain degrees of freedom. It is sufficient to choose the atoms ${}^{(t)}D^\tau$ to be of equal measure $\frac{1}{t}$ and not to cut into the two sets S and $\{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\vartheta}}\}$ that are given by Lemma 1.7.

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Combining our results we obtain relations between dominance w.r.t $\boldsymbol{v}$ and dominance w.r.t. $\boldsymbol{v}^{(t)}$. The following is a version of lower hemicontinuity.

Theorem 1.10 (**Dominance is lhc.**). *Let $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ be imputations and denote $\boldsymbol{\xi}^{(t)} := P\left\{\boldsymbol{\xi} \,\middle|\, \underline{\underline{\mathbf{F}}}^{(t)}\right\}(\bullet)$. Also, let S be a coalition such that $\boldsymbol{\xi} \operatorname{dom}_S^{\boldsymbol{v}} \boldsymbol{\eta}$. There exists $t_0 \in \mathbb{N}$ and for all multiples $t = rt_0 \in \mathbb{N}$ some uniform model ${}^{(t)}\underline{\underline{\mathbf{D}}}$ as well as a vector $\boldsymbol{a}^{(t)}$ relevant with respect to ${}^{(t)}\underline{\underline{\mathbf{D}}}$ such that $T^{\boldsymbol{a}^{(t)}}$ satisfies*

1. $T^{\boldsymbol{a}(t)} \subseteq S$,

2. $\boldsymbol{\xi} \operatorname{dom}^{\boldsymbol{v}}_{T^{\boldsymbol{a}(t)}} \boldsymbol{\eta}$, $\boldsymbol{\xi} \operatorname{dom}^{\boldsymbol{v}^{(t)}}_{T^{\boldsymbol{a}(t)}} \boldsymbol{\eta}$, *and* $\boldsymbol{\xi}^{(t)} \operatorname{dom}^{\boldsymbol{v}^{(t)}}_{T^{\boldsymbol{a}(t)}} \boldsymbol{\eta}$.

Proof:

Choose $t_0 \in \mathbb{N}$ and some multiple $t = rt_0$ as well as ${}^{(t)}\underline{\underline{\mathbf{D}}}$ and some $R \subseteq S$ and $\boldsymbol{\vartheta}$ according to Lemma 1.8. That is, $\boldsymbol{\xi} = \boldsymbol{\vartheta}$ on R and $\{\boldsymbol{\xi} > \boldsymbol{\eta}\}$ as well as R are $\underline{\underline{\mathbf{F}}}^{(t)}$-measurable. Then we obtain

$$\boldsymbol{v}^{(t)}(R) \;=\; \boldsymbol{v}(R) \tag{1.66}$$

and hence, in view of Lemma 1.1,

$$\boldsymbol{\xi} \operatorname{dom}^{\boldsymbol{v}}_R \boldsymbol{\vartheta}, \quad \boldsymbol{\xi} \operatorname{dom}^{\boldsymbol{v}^{(t)}}_R \boldsymbol{\vartheta}, \quad \text{and} \quad \boldsymbol{\xi}^{(t)} \operatorname{dom}^{\boldsymbol{v}^{(t)}}_R \boldsymbol{\vartheta} \ . \tag{1.67}$$

Now by the "ε–free" Inheritance Theorem 1.2, there exists a vector $\boldsymbol{a} = \boldsymbol{a}^{(t)}$ relevant w.r.t $\boldsymbol{v}^{(t)}$ such that $T^{\boldsymbol{a}^{(t)}} \subseteq R$ and

$$\boldsymbol{\xi} \operatorname{dom}^{\boldsymbol{v}^{(t)}}_{T^{\boldsymbol{a}^{(t)}}} \boldsymbol{\vartheta} \ .$$

Again by Lemma 1.1 we have

$$\boldsymbol{\xi} \operatorname{dom}^{\boldsymbol{v}}_{T^{\boldsymbol{a}^{(t)}}} \boldsymbol{\eta}, \quad \boldsymbol{\xi} \operatorname{dom}^{\boldsymbol{v}^{(t)}}_{T^{\boldsymbol{a}^{(t)}}} \boldsymbol{\eta}, \quad \text{and} \quad \boldsymbol{\xi}^{(t)} \operatorname{dom}^{\boldsymbol{v}^{(t)}}_{T^{\boldsymbol{a}^{(t)}}} \boldsymbol{\eta} \ . \tag{1.68}$$

q.e.d.

2 Truncating the Central Factor

In the framework of the uniform model the pre sub–imputation $\bar{\boldsymbol{x}}$ is constructed via some truncation procedure of the central measure. It provides the basis for the construction of a vNM–Stable Set via the convex hull with the core. We refer to this pre sub–imputation as to the ***standard truncation***.

We provide the procedure for the continuous model. We will suitably truncate the central density and augment it in order to receive an imputation. This imputation together with the core will allow the definition of a vNM–Stable Set as the convex hull of both.

Simultaneously we are dealing with a continuous model and a uniform one. The first one is represented by a game $(\boldsymbol{I}, \underline{\underline{\mathbf{F}}}, \boldsymbol{v})$ with coalitional function

$$\boldsymbol{v} = \bigwedge \{\boldsymbol{\lambda}^0, \boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r\} \;=\; \bigwedge_{\rho \in \mathbf{R}_0} \boldsymbol{\lambda}^\rho \;,$$

where $\boldsymbol{\lambda}^0$ is an arbitrary ("continuous") central distribution. The second we indicate by some family

$${}^{(t)}\underline{\underline{\mathbf{D}}} \;=\; \left\{{}^{(t)}\boldsymbol{D}^\tau\right\}_{\tau \in \mathbf{T}} \text{ with } \mathbf{T} \;=\; \mathbf{T}^{(t)} := \{1, \ldots, rt\})$$

The central distribution is the conditional probability $\boldsymbol{\lambda}^{(t)}$ w.r.t the corresponding field $\underline{\underline{\mathbf{F}}}^{(t)}$. The game $\boldsymbol{v}^{(t)}$ is defined accordingly.

We have to recall notation in order to relate the uniform setup of *Part IV* and the continuous one.

First of all the sets $\overset{\vee}{\mathbf{T}}{}^\sigma$ and $\overset{\wedge}{\mathbf{T}}{}^\sigma$ are provided by *Definition 2.1* of *Part IV*. Naturally, the corresponding sets with a model indexed with ${}^{(t)}$ are ${}^{(t)}\overset{\vee}{\mathbf{T}}{}^\sigma$ and ${}^{(t)}\overset{\vee}{\mathbf{T}}{}^\sigma$. The corresponding coalitions are written

$$\begin{aligned} {}^{(t)}\overset{\vee}{\boldsymbol{T}}{}^\sigma &:= \bigcup_{\tau \in {}^{(t)}\overset{\vee}{\mathbf{T}}{}^\sigma} {}^{(t)}\boldsymbol{D}^\tau \;, & {}^{(t)}\overset{\wedge}{\boldsymbol{T}}{}^\sigma &:= \bigcup_{\tau \in {}^{(t)}\overset{\wedge}{\mathbf{T}}{}^\sigma} {}^{(t)}\boldsymbol{D}^\tau \\ \overset{\vee}{\boldsymbol{T}}{}^{(t)} &:= \bigcup_{\rho \in \mathbf{R}} {}^{(t)}\overset{\vee}{\boldsymbol{T}}{}^\rho \;, & \overset{\wedge}{\boldsymbol{T}}{}^{(t)} &:= \bigcup_{\rho \in \mathbf{R}} {}^{(t)}\overset{\wedge}{\boldsymbol{T}}{}^\rho \;. \end{aligned} \tag{2.1}$$

In the uniform context (*Part IV*) we use discrete values h_τ of the central density as well as quantities

$$h^0_\rho \;:=\; \min\{h_\tau \,\big|\, \tau \in \mathbf{T}^\rho\} \quad \rho \in \mathbf{R} \tag{2.2}$$

and

$$h^\star_\sigma \;=\; \sum_{\rho \in \mathbf{R} \setminus \{\sigma\}} h^0_\rho. \tag{2.3}$$

Now with respect to some ${}^{(t)}\underline{\underline{\mathbf{D}}}$ and the central distribution $\boldsymbol{\lambda}^{(t)}$ derived from $\boldsymbol{\lambda}^0$, we denote the values by

$$(2.4) \qquad {}^{(t)}h_\tau \;=\; \mathbb{P}\left\{\boldsymbol{\lambda}^0 \,\middle|\, {}^{(t)}\boldsymbol{D}^\tau \right\} \quad (\tau \in \mathbf{T}^{(t)}).$$

Accordingly we have data

$$(2.5) \qquad {}^{(t)}h^0_\rho \;:=\; \min\{{}^{(t)}h_\tau \,\big|\, \tau \in^{(t)} \mathbf{T}^\rho\} \quad \rho \in \mathbf{R}$$

and

$$(2.6) \qquad {}^{(t)}h^\star_\sigma \;=\; \sum_{\rho\in\mathbf{R}\setminus\{\sigma\}} {}^{(t)}h^0_\rho.$$

Now recall the definition

$$(2.7) \qquad l^0_\sigma \;:=\; \text{ess inf}_{\boldsymbol{C}^\sigma} \dot{\boldsymbol{\lambda}}^0 \;, \quad l^\star_\sigma \;:=\; \sum_{\rho\in\mathbf{R}\setminus\{\sigma\}} l^0_\rho \;\; (\sigma \in \mathbf{R})$$

and the assumption

$$(2.8) \qquad \sum_{\rho\in\mathbf{R}} l^0_\rho \le 1$$

imposed on the continuous model. Clearly we have

$$(2.9) \qquad {}^{(t)}h^0_\rho \ge l^0_\rho \;, \quad {}^{(t)}h^\star_\rho \ge l^\star_\rho \;, \quad \text{and} \quad 1 - {}^{(t)}h^\star_\rho \le 1 - l^\star_\rho \quad (\rho \in \mathbf{R}) \;.$$

Define for $\sigma \in \mathbf{R}$

(2.10)

$$\overset{\vee}{\boldsymbol{E}}{}^\sigma \;:=\; \left\{\omega \,\middle|\, \dot{\boldsymbol{\lambda}}^0(\omega) + \sum_{\rho\in\mathbf{R}\setminus\{\sigma\}} l^0_\rho < 1\right\} \cap \boldsymbol{C}^\sigma \;=\; \left\{\omega \,\middle|\, \dot{\boldsymbol{\lambda}}^0(\omega) + l^\star_\sigma < 1\right\} \cap \boldsymbol{C}^\sigma \;,$$

$$\overset{\vee}{\boldsymbol{E}} \;:=\; \bigcup_{\rho\in\mathbf{R}} \overset{\vee}{\boldsymbol{E}}{}^\rho \;,$$

as well as

$$(2.11) \qquad \begin{aligned} \overset{\wedge}{\boldsymbol{E}}{}^\rho \;&:=\; \boldsymbol{C}^\rho \setminus \overset{\vee}{\boldsymbol{E}}{}^\rho \quad (\rho \in \mathbf{R}) \;, \\ \overset{\wedge}{\boldsymbol{E}} \;&:=\; \bigcup_{\rho\in\mathbf{R}} \overset{\wedge}{\boldsymbol{E}}{}^\rho \;=\; \bigcup_{\rho\in\mathbf{R}} \left\{\omega \,\middle|\, \dot{\boldsymbol{\lambda}}^0(\omega) \ge 1 - l^\star_\rho\right\} \cap \boldsymbol{C}^\rho \;, \end{aligned}$$

The notation for the "***standard truncation***" vector $\bar{\boldsymbol{x}}$ established by *Definition 2.4* of *Part IV* to our present set–up is straightforward. We write

$$(2.12) \qquad \bar{\boldsymbol{x}}^{(t)} \;=\; \left\{\overline{x}^{(t)}\right\}_{\tau\in\mathbf{T}^{(t)}} .$$

The pre imputation generated is the measure given by

$$(2.13)\qquad \overline{\overset{\bullet}{\boldsymbol{\vartheta}}}^{(t)} := \overset{\bullet}{\boldsymbol{\vartheta}}^{\overline{\boldsymbol{x}}^{(t)}} = \sum_{\tau\in\mathbf{T}^{(t)}} \overline{x}_\tau^{(t)} \mathbb{1}_{{}^t\boldsymbol{D}^\tau}\ , \quad \text{that is}\ , \quad \overline{\boldsymbol{\vartheta}}^{(t)} = \sum_{\tau\in\mathbf{T}^{(t)}} \overline{x}_\tau^{(t)} \boldsymbol{\lambda}_{\big|{}^t\boldsymbol{D}^\tau}\ .$$

Now we present the version for the continuous model. First of all we put

$$(2.14)\qquad \bar{\boldsymbol{\xi}} := \boldsymbol{\lambda}^0 \quad \text{on} \quad \overset{\vee}{\boldsymbol{E}}\ .$$

Then, we define

$$(2.15)\qquad \overset{\bullet}{\bar{\boldsymbol{\xi}}} := 1 - l_\sigma^\star = 1 - \sum_{\rho\in\mathbf{R}\setminus\{\sigma\}} l_\rho^0 \text{ on } \overset{\wedge}{\boldsymbol{E}}{}^\sigma \quad (\sigma\in\mathbf{R})\ .$$

Using the indicator function $\mathbb{1}_\bullet$ we can also write

$$(2.16)\qquad \overset{\bullet}{\bar{\boldsymbol{\xi}}} = \sum_{\rho\in\mathbf{R}} \overset{\bullet}{\boldsymbol{\lambda}}{}^0 \mathbb{1}_{\overset{\vee}{\boldsymbol{E}}{}^\rho} + \sum_{\rho\in\mathbf{R}} (1-l_\rho^\star)\mathbb{1}_{\overset{\wedge}{\boldsymbol{E}}{}^\rho} = \overset{\bullet}{\boldsymbol{\lambda}}{}^0 \mathbb{1}_{\overset{\vee}{\boldsymbol{E}}} + \sum_{\rho\in\mathbf{R}} (1-l_\rho^\star)\mathbb{1}_{\overset{\wedge}{\boldsymbol{E}}{}^\rho}\ .$$

As it turns out, we can assume that $\bar{\boldsymbol{\xi}}(\boldsymbol{I}) \le 1$, i.e., $\bar{\boldsymbol{\xi}}$ is a subimputation. For otherwise the core is the (unique) vNM–Stable Set. This is established as in SECTION 4 of *Part IV*.

Definition 2.1. *The sub–imputation* $\bar{\boldsymbol{\xi}}$ *is called the* ***standard truncation*** *measure.*

Figure 2.1 indicates the shape of the measure $\bar{\boldsymbol{\xi}}$.

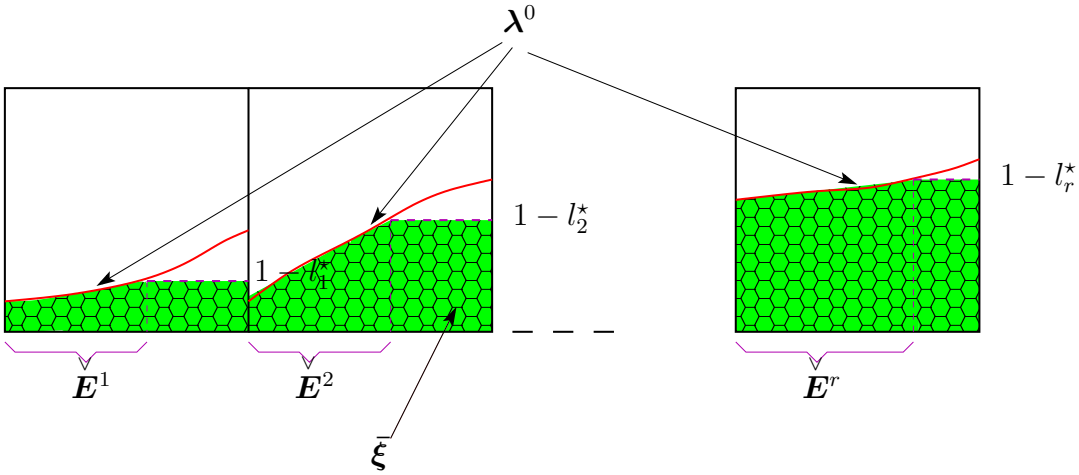

Figure 2.1: The Continuous Case – The Shape of $\bar{\boldsymbol{\xi}}$

Now the standard truncation measure $\bar{\boldsymbol{\xi}}$ may be projected to the uniform model induced by some $\underline{\underline{\mathbf{D}}}^{(t)}$.

Thus, we denote the conditional measure (sub–imputation)

$$\bar{\boldsymbol{\xi}}^{(t)} := \mathbb{P}\left\{\overset{\bullet}{\bar{\boldsymbol{\xi}}} \middle| \underline{\underline{\mathbf{F}}}^{(t)}\right\} , \tag{2.17}$$

which has a piece–wise constant density $\overset{\bullet}{\bar{\boldsymbol{\xi}}}^{(t)}$. The values attained by this density are the quantities

$$\bar{\boldsymbol{\xi}}_{\tau}^{(t)} := \mathbb{E}\left\{\overset{\bullet}{\bar{\boldsymbol{\xi}}} \middle| {}^{(t)}\boldsymbol{D}^{\tau}\right\} \quad (\tau \in \mathbf{T}^{(t)}) . \tag{2.18}$$

This result, of course, has to be well distinguished from the standard truncation measure that is obtained by the standard truncation vector of the uniform model which is $\overset{\bullet}{\bar{\boldsymbol{\vartheta}}}^{(t)}$ as given by (2.13).

The following establishes a condition for the two sub–imputations provided by (2.17) and (2.13) to be identical. This condition will be maintained throughout this SECTION and the following one. Eventually we devoe that last SECTION) to the task of removing it.

Definition 2.2. *We shall say that $\boldsymbol{\lambda}^0$ is **locally flat** if there are positive rational numbers $\frac{p}{t}$, $\frac{q}{t}$ and $\frac{r_\rho}{t}$ $(\rho \in \mathbf{R})$ such that the following are satisfied.*

1. *For some rational number r we have*

$$\boldsymbol{\lambda}\left(\left\{\overset{\bullet}{\boldsymbol{\lambda}}{}^0 = l_\rho^0\right\}\right) = \frac{r}{t} \quad (\rho \in \mathbf{R}) \quad . \tag{2.19}$$

2.

$$\boldsymbol{\lambda}\left(\overset{\vee}{\boldsymbol{E}}\right) = \frac{p}{t} \text{ and } \boldsymbol{\lambda}\left(\overset{\wedge}{\boldsymbol{E}}\right) = \frac{q}{t} \tag{2.20}$$

Lemma 2.3. *Let $\boldsymbol{\lambda}^0$ be locally flat. Then there exist a uniform model ${}^{(t)}\underline{\underline{\mathbf{D}}}$ such that the following is satisfied.*

1.

$${}^{(t)}h_\rho^0 = l_\rho^0 \quad (\rho \in \mathbf{R}) \; ; \qquad {}^{(t)}h_\sigma^\star = l_\sigma^\star \quad (\sigma \in \mathbf{R}). \tag{2.21}$$

2.

$${}^{(t)}\overset{\vee}{\boldsymbol{T}} = \overset{\vee}{\boldsymbol{E}} \; , \quad {}^{(t)}\overset{\wedge}{\boldsymbol{T}} = \overset{\wedge}{\boldsymbol{E}} \quad (t \in \mathbb{N}_0) \tag{2.22}$$

Proof:

1stSTEP :

$\left\{\dot{\boldsymbol{\lambda}}^0 = l^0_\rho\right\}$ has positive and rational measure. Making use of the considerable degrees of freedom available (Remark 1.9), we can arrange for some ${}^{(t)}\underline{\underline{\mathbf{D}}}$ to satisfy for all $\rho \in \mathbf{R}$

$$\left\{\dot{\boldsymbol{\lambda}}^0 = l^0_\rho\right\} \in \underline{\underline{\mathbf{F}}}^{(t)} \ , \qquad \boldsymbol{\lambda}\left(\left\{\dot{\boldsymbol{\lambda}}^0 = l^0_\rho\right\}\right) \geq \frac{1}{t}$$

as well as

$$\overset{\vee}{\boldsymbol{E}}, \overset{\wedge}{\boldsymbol{E}} \in \underline{\underline{\mathbf{F}}}^{(t)} \ .$$

Then, for all ρ and at least one $\tau \in \mathbf{T}^{(t)}$ we have ${}^{(t)}\boldsymbol{D}^\tau \subseteq \left\{\dot{\boldsymbol{\lambda}}^0 = l^0_\rho\right\}$. Hence $h^{(t)}_\tau = l^\star_\rho$ and in view of ${}^{(t)}h^0_\rho \geq l^0_\rho$ we have (2.21). Next, (2.22) follows at once.

2ndSTEP :

Let ${}^{(t)}\boldsymbol{D}^\tau \subseteq {}^{(t)}\overset{\vee}{\boldsymbol{T}}{}^\sigma$ for some $\sigma \in \mathbf{R}$. Then

$$h^{(t)}_\tau < 1 - h^{\star(t)}_\sigma = 1 - l^\star_\sigma \ .$$

Hence

$$ {}^{(t)}\boldsymbol{D}^t \cap \left\{\omega \,\middle|\, \dot{\boldsymbol{\lambda}}^0(\omega) < 1 - l^\star_\sigma\right\} \ = \ {}^{(t)}\boldsymbol{D}^t \cap \overset{\vee}{\boldsymbol{E}}{}^\sigma$$

has positive measure as $h^{(t)}_\tau$ is the conditional expectation of $\dot{\boldsymbol{\lambda}}^0$. As we assume that $\overset{\vee}{\boldsymbol{E}}{}^\sigma$ is measurable, we conclude ${}^{(t)}\boldsymbol{D}^\tau \subseteq \overset{\vee}{\boldsymbol{E}}{}^\sigma$.

On the other hand, if ${}^{(t)}\boldsymbol{D}^\tau \subseteq \overset{\vee}{\boldsymbol{E}}{}^\sigma$, then

$$ {}^{(t)}\boldsymbol{D}^\tau \subseteq \left\{\omega \,\middle|\, \dot{\boldsymbol{\lambda}}^0(\omega) < 1 - l^\star_\sigma = 1 - h^{\star(t)}_\sigma\right\} \ ,$$

hence $h^{(t)}_\tau < 1 - h^{\star(t)}_\sigma$ and therefore ${}^{(t)}\boldsymbol{D}^\tau \subseteq {}^{(t)}\overset{\vee}{\boldsymbol{T}}{}^\sigma$. This implies (2.22).

q.e.d.

Lemma 2.4. *Let $\boldsymbol{\lambda}^0$ be locally flat and let ${}^{(t)}\underline{\underline{\mathbf{D}}}$ be satisfying conditions* (2.21) *and* (2.22). *Then*

$$\bar{\boldsymbol{\xi}}^{(t)} \ = \ \mathbb{P}\left\{\bar{\boldsymbol{\xi}} \,\middle|\, \underline{\underline{\mathbf{F}}}^{(t)}\right\} \ = \ \bar{\boldsymbol{\vartheta}}^{(t)} \ = \ \boldsymbol{\vartheta}^{\bar{\boldsymbol{x}}^{(t)}} \ . \tag{2.23}$$

Proof: Recall

$$\dot{\bar{\boldsymbol{\xi}}}^{(t)} \ = \ \mathbb{E}\left\{\dot{\bar{\boldsymbol{\xi}}} \,\middle|\, \underline{\underline{\mathbf{F}}}^{(t)}\right\} \ = \ \mathbb{E}\left\{\dot{\boldsymbol{\lambda}}^{(0)}\mathbb{1}_{\mid \overset{\vee}{\boldsymbol{E}}} + \sum_{\rho\in\mathbf{R}}(1 - l^\star_\rho)\mathbb{1}_{\mid \overset{\wedge}{\boldsymbol{E}}{}^\rho} \,\middle|\, \underline{\underline{\mathbf{F}}}^{(t)}\right\} \tag{2.24}$$

and

$$\dot{\bar{\vartheta}}^{(t)} = \dot{\vartheta}^{\bar{x}^{(t)}} = \dot{\lambda}^{(t)}\mathbb{1}_{\mid \check{T}} + \sum_{\rho\in\mathbf{R}}(1-{}^{(t)}h^{\star}_{\rho})\dot{\lambda}^{\rho}_{\mid \hat{T}^{\rho}} ; \tag{2.25}$$

Note that $\check{T}$ and $\hat{T}^{\rho}$ carry an index t, we omit writing ${}^{(t)}\hat{T}^{\rho}$ etc. for the sake of clarity. Now, as we assume local flatness. we can employ Lemma 2.3. Therefore, for the first two terms in (2.24) and (2.25) we obtain:

$$\begin{aligned}\mathbb{E}\left\{\dot{\lambda}^{(0)}\mathbb{1}_{\mid \check{E}}\middle|\underline{\underline{\mathbf{F}}}^{(t)}\right\} &= \mathbb{E}\left\{\dot{\lambda}^{(0)}\mathbb{1}_{\mid \check{T}}\middle|\underline{\underline{\mathbf{F}}}^{(t)}\right\}\\ &= \mathbb{E}\left\{\dot{\lambda}^{(0)}\middle|\underline{\underline{\mathbf{F}}}^{(t)}\right\}\mathbb{1}_{\mid \check{T}} = \dot{\lambda}^{(t)}\mathbb{1}_{\mid \check{T}}\\ &\quad \text{as } \mathbb{1}_{\mid \check{T}} \text{ is } \underline{\underline{\mathbf{F}}}^{(t)} \text{ measurable.}\end{aligned} \tag{2.26}$$

For the second terms in (2.24) and (2.25) respectively we find

$$\begin{aligned}&\mathbb{E}\left\{\sum_{\rho\in\mathbf{R}}(1-l^{\star}_{\rho})\mathbb{1}_{\mid \hat{E}^{\rho}}\middle|\underline{\underline{\mathbf{F}}}^{(t)}\right\}\\ &\sum_{\rho\in\mathbf{R}}(1-l^{\star}_{\rho})(\mathbb{1}_{\mid \hat{E}^{\rho}}) = \sum_{\rho\in\mathbf{R}}(1-{}^{(t)}h^{\star}_{\rho})\mathbb{1}_{\mid \hat{T}^{\rho}} ,\end{aligned} \tag{2.27}$$

again in view of Lemma 2.3. Now in view of (2.26) and (2.27) we obtain our result.

q.e.d.

Combining we obtain

Lemma 2.5. *Given the continuous model there exists a uniform model* ${}^{(t)}\underline{\underline{\mathbf{D}}}$ *such that the following holds true:*

1. ${}^{(t)}\check{T}^{\rho} = \check{E}^{\rho}$, ${}^{(t)}\hat{T}^{\rho} = \hat{E}^{\rho}$ $(\rho \in \mathbf{R})$.
2. ${}^{(t)}h^{0}_{\rho} = l^{0}_{\rho}$ $(\rho \in \mathbf{R})$; ${}^{(t)}h^{\star}_{\sigma} = l^{\star}_{\sigma}$ $(\sigma \in \mathbf{R})$.
3. $\bar{\xi}^{(t)} = \mathbb{P}\left\{\bar{\xi}\,\middle|\,\underline{\underline{\mathbf{F}}}^{(t)}\right\} = \bar{\vartheta}^{(t)} = \vartheta^{\bar{x}^{(t)}}$.

Definition 2.6. *Let* ${}^{(t)}\underline{\underline{\mathbf{D}}}$ *be a uniform model derived from the continuous model* $(\boldsymbol{I}, \underline{\underline{\mathbf{F}}}, \boldsymbol{v})$ *via the conditional measures* $\lambda^{(t)}$ *of* $\lambda^{(0)}$. *We call* ${}^{(t)}\underline{\underline{\mathbf{D}}}$ ***compatible*** *if the conditions listed in Lemma 2.5 are satisfied.*

We are now in the position to formulate a version of *u.h.c.* continuity complementing *l.h.c.* continuity presented in Theorem 1.10.

Theorem 2.7 (Dominance is uhc.). *Let* λ^{0} *be locally flat. Let* ξ *and* η *be imputations. Suppose that there is some compatible uniform model* ${}^{(t)}\underline{\underline{\mathbf{D}}}$ *and*

a relevant vector $\boldsymbol{a}^{(t)}$ *such that there is a coalition* $S^{(t)} = T^{\boldsymbol{a}^{(t)}}$ *obtained by Theorem 1.2 and hence satisfying Remark 1.3.*

Assume furthermore, that the conditional expectation $\boldsymbol{\xi}^{(t)} = \mathbb{P}\left\{\boldsymbol{\xi} \,\middle|\, \underline{\underline{\mathbf{F}}}^{(t)}\right\}$ *satisfies the following:*

1. $\boldsymbol{\xi}^{(t)} \operatorname{dom}_{S^t}^{\boldsymbol{v}^{(t)}} \boldsymbol{\eta}$.
2. $\{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\eta}}\} \in \underline{\underline{\mathbf{F}}}^{(t)}, \qquad t \in \mathbb{N}_0.$

Then, there is $t_0 \in \mathbb{N}_0$ *such that for all* $t \in \mathbb{N}_0,\ t \geq t_0$

$$\boldsymbol{\xi} \operatorname{dom}_{S^{(t)}}^{\boldsymbol{v}} \boldsymbol{\eta} \ .$$

Proof:

1st STEP :

We proceed as in the proof of Lemma 1.7. As $\{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\eta}}\} \in \underline{\underline{\mathbf{F}}}^{(t)}$, and $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are imputations, it follows that $\{\dot{\boldsymbol{\xi}} \leq \dot{\boldsymbol{\eta}}\} \subseteq \boldsymbol{I} \setminus S$ has positive measure.

Denote $\boldsymbol{T}_> := \{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\eta}}\}$ and $\boldsymbol{T}_\leq := \{\dot{\boldsymbol{\xi}} \leq \dot{\boldsymbol{\eta}}\}$. Choose $R^+ \subseteq \boldsymbol{T}_>$ and $R^- \subseteq \boldsymbol{T}_\leq$ of sufficiently small but positive measure such that $\boldsymbol{\lambda}(\{\boldsymbol{T}_> \setminus R^+\} \cap \boldsymbol{C}^\rho)$ is rational for all $\rho \in \mathbf{R}$ and

$$\int_{R^+} (\dot{\boldsymbol{\xi}} - \dot{\boldsymbol{\eta}}) d\boldsymbol{\lambda} = \int_{R^-} (\dot{\boldsymbol{\eta}} - \dot{\boldsymbol{\xi}}) d\boldsymbol{\lambda} \tag{2.28}$$

holds true. Decrease R^+ if necessary in a way such that for some $t_{00} \in \mathbb{N}_0$ we have $R^+ \in \underline{\underline{\mathbf{F}}}^{(t)} (t \in \mathbb{N}_0,\ t \geq t_{00})$. Decrease R^+ if necessary in a way such that for some $t_{11} \geq t_{00}$ we have $R^+ \cap S^t = \emptyset \ \ (t \in \mathbb{N}_0,\ t \geq t_{11})$. In view of *item 2* of our assumptions, we can now find $t_{22} \in \mathbb{N}, t_{22} \geq t_{11}$, such that for $t \in \mathbb{N}_0,\ t \geq t_{22}$, we have $(\{\boldsymbol{\xi} > \boldsymbol{\eta}\} \setminus \mathbb{R}^+) \in \underline{\underline{\mathbf{F}}}^{(t)}$. Combining these requirements, we have for $t \in \mathbb{N}_0,\ t \geq t_{22}$

$$(\{\boldsymbol{\xi} > \boldsymbol{\eta}\} \setminus \mathbb{R}^+) \in \underline{\underline{\mathbf{F}}}^{(t)} \quad , \quad R^+ \cap S^t = \emptyset. \tag{2.29}$$

Now consider the imputation

$$\boldsymbol{\vartheta} := \boldsymbol{\eta} + (\boldsymbol{\xi} - \boldsymbol{\eta}) \mathbb{1}_{R^+} - (\boldsymbol{\eta} - \boldsymbol{\xi}) \mathbb{1}_{R^-}. \tag{2.30}$$

We know that $\boldsymbol{\vartheta} = \boldsymbol{\xi} + (\boldsymbol{\eta} - \boldsymbol{\xi}) = \boldsymbol{\eta} > \boldsymbol{\xi}$ on R^+. Similarly, $\boldsymbol{\vartheta} = \boldsymbol{\eta} + (\boldsymbol{\xi} - \boldsymbol{\eta}) = \boldsymbol{\xi}$ on R^-.

Thus

$$\{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\vartheta}}\} = \boldsymbol{T}_> \setminus R + \ \subseteq \{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\eta}}\} \tag{2.31}$$

and $\{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\vartheta}}\} \in \underline{\underline{\mathbf{F}}}^{(t)}$ for $t \in \mathbb{N}_0, t \geq t_{22}$. Finally, $\boldsymbol{\vartheta} = \boldsymbol{\eta}$ holds true on S^t. Therefore we have clearly $\boldsymbol{\xi}^{(t)} \operatorname{dom}_S^{\boldsymbol{v}^{(t)}} \boldsymbol{\vartheta}$ as well.

By Theorem 1.2 we know that

$$\boldsymbol{\xi}^{(t)}(S^t) \leq \boldsymbol{v}^{(t)}(S^t). \tag{2.32}$$

Moreover, any atom/block ${}^{(t)}\boldsymbol{D}^{\tau}$ of $\underline{\underline{\mathbf{F}}}^{(t)}$ is either contained in or disjoint to $\{\boldsymbol{\xi} > \boldsymbol{\vartheta}\}$. For every atom ${}^{(t)}\boldsymbol{D}^{\tau}$ of $S^{(t)}$ we know that ${}^{(t)}\boldsymbol{D}^{\tau} \cap \{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\vartheta}}\}$ has positive measure, hence ${}^{(t)}\boldsymbol{D}^{\tau} \subseteq \{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\vartheta}}\}$.

Therefore, from $\boldsymbol{\xi}^{(t)} > \boldsymbol{\vartheta}$ on S^t it follows that

$$\boldsymbol{\xi} > \boldsymbol{\vartheta} \quad \text{on} \quad S^t. \tag{2.33}$$

2ndSTEP :

Typically we shall treat the case that relevant vector $\boldsymbol{a}^{(t)}$ is of the third type, $\boldsymbol{a}^{(t)} = {}^{(t)}\boldsymbol{a}^{\ominus}$, the other types can be treated in exactly the same way. Thus our relevant coalition is

$$S_t = T^{\frac{1}{t}\boldsymbol{a}^{(t)}} = T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^{\ominus}} .$$

As previously, assume that the critical coordinate for the construction of ${}^{(t)}\boldsymbol{a}^{\ominus}$ is r. That is, ${}^{(t)}\boldsymbol{a}^{\ominus}$ is of the shape

$${}^{(t)}\boldsymbol{a}^{\ominus} = (1, \ldots, 1, \ldots, 1, \ldots\ldots, \alpha_r, .., \beta_r) \tag{2.34}$$

with

$$\begin{aligned} \alpha_r &= a^{\ominus(t)}_{\widehat{\tau}_r} = \frac{(h_{\overline{\tau}_1} + \ldots + h_{\overline{\tau}_{r-1}} + h_{\overline{\overline{\tau}}_r}) - 1}{h_{\overline{\overline{\tau}}_r} - h_{\overline{\tau}_r}} , \\ \beta_r &= a^{\ominus(t)}_{\overline{\overline{\tau}}_r} = \frac{1 - (h_{\overline{\tau}_1} + \ldots + h_{\overline{\tau}_r})}{h_{\overline{\overline{\tau}}_r} - h_{\overline{\tau}_r}} ; \end{aligned} \tag{2.35}$$

according to the definition of relevant vectors (Theorem 3.5. of *Part I*). Note that $\alpha_r + \beta_r = 1$. Let

$$\widehat{\boldsymbol{\tau}} = (\widehat{\tau}_1, \ldots, \widehat{\tau}_r, \overline{\overline{\tau}}_r)$$

denote the sequence corresponding to the positive coordinates of ${}^{(t)}\boldsymbol{a}^{\ominus}$. For $\rho \in \mathbf{R} \setminus \{r\}$, the coalition

$$T^{\frac{1}{t}\boldsymbol{a}^{(t)}} \cap \boldsymbol{C}^{\rho} = {}^{(t)}\boldsymbol{D}^{\widehat{\tau}_\rho} \quad (\rho \in \mathbf{R} \setminus \{r\}) . \tag{2.36}$$

yields

$$\bar{\boldsymbol{\xi}}({}^{(t)}\boldsymbol{D}^{\widehat{\tau}_\rho}) = \boldsymbol{\lambda}^0({}^{(t)}\boldsymbol{D}^{\widehat{\tau}_\rho}) , \quad {}^{(t)}\bar{\boldsymbol{\xi}}({}^{(t)}\boldsymbol{D}^{\widehat{\tau}_\rho}) = \boldsymbol{\lambda}^{(t)}({}^{(t)}\boldsymbol{D}^{\widehat{\tau}_\rho}) , \tag{2.37}$$

as ${}^{(t)}\boldsymbol{D}^{\widehat{\tau}_\rho} \in \overset{\vee}{\boldsymbol{T}}{}^{(t)} = \overset{\vee}{\boldsymbol{E}}$. moreover, as the ${}^{(t)}$ quantities within the right hand equation are the conditional probabilities of the quantities on the left hand equation, we find that *all* quantities involved are equal, hence we have

$$\bar{\boldsymbol{\xi}}({}^{(t)}\boldsymbol{D}^{\widehat{\tau}_\rho}) = \boldsymbol{\lambda}^0({}^{(t)}\boldsymbol{D}^{\widehat{\tau}_\rho}) = {}^{(t)}\bar{\boldsymbol{\xi}}({}^{(t)}\boldsymbol{D}_{\widehat{\tau}_\rho}) = \boldsymbol{\lambda}^{(t)}({}^{(t)}\boldsymbol{D}^{\widehat{\tau}_\rho}) \quad \rho \in \mathbf{R} \subseteq \{r\} . \tag{2.38}$$

3rdSTEP :

Next turn to $\boldsymbol{C}^r$, we write for short

$$T^{\alpha_r} \;:=\; T^{\frac{1}{t}\boldsymbol{a}^{(t)}} \cap^{(t)} \boldsymbol{D}^{\widehat{\tau}_r} \quad \text{and} \quad T^{\beta_r} \;:=\; T^{\frac{1}{t}\boldsymbol{a}^{(t)}} \cap^{(t)} \boldsymbol{D}^{\overline{\overline{\tau}}_r}$$

Again, regarding $\widehat{\boldsymbol{\tau}}_r$, we have ${}^{(t)}\boldsymbol{D}^{\widehat{\tau}_r} \in \overset{\vee}{\boldsymbol{E}} \;=\; \overset{\vee}{\boldsymbol{T}}$ hence

$$\bar{\boldsymbol{\xi}} = \boldsymbol{\lambda}^0 \quad \text{and} \quad \bar{\boldsymbol{\xi}}^{(t)} \;=\; \boldsymbol{\lambda}^{(t)} \quad \text{on} \quad {}^{(t)}\boldsymbol{D}^{\widehat{\tau}_r} \tag{2.39}$$

Now $\boldsymbol{\lambda}^0$ may vary on that set and we cannot argue with measurability as T^{α_r} is not measurable. However, according to Remark 1.3 we do have complete freedom for choosing T^{α_r} up to preserving its measure α_r. Hence, as $\boldsymbol{\lambda}^{(t)}$ is the conditional probability of $\boldsymbol{\lambda}^0$ on ${}^{(t)}\boldsymbol{D}^{\widehat{\tau}_\rho}$ we can *choose* T^{α_r} in such a way that $\boldsymbol{\lambda}^0(T^{\alpha_r}) \;=\; \boldsymbol{\lambda}^{(t)}(T^{\alpha_r})$ holds true. Thus, using (2.39), we have again (the analogue to (2.38))

$$\bar{\boldsymbol{\xi}}(T^{\alpha_r}) \;=\; \boldsymbol{\lambda}^0(T^{\alpha_r}) \;=\; {}^{(t)}\bar{\boldsymbol{\xi}}(T^{\alpha_r}) \;=\; \boldsymbol{\lambda}^{(t)}(T^{\alpha_r}) \quad . \tag{2.40}$$

4thSTEP :

Regarding $\overline{\overline{\tau}}_r$, we now have ${}^{(t)}\boldsymbol{D}^{\overline{\overline{\tau}}_r} \in \overset{\wedge}{\boldsymbol{E}} \;=\; \overset{\wedge}{\boldsymbol{T}}{}^{(t)}$, hence

$$\bar{\boldsymbol{\xi}} = l_r^\star \;=\; h_r^\star \;=\; \bar{\boldsymbol{\xi}}^{(t)} \quad \text{on} \quad {}^{(t)}\boldsymbol{D}_{\overline{\overline{\tau}}_r} \;. \tag{2.41}$$

This time we *choose* T^{β_r} to be such that

$$\boldsymbol{\lambda}^0(T^{\alpha_r}) \;=\; \boldsymbol{\lambda}^{(t)}(T^{\alpha_r}) \tag{2.42}$$

which is possible as above via the conditional probability argument. Combining (2.38), (2.40), and (2.41) we obtain

$$\bar{\boldsymbol{\xi}}(T^{\frac{1}{t}\boldsymbol{a}^{(t)}}) \;=\; \bar{\boldsymbol{\xi}}^{(t)}(T^{\frac{1}{t}\boldsymbol{a}^{(t)}}) \tag{2.43}$$

and combining (2.38), (2.40), and (2.42) we have

$$\boldsymbol{\lambda}^0(T^{\frac{1}{t}\boldsymbol{a}^{(t)}}) \;=\; \boldsymbol{\lambda}^{(t)}(T^{\frac{1}{t}\boldsymbol{a}^{(t)}}) \tag{2.44}$$

which implies

$$\boldsymbol{v}(T^{\frac{1}{t}\boldsymbol{a}^{(t)}}) \;=\; \boldsymbol{v}^{(t)}(T^{\frac{1}{t}\boldsymbol{a}^{(t)}}) \quad . \tag{2.45}$$

Therefore

$$\bar{\boldsymbol{\xi}}(T^{\frac{1}{t}\boldsymbol{a}^{(t)}}) \;=\; \bar{\boldsymbol{\xi}}^{(t)}(T^{\frac{1}{t}\boldsymbol{a}^{(t)}}) \;\leq \boldsymbol{v}^{(t)}(T^{\frac{1}{t}\boldsymbol{a}^{(t)}}) \;=\; \boldsymbol{v}(T^{\frac{1}{t}\boldsymbol{a}^{(t)}}) \;. \tag{2.46}$$

Now combining (2.33) and (2.46) and observing that $\boldsymbol{\eta}$ and $\boldsymbol{\vartheta}$ coincide on $S^t \;=\; T^{\frac{1}{t}\boldsymbol{a}^{(t)}}$, we obtain

$$\boldsymbol{\xi}\,\mathrm{dom}^{\boldsymbol{v}}_{S^t}\,\boldsymbol{\eta} \quad (t \geq t_0),$$

q.e.d.

For the remainder of this SECTION and for SECTION 3 we assume that $\boldsymbol{\lambda}^0$ is locally flat, hence Lemma 2.5 applies. Thuse we have we have (2.21), (2.22), and (2.23) for some compatible ${}^{(t)}\underline{\underline{\mathbf{D}}}$.

Next we have to augment the sub–imputation $\bar{\boldsymbol{\xi}}$ to an imputation. The development is quite similar to the one in *Part IV*. We have to distribute that missing chunk of mass over $\hat{\boldsymbol{E}}$ – as in the uniform model the mass missing is distributed over $\hat{\boldsymbol{T}}$.

Recall the construction for a uniform model. A vector $\boldsymbol{\Delta} := (\Delta_1, \ldots, \Delta_r)$ is called an ***admissible distribution of mass*** (Definition 4.5. of *Part IV*, [4]) if it satisfies the conditions

$$\Delta_\rho < h^\star_\rho \quad \text{and} \quad \sum_{\rho \in \mathbf{R}} \underline{\Delta}_\rho \boldsymbol{\lambda}(\hat{\boldsymbol{T}}^\rho) \;=\; 1 - \bar{\boldsymbol{x}}(\boldsymbol{I}) \tag{2.47}$$

(see equation 4.54 of *Part IV*). Naturally the analogue for the continuous model leads to

Definition 2.8. *A vector* $\underline{\boldsymbol{\Delta}} = (\underline{\Delta}_1, \ldots, \underline{\Delta}_r)$, *is said to represent an* ***admissible distribution of mass*** *if*

$$\underline{\Delta}_\rho < l^\star_\rho \;\; (\rho \in \mathbf{R}) \;\; ; \sum_{\rho \in \mathbf{R}} \underline{\Delta}_\rho \boldsymbol{\lambda}(\hat{\boldsymbol{E}}^\rho) \;=\; 1 - \bar{\boldsymbol{\xi}}(\boldsymbol{I}) \tag{2.48}$$

is satisfied.

Now if a the uniform model is given by ${}^{(t)}\underline{\underline{\mathbf{D}}}$, then $\bar{\boldsymbol{x}}^{(t)}$ is established by (2.12). Given an admissible distribution of mass, we obtain a pre–imputation

$$\overset{\circ}{\bar{\boldsymbol{x}}}{}^{(t)} \;:=\; \bar{\boldsymbol{x}}^{(t)} + \sum_{\rho \in \mathbf{R}} \Delta_\rho \sum_{\tau \in {}^{(t)}\hat{\mathbf{T}}^\rho} \boldsymbol{e}^\tau \;=:\; \bar{\boldsymbol{x}}^{(t)} +^{(t)} \bar{\boldsymbol{x}}^{\boldsymbol{\Delta}} \;. \tag{2.49}$$

The corresponding imputation is constructed via

$$\overset{\circ}{\bar{\boldsymbol{\vartheta}}}{}^{(t)} \;:=\; \boldsymbol{\vartheta}^{\overset{\circ}{\bar{\boldsymbol{x}}}{}^{(t)}} \quad \text{and} \quad {}^{(t)}\boldsymbol{\vartheta}^{\boldsymbol{\Delta}} \;:=\; \boldsymbol{\vartheta}^{{}^{(t)}\bar{\boldsymbol{x}}^{\boldsymbol{\Delta}}} \;=\; \boldsymbol{\vartheta}^{{}^{(t)}\bar{\boldsymbol{x}}^{\boldsymbol{\Delta}}{}^{(t)}} \tag{2.50}$$

so that (2.49) results in

$$\overset{\circ}{\bar{\boldsymbol{\vartheta}}}{}^{(t)} \;=\; \bar{\boldsymbol{\vartheta}}^{(t)} +^{(t)} \boldsymbol{\vartheta}^{\boldsymbol{\Delta}} \tag{2.51}$$

The imputation ${}^{(t)}\boldsymbol{\vartheta}^{\boldsymbol{\Delta}}$ has a step function density, hence resembles a multiple of uniform distribution on every ${}^{(t)}\hat{\mathbf{T}}^\rho$ (see *Part IV*). The continuous counterpart (viewing $\bar{\boldsymbol{\xi}}$) of $\overset{\circ}{\bar{\boldsymbol{\vartheta}}}{}^{(t)}$ is therefore the measure

$$\overset{\circ}{\bar{\boldsymbol{\xi}}} \;:=\; \bar{\boldsymbol{\xi}} + \sum_{\rho \in \mathbf{R}} \underline{\Delta}_\rho \boldsymbol{\lambda}^\rho{}_{|\hat{\boldsymbol{E}}} \;:=\; \bar{\boldsymbol{\xi}} + \boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}} \tag{2.52}$$

with an admissible distribution of mass $\underline{\boldsymbol{\Delta}}$. We have to make sure that the non–discriminating use of the term is justified.

Corollary 2.9. *Let $\underline{\boldsymbol{\Delta}}$ be an admissible distribution of mass (within the continuous model) and let ${}^{(t)}\underline{\underline{\mathbf{D}}}$ be a compatible uniform model. Then, $\underline{\boldsymbol{\Delta}}$ is an admissible distribution of mass for ${}^{(t)}\underline{\underline{\mathbf{D}}}$ and vice versa.*

Proof: As $\boldsymbol{\xi}^{(t)}$ is the conditional expectation of $\bar{\boldsymbol{\xi}}$ we know that $\boldsymbol{\xi}^{(t)}(\boldsymbol{I}) = \bar{\boldsymbol{\xi}}(\boldsymbol{I})$. Hence,

$$\Delta^{(t)} = t\Delta = t(1-\bar{\boldsymbol{\xi}}(\boldsymbol{I})) = t\left(1-\boldsymbol{\xi}^{(t)}(\boldsymbol{I})\right) = t - t\boldsymbol{\xi}^{(t)}(\boldsymbol{I}) \ . \tag{2.53}$$

In view of (2.21) of Lemma 2.3 we see that

$$\underline{\Delta}_\rho < l^\star_\rho = h^{\star(t)}_\rho \tag{2.54}$$

for $\rho \in \mathbf{R}$ and $t \in \mathbb{N}_0$.

Both equations (2.53) and (2.54) show that $\boldsymbol{\Delta}$ is an admissible distribution of mass within the context of the continuous model as well as in the context of ${}^{(t)}\underline{\underline{\mathbf{D}}}$.

q.e.d.

Lemma 2.10. *Let ${}^{(t)}\underline{\underline{\mathbf{D}}}$ be a compatible uniform model.*

1. *Let $\underline{\boldsymbol{\Delta}}$ be an admissible distribution of mass (for $\bar{\boldsymbol{\xi}}$ as well as for $\bar{\boldsymbol{\vartheta}}^{(t)}$). Then for $\overset{\circ}{\bar{\boldsymbol{\xi}}} = \bar{\boldsymbol{\xi}} + \boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}}$ and $\overset{\circ}{\bar{\boldsymbol{x}}}^{(t)} = \bar{\boldsymbol{x}}^{(t)} + \boldsymbol{e}^{\underline{\boldsymbol{\Delta}}}$ we have*

$$\overset{\circ}{\bar{\boldsymbol{\xi}}}^{(t)} = \mathbb{P}\left\{\overset{\circ}{\bar{\boldsymbol{\xi}}} \,\middle|\, \underline{\underline{\mathbf{F}}}^{(t)}\right\} = \overset{\circ}{\bar{\boldsymbol{\vartheta}}}^{(t)} = \boldsymbol{\vartheta}^{\overset{\circ}{\bar{\boldsymbol{x}}}^{(t)}} \tag{2.55}$$

2. *Let $\underline{\boldsymbol{\Delta}}$ be admissible and let $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_r)$ be a "convex" set of weights, i.e., nonnegative and summing up to 1. Let*

$$\begin{aligned} \widehat{\boldsymbol{\xi}} = \widehat{\boldsymbol{\xi}}^{\boldsymbol{\alpha}\underline{\boldsymbol{\Delta}}} &:= \alpha_0 \overset{\circ}{\bar{\boldsymbol{\xi}}} + \sum_{\rho\in\mathbf{R}} \alpha_\rho \boldsymbol{\lambda}^\rho \\ &= \alpha_0\left(\bar{\boldsymbol{\xi}} + \boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}}\right) + \sum_{\rho\in\mathbf{R}} \alpha_\rho \boldsymbol{\lambda}^\rho \in \mathcal{H} \end{aligned} \tag{2.56}$$

and

$$\begin{aligned} \widehat{\boldsymbol{x}}^{(t)} = \widehat{\boldsymbol{x}}^{(t)\boldsymbol{\alpha}\underline{\boldsymbol{\Delta}}} &:= \alpha_0 \overset{\circ}{\bar{\boldsymbol{x}}}^{(t)} + \sum_{\rho\in\mathbf{R}} \alpha_\rho \boldsymbol{e}^{\boldsymbol{C}^\rho} \\ &= \alpha_0\left(\bar{\boldsymbol{x}}^{(t)} + \boldsymbol{e}^{\underline{\boldsymbol{\Delta}}}\right) + \sum_{\rho\in\mathbf{R}} \alpha_\rho \boldsymbol{e}^{\boldsymbol{C}^\rho} \ . \end{aligned} \tag{2.57}$$

Then we have

$$\widehat{\boldsymbol{\xi}}^{(t)} = \mathbb{P}\left\{\widehat{\boldsymbol{\xi}} \,\middle|\, \underline{\underline{\mathbf{F}}}^{(t)}\right\} = \widehat{\boldsymbol{\vartheta}}^{(t)} = \boldsymbol{\vartheta}^{\widehat{\boldsymbol{x}}^{(t)}} \ . \tag{2.58}$$

Proof: Essentially this follows from (2.23) of Lemma 2.3 as the $\boldsymbol{\lambda}^{\rho}$ are $\underline{\underline{\mathbf{F}}}^{(t)}$ measurable. More precisely,

$$
\begin{aligned}
\overset{\circ}{\overline{\boldsymbol{\xi}}}^{(t)} = \overset{\circ}{\overline{\boldsymbol{\xi}}}^{\underline{\boldsymbol{\Delta}}(t)} &= \mathbb{P}\left\{\overset{\circ}{\overline{\boldsymbol{\xi}}} \,\middle|\, \underline{\underline{\mathbf{F}}}^{(t)}\right\} \\
&= \mathbb{P}\left\{\overline{\boldsymbol{\xi}} + \boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}} \,\middle|\, \underline{\underline{\mathbf{F}}}^{(t)}\right\} = \overline{\boldsymbol{\xi}}^{(t)} + \mathbb{P}\left\{\boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}} \,\middle|\, \underline{\underline{\mathbf{F}}}^{(t)}\right\} \\
&= \overline{\boldsymbol{\xi}}^{(t)} + \sum_{\rho\in\mathbf{R}} \underline{\Delta}_{\rho}\mathbb{P}\left\{\boldsymbol{\lambda}_{\mid \hat{\boldsymbol{E}}^{\rho}} \,\middle|\, \underline{\underline{\mathbf{F}}}^{(t)}\right\} \\
&= \overline{\boldsymbol{\xi}}^{(t)} + \sum_{\rho\in\mathbf{R}} \underline{\Delta}_{\rho}\boldsymbol{\lambda}_{\mid \hat{\boldsymbol{E}}^{\rho}} \\
&= \overline{\boldsymbol{\vartheta}}^{(t)} + \sum_{\rho\in\mathbf{R}} \underline{\Delta}_{\rho}\boldsymbol{\lambda}_{\mid \hat{\boldsymbol{T}}^{\rho}} \\
&= \overset{\circ}{\overline{\boldsymbol{\vartheta}}}^{(t)}
\end{aligned}
\tag{2.59}
$$

proves (2.55). Obviously, the proof of (2.58) is running along the same lines,

q.e.d.

3 The Standard vNM–Stable Set

We focus on the continuous model $(\boldsymbol{I}, \underline{\underline{\mathbf{F}}}, \boldsymbol{v})$ assuming that $\boldsymbol{\lambda}^0$ is locally flat. Let $\underline{\boldsymbol{\Delta}}$ be an admissible distribution of mass and consider the set

$$\mathcal{H} \;:=\; \mathbf{ConvH}\left\{\overset{\circ}{\bar{\boldsymbol{\xi}}}{}^{\underline{\boldsymbol{\Delta}}}, \boldsymbol{\lambda}^{\rho} \;\; (\rho \in \mathbf{R})\right\} . \tag{3.1}$$

We will prove within this section that $\mathcal{H}$ is a vNM–Stable Set. The assumption of local flatness serves to avoid a limiting procedure as it implies that – for sufficiently large t and some compatible ${}^{(t)}\underline{\underline{\mathbf{D}}}$ – the "flat" part of the central measure $\boldsymbol{\lambda}^0$ provides the "minimal" undercutting sequences of $\boldsymbol{\lambda}^{(t)}$. To remove this basic assumption will be another task to be tackled in SECTION 4.

Remark 3.1. The role of the admissile distribution of mass $\boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}}$ is discussed in the uniform context in SECTION 5 of *Part IV* ([4]). As is seen in the proof of *Theorem 5.4.* (see the $3^{rd}STEP$ in particular), the quantity $\boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}}$ serves to render $\overset{\circ}{\bar{\boldsymbol{\xi}}}$ to be an imputation. However the dominance relations are actually provided w.r.t the sub–imputation obtained by omitting $\boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}}$. The measure $\boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}}$ lives on $\overset{\wedge}{\boldsymbol{E}}$ and dominance w.r.t some relevant coalition involves $\overset{\wedge}{\boldsymbol{E}}$ only for a relevant vector $\boldsymbol{a}^{\ominus}$. But it is seen, that the dominance relations are actually constructed w.r.t. the standard truncation measure (sub–imputation) $\bar{\boldsymbol{\xi}}$, the measure $\boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}}$ is actually not involved.

Therefore, frequently we can without loss of generality assume that $\overset{\circ}{\bar{\boldsymbol{\xi}}}$ is an imputation hence $\boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}} = 0$, $\overset{\circ}{\bar{\boldsymbol{\xi}}} \;=\; \bar{\boldsymbol{\xi}}$. Consequently the typical element of $\mathcal{H}$ writes

$$\widehat{\boldsymbol{\xi}} \;:=\; \alpha_0\, \bar{\boldsymbol{\xi}} + \sum_{\rho \in \mathbf{R}} \alpha_\rho \boldsymbol{\lambda}^{\rho} . \tag{3.2}$$

We will base our discussion on this assumption in SECTION 4. In the present section we will continue to carry on $\boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}}$. We make it clear that $\boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}}$ does not harm the approximation procedure. In passing, this procedure serves also to demonstrate that our above arguments about the negligible role of $\boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}}$ hold true for the continuous model as well.

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Remark 3.2. The *Standard vNM–Stable Set* is defined in [5] and [6] with respect to a purely orthogonal production game. It is the convex hull of r imputations, say $\boldsymbol{\vartheta}^{\rho}$ $(\rho \in \mathbf{R})$. Each one is restricted to a carrier $\boldsymbol{C}^{\rho}$ and satisfies two conditions:

$\boldsymbol{\vartheta}^{\rho}$ is absolutely continous w.r.t $\boldsymbol{\lambda}^{\rho}$.

The density $\overset{\bullet}{\boldsymbol{\vartheta}}{}^{\rho}$ w.r.t $\boldsymbol{\lambda}^{\rho}$ is restricted by 1.

In particular, if $\boldsymbol{\lambda}^\rho(\boldsymbol{I}) = 1 = \boldsymbol{v}(\boldsymbol{I})$ is normalized (hence an extremal of the core), then $\boldsymbol{\vartheta}^\rho = \boldsymbol{\lambda}^\rho$.

The framework of a continuous model involves the central measure $\boldsymbol{\lambda}^{(0)}$. The set $\mathcal{H}$ is the convex hull of $r+1$ imputations. The measures $\boldsymbol{\lambda}^\rho$ seen as imputations (the extremals of the core) naturally satisfy the two conditions above. In this respect, the situation is exactly the same as in the purely orthogonal case.

The imputation $\overset{\circ}{\bar{\boldsymbol{\xi}}}{}^{\underline{\boldsymbol{\Delta}}}$ is a sum of the standard truncation $\bar{\boldsymbol{\xi}}$ and some fractions of the $\boldsymbol{\lambda}^\rho$. Let us assume (in view of Remark 3.1) that $\bar{\boldsymbol{\xi}}$ is an imputation and $\boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}} = 0$, hence the elements of $\mathcal{H}$ have the form (3.2), that is

$$\widehat{\boldsymbol{\xi}} \;:=\; \alpha_0\, \bar{\boldsymbol{\xi}} + \sum_{\rho\in\mathbf{R}} \alpha_\rho \boldsymbol{\lambda}^\rho \;.$$

Now, with respecto to some $\boldsymbol{\lambda}^\rho$, the density of $\bar{\boldsymbol{\xi}}$ on $\boldsymbol{C}^\rho \subseteq \overset{\vee}{\boldsymbol{T}}$ is the one of $\bar{\boldsymbol{\xi}} = \boldsymbol{\lambda}^0$ wich is bounded by $1 - l^\star_\rho \le 1$. The density on some $\boldsymbol{C}^\rho \subseteq \overset{\wedge}{\boldsymbol{T}}$ equals $1 - l^\star_\rho + \underline{\Delta}_\rho \le 1$ by (2.48) which is $1 - l^\star_\rho < 1$ when $\boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}} = 0$. Therefore

$$\frac{d\bar{\boldsymbol{\xi}}}{d\boldsymbol{\lambda}^\rho} \le \alpha_0(1 - l^\star_\rho) + \alpha_\rho \le 1 \quad (\rho \in \mathbf{R}) \;.$$

Next, $\frac{d\bar{\boldsymbol{\xi}}}{d\boldsymbol{\lambda}^0}$ is directly computet in view of Definition 2.1 in SECTION 2, that is (2.14) and (2.15); we have

$$\frac{d\bar{\boldsymbol{\xi}}}{d\boldsymbol{\lambda}^0} \;=\; \left\{ \begin{array}{rcl} 1 & \text{on} & \overset{\vee}{\boldsymbol{T}} \\ 1 - l^\star_\rho & \text{on} & \overset{\wedge}{\boldsymbol{T}} \end{array} \right\} \;\le\; 1 \;.$$

Thus, $\bar{\boldsymbol{\xi}}$ satisfies the above two conditions in a suitably modified sense: it is absolutely continuous w.r.t Lebesgue measure $\boldsymbol{\lambda}$ (or all the $\boldsymbol{\lambda}^\rho$) and the density w.r.t *all* measures $\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r, \boldsymbol{\lambda}^0$ is bounded by 1. This justifies the following definition.

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Definition 3.3. *The set $\mathcal{H}$ is called a* ***Standard vNM–Stable Set****.*

Now we prove that $\mathcal{H}$ indeed deserves to be called stable.

Theorem 3.4. *Let $\underline{\boldsymbol{\Delta}}$ be an admissible distribution of mass and let*

$$\overset{\circ}{\bar{\boldsymbol{\xi}}} \;=\; \overset{\circ}{\bar{\boldsymbol{\xi}}}{}^{\underline{\boldsymbol{\Delta}}} \;=\; \bar{\boldsymbol{\xi}} + \boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}}$$

be the resulting imputation. Then

$$\mathcal{H} \;:=\; \boldsymbol{ConvH}\left\{ \overset{\circ}{\bar{\boldsymbol{\xi}}}{}^{\underline{\boldsymbol{\Delta}}}, \boldsymbol{\lambda}^\rho \;\; (\rho \in \mathbf{R}) \right\} \tag{3.3}$$

is internally stable.

Proof:

1stSTEP :

By Lemma 2.5 we can find a compatible uniform model ${}^{(t)}\underline{\underline{\mathbf{D}}}$ such that ${}^{(t)}\overset{\vee}{\boldsymbol{T}}$ and ${}^{(t)}\overset{\wedge}{\boldsymbol{T}}$ are $\underline{\underline{\mathbf{F}}}^{(t)}$ measurable. Hence for every $\rho \in \mathbf{R}$ we have either ${}^{(t)}\boldsymbol{D}^{\tau} \subseteq^{(t)} \overset{\vee}{\boldsymbol{T}}^{\rho}$ or else ${}^{(t)}\boldsymbol{D}^{\tau} \subseteq^{(t)} \overset{\wedge}{\boldsymbol{T}}^{\rho}$.

1stSTEP : Let $\widehat{\boldsymbol{\xi}},\ \widehat{\boldsymbol{\eta}} \in \mathcal{H}$ be imputations and let S be a coalition satisfying

$$\widehat{\boldsymbol{\xi}}\, \mathrm{dom}_S\, \widehat{\boldsymbol{\eta}}$$

Then there are sets of "convex" coefficients $\{\alpha_\rho\}_{\rho\in\mathbf{R}_0},\ \{\beta_\rho\}_{\rho\in\mathbf{R}_0}$ such that

$$\widehat{\boldsymbol{\xi}} \;=\; \alpha_0 \overset{\circ}{\overline{\boldsymbol{\xi}}}{}^{\underline{\boldsymbol{\Delta}}} + \sum_{\rho\in\mathbf{R}} \alpha_\rho \boldsymbol{\lambda}^\rho \;\text{ and }\; \widehat{\boldsymbol{\eta}} \;=\; \beta_0 \overset{\circ}{\overline{\boldsymbol{\xi}}}{}^{\underline{\boldsymbol{\Delta}}} + \sum_{\rho\in\mathbf{R}} \beta_\rho \boldsymbol{\lambda}^\rho$$

holds true. As $\overset{\wedge}{\boldsymbol{E}}$ is measurable w.r.t $\underline{\underline{\mathbf{F}}}^{(t)}$, we have

$$\overset{\circ}{\overline{\boldsymbol{\xi}}}{}^{(t)} \;=\; \overset{\circ}{\overline{\boldsymbol{\xi}}}{}^{\underline{\boldsymbol{\Delta}}(t)} \;+\; \sum_{\rho\in\mathbf{R}} \underline{\Delta}_\rho \boldsymbol{\lambda} \;\Big|\; \overset{\wedge}{\boldsymbol{E}}{}^{\rho}$$

Accordingly, the two imputations $\widehat{\boldsymbol{\xi}},\ \widehat{\boldsymbol{\eta}} \in \mathcal{H}$ involved yield conditional versions

$$\widehat{\boldsymbol{\xi}}^{(t)} \;=\; \alpha_0 \overset{\circ}{\overline{\boldsymbol{\xi}}}{}^{(t)} + \sum_{\rho\in\mathbf{R}} \alpha_\rho \boldsymbol{\lambda}^\rho$$

as well as

$$\widehat{\boldsymbol{\eta}}^{(t)} \;=\; \beta_0 \overset{\circ}{\overline{\boldsymbol{\xi}}}{}^{(t)} + \sum_{\rho\in\mathbf{R}} \beta_\rho \boldsymbol{\lambda}^\rho \;.$$

On the other hand, a second set of quantities adapted results from the construction of the vector $\bar{\boldsymbol{x}}^{(t)}$ as obtained within the framework of the game generated by

$$\boldsymbol{\lambda}^{(t)} = \mathbb{P}\left\{\boldsymbol{\lambda}^{(0)} \,\middle|\, \underline{\underline{\mathbf{F}}}^{(t)}\right\}$$

via the construction provided in *Part IV*. The corresponding imputation is $\overline{\boldsymbol{\vartheta}}^{(t)} \;=\; \boldsymbol{\vartheta}^{\bar{\boldsymbol{x}}(t)}$. Using the same $\underline{\boldsymbol{\Delta}}$ we obtain the imputation

$$\overset{\circ}{\overline{\boldsymbol{\vartheta}}}{}^{(t)} \;=\; \overline{\boldsymbol{\vartheta}}^{(t)} + \boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}} \;. \tag{3.4}$$

which is obtained in the usual manner by

$$\overset{\circ}{\bar{\boldsymbol{x}}}{}^{(t)} \;=\; \bar{\boldsymbol{x}}^{(t)} + \sum_{\rho\in\overset{\wedge}{\boldsymbol{T}}{}^{(t)}} \underline{\Delta}_\rho \boldsymbol{e}^{C^\rho} \;=\; \bar{\boldsymbol{x}}^{(t)} + \boldsymbol{e}^{\underline{\boldsymbol{\Delta}}}$$

via

$$\overset{\circ}{\overline{\boldsymbol{\vartheta}}}{}^{(t)} \;=\; \boldsymbol{\vartheta}^{\overset{\circ}{\bar{\boldsymbol{x}}}{}^{(t)}} \;.$$

This construction canonically results in sub–imputations

$$\widehat{\boldsymbol{x}}^{(t)} \;=\; \alpha_0 \overset{\circ}{\overline{\boldsymbol{\vartheta}}}^{(t)} + \sum_{\rho\in\mathbf{R}} \alpha_\rho \boldsymbol{e}^{\boldsymbol{C}^\rho} \;, \quad \widehat{\boldsymbol{y}}^{(t)} \;=\; \beta_0 \overset{\circ}{\overline{\boldsymbol{\vartheta}}}^{(t)} + \sum_{\rho\in\mathbf{R}} \beta_\rho \boldsymbol{e}^{\boldsymbol{C}^\rho}$$

and imputations

$$\boldsymbol{\vartheta}^{\widehat{\boldsymbol{x}}(t)} \;:=\; \alpha_0 \overset{\circ}{\overline{\boldsymbol{\vartheta}}}^{(t)} + \sum_{\rho\in\mathbf{R}} \alpha_\rho \boldsymbol{\lambda}^\rho \quad \text{and} \quad \boldsymbol{\vartheta}^{\widehat{\boldsymbol{y}}(t)} \;:=\; \beta_0 \overset{\circ}{\overline{\boldsymbol{\vartheta}}}^{(t)} + \sum_{\rho\in\mathbf{R}} \beta_\rho \boldsymbol{\lambda}^\rho \tag{3.5}$$

Recall that $\underline{\boldsymbol{\Delta}}$ is an admissible distribution of mass in the context of the uniform model induced by ${}^{(t)}\underline{\underline{\mathbf{D}}}$ as well in the context of the continuous model.

2ndSTEP :

Now consider the *l.h.c.*–Theorem 1.10. Accordingly, for some $t \in \mathbb{N}_0$, choose $\boldsymbol{a}^{(t)}$ such that

$$T^{\boldsymbol{a}(t)} \subseteq S$$

and the relations

$$\widehat{\boldsymbol{\xi}} \operatorname{dom}^{\boldsymbol{v}}_{T^{\boldsymbol{a}(t)}} \widehat{\boldsymbol{\eta}}, \;\; \widehat{\boldsymbol{\xi}} \operatorname{dom}^{\boldsymbol{v}^{(t)}}_{T^{\boldsymbol{a}(t)}} \widehat{\boldsymbol{\eta}}, \;\; \text{and} \;\; \widehat{\boldsymbol{\xi}}^{(t)} \operatorname{dom}^{\boldsymbol{v}^{(t)}}_{T^{\boldsymbol{a}(t)}} \widehat{\boldsymbol{\eta}}^{(t)} \tag{3.6}$$

are satisfied.

However, by (2.58) in Lemma 2.10 we know that

$$\widehat{\boldsymbol{\xi}}^{(t)} \;=\; \mathbb{P}\left\{\widehat{\boldsymbol{\xi}} \,\middle|\, \underline{\underline{\mathbf{F}}}^{(t)}\right\} \;=\; \widehat{\boldsymbol{\vartheta}}^{(t)} \;=\; \boldsymbol{\vartheta}^{\widehat{\boldsymbol{x}}^{(t)}} \tag{3.7}$$

and analogously

$$\widehat{\boldsymbol{\eta}}^{(t)} \;=\; \mathbb{P}\left\{\widehat{\boldsymbol{\eta}} \,\middle|\, \underline{\underline{\mathbf{F}}}^{(t)}\right\} \;=\; \boldsymbol{\vartheta}^{\widehat{\boldsymbol{y}}^{(t)}} \;. \tag{3.8}$$

Thus, (3.6) implies

$$\widehat{\boldsymbol{\vartheta}}^{(t)} \operatorname{dom}^{\boldsymbol{v}^{(t)}}_{T^{\boldsymbol{a}(t)}} \boldsymbol{\vartheta}^{\widehat{\boldsymbol{y}}^{(t)}} \;.$$

On the other hand,

$$\boldsymbol{\vartheta}^{\widehat{\boldsymbol{x}}^{(t)}}, \boldsymbol{\vartheta}^{\widehat{\boldsymbol{y}}^{(t)}} \in \mathcal{H}^{(t)} \;:=\; \mathbf{ConvH}\left\{\overset{\circ}{\overline{\boldsymbol{\vartheta}}}^{(t)}, \boldsymbol{\lambda}^\rho \;\; (\rho \in \mathbf{R})\right\}$$

contradicting the result of the main Theorem of *Part IV* according to which $\mathcal{H}^{(t)}$ is a vNM–Stable Set in the context of ${}^{(t)}\underline{\underline{\mathbf{D}}}$. **q.e.d.**

Lemma 3.5 (The Extended ε–free Inheritance Theorem). *Let $\boldsymbol{\lambda}^0$ be locally flat. and let ${}^{t}\underline{\underline{\mathbf{D}}}$ be a compatible uniform model.*

Let $\underline{\boldsymbol{\Delta}}$ be admissible and let $\{\alpha_\rho\}_{\rho\in\mathbf{R}_0}$, $\{\beta_\rho\}_{\rho\in\mathbf{R}_0}$ be a set of "convex" coefficients such that

$$\widehat{\boldsymbol{\xi}} \;=\; \widehat{\boldsymbol{\xi}}^{\boldsymbol{\alpha}\underline{\boldsymbol{\Delta}}} \;=\; \alpha_0 \overset{\circ}{\overline{\boldsymbol{\xi}}}^{\underline{\boldsymbol{\Delta}}} + \sum_{\rho\in\mathbf{R}} \alpha_\rho \boldsymbol{\lambda}^\rho \;\in \mathcal{H} \;.$$

Let $\boldsymbol{\eta}$ *be an imputation such that* $T_{>} := \{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\eta}}\}$ *is* $\underline{\underline{\mathbf{F}}}^{(t)}$*–measurable and let* $S \in \underline{\underline{\mathbf{F}}}^{(t)}$ *be a coalition such that*

$$\widehat{\boldsymbol{\xi}}^{(t)} \operatorname{dom}_S^{\boldsymbol{v}^{(t)}} \boldsymbol{\eta} \ . \tag{3.9}$$

is satisfied. Then there exists a relevant vector $\boldsymbol{a}^{(t)}$ *(w.r.t* ${}^{(t)}\underline{\underline{\mathbf{D}}}$*) and a corresponding coalition* $T^{\frac{1}{t}\boldsymbol{a}^{(t)}}$ *such that*

$$\widehat{\boldsymbol{\xi}} \operatorname{dom}^{\boldsymbol{v}}_{T^{\frac{1}{t}\boldsymbol{a}^{(t)}}} \boldsymbol{\eta} \tag{3.10}$$

holds true.

Proof:

1st STEP :

Consider the "ε–free" Inheritance Theorem 1.2 to $\bar{\boldsymbol{\xi}}^{(t)}$, $S \in \underline{\underline{\mathbf{F}}}^{(t)}$, and $\boldsymbol{\eta}$ and apply it to $\bar{\boldsymbol{\xi}}^{(t)}$. Repeating the four steps of Theorem 1.2 w.r.t. $\bar{\boldsymbol{\xi}}^{(t)}$, we come up with a relevant vector $\boldsymbol{a}^{(t)}$ and a coalition $T^{\frac{1}{t}\boldsymbol{a}^{(t)}}$ such that equation (1.25) and (1.27) are satisfied. Again we focus on some vector ${}^{(t)}\boldsymbol{a}^{\ominus}$ of the shape

$${}^{(t)}\boldsymbol{a}^{\ominus} = (1, \ldots, 1, \ldots, 1, \ldots\ldots, \alpha_r, \beta_r) \ . \tag{3.11}$$

That is, (1.25) and (1.27) now read

$$\vec{\boldsymbol{\lambda}}(T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^{\ominus}}) \ = \ \frac{1}{t}\boldsymbol{a}^{(t)} \quad , \quad \boldsymbol{\lambda}^0(T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^{\ominus}}) \ = \ \frac{1}{t} \ . \tag{3.12}$$

Let

$$\widehat{\boldsymbol{\tau}} \ = \ (\widehat{\tau}_1, \ldots, \widehat{\tau}_r, \overline{\overline{\tau}}_r)$$

denote the sequence corresponding to the positive coordinates of ${}^{(t)}\boldsymbol{a}^{\ominus}$ such that

$$\begin{aligned} \alpha_r \ &= \ a^{\ominus(t)}_{\widehat{\tau}_r} = \frac{(h_{\overline{\tau}_1} + \ldots + h_{\overline{\tau}_{r-1}} + h_{\overline{\overline{\tau}}_r}) - 1}{h_{\overline{\overline{\tau}}_r} - h_{\overline{\tau}_r}} \ , \\ \beta_r \ &= \ a^{\ominus(t)}_{\overline{\overline{\tau}}_r} = \frac{1 - (h_{\overline{\tau}_1} + \ldots + h_{\overline{\tau}_r})}{h_{\overline{\overline{\tau}}_r} - h_{\overline{\tau}_r}} \end{aligned} \tag{3.13}$$

according to the definition of relevant vectors (Theorem 3.5. of *Part I*). Recall that $\alpha_r + \beta_r = 1$.

Observe that the choice of the coalition $T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^{\ominus}}$ according to Theorem 1.2 and Remark 1.3 at this stage allows for a free choice of the subsets of $\boldsymbol{C}^r$, i.e. in the notation of the proof of Theorem 1.2, of

$$T^{\frac{1}{t}\alpha_r} = {}^{(t)}\boldsymbol{D}_{\widehat{\tau}_r} \cap T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^{\ominus}} \quad \text{and} \quad T^{\frac{1}{t}\beta_r} = {}^{(t)}\boldsymbol{D}_{\overline{\overline{\tau}}_r} \cap T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^{\ominus}}$$

within ${}^{(t)}\boldsymbol{D}^{\widehat{\tau}_r}$ and ${}^{(t)}\boldsymbol{D}^{\overline{\overline{\tau}}_r}$ respectively.

2ndSTEP :

Again the assumption $\{\boldsymbol{\xi} > \boldsymbol{\eta}\} \in \underline{\underline{\mathbf{F}}}^{(t)}$ implies that $\boldsymbol{\xi} > \boldsymbol{\eta}$ on $T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^{\ominus}}$ – no matter the choice of $T^{\frac{1}{t}\alpha_r}$ and $T^{\frac{1}{t}\beta_r}$. For, indeed following the construction in Theorem 1.2, we have

$$\bar{\boldsymbol{\xi}}^{(t)} \operatorname{dom}_{T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^{\ominus}}} \boldsymbol{\eta} \ ,$$

Hence $\bar{\boldsymbol{\xi}}^{(t)}$ exceeds $\boldsymbol{\eta}$ on every atom $\boldsymbol{D}^{\tau}$ involved in $T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^{\ominus}}$. Hence $\{\boldsymbol{\xi} > \boldsymbol{\eta}\} \cap \boldsymbol{D}^{\tau}$ has positive measure in every atom $\boldsymbol{D}^{\tau}$ cutting into $T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^{\ominus}}$, for $\bar{\boldsymbol{\xi}}^{(t)} > \boldsymbol{\eta}$ on $\boldsymbol{D}^{\widehat{\tau}_r}$ and $\bar{\boldsymbol{\xi}}^{(t)}$ is the conditional probability of $\bar{\boldsymbol{\xi}}$ over $\boldsymbol{D}^{\widehat{\tau}_r}$.

Therefore

$$\bar{\boldsymbol{\xi}} > \boldsymbol{\eta} \quad \text{on} \quad \boldsymbol{D}^{\tau} \tag{3.14}$$

as $\{\boldsymbol{\xi} > \boldsymbol{\eta}\} \in \underline{\underline{\mathbf{F}}}^{(t)}$. Consequently

$$\bar{\boldsymbol{\xi}} > \boldsymbol{\eta} \quad \text{on} \quad T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^{\ominus}}, \tag{3.15}$$

regardless the choice of $T^{\frac{1}{t}\alpha_r}$ and $T^{\frac{1}{t}\alpha_r}$.

3rdSTEP :

We now attempt to modify $T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^{\ominus}}$ such that

$$\bar{\boldsymbol{\xi}}(T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^{\ominus}}) \ = \ \bar{\boldsymbol{\xi}}^{(t)}(T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^{\ominus}}) \ = \ \bar{\boldsymbol{x}}^{(t)}{}^{(t)}\boldsymbol{a}^{\ominus} \tag{3.16}$$

and simultaneously

$$\boldsymbol{v}(T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^{\ominus}}) \ = \ \boldsymbol{v}^{(t)}(T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^{\ominus}}) \ = \ v^{(t)}({}^{(t)}\boldsymbol{a}^{\ominus}) \tag{3.17}$$

holds true. This will be done by *choosing* components $T^{\frac{1}{t}\alpha_r}$ and $T^{\frac{1}{t}\alpha_r}$ that have allowed for a degree of freedom so far, appropriately. Essentially, we repeat and extend the argument that was given for taking out "equal chunks" in Theorem 1.4 for proving (1.32).

We know that $\widehat{\boldsymbol{\tau}}$ is overshooting and the truncated sequence $(\widehat{\tau}_1, \ldots, \widehat{\tau}_r)$ is undercutting, that is

$$\sum_{\rho \in \mathbf{R}} h^{(t)}_{\widehat{\tau}_\rho} < 1 \leq \sum_{\rho \in \mathbf{R} \setminus \{r\}} h^{(t)}_{\widehat{\tau}_\rho} + h^{(t)}_{\overline{\widehat{\tau}}_r} \ .$$

Consequently $\widehat{\tau}_\rho \subseteq {}^{(t)}\overset{\vee}{\mathbf{T}}$ $(\rho \in \mathbf{R})$ and $\overline{\widehat{\tau}}_r \in {}^{(t)}\overset{\wedge}{\mathbf{T}}$. As ${}^{(t)}\overset{\vee}{\boldsymbol{T}} \ = \ \overset{\vee}{\boldsymbol{E}}$ we conclude that $\boldsymbol{D}^{\widehat{\tau}_\rho} \subseteq \overset{\vee}{\boldsymbol{E}}$, hence for $\rho \in \mathbf{R} \setminus \{r\}$.

$$\begin{aligned} \bar{\boldsymbol{\xi}}(\boldsymbol{D}^{\widehat{\tau}_\rho} \cap T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^{\ominus}}) \ &= \ \bar{\boldsymbol{\xi}}(\boldsymbol{D}\widehat{\tau}_\rho) \ = \ \bar{\boldsymbol{\xi}}^{(t)}(\boldsymbol{D}\widehat{\tau}_\rho) \\ &= \ \boldsymbol{\lambda}^{(t)}(\boldsymbol{D}^{\widehat{\tau}_\rho}) = \boldsymbol{\lambda}^{0}(\boldsymbol{D}^{\widehat{\tau}_\rho}) \\ &= \ {}^{(t)}\boldsymbol{a}^{\ominus}{}_{\widehat{\tau}_\rho}\bar{\boldsymbol{x}}^{(t)}_{\widehat{\tau}_\rho} \ = \ {}^{(t)}\boldsymbol{a}^{\ominus}{}_{\widehat{\tau}_\rho} h^{(t)}_{(\widehat{\tau}_\rho)} \\ &= \ \frac{1}{t} h^{(t)}_{(\widehat{\tau}_\rho)} \ . \end{aligned} \tag{3.18}$$

Therefore, with respect to the first $(r-1)$ positive coordinates of ${}^{(t)}\boldsymbol{a}^{\ominus}$, the sets

$$T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^{\ominus}} \cap \boldsymbol{C}^{\rho} \;=\; \boldsymbol{D}^{\widehat{\tau}_\rho} \quad (\rho \in \mathbf{R} \setminus \{r\})$$

satisfy

$$\bar{\boldsymbol{\xi}}(\boldsymbol{D}^{\widehat{\tau}_\rho} \cap T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^{\ominus}}) \;=\; \frac{1}{t} a^{(t)}_{\tau_\rho} h^{(t)}_{(\widehat{\tau}_\rho)} \tag{3.19}$$

4thSTEP :

Now we turn to the non-atoms of $T^{\frac{1}{t}\boldsymbol{a}^{(t)}}$, i.e. to $\boldsymbol{C}^r \cap T^{\frac{1}{t}\boldsymbol{a}^{(t)}} = T^{\frac{1}{t}\alpha_r} \cup T^{\frac{1}{t}\beta_r}$. First of all consider ${}^{(t)}\boldsymbol{D}_{\widehat{\tau}_r}$. In the context of Theorem 1.2 we can choose the set $T^{\frac{1}{t}\alpha_r}$ freely in the above sense.

Now, as $\frac{1}{t}h^{(t)}_{(\widehat{\tau}_\rho)}$ is the conditional probability of $\boldsymbol{\lambda}^0$ over $\boldsymbol{D}^{\widehat{\tau}_\rho}$ it follows that

$$\int\limits_{\left\{\dot{\boldsymbol{\lambda}}^0 > h^{(t)}_{(\widehat{\tau}_\rho)}\right\}} \dot{\boldsymbol{\lambda}}^0 - h^{(t)}_{(\widehat{\tau}_\rho)} \;=\; \int\limits_{\left\{\dot{\boldsymbol{\lambda}}^0 < h^{(t)}_{(\widehat{\tau}_\rho)}\right\}} h^{(t)}_{(\widehat{\tau}_\rho)} - \dot{\boldsymbol{\lambda}}^0 \;. \tag{3.20}$$

Now, for $\varepsilon > 0$ we can choose

$$F^{\varepsilon}_{>} \subseteq \left\{\dot{\boldsymbol{\lambda}}^0 > h^{(t)}_{(\widehat{\tau}_\rho)}\right\} \quad , \quad F^{\varepsilon}_{=} \subseteq \left\{\dot{\boldsymbol{\lambda}}^0 = h^{(t)}_{(\widehat{\tau}_\rho)}\right\} \quad , \quad F^{\varepsilon}_{<} \subseteq \left\{\dot{\boldsymbol{\lambda}}^0 < h^{(t)}_{(\widehat{\tau}_\rho)}\right\}$$

such that

$$\boldsymbol{\lambda}\left(F^{\varepsilon}_{>} \cup F^{\varepsilon}_{=} \cup F^{\varepsilon}_{<}\right) = \varepsilon \tag{3.21}$$

and

$$\int\limits_{F^{\varepsilon}_{>}} \left(\dot{\boldsymbol{\lambda}}^0 - h^{(t)}_{(\widehat{\tau}_\rho)}\right) d\boldsymbol{\lambda} \;=\; \int\limits_{F^{\varepsilon}_{<}} \left(h^{(t)}_{(\widehat{\tau}_\rho)} - \dot{\boldsymbol{\lambda}}^0\right) d\boldsymbol{\lambda} \;. \tag{3.22}$$

via Ljapounoffs Theorem. Then we have

(3.23)

$$\begin{aligned}
&\int\limits_{F^{\varepsilon}_{>}} \dot{\boldsymbol{\lambda}}^0 d\boldsymbol{\lambda} + \int\limits_{F^{\varepsilon}_{=}} \dot{\boldsymbol{\lambda}}^0 d\boldsymbol{\lambda} + \int\limits_{F^{\varepsilon}_{<}} \dot{\boldsymbol{\lambda}}^0 d\boldsymbol{\lambda} \\
&= \int\limits_{F^{\varepsilon}_{>} \cup F^{\varepsilon}_{=} \cup F^{\varepsilon}_{<}} h^{(t)}_{(\widehat{\tau}_\rho)} d\boldsymbol{\lambda} \\
&\quad + \int\limits_{F^{\varepsilon}_{>}} \left(\dot{\boldsymbol{\lambda}}^0 - h^{(t)}_{(\widehat{\tau}_\rho)}\right) d\boldsymbol{\lambda} + \int\limits_{F^{\varepsilon}_{=}} \left(\dot{\boldsymbol{\lambda}}^0 d\boldsymbol{\lambda} - h^{(t)}_{(\widehat{\tau}_\rho)}\right) d\boldsymbol{\lambda} + \int\limits_{F^{\varepsilon}_{<}} \left(\dot{\boldsymbol{\lambda}}^0 d\boldsymbol{\lambda} - h^{(t)}_{(\widehat{\tau}_\rho)}\right) d\boldsymbol{\lambda} \\
&= \varepsilon h^{(t)}_{(\widehat{\tau}_\rho)} d\boldsymbol{\lambda} + \int\limits_{F^{\varepsilon}_{>}} \left(\dot{\boldsymbol{\lambda}}^0 - h^{(t)}_{(\widehat{\tau}_\rho)}\right) d\boldsymbol{\lambda} - \int\limits_{F^{\varepsilon}_{<}} \left(\dot{\boldsymbol{\lambda}}^0 d\boldsymbol{\lambda} - h^{(t)}_{(\widehat{\tau}_\rho)}\right) d\boldsymbol{\lambda} \\
&= \varepsilon h^{(t)}_{(\widehat{\tau}_\rho)} d\boldsymbol{\lambda} \;.
\end{aligned}$$

the last equation following from (3.22). That is, for $\varepsilon = \frac{\alpha_r}{t}$ we obtain

$$\boldsymbol{\lambda}^0(F^\varepsilon_> \cup F^\varepsilon_= \cup F^\varepsilon_<) \;=\; \int\limits_{F^\varepsilon_> \cup F^\varepsilon_= \cup F^\varepsilon_<} \dot{\boldsymbol{\lambda}}^0 d\boldsymbol{\lambda} \;=\; \frac{\alpha_r h^{(t)}_{(\widehat{\tau}_\rho)}}{t}\,. \tag{3.24}$$

So that, when we choose $T^{\frac{1}{t}\alpha_r} \;:=\; F^\varepsilon_> \cup F^\varepsilon_= \cup F^\varepsilon_<$, then

$$\boldsymbol{\lambda}^0(T^{\frac{1}{t}\alpha_r}) = \frac{\alpha_r h^{(t)}_{(\widehat{\tau}_\rho)}}{t} = \frac{\overline{x}^{(t)}_{\widehat{\tau}_\rho} h^{(t)}_{(\widehat{\tau}_\rho)}}{t} = \boldsymbol{\lambda}^{(t)}(T^{\frac{1}{t}\alpha_r})\,. \tag{3.25}$$

That is

$$\boldsymbol{\lambda}^0(\boldsymbol{D}^{\widehat{\tau}_\rho} \cap T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^\ominus}) \;=\; \boldsymbol{\lambda}(\boldsymbol{D}^{\widehat{\tau}_r}) h^{(t)}_{(\widehat{\tau}_r)} \;=\; \frac{\alpha_r}{t} h^{(t)}_{(\widehat{\tau}_r)} \;=\; \frac{1}{t}\bar{\boldsymbol{x}}^{(t)}_{\widehat{\tau}_r}{}^{(t)}\boldsymbol{a}^\ominus_{\widehat{\tau}_r} \;=\; \frac{1}{t}{}^{(t)}\boldsymbol{a}^\ominus_{\widehat{\tau}_r} h^{(t)}_{(\widehat{\tau}_r)}$$

and

$$\begin{aligned} \bar{\boldsymbol{\xi}}^{(t)}(\boldsymbol{D}^{\widehat{\tau}_r} \cap T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^\ominus}) \;&=\; \bar{\boldsymbol{\xi}}^{(t)}(\boldsymbol{D}^{\widehat{\tau}_r} \cap T^{\frac{1}{t}\alpha_r}) \\ &=\; \alpha_r \boldsymbol{\lambda}^{(t)}(\boldsymbol{D}^{\widehat{\tau}_\rho}) = \alpha_r \boldsymbol{\lambda}^0(\boldsymbol{D}^{\widehat{\tau}_r}) \\ &=\; \frac{1}{t}\bar{\boldsymbol{x}}^{(t)}_{\widehat{\tau}_r}{}^{(t)}\boldsymbol{a}^\ominus_{\widehat{\tau}_r} \;=\; \frac{1}{t}{}^{(t)}\boldsymbol{a}^\ominus_{\widehat{\tau}_r} h^{(t)}_{(\widehat{\tau}_r)} \\ &=\; \frac{\alpha_r}{t} h^{(t)}_{(\widehat{\tau}_r)} \end{aligned} \tag{3.26}$$

i.e.,

$$\bar{\boldsymbol{\xi}}(T^{\frac{1}{t}\alpha_r}) \;=\; \bar{\boldsymbol{\xi}}^{(t)}(T^{\frac{1}{t}\alpha_r}) \;=\; \frac{\alpha_r}{t} h^{(t)}_{(\widehat{\tau}_r)}\,.$$

5thSTEP :

Finally for $\overline{\overline{\tau}}_r$ we have ${}^{(t)}\boldsymbol{D}^{\overline{\overline{\tau}}_r} \subseteq \overset{\wedge}{\boldsymbol{E}}$, hence

$$\dot{\bar{\boldsymbol{\xi}}} \;=\; 1 - l^\star_r \;=\; 1 - h^{(t)\star}_r = \boldsymbol{x}^{(t)}_{\overline{\overline{\tau}}_r} \quad \text{on} \quad {}^{(t)}\boldsymbol{D}^{\overline{\overline{\tau}}_r}.$$

Therefore, no matter the choice of $T^{\frac{1}{t}\beta_r}$, we have

$$\bar{\boldsymbol{\xi}}(T^{\frac{1}{t}\beta_r}) \;=\; \bar{\boldsymbol{\xi}}^{(t)}(T^{\frac{1}{t}\beta_r}) \;=\; \beta_r \frac{1 - l^\star_r}{t} \;=\; \beta_r \frac{1 - h^{(t)\star}_r}{t} \;=\; \beta_r \frac{1}{t}\bar{\boldsymbol{x}}^{(t)}_{\overline{\overline{\tau}}_r}.$$

Now, exactly as in the $4^{th}STEP$ we can *choose* $T^{\frac{1}{t}\beta_r} \subseteq \boldsymbol{D}^{\overline{\overline{\tau}}_r}$ via Ljapounoffs Theorem such that

$$\boldsymbol{\lambda}^0(T^{\frac{1}{t}\beta_r}) \;=\; \beta_r \boldsymbol{\lambda}^0(\boldsymbol{D}^{\overline{\overline{\tau}}_r})$$

is true. Then again

$$\boldsymbol{\lambda}^0(\boldsymbol{D}^{\overline{\overline{\tau}}_r} \cap T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^\ominus}) \;=\; \frac{\beta_r}{t} h^{(t)}_{(\overline{\overline{\tau}}_r)} \;=\; \frac{1}{t}{}^{(t)}\boldsymbol{a}^\ominus_{\overline{\overline{\tau}}_r} h^{(t)}_{(\overline{\overline{\tau}}_r)} \;=\; \frac{1}{t}\bar{\boldsymbol{x}}^{(t)}_{\overline{\overline{\tau}}_r}{}^{(t)}\boldsymbol{a}^\ominus_{\overline{\overline{\tau}}_r} \tag{3.27}$$

Now we combine out results regarding $\bar{\boldsymbol{\xi}}$, that is (3.19), (3.26), and (3.27). We obtain

$$\bar{\boldsymbol{\xi}}(T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^\ominus}) \;=\; \bar{\boldsymbol{\xi}}^{(t)}(T^{\frac{1}{t}{}^{(t)}\boldsymbol{a}^\ominus}) \;=\; \bar{\boldsymbol{x}}^{(t)}{}^{(t)}\boldsymbol{a}^\ominus \tag{3.28}$$

i.e., the desired relation (3.16) as well as

$$\boldsymbol{v}(T^{(t)}\boldsymbol{a}^{\ominus}) \;=\; \boldsymbol{v}^{(\tau)}(T^{(t)}\boldsymbol{a}^{\ominus}) \;=\; v({}^{(t)}\boldsymbol{a}^{\ominus}) \;=\; \frac{1}{t}\;, \tag{3.29}$$

that is, the desired relation (3.17).

6thSTEP : Next we extend this result to $\widehat{\boldsymbol{\xi}}$. This is rather obvious as the remaining summands are multiples of $\boldsymbol{\lambda}^{\rho}$ and hence do not change when one turns to the stepfunction case, more precisely we proceed as follows. We have to consider

$$\widehat{\boldsymbol{\xi}} \;:=\; \bar{\boldsymbol{\xi}}^{\underline{\boldsymbol{\Delta}},\boldsymbol{\alpha}}$$

and we recall the vector/pre–imputation

$$\widehat{\boldsymbol{x}}^{(t)} \;=\; \bar{\boldsymbol{x}}^{(t)} + \sum_{\rho\in\overset{\wedge}{\boldsymbol{T}}^{(t)}} \underline{\Delta}_{\rho}\boldsymbol{e}^{\boldsymbol{C}^{\rho}} + \sum_{\rho\in\mathbf{R}} \alpha_{\rho}\boldsymbol{\lambda}^{\rho} \;=\; \bar{\boldsymbol{x}}^{(t)} + \boldsymbol{e}^{\underline{\boldsymbol{\Delta}}} + \boldsymbol{e}^{\boldsymbol{\alpha}}$$

which induces the imputation

$$\widehat{\boldsymbol{\vartheta}}^{(t)} \;=\; \boldsymbol{\vartheta}^{\widehat{\boldsymbol{x}}^{(t)}} \qquad (t\in\mathbb{N}_0)\;.$$

Now because of

$$\boldsymbol{\lambda}^{\rho}(T^{\frac{1}{t}\boldsymbol{a}^{(t)}}) = \boldsymbol{e}^{\boldsymbol{C}^{\rho}(t)}\boldsymbol{a}^{\ominus}$$

and in view of (3.16) it follow that

$$\widehat{\boldsymbol{\xi}}(T^{(t)}\boldsymbol{a}^{\ominus}) \;=\; \widehat{\boldsymbol{\xi}}^{(t)}(T^{(t)}\boldsymbol{a}^{\ominus}) \;=\; \widehat{\boldsymbol{\vartheta}}^{(t)}(T^{(t)}\boldsymbol{a}^{\ominus})\;, \tag{3.30}$$

which together with (3.15) shows

$$\widehat{\boldsymbol{\xi}}\,\mathrm{dom}^{\boldsymbol{v}}_{T^{\frac{1}{t}\boldsymbol{a}^{(t)}}}\,\boldsymbol{\eta}\;.$$

q.e.d.

Theorem 3.6 (***Dominance is u.h.c. – for relevant vectors***). *Let $\boldsymbol{\lambda}^0$ be locally flat. Let ${}^{(t)}\underline{\underline{\mathbf{D}}}$ be a compatible uniform model. Let $\underline{\boldsymbol{\Delta}}$ be an admissible distribution of mass and let*

$$\overset{\circ}{\bar{\boldsymbol{\xi}}} \;=\; \overset{\circ}{\bar{\boldsymbol{\xi}}}^{\underline{\boldsymbol{\Delta}}} \;=\; \bar{\boldsymbol{\xi}} + \boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}}$$

be the resulting imputation. Also, for some "convex set of coefficients" $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_r)$ let

$$\widehat{\boldsymbol{\xi}} \;:=\; \widehat{\boldsymbol{\xi}}^{\underline{\boldsymbol{\Delta}}\boldsymbol{\alpha}} \;:=\; \alpha_0\overset{\circ}{\bar{\boldsymbol{\xi}}} + \sum_{\rho\in\mathbf{R}} \alpha_{\rho}\boldsymbol{\lambda}^{\rho} \in \mathcal{H}$$

Assume that for some imputation $\boldsymbol{\eta}$ there exists for all a relevant vector $\boldsymbol{a}^{(t)}$ and a corresponding coalition $T^{\frac{1}{t}\boldsymbol{a}^{(t)}}$ such that the conditional expectations $\widehat{\boldsymbol{\xi}}^{(t)}$ satisfy

$$\widehat{\boldsymbol{\xi}}^{(t)}\,\mathrm{dom}_{T^{\boldsymbol{a}^{(t)}}}\,\boldsymbol{\eta} \quad (t\in\mathbb{N}_0)\;. \tag{3.31}$$

Then there there is some relevant vector $\boldsymbol{a}^{(t)}$ *and a corresponding coalition* $T^{\boldsymbol{a}^{(t)}}$ *such that*

$$\widehat{\boldsymbol{\xi}}\,\mathrm{dom}_{T^{\frac{1}{t}\boldsymbol{a}^{(t)}}}\,\boldsymbol{\eta} \tag{3.32}$$

holds true.

Proof: Follows immediately by Lemma 3.5. Note that essentially one has to rearrange the non–atoms of $T^{\frac{1}{t}\boldsymbol{a}^{(t)}} \cap \boldsymbol{C}^r$ in a way described in the $\mathbf{4^{th}}$**STEP** and $\mathbf{5^{th}}$**STEP** of Lemma 3.5 in order to receive the desired properties of $T^{\frac{1}{t}\boldsymbol{a}^{(t)}}$.

q.e.d.

Theorem 3.7. *Let* $(\boldsymbol{I}, \underline{\underline{\mathbf{F}}}, \boldsymbol{v})$ *be a continuous model. Let* $\boldsymbol{\lambda}^0$ *be locally flat. Let* $\underline{\boldsymbol{\Delta}}$ *be an admissible distribution of mass and let*

$$\overset{\circ}{\overline{\boldsymbol{\xi}}} \;=\; \overset{\circ}{\overline{\boldsymbol{\xi}}}^{\underline{\boldsymbol{\Delta}}} \;=\; \overline{\boldsymbol{\xi}} + \boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}}$$

be the resulting imputation.

Then the set $\mathcal{H}$ *defined by* (3.1), *i.e.,*

$$\mathcal{H} \;:=\; \boldsymbol{ConvH}\left\{\overset{\circ}{\overline{\boldsymbol{\xi}}}^{\underline{\boldsymbol{\Delta}}}, \boldsymbol{\lambda}^{\rho} \quad (\rho \in \mathbf{R})\right\} \tag{3.33}$$

is externally stable, hence a vNM–Stable Set.

Proof:

$\mathbf{1^{st}}$**STEP** : Let $\boldsymbol{\eta} \notin \mathcal{H}$.

Use Definition 2.6 and Lemma 2.5 in order to find a uniform model ${}^{(t)}\underline{\underline{\mathbf{D}}}$. By Lemma 1.7 and Corollary 1.8, we can assume that $\left\{\dot{\boldsymbol{\xi}} > \dot{\boldsymbol{\eta}}\right\}$ is measurable w.r.t. $\underline{\underline{\mathbf{F}}}^{(t)}$.

$\mathbf{2^{nd}}$**STEP** :

Let $\mathcal{H}^{(t)}$ be the vNM–Stable Set constructed by means of $\bar{\boldsymbol{x}}^{(t)}$ as described in the $1^{st}STEP$ of the proof of Theorem 3.4, that is by means of

$$\overset{\circ}{\bar{\boldsymbol{x}}}^{(t)} \;=\; \bar{\boldsymbol{x}}^{(t)} + \sum_{\rho \in \overset{\wedge}{\boldsymbol{T}}^{(t)}} \underline{\Delta}_{\rho} \boldsymbol{e}^{C^{\rho}} \;=\; \bar{\boldsymbol{x}}^{(t)} + \boldsymbol{e}^{\underline{\boldsymbol{\Delta}}}$$

and

$$\overset{\circ}{\overline{\boldsymbol{\vartheta}}}^{(t)} \;=\; \boldsymbol{\vartheta}^{\overset{\circ}{\bar{\boldsymbol{x}}}^{(t)}} \qquad (t \in \mathbb{N}_0)\ .$$

If $\boldsymbol{\eta} \in \mathcal{H}^{(t)}$ for a sequence ${}^{(t)}\underline{\underline{\mathbf{D}}}$ $(t \in \mathbb{N}_1 \in \mathbb{N})$, then by an limiting argument, we would have $\boldsymbol{\eta} \in \mathcal{H}$ as well, hence, we can assume that $\boldsymbol{\eta} \notin \mathcal{H}^{(t)}$.

Therefore, we find $\widehat{\boldsymbol{x}}^{(t)} \in \boldsymbol{H}^{(t)}$ and a relevant vector $\boldsymbol{a}^{(t)}$ such that $\boldsymbol{\vartheta}^{\widehat{\boldsymbol{x}}^{(t)}} \in \mathcal{H}^{(t)}$ satisfies

$$\boldsymbol{\vartheta}^{\widehat{\boldsymbol{x}}^{(t)}} \operatorname{dom}_{T^{\boldsymbol{a}^{(t)}}} \boldsymbol{\eta} \quad (t \in \mathbb{N}_0) \ . \tag{3.34}$$

By Lemma 2.3, (2.58) we know that

$$\widehat{\boldsymbol{\xi}}^{(t)} \ = \ \boldsymbol{\vartheta}^{\widehat{\boldsymbol{x}}^{(t)}}$$

hence

$$\widehat{\boldsymbol{\xi}}^{(t)} \operatorname{dom}_{T^{\boldsymbol{a}^{(t)}}} \boldsymbol{\eta} \quad (t \in \mathbb{N}_0) \ . \tag{3.35}$$

Now from the *u.h.c.* Theorem 3.6 we finally conclude that there is some $t_0 \in \mathbb{N}_0$ such that for all $t > t_0$

$$\widehat{\boldsymbol{\xi}} \operatorname{dom}_{T^{\boldsymbol{a}^{(t)}}} \boldsymbol{\eta} \tag{3.36}$$

holds true,

q.e.d.

4 All Games Have Standard Stable sets

We shall dispose of the locally flat requirement for $\boldsymbol{\lambda}^0$. That is, we consider a continuous model $(\boldsymbol{I}, \underline{\underline{\mathbf{F}}}, \boldsymbol{v})$ with arbitrary $\boldsymbol{\lambda}^0$. Then we focus on the same candidate given by (3.1), i.e.,

$$\mathcal{H} \;:=\; \mathbf{ConvH}\left\{\overset{\circ}{\overline{\boldsymbol{\xi}}}{}^{\underline{\boldsymbol{\Delta}}}, \boldsymbol{\lambda}^{\rho} \;\; (\rho \in \mathbf{R})\right\} \;. \tag{4.1}$$

Again we attempt to prove that $\mathcal{H}$ is externally stable.

Remark 4.1. For simplicity we will assume that $\boldsymbol{\lambda}^0$ has "no levels" on *all* carriers $\boldsymbol{C}^{\rho}$ $(\rho \in \mathbf{R})$, that is $\boldsymbol{\lambda}^0(\{\overset{\bullet}{\boldsymbol{\lambda}}{}^0 = l_0^{\rho}\} \cap \boldsymbol{C}^{\rho}) = 0$ $(\rho \in \mathbf{R})$ holds true. The slight changes to be made in our procedure in order to deal with a deviation from this principle are rather obvious but tedious and would actually cloud our view.

Also, we will assume that $\overset{\circ}{\overline{\boldsymbol{\xi}}}$ is an imputation hence $\boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}} = 0$, $\overset{\circ}{\overline{\boldsymbol{\xi}}} \;=\; \overline{\boldsymbol{\xi}}$, and hence the typical element of $\mathcal{H}$ writes

$$\widehat{\boldsymbol{\xi}} \;:=\; \alpha_0\, \overline{\boldsymbol{\xi}} + \sum_{\rho \in \mathbf{R}} \alpha_{\rho} \boldsymbol{\lambda}^{\rho} \;. \tag{4.2}$$

The argument for this assumption has been provided in Remark 3.1.

∘ ~~~~~~ ∘

Theorem 4.2. *Let $(\boldsymbol{I}, \underline{\underline{\mathbf{F}}}, \boldsymbol{v})$ be a continuous model. Let $\underline{\boldsymbol{\Delta}}$ be an admissible distribution of mass and let*

$$\overset{\circ}{\overline{\boldsymbol{\xi}}} \;=\; \overset{\circ}{\overline{\boldsymbol{\xi}}}{}^{\underline{\boldsymbol{\Delta}}} \;=\; \overline{\boldsymbol{\xi}} + \boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}}$$

be the resulting imputation. Then the set $\mathcal{H}$ defined by (3.1) is externally stable.

Proof:

In view of Remark 4.1 we assume $\boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}} \;=\; 0$.

For small $\varepsilon > 0$ we will construct a central measure ${}^{\varepsilon}\boldsymbol{\lambda}^0$ which satisfies the locally flat condition. The resulting continuous model is denoted $(\boldsymbol{I}, \underline{\underline{\mathbf{F}}}, {}^{\varepsilon}\boldsymbol{v})$. The dominance relations of this model will be seen to induce dominance also with regard to the original model $(\boldsymbol{I}, \underline{\underline{\mathbf{F}}}, \boldsymbol{v})$.

1stSTEP : Choose $\varepsilon > 0$ such that for all $\sigma \in \mathbf{R}$

$$\left(\frac{1 - l_{\sigma}^{\star}}{l_{\sigma}^0}\right) \frac{(1 - l_{\sigma}^{\star}) - (r-1)\varepsilon}{(1 - l_{\sigma}^{\star}) + \varepsilon \frac{1 - l_{\sigma}^{\star}}{l_{\sigma}^0}} \;>\; 1 \;. \tag{4.3}$$

In addition, choose ε sufficiently small such that the following quantities are well defined. For $\rho \in \mathbf{R}$ define

$$(4.4) \qquad {}^{\varepsilon}U^{\rho} \subseteq \left\{ \overset{\bullet}{\boldsymbol{\lambda}}{}^{0} < l^{0}_{\rho} + \varepsilon \right\} \cap \boldsymbol{C}^{\rho} \subseteq \overset{\vee}{\boldsymbol{E}}{}^{\rho}$$

such that $\boldsymbol{\lambda}\{{}^{\varepsilon}U^{\rho}\} < \varepsilon$ is a rational number. Similarly, define a set

$$(4.5) \qquad {}^{\varepsilon}V^{\rho} \subseteq \left\{ \overset{\bullet}{\boldsymbol{\lambda}}{}^{0} < \min_{\overset{\wedge}{\boldsymbol{E}}} \overset{\bullet}{\boldsymbol{\lambda}}{}^{0} + \varepsilon \right\} \cap \boldsymbol{C}^{\rho} \subseteq \overset{\wedge}{\boldsymbol{E}}{}^{\rho}$$

such that $\boldsymbol{\lambda}\{{}^{\varepsilon}V^{\rho}\} < \varepsilon$ and $\boldsymbol{\lambda}\{\overset{\wedge}{\boldsymbol{E}}{}^{\rho} \setminus {}^{\varepsilon}V^{\rho}\}$ is rational. Define

$$(4.6) \qquad {}^{\varepsilon}\overset{\bullet}{\boldsymbol{\lambda}}{}^{0} := \sum_{\rho \in \mathbf{R}} l^{0}_{\rho} \mathbb{1}_{{}^{\varepsilon}U^{\rho} \subseteq} + (1 - l^{\star}_{\rho} - \varepsilon) \mathbb{1}_{{}^{\varepsilon}V^{\rho} \subseteq} \overset{\bullet}{\boldsymbol{\lambda}}{}^{0} \mathbb{1}_{\boldsymbol{C}^{\rho} \setminus ({}^{\varepsilon}U^{\rho} \cup {}^{\varepsilon}V^{\rho})} \ .$$

If necessary, rearrange ε such that ${}^{\varepsilon}\boldsymbol{\lambda}^{0}(\boldsymbol{I}) > 1$. Then consider the continuous model $(\boldsymbol{I}, \underline{\underline{\mathbf{F}}}, {}^{\varepsilon}\boldsymbol{v})$. Clearly, the minima are unchanged, that is (with obvious notational adaptation)

$$(4.7) \qquad {}^{\varepsilon}l^{0}_{\rho} = l^{0}_{\rho} \quad (\rho \in \mathbf{R}), \qquad {}^{\varepsilon}l^{\star}_{\rho} = l^{\star}_{\rho} \quad (\rho \in \mathbf{R}).$$

Also,

$$(4.8) \qquad {}^{\varepsilon}\overset{\vee}{\boldsymbol{E}}{}^{\rho} = \overset{\vee}{\boldsymbol{E}}{}^{\rho} \cup {}^{\varepsilon}V^{\rho} \qquad {}^{\varepsilon}\overset{\wedge}{\boldsymbol{E}}{}^{\rho} = \overset{\wedge}{\boldsymbol{E}}{}^{\rho} \setminus {}^{\varepsilon}V^{\rho},$$

from which it follows that $\boldsymbol{\lambda}({}^{\varepsilon}\overset{\vee}{\boldsymbol{E}}{}^{\rho})$ and $\boldsymbol{\lambda}({}^{\varepsilon}\overset{\wedge}{\boldsymbol{E}}{}^{\rho})$ are rational. Therefore, ${}^{\varepsilon}\boldsymbol{\lambda}^{0}$ is locally flat and we may apply the results of the previous sections to this central measure. Obviously we have

$$(4.9) \qquad \boldsymbol{\lambda}^{0} \ \geq \ {}^{\varepsilon}\boldsymbol{\lambda}^{0} \ .$$

Now let ${}^{\varepsilon}\bar{\boldsymbol{\xi}}$ denote the standard truncation measure derived from the model $(\boldsymbol{I}, \underline{\underline{\mathbf{F}}}, {}^{\varepsilon}\boldsymbol{\lambda}^{0})$. Then, from (4.9) and (4.7) it follows that

$$(4.10) \qquad \bar{\boldsymbol{\xi}} \geq^{\varepsilon} \bar{\boldsymbol{\xi}} \ .$$

2ndSTEP : Now let $\boldsymbol{\eta} \notin \mathcal{H}$. Let ${}^{\varepsilon}\mathcal{H}$ denote the standard–T vNM Stable Set for ${}^{\varepsilon}\boldsymbol{v}$. By some continuity argument (sic!) one observes that $\boldsymbol{\eta} \in {}^{\varepsilon}\mathcal{H}$ cannot be true for all ε. Hence, by rearranging ε if necessary we can assume that $\boldsymbol{\eta} \notin^{\varepsilon} \mathcal{H}$.

Therefore, there exists for large $t \in \mathbb{N}$ some compatible model ${}^{(t)}\underline{\underline{\mathbf{D}}}$ and some relevant vector $\boldsymbol{a}^{(t)}$ plus a set of convexifying coefficients $\{\alpha_{\rho}\}_{\rho \in \mathbf{R}_0}$ such that

$$(4.11) \qquad {}^{\varepsilon}\widehat{\boldsymbol{\xi}} := \alpha_0 \, {}^{\varepsilon}\bar{\boldsymbol{\xi}} + \sum_{\rho \in \mathbf{R}} \alpha_{\rho} \boldsymbol{\lambda}^{\rho}$$

yields

$$(4.12) \qquad {}^{\varepsilon}\widehat{\boldsymbol{\xi}} \operatorname{dom}^{{}^{\varepsilon}\boldsymbol{v}}_{T^{\boldsymbol{a}^{(t)}}} \boldsymbol{\eta} \ ;$$

here dominance is established w.r.t. ${}^{\varepsilon}\boldsymbol{v}$. Also we have assumed that ${}^{\varepsilon}\bar{\boldsymbol{\xi}}$ is an imputation following Remark 4.1. Naturally we put

$$(4.13) \qquad \widehat{\boldsymbol{\xi}} \ := \ \alpha_0 \ \bar{\boldsymbol{\xi}} + \sum_{\rho \in \mathbf{R}} \alpha_\rho \boldsymbol{\lambda}^\rho \ .$$

Because of (4.10) we conclude that

$$(4.14) \qquad \widehat{\boldsymbol{\xi}} \geq^{\varepsilon} \widehat{\boldsymbol{\xi}},$$

holds true as well. Hence we have at once

$$(4.15) \qquad \widehat{\boldsymbol{\xi}} > \boldsymbol{\eta} \qquad \text{on } T^{\boldsymbol{a}^{(t)}}.$$

It is therefore essentially our task to establish some kind of ("relevant") subcoalition of $T^{\boldsymbol{a}^{(t)}}$ which is effective for $\boldsymbol{v}$ with respect to $\widehat{\boldsymbol{\xi}}$. To this end we have to discuss the three familiar types of relevant vectors, that is,

$$\boldsymbol{a}^{(t)\odot} \ , \ \boldsymbol{a}^{(t)\ominus} \quad \text{and} \quad \boldsymbol{a}^{(t)\oplus} \ .$$

For short and to ease up our notation we shall henceforth omit the index $^{(t)}$, i. e., just write

$$\boldsymbol{a}, \quad T^{\boldsymbol{a}}, \quad \boldsymbol{a}^{\odot} \ , \ \boldsymbol{a}^{\ominus} \quad \text{and} \quad \boldsymbol{a}^{\oplus} \ .$$

3rdSTEP : First of all assume that the relevant vector is $\boldsymbol{a} \ = \ \boldsymbol{a}^{(\odot)}$. Then ${}^{\varepsilon}\boldsymbol{\lambda}^0$ is not involved in forming ${}^{\varepsilon}\boldsymbol{v}$, that is

$$(4.16) \qquad {}^{\varepsilon}\boldsymbol{v}(T^{\boldsymbol{a}}) \ = \ \boldsymbol{\lambda}^1(T^{\boldsymbol{a}}) \ = \ \ldots \ = \ \boldsymbol{\lambda}^r(T^{\boldsymbol{a}}) \ \leq \ {}^{\varepsilon}\boldsymbol{\lambda}^0(T^{\boldsymbol{a}}) \ .$$

Also, if the last inequality in (4.16) is a strict one, then the coefficient α_0 in the representation (4.13) equals 0, that is we have

$$(4.17) \qquad {}^{\varepsilon}\widehat{\boldsymbol{\xi}} \ = \ \sum_{\rho \in \mathbf{R}} \alpha_\rho \boldsymbol{\lambda}^\rho \ = \ \widehat{\boldsymbol{\xi}}$$

In view of (4.9) we have all the more

$$(4.18) \qquad \boldsymbol{v}(T^{\boldsymbol{a}}) \ = \ \boldsymbol{\lambda}^1(T^{\boldsymbol{a}}) \ = \ \ldots \ = \ \boldsymbol{\lambda}^r(T^{\boldsymbol{a}}) \ \leq \ \boldsymbol{\lambda}^0(T^{\boldsymbol{a}}) \ ,$$

and consequently

$$(4.19) \qquad \widehat{\boldsymbol{\xi}}(T^{\boldsymbol{a}}) \ = \ \boldsymbol{v}(T^{\boldsymbol{a}}) \ = \ {}^{\varepsilon}\boldsymbol{v}(T^{\boldsymbol{a}}).$$

Now by (4.19) and (4.15) we obtain

$$\widehat{\boldsymbol{\xi}}\ dom^{\boldsymbol{v}}_{T^{\boldsymbol{a}}}\ \boldsymbol{\eta}$$

(f the last inequality in (4.16) happens to be an equation the argument is obviously the same). This way we have established domination for the case that $\boldsymbol{a}$ is of the type $\boldsymbol{a}^{\odot}$.

4thSTEP : Now assume that the relevant vector is $\boldsymbol{a} = \boldsymbol{a}^{\oplus}$. W.l.o.g the critical index is as always r. That is $\boldsymbol{\lambda}^r$ is not involved in forming ${}^{\varepsilon}\boldsymbol{v}$, more precisely:

$$\begin{aligned} {}^{\varepsilon}\boldsymbol{v}(T^{\boldsymbol{a}}) &= \boldsymbol{\lambda}^1(T^{\boldsymbol{a}}) = \ldots = \boldsymbol{\lambda}^{(r-1)}(T^{\boldsymbol{a}}) \\ &= {}^{\varepsilon}\boldsymbol{\lambda}^0(T^{\boldsymbol{a}}) = \frac{1}{t} < \boldsymbol{\lambda}^{(r)}(T^{\boldsymbol{a}}) \end{aligned} \tag{4.20}$$

Also in (4.2) we have w.l.g. $\alpha_r = 0$, that is

$${}^{\varepsilon}\widehat{\boldsymbol{\xi}} := \alpha_0\ {}^{\varepsilon}\overline{\boldsymbol{\xi}} + \sum_{\rho\in\mathbf{R}\setminus\{r\}} \alpha_\rho \boldsymbol{\lambda}^\rho\ . \tag{4.21}$$

We know that by the construction of $\boldsymbol{a}^{\oplus}$ with reference to ${}^{\varepsilon}\boldsymbol{\lambda}^0$, the coalitions $T^{\boldsymbol{a},\rho} := T^{\boldsymbol{a}} \cap \boldsymbol{C}^\rho$ are "minimal" (belong to some minimizing undercutting sequence in the language of Part IV), hence yield

$$T^{\boldsymbol{a},\rho} \subseteq^{\varepsilon} U^\rho \quad (\rho \in \mathbf{R})$$

and therefore

$${}^{\varepsilon}\boldsymbol{\lambda}^0(T^{\boldsymbol{a},r}) = \frac{1}{t} l_0^\rho \quad (\rho \in \mathbf{R} \setminus \{r\}).$$

as well as

$$\begin{aligned} \boldsymbol{\lambda}^{(r)}(T^{\boldsymbol{a},r}) &= \frac{1}{t}\frac{1-\sum_{\rho\in\mathbf{R}\setminus\{r\}} l_\rho^0}{l_\rho^0} = \frac{1}{t}\frac{1-l_r^\star}{l_r^0} > \frac{1}{t} \\ \boldsymbol{\lambda}^{(0)}(T^{\boldsymbol{a},r}) &= \frac{1}{t}\left(1 - \sum_{\rho\in\mathbf{R}\setminus\{r\}} l_\rho^0\right) = \frac{1}{t}\ \frac{1-l_r^\star}{l_r^0} > \frac{1}{t}\ . \end{aligned} \tag{4.22}$$

Now we switch to the context of the original model $(\boldsymbol{I}, \underline{\underline{\mathbf{F}}}, \boldsymbol{v})$. For some $\delta > 0$ sufficiently small we choose $S^\rho \subseteq T^{\boldsymbol{a},\rho}$ such that

$$\boldsymbol{\lambda}^\rho(S^\rho) = \delta \quad \rho \in \mathbf{R} \setminus \{r\}\ . \tag{4.23}$$

Also choose ${}^\star S^r \in T^{\boldsymbol{a},r}$ such that

$$\boldsymbol{\lambda}^r({}^\star S^r) = \delta\frac{1-l_\rho^\star}{l_\rho^0} > \delta\ , \tag{4.24}$$

then

$$^{\varepsilon}\boldsymbol{\lambda}^0(^{\star}S^r) = \delta(1 - l_r^{\star}) \tag{4.25}$$

as $\overset{\bullet}{^{\varepsilon}\boldsymbol{\lambda}}{}^0 = l_r^0$ on $T^{\boldsymbol{a},r}$.

Now, as $T^{\boldsymbol{a}} \subseteq \overset{\vee}{\boldsymbol{E}}$, we know that

$$\widehat{\boldsymbol{\xi}} = \boldsymbol{\lambda}^0 \geq^{\varepsilon} \boldsymbol{\lambda}^0 \ =^{\varepsilon} \bar{\boldsymbol{\xi}} > \boldsymbol{\eta}$$

holds true on $T^{\boldsymbol{a}}$. Then clearly

$$\widehat{\boldsymbol{\xi}} > \boldsymbol{\eta} \quad \text{on} \quad {}^{\star}S \ = \ S^1 \cup \ldots \cup S^{r-1} \cup \ {}^{\star}S^r \ \subseteq T^{\boldsymbol{a}}. \tag{4.26}$$

Now denote tentatively

$$L^r \ := \ \boldsymbol{\lambda}(^{\star}S^r) \ = \ \boldsymbol{\lambda}^r(^{\star}S^r) \ , \tag{4.27}$$

then

$$L^r \ = \ \delta\left(\frac{1 - l_r^{\star}}{l_r^0}\right) > \delta \tag{4.28}$$

Next let

$$H^r \ := \ ^{\varepsilon}\boldsymbol{\lambda}^0(^{\star}S^r) \ , \quad G^r \ := \ \boldsymbol{\lambda}^0(^{\star}S^r) \ , \tag{4.29}$$

then

$$H^r \ = \ \delta(1 - l_r^{\star}) \ = \ \delta - \sum_{\rho \in \mathbf{R}\setminus\{r\}} {}^{\varepsilon}\boldsymbol{\lambda}^0(S^{\rho}) \tag{4.30}$$

as $^{\varepsilon}\boldsymbol{\lambda}^0$ is constant and equals l_r^0 on $^{\varepsilon}U^{\rho}$. Because of (4.24) and

$$l_r^0 \leq \overset{\bullet}{\boldsymbol{\lambda}}{}^0 \leq l_{\rho}^0 + \varepsilon \quad \text{on} \quad {}^{\varepsilon}U^{\rho} \quad (\rho \in \mathbf{R})$$

we obtain

$$\begin{aligned} H^r \leq G^r &\leq \delta\frac{1 - l_r^{\star}}{l_r^0}(l_r^0 + \varepsilon) \\ &= \delta(1 - l_r^{\star}) + \delta\varepsilon(\frac{1 - l_r^{\star}}{l_r^0}) \\ &= \ H^r + \delta\varepsilon(\frac{1 - l_r^{\star}}{l_r^0}) \ . \end{aligned} \tag{4.31}$$

Finally we define

$$G_0 \ := \ \delta - \sum_{\rho \in \mathbf{R}\setminus\{r\}} \boldsymbol{\lambda}^0(S^{\rho}) \ ,$$

then

$$\begin{aligned} G^0 \ &\geq \ \delta - \delta \sum_{\rho \in \mathbf{R}\setminus\{r\}} (l_{\rho}^0 - \varepsilon) \\ &= \ \delta(1 - \sum_{\rho \in \mathbf{R}\setminus\{r\}} l_{\rho}^0) - \delta\varepsilon(r-1) \ = \ H^r - \delta\varepsilon(r-1) \ . \end{aligned} \tag{4.32}$$

Now, in $\mathbb{R}^2$ consider the triangle

$$(0,0),(L^r,0),(L^r,G^r)$$

as well as the straight line

$$\{(x_1,G^0)\,|\,0\le x_1\le L^r\}$$

(see Figure 4.1).

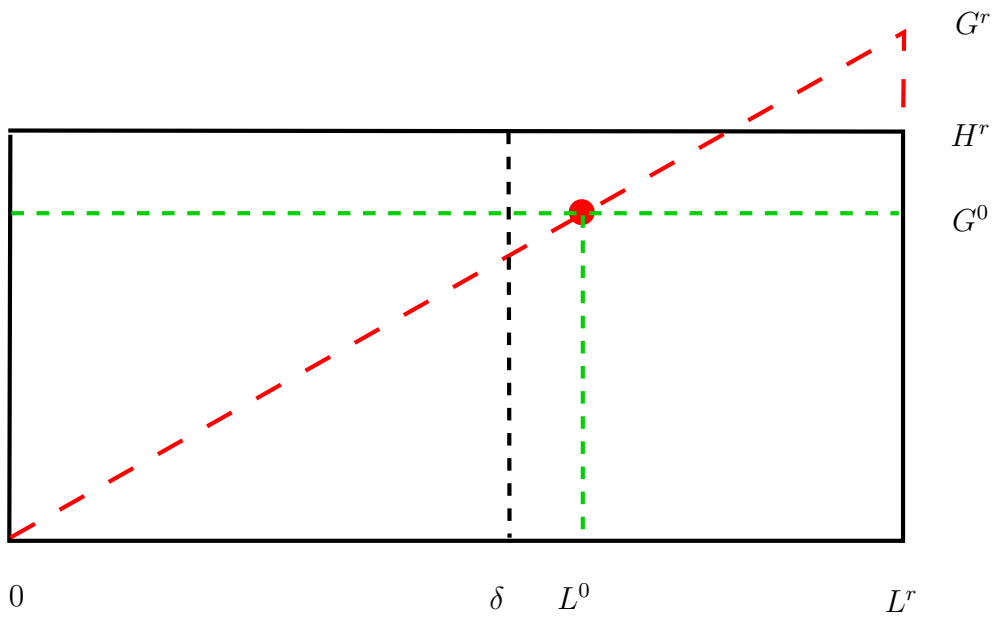

Figure 4.1: Reducing a Relevant Coalition

Because of $G^0 \le H^r \le G^r$ this straight line and the line connecting $(0,0)$ and (L^r,G^r) intersect at some point (L^0,G^0). As the resulting triangle

$$(0,0),(L^0,0),(L^0,G^0)$$

is similar to the one above we have

$$\frac{L^0}{G^0} = \frac{L^r}{G^r}$$

hence

$$
\begin{aligned}
L^0 = G^0 \frac{L^r}{G^r} &= \delta \left(\frac{1 - l_r^\star}{l_r^0} \right) \frac{G^0}{G^r} \\
&\text{by (4.28)} \\
&\geq \delta \left(\frac{1 - l_r^\star}{l_r^0} \right) \frac{H^r - \delta(r-1)\varepsilon}{H^r + \delta\varepsilon \frac{1 - l_r^\star}{l_r^0}} \\
&\text{by (4.32) and (4.31) .} \\
&= \delta \left(\frac{1 - l_r^\star}{l_r^0} \right) \frac{\delta(1 - l^\star) - \delta(r-1)\varepsilon}{\delta(1 - l^\star) + \delta\varepsilon \frac{1 - l_r^\star}{l_r^0}} \qquad (4.33) \\
&\text{by (4.30)} \\
&= \delta \left(\frac{1 - l_r^\star}{l_r^0} \right) \frac{(1 - l^\star) - (r-1)\varepsilon}{(1 - l^\star) + \varepsilon \frac{1 - l_r^\star}{l_r^0}} \\
&> \delta \\
&\text{by assumption (4.3) .}
\end{aligned}
$$

5thSTEP : By definition the point (L^r, G^r) is an element within the range of the vector–valued measure $(\boldsymbol{\lambda}^r, \boldsymbol{\lambda}^0)$. Using Ljapounoffs Theorem we can therefore find a coalition $S^r \subseteq {}^\star S^r$ such that

$$
(\boldsymbol{\lambda}^r, \boldsymbol{\lambda}^0)(S^r) = (L^0, G^0) \tag{4.34}
$$

holds true. Then

$$
\boldsymbol{\lambda}^r(S^r) = L^0 > \delta \ , \tag{4.35}
$$

while the coalition $S := S^1 \cup \ldots \cup S^r$ Satisfies

$$
\begin{aligned}
\boldsymbol{\lambda}^0(S) &= \sum_{\rho \in \mathbf{R} \setminus \{r\}} \boldsymbol{\lambda}^0(S^\rho) + \boldsymbol{\lambda}^0(S^r) \\
&= \sum_{\rho \in \mathbf{R} \setminus \{r\}} \boldsymbol{\lambda}^0(S^\rho) + G^0 = \delta \ .
\end{aligned} \tag{4.36}
$$

hence

$$
\boldsymbol{v}(S) = \boldsymbol{\lambda}^1(S) = \ldots = \boldsymbol{\lambda}^{r-1}(S) = \boldsymbol{\lambda}^0(S) = \delta < \boldsymbol{\lambda}^r(S) \ . \tag{4.37}
$$

As $\bar{\boldsymbol{\xi}}$ coincides with $\boldsymbol{\lambda}^0$ on $\overset{\vee}{\boldsymbol{E}}$ and $S \subseteq \overset{\vee}{\boldsymbol{E}}$, we have

$$
\bar{\boldsymbol{\xi}}(S) = \boldsymbol{\lambda}^0(S) = \boldsymbol{v}(S) = \delta \ , \tag{4.38}
$$

and thus

$$
\widehat{\boldsymbol{\xi}}(S) := \alpha_0 \, \bar{\boldsymbol{\xi}}(S) + \sum_{\rho \in \mathbf{R} \setminus \{r\}} \alpha_\rho \boldsymbol{\lambda}^\rho(S) = \boldsymbol{v}(S) = \delta \ . \tag{4.39}
$$

In addition $S \subseteq {}^{\star}S$ and (4.26) show that

$$\widehat{\boldsymbol{\xi}} > \boldsymbol{\eta} \quad \text{on} \quad S \ . \tag{4.40}$$

Consequently, by (4.39) and (4.40) we obtain

$$\widehat{\boldsymbol{\xi}} \operatorname{dom}_S \boldsymbol{\eta} \ , \tag{4.41}$$

6thSTEP : It remains to treat the case $\boldsymbol{a} = \boldsymbol{a}^{\ominus}$. This we will achieve by following the same path as indicated within the last two steps.

Hence for sufficiently small $\delta > 0$ we choose again $S^{\rho} \subseteq T^{\boldsymbol{a},\rho}$ as in (4.23) such that

$$\boldsymbol{\lambda}^{\rho}(S^{\rho}) = \delta \quad \rho \in \mathbf{R} \setminus \{r\} \ . \tag{4.42}$$

Now recall that as $\boldsymbol{a} = \boldsymbol{a}^{\ominus}$ the coalition $T^{\boldsymbol{a},r}$ decomposes into two components, say $T^{\alpha,r} = T^{\boldsymbol{a},r} \cap \overset{\vee}{\boldsymbol{E}}{}^{r}$ and $T^{\beta,r} = T^{\boldsymbol{a},r} \cap \overset{\wedge}{\boldsymbol{E}}{}^{r}$. In $T^{\alpha,r}$ we have the same situation as in the $4^{th}STEP$ (as we are dealing with a coalition in $\overset{\vee}{\boldsymbol{E}}$) and hence we can choose some coalition ${}^{\star}S^{\alpha} \subseteq T^{\alpha,r}$ such that the situation of Figure 4.1 prevails when we replace ${}^{\star}S^{r}$ by ${}^{\star}S^{\alpha}$ (that is , put $H^{r} := {}^{\varepsilon}\boldsymbol{\lambda}^{0}({}^{\star}S^{\alpha})$). However, this time we focus on the point (δ, H^{0}) located on the intersection of the line $(0,0), (L^{r}, G^{r})$ and the line

$$\{(x_1, x_2) \,|\, x, 1 = \delta\} \ .$$

This point obviously yields

$$H^{0} \leq G^{0} \ := \ \delta - \sum_{\rho \in \mathbf{R} \setminus \{r\}} \boldsymbol{\lambda}^{0}(S^{\rho}) \ .$$

(see Figure 4.2). Now via Ljapounoffs Theorem we choose a coalition $\overset{\circ}{S}{}^{\alpha} \subseteq T^{\alpha,r}$ such that

$$\begin{aligned} (\boldsymbol{\lambda}^{r}, \boldsymbol{\lambda}^{0})(\overset{\circ}{S}{}^{)}\alpha &= (\delta, H^{0}) \\ \boldsymbol{\lambda}^{0}(\overset{\circ}{S}{}^{\alpha}) &= H^{0} \leq \delta \end{aligned} \tag{4.43}$$

is satisfied.

On $T^{\beta,r}$ the situation differs as we have a coalition in $\overset{\wedge}{\boldsymbol{E}}$. On $T^{\beta,r}$ we have

$$\boldsymbol{\lambda}^{0} = {}^{\varepsilon}\boldsymbol{\lambda}^{0} \ , \quad \text{and} \quad {}^{\varepsilon}\bar{\boldsymbol{\xi}} = \bar{\boldsymbol{\xi}} \tag{4.44}$$

as the minima have not changed, see (4.7). Now we choose $\overset{\circ}{S}{}^{\beta} \subseteq T^{\beta,r}$ Such that

$${}^{\varepsilon}\boldsymbol{\lambda}^{0}(\overset{\circ}{S}{}^{\beta}) = \boldsymbol{\lambda}^{0}(\overset{\circ}{S}{}^{\beta}) = \delta \ . \tag{4.45}$$

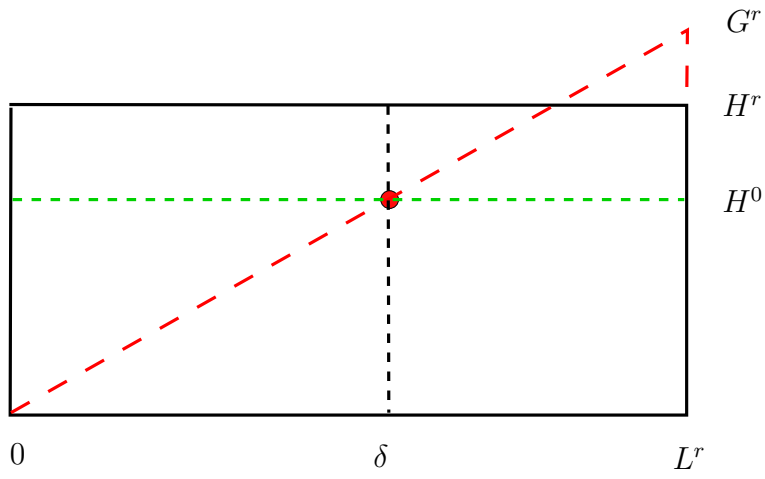

Figure 4.2: Modification: The Construction of (δ, H^0)

Then, as

$$\dot{\boldsymbol{\lambda}}^0 \geq 1 - l_r^\star \quad on \ \mathring{S}^\beta \subseteq T^{\beta,r}$$

we have

$$\begin{aligned} \boldsymbol{\lambda}^0(\mathring{S}^\beta) &\geq \delta(1 - l_r^\star) \\ &= \delta - \sum_{\rho \in \mathbf{R}\setminus\{r\}} \delta l_\rho^0 \\ &\geq \delta - \sum_{\rho \in \mathbf{R}\setminus\{r\}} \boldsymbol{\lambda}^0(S^\rho) \ = \ G^0 \ . \end{aligned} \tag{4.46}$$

Thus, we have found a second point within the range of $(\boldsymbol{\lambda}^r, \boldsymbol{\lambda}^0)$ which yields

$$\begin{aligned} (\boldsymbol{\lambda}^r, \boldsymbol{\lambda}^0)(\mathring{S}^\beta) &= \ : (\delta, H^1) \\ \boldsymbol{\lambda}^0(\mathring{S}^\beta) &= H^1 \geq \delta \end{aligned} \tag{4.47}$$

Naturally we combine the two points (4.43) and (4.47) in order to render the second coordinate to be δ as well. That is, we find $\alpha, \beta \geq 0$ adding up to 1 (a "convex combination") such that

$$\alpha(\delta, H^0) + \beta(\delta, H^1) \ = \ (\delta, \delta) \ . \tag{4.48}$$

Accordingly, we again use Ljapounoff's Theorem to find $S^\alpha \subseteq \mathring{S}^\alpha$, $S^\beta \subseteq \mathring{S}^\beta$ such that in a way that

$$(\boldsymbol{\lambda}^r, \boldsymbol{\lambda}^0)(S^\alpha) = (\delta, \alpha\delta) \ , \quad (\boldsymbol{\lambda}^r, \boldsymbol{\lambda}^0)(S^\beta) = (\delta, \beta\delta) \ . \tag{4.49}$$

and

$$(\boldsymbol{\lambda}^r, \boldsymbol{\lambda}^0)(S^\alpha \cup S^\beta) \;=\; (\delta, \delta)\ , \tag{4.50}$$

Then

$$S \;:=\; \left(\bigcup_{\rho \in \mathbf{R} \setminus \{r\}} S^\rho \right) \cup \left(S^\alpha \cup S^\beta \right) \tag{4.51}$$

satisfies

$$\boldsymbol{\lambda}^1(S) \ldots \;=\; \boldsymbol{\lambda}^r(S) \;=\; \boldsymbol{\lambda}^0(S) \;=\; \boldsymbol{v}(S)\ . \tag{4.52}$$

7thSTEP :

Now we turn to $\bar{\boldsymbol{\xi}}$ and $\widehat{\boldsymbol{\xi}}$. We know that $\bar{\boldsymbol{\xi}}$ coincides with $\boldsymbol{\lambda}^0$ on

$$S \;:=\; \left(\bigcup_{\rho \in \mathbf{R} \setminus \{r\}} S^\rho \right) \cup S^\alpha\ ,$$

while $\bar{\boldsymbol{\xi}} = 1 - l_r^\star \leq \boldsymbol{\lambda}^0$ on S^β. Therefore we have

$$\bar{\boldsymbol{\xi}}(S) \leq \boldsymbol{\lambda}^0(S) = \boldsymbol{v}(S) \tag{4.53}$$

and hence it follows from (4.52) that

$$\widehat{\boldsymbol{\xi}}(S) \leq \boldsymbol{\lambda}^0(S) = \boldsymbol{v}(S)\ . \tag{4.54}$$

But $\bar{\boldsymbol{\xi}} > \boldsymbol{\eta}$ follows exactly the same way as in the previous steps (compare (4.26)). Combining this and (4.54) we obtain indeed

$$\widehat{\boldsymbol{\xi}} \operatorname{dom}_S \boldsymbol{\eta}\ ,$$

q.e.d.

The above proof gives rise to consider a slightly modified version of a relevant coalition. We have constructed ε–relevant coalitions for the case that $\boldsymbol{\lambda}^0$ is not locally flat. However the description – other than in the discrete model – is not "canonical": we cannot just determine "relevant" coalition by specifying its values of $\boldsymbol{\lambda}$, the stringent form that is governing the uniform case is missing.

Definition 4.3. *Consider the vectorvalued measure* $\vec{\boldsymbol{\lambda}} = (\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r, \boldsymbol{\lambda}^0)$. *Let* $\varepsilon > 0$. *We shall say that a coalition* S *is* ***ε–relevant*** *if one of the following three alternatives is satisfied:*

1.

$$\vec{\boldsymbol{\lambda}}(S) = (\varepsilon, \ldots, \varepsilon, \boldsymbol{\lambda}^0(S))\ , \quad \boldsymbol{\lambda}^0(S) > \varepsilon\ . \tag{4.55}$$

2.

$$(4.56)\qquad \begin{aligned} S &\subseteq \overset{\vee}{\boldsymbol{E}} \\ \overrightarrow{\boldsymbol{\lambda}}(S) &= (\varepsilon,\ldots,\varepsilon,\boldsymbol{\lambda}^r(S),\varepsilon)\ , \quad \boldsymbol{\lambda}^r(S) > \varepsilon\ . \end{aligned}$$

3.

$$(4.57)\qquad \begin{aligned} S^r &= S^\alpha \cup S^\beta\ , \quad S^1 \cup \ldots \cup S^{r-1} \cup S^\alpha \subseteq \overset{\vee}{\boldsymbol{E}}\ , \quad S^\beta \subseteq \overset{\wedge}{\boldsymbol{E}}\ , \\ \overrightarrow{\boldsymbol{\lambda}}(S) &= (\varepsilon,\ldots,\varepsilon,\varepsilon)\ . \end{aligned}$$

Presently we formulate a version of the Inheritance Theorem for arbitrary $\boldsymbol{\lambda}^0$ (i.e., not necessarily locally flat) but restricted to the dominating imputation being an element of $\mathcal{H}$ as follows

Theorem 4.4. *Let $\widehat{\boldsymbol{\xi}} \in \mathcal{H}$ and $\boldsymbol{\eta}$ be imputations. Then $\widehat{\boldsymbol{\xi}}\,\mathrm{dom}\,\boldsymbol{\eta}$ if and only if there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ there is a relevant coalition S^ε such that $\widehat{\boldsymbol{\xi}}\,\mathrm{dom}_{S^\varepsilon}\,\boldsymbol{\eta}$ holds true.*

Proof: The "if" part is obvious. Assume, therefore, that there is some coalition T such that $\widehat{\boldsymbol{\xi}}\,\mathrm{dom}_T\,\boldsymbol{\eta}$ holds true.

W.l.o.g we assume again that $\bar{\boldsymbol{\xi}}$ is an imputation, thus $\widehat{\boldsymbol{\xi}}$ admits of a representation

$$(4.58)\qquad \widehat{\boldsymbol{\xi}} := \alpha_0\,\bar{\boldsymbol{\xi}} + \sum_{\rho\in\mathbf{R}} \alpha_\rho \boldsymbol{\lambda}^\rho\ .$$

We distinguish the following three cases

1stSTEP : Assume $\boldsymbol{v}(T) < \boldsymbol{\lambda}^0(T)$. Then w.l.g we have $\boldsymbol{v}(T) = \boldsymbol{\lambda}^1(T)$. Decrease (if necessary) T^2 to T'^2 such that $\boldsymbol{\lambda}^2(T'^2) = \boldsymbol{\lambda}^1(T) = \boldsymbol{v}(T)$. Continue this way until T^r has been decreased to T'^r such that $\boldsymbol{\lambda}^r(T'^r) = \boldsymbol{\lambda}^1(T) = \boldsymbol{v}(T)$ such that we have for $T' := T^1 \cup T'^2 \cup \ldots \cup T'^r$

$$(4.59)\qquad \boldsymbol{v}(T') = \boldsymbol{\lambda}^1(T') = \ldots = \boldsymbol{\lambda}^r(T') < \boldsymbol{\lambda}^0(T')\ .$$

If during this process it happens that

$$\boldsymbol{v}(T') = \boldsymbol{\lambda}^0(T')\ .$$

occurs, then turn to the 2nd and $3^{rd}STEP$. Otherwise clearly $\boldsymbol{v}$ has not changed but $\widehat{\boldsymbol{\xi}}$ has been diminished at most, thus effectiveness is being preserved. Hence $\widehat{\boldsymbol{\xi}}\,\mathrm{dom}_{T'}\,\boldsymbol{\eta}$ and we have *item* 1 for some $\varepsilon = \boldsymbol{v}(T)$. In order to proceed to arbitrary small ε we continue similarly as in previous versions of the Inheritance Theorem (see [5] and [1]): using Ljapounoff regarding the vectorvalued measure $\boldsymbol{\lambda}^1,\ldots,\boldsymbol{\lambda}^r,\boldsymbol{\lambda}^0$. we cut T' into equal pieces $\overline{T}$ an $\overline{\overline{T}}$ such that

$$(\boldsymbol{\lambda}^1,\ldots,\boldsymbol{\lambda}^r)(\overline{T}) = (\boldsymbol{\lambda}^1,\ldots,\boldsymbol{\lambda}^r)(\overline{\overline{T}}) = \frac{1}{2}(\boldsymbol{\lambda}^1,\ldots,\boldsymbol{\lambda}^r)(T')$$

and $\boldsymbol{v}(\overline{T}) = \boldsymbol{\lambda}^1(\overline{T}) < \boldsymbol{\lambda}^0(\overline{T})$ as well as $\boldsymbol{v}(\overline{\overline{T}}) = \boldsymbol{\lambda}^1(\overline{\overline{T}}) < \boldsymbol{\lambda}^0(\overline{\overline{T}})$. Then we must have either $\widehat{\boldsymbol{\xi}}(\overline{T}) \leq \boldsymbol{\lambda}^1(\overline{T}) = \boldsymbol{v}(\overline{T})$ or else $\widehat{\boldsymbol{\xi}}(\overline{\overline{T}}) \leq \boldsymbol{\lambda}^1(\overline{\overline{T}}) = \boldsymbol{v}(\overline{\overline{T}})$ for both reverse strict inequalities holding true would establish a contradiction. Hence one of these coalitions is effective and domination takes place.

Note that this procedure actually works for arbitrary $\boldsymbol{\xi}$ dominating $\boldsymbol{\eta}$ – we do not refer to the fact that $\boldsymbol{\xi} \in \mathcal{H}$.

2ndSTEP :

Assume now that $\boldsymbol{v}(T) = \boldsymbol{\lambda}^0(T)$ and $T \subseteq \overset{\vee}{\boldsymbol{E}}$. W.l.g we have $\boldsymbol{\lambda}^r(T^r) > \boldsymbol{\lambda}^0(T)$ and $\alpha_r = 0$ in the representation (4.58), otherwise we are done. Then, choose $\varepsilon > 0$ such that

$$\varepsilon \boldsymbol{\lambda}^0(T) < \boldsymbol{\lambda}^0(T^r) \ . \tag{4.60}$$

Applying Ljapounoff on the vectorvalued measure $\overrightarrow{\boldsymbol{\lambda}} = (\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r, \boldsymbol{\lambda}^0)$ choose a coalition $T^\varepsilon = T^{\varepsilon 1} \cup \cdots \cup T^{\varepsilon r} \subseteq T$ such that $\overrightarrow{\boldsymbol{\lambda}}(T^\varepsilon) = \varepsilon \overrightarrow{\boldsymbol{\lambda}}(T)$. Then of course we have

$$\boldsymbol{v}(T^\varepsilon) = \boldsymbol{\lambda}^0(T^\varepsilon) \leq \boldsymbol{\lambda}^\rho(T^{\varepsilon\rho}) \quad (\rho \in \mathbf{R} \setminus \{r\}) \ , \quad \boldsymbol{v}(T^\varepsilon) = \boldsymbol{\lambda}^0(T^\varepsilon) < \boldsymbol{\lambda}^r(T^{\varepsilon r}) \ .$$

Next choose coalitions $S^1 \subseteq T^{\varepsilon 1}, \ldots, S^{r-1} \subseteq T^{\varepsilon r-1}$ such that $\boldsymbol{\lambda}^1(S^1) = \ldots = \boldsymbol{\lambda}^{r-1}(T) = \boldsymbol{v}(T^\varepsilon)$. In doing this, we naturally decrease $\boldsymbol{\lambda}^0$ by exactly the quantity

$$G^{r-1} := \sum_{\{\rho \in \mathbf{R} \setminus \{r\}\}} \left(\boldsymbol{\lambda}^0(T^{\varepsilon\rho} \setminus S^\rho)\right) \ .$$

Now we compensate this loss of matter by enlarging $\boldsymbol{\lambda}^0$ on $T^{\varepsilon r}$ by exactly this amount. That is, we choose some coalition $\overline{T}^r$ satisfying $T^{\varepsilon r} \subseteq \overline{T}^r \subseteq T^r$ such that

$$\boldsymbol{\lambda}^0(\overline{T}^r) = G^{r-1} \ ;$$

we choose some coalition $\overline{T}^r$ satisfying $T^{\varepsilon r} \subseteq \overline{T}^r \subseteq T^r$ such that

$$\boldsymbol{\lambda}^0(\overline{T}^r) = G^{r-1} \ .$$

This can be achieved as we are sure that

$$\boldsymbol{\lambda}^0(T^{\varepsilon r}) + \boldsymbol{\lambda}^0(\overline{T}^r) = \boldsymbol{\lambda}^0(T^{\varepsilon r}) + G^{r-1} \leq \boldsymbol{\lambda}^0(T^{\varepsilon r}) + \sum_{\{\rho \in \mathbf{R} \setminus \{r\}\}} \boldsymbol{\lambda}(T^{\varepsilon\rho}) = \boldsymbol{\lambda}^0(T^\varepsilon) = \varepsilon \boldsymbol{\lambda}^0(T) < \boldsymbol{\lambda}(T^r)$$

by (4.60).

Now we have

$$\boldsymbol{\lambda}^0(T^{\varepsilon r} \cup \overline{T}^r) = \boldsymbol{\lambda}^0(T^{\varepsilon r}) + G^{r-1} \ . \tag{4.61}$$

Therefore, if we denote $S^r := T^{\varepsilon r} \cup \overline{T}^r$ then the coalition $S := S^1 \cup \ldots \cup S^r$ obviously satisfies

$$\boldsymbol{\lambda}^0(S^r) = \boldsymbol{\lambda}^0(T^{\varepsilon r}) + G^{r-1} \ , \quad \boldsymbol{\lambda}^0(S^1 \cup \ldots \cup S^{r-1}) = \boldsymbol{\lambda}^0(T^{\varepsilon 1} \cup \ldots \cup T^{\varepsilon r-1}) - G^{r-1} \ ,$$

hence

$$(4.62)\quad \boldsymbol{\lambda}^0(S) \;=\; \boldsymbol{\lambda}^0(T^\varepsilon) \;=\; \boldsymbol{v}(T^\varepsilon) \;=\; \boldsymbol{\lambda}^1(S^1) = \ldots \;=\; \boldsymbol{\lambda}(S^{r-1}) \;=\; \boldsymbol{v}(S)\ .$$

Clearly

$$\boldsymbol{\lambda}^r(S) = \boldsymbol{\lambda}^r(S^r) \geq \boldsymbol{\lambda}^r(T^{\varepsilon r}) \geq \boldsymbol{v}(T^\varepsilon) = \boldsymbol{v}(S)\ .$$

Now, we know that $S \subseteq \overset{\vee}{\boldsymbol{E}}$, hence $\bar{\boldsymbol{\xi}} = \boldsymbol{\lambda}^0$ on S. Then, in view of (4.58) we can establish that

$$(4.63)\qquad \widehat{\boldsymbol{\xi}}(S) \;:=\; \alpha_0\, \boldsymbol{\lambda}^0(S) + \sum_{\{\rho\in\mathbf{R}\setminus\{r\}\}} \alpha_\rho \boldsymbol{\lambda}^\rho(S) = \boldsymbol{v}(S)$$

and as all sets constructed are subsets of T we have

$$(4.64)\qquad \widehat{\boldsymbol{\xi}} > \boldsymbol{\eta} \quad \text{on} \quad S\ .$$

Thus

$$\widehat{\boldsymbol{\xi}}\, \mathrm{dom}_S\, \boldsymbol{\eta}.$$

The generalization to smaller ε is obvious.

3rdSTEP :

Assume now that $\boldsymbol{v}(T) = \boldsymbol{\lambda}^0(T)$ and $T \cap \overset{\wedge}{\boldsymbol{E}} \neq \emptyset$. First note that, for $\rho \in \mathbf{R}$, we can outrule $T^\rho \subseteq \overset{\wedge}{\boldsymbol{E}}$. Indeed, if this holds true say for $\rho = r$, then $\boldsymbol{\lambda}^0(T^t) \geq 1 - l_r^\star$ and as $\boldsymbol{\lambda}^0 \geq l_\rho^0$ for $\rho \neq r$, we have

$$\begin{aligned}
\boldsymbol{v}(T) \;=\; \boldsymbol{\lambda}(T) \;&\geq \sum_{\{\rho\in\mathbf{R}\setminus\{r\}\}} l_\rho^0 \boldsymbol{\lambda}(T^\rho) + (1 - l_r^\star)\boldsymbol{\lambda}(T^r) \\
&\geq \sum_{\{\rho\in\mathbf{R}\setminus\{r\}\}} l_\rho^0 \boldsymbol{v}(T^\rho) + (1 - l_r^\star)\boldsymbol{v}(T^r) \\
&= \boldsymbol{v}(T)\ .
\end{aligned}$$

Hence all inequalities are equations and

$$\boldsymbol{\lambda}^0 \;=\; l_\rho^0 \quad \text{on} \quad T^\rho\ ,(\{\rho \in \mathbf{R} \setminus \{r\}\}) \quad , \quad \boldsymbol{\lambda}^0 \;=\; 1 - l_r^\star \quad \text{on} \quad T^r\ .$$

That is $\boldsymbol{\lambda}^0$ has minimum levels on each $\boldsymbol{C}^\rho$. We would then have a locally flat $\boldsymbol{\lambda}^0$, a case which we have ruled out for the present discussion.

4thSTEP :

Therefore we can assume that w.l.g T^r has positive measure in $\overset{\vee}{\boldsymbol{E}}$ as well as in $\overset{\wedge}{\boldsymbol{E}}$. Accordingly we write

$$T^r = T^\alpha \cup T^\beta\ , \quad T^\alpha \subseteq \overset{\vee}{\boldsymbol{E}}\ , \quad T^\beta \subseteq \overset{\wedge}{\boldsymbol{E}}\ .$$

Now, for sufficiently small $\varepsilon > 0$ we first choose (via Ljapounoff) $T^\varepsilon \subseteq T$ such that the vectorvalued measure $\overrightarrow{\boldsymbol{\lambda}}$ satisfies

$$\overrightarrow{\boldsymbol{\lambda}}(T^\varepsilon) = \frac{\varepsilon}{\boldsymbol{v}(T)} \overrightarrow{\boldsymbol{\lambda}}(T)$$

that is

$$\boldsymbol{\lambda}^0(T^\varepsilon) = \varepsilon \leq \boldsymbol{\lambda}^\rho(T^\rho) \quad (\rho \in \mathbf{R}) \ .$$

Next we can choose

$$S^\rho \subseteq T^\rho \quad (\{\rho \in \mathbf{R} \setminus \{r\}\}) \quad \text{such that} \quad \boldsymbol{\lambda}^\rho(S^\rho) = \varepsilon$$

holds true. Also choose $\overline{S}^r \subseteq T^{\varepsilon r}$ such that $(\boldsymbol{\lambda}^r, \boldsymbol{\lambda}^0)(\overline{S}^r)$ is located on the ray $(0,0), (\boldsymbol{\lambda}^r, \boldsymbol{\lambda}^0)(T^r)$ and satisfies $\boldsymbol{\lambda}^r(\overline{S}^r) = \varepsilon$. Then clearly

$$\boldsymbol{\lambda}^0(S^1 \cup \ldots \cup S^{r-1} \cup \overline{S}^r) \leq \boldsymbol{\lambda}^r(\overline{S}^r) \ = \ \varepsilon \ . \tag{4.65}$$

We write $F^r \ := \ \varepsilon - \sum_{\{\rho \in \mathbf{R} \setminus \{r\}\}} \boldsymbol{\lambda}^\rho(S^\rho)$ then

$$\boldsymbol{\lambda}^0(\overline{S}^r) \leq F^r \quad , \quad \boldsymbol{\lambda}^r(\overline{S}^r) = \varepsilon \ . \tag{4.66}$$

Next choose $S^\beta \subseteq T^\beta$ such that $\boldsymbol{\lambda}^r(S^\beta) = \varepsilon$ holds true. Then, as $T^\beta \subseteq \hat{\boldsymbol{E}}$, we have $\dot{\boldsymbol{\lambda}}^0 \geq 1 - l_r^\star$ on T^β and $\dot{\boldsymbol{\lambda}}^0 \leq l_\rho^0$ on S^ρ.

Hence

$$\begin{aligned} \boldsymbol{\lambda}^0(S^\beta) \ &\geq \ \varepsilon(1 - l_r^\star) \\ &= \ \varepsilon - \sum_{\{\rho \in \mathbf{R} \setminus \{r\}\}} l_\rho^0 \\ &\geq \ \varepsilon - \sum_{\{\rho \in \mathbf{R} \setminus \{r\}\}} \boldsymbol{\lambda}^\rho(S^\rho) \\ &= \ F^r \ , \end{aligned} \tag{4.67}$$

that is

$$\boldsymbol{\lambda}^0(S^\beta) \geq F^r \quad , \quad \boldsymbol{\lambda}^r(S^\beta) = \varepsilon \ . \tag{4.68}$$

Again using Ljapounoff we use (4.66) and (4.68) to construct some coalition $S^r \subseteq T^r$ such that

$$(\boldsymbol{\lambda}^r, \boldsymbol{\lambda}^0)(S^r) \ = \ (\varepsilon, F^r) \tag{4.69}$$

holds true. Then, with $S \ := \ S^1 \cup \ldots \cup S^r$ we obtain

$$\boldsymbol{\lambda}^0(S) \ = \ \sum_{\{\rho \in \mathbf{R} \setminus \{r\}\}} \boldsymbol{\lambda}^\rho(S^\rho) + F^r \ = \ \varepsilon \ , \tag{4.70}$$

meaning

$$\boldsymbol{\lambda}^1(S) = \ldots = \boldsymbol{\lambda}^r(S) = \boldsymbol{\lambda}^0(S) = \boldsymbol{v}(S) \ .$$

Now we have generally $\bar{\boldsymbol{\xi}} \leq \boldsymbol{\lambda}^0$ and hence by (4.58)

$$\widehat{\boldsymbol{\xi}}(S) \ \leq \ \alpha_0 \, \boldsymbol{\lambda}^0(S) + \sum_{\rho \in \mathbf{R}} \alpha_\rho \boldsymbol{\lambda}^\rho(S) = \boldsymbol{v}(S) \ . \tag{4.71}$$

As all the sets constructed are subsets of T we have of course $\widehat{\boldsymbol{\xi}} > \boldsymbol{\eta}$ on S, hence $\widehat{\boldsymbol{\xi}} \ dom_S \ \boldsymbol{\eta}$, the generalization to smaller ε runs as above.

q.e.d.

Theorem 4.5 (All Games have a Standard vNM–Stable Set). *Let* $(\boldsymbol{I}, \underline{\underline{\mathbf{F}}}, \boldsymbol{v})$ *be a continuous model. Let* $\underline{\boldsymbol{\Delta}}$ *be an admissible distribution of mass and let*

$$\overset{\circ}{\overline{\boldsymbol{\xi}}} \;=\; \overset{\circ}{\overline{\boldsymbol{\xi}}}^{\underline{\boldsymbol{\Delta}}} \;=\; \overline{\boldsymbol{\xi}} + \boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}}$$

be the resulting imputation. Then the set $\mathcal{H}$ *defined by* (3.1) *is internally stable, hence a vNM–Stable Set.*

Proof:

1stSTEP :

Again w.l.g $\boldsymbol{\lambda}^{\underline{\boldsymbol{\Delta}}} = 0$ and $\bar{\boldsymbol{\xi}}$ is an imputation. Assume that for some $\widehat{\boldsymbol{\xi}},\ \widehat{\boldsymbol{\eta}} \in \mathcal{H}$ we have

$$\widehat{\boldsymbol{\xi}} \operatorname{dom}_S \widehat{\boldsymbol{\eta}} \tag{4.72}$$

for some coalition S satisfying one of the above three *items*. Let

$$\widehat{\boldsymbol{\xi}} \;=\; \alpha_0 \bar{\boldsymbol{\xi}} + \sum_{\rho \in \mathbf{R}} \alpha_\rho \boldsymbol{\lambda}^r, \quad \widehat{\boldsymbol{\eta}} \;=\; \beta_0 \bar{\boldsymbol{\xi}} + \sum_{\rho \in \mathbf{R}} \beta_\rho \boldsymbol{\lambda}^r \quad .$$

We can rewrite these equations such that

$$\widehat{\boldsymbol{\xi}} = \alpha_0 \bar{\boldsymbol{\xi}} + (1 - \alpha_0) \boldsymbol{e}^\alpha \ , \tag{4.73}$$

where

$$\boldsymbol{e}^\alpha \;:=\; \sum_{\sigma \in \mathbf{R}} \left(\frac{\alpha_\sigma}{\sum_{\rho \in \mathbf{R}} \alpha_\rho} \right) \boldsymbol{\lambda}^\sigma$$

is a core element. Similarly we write

$$\widehat{\boldsymbol{\eta}} = \beta_0 \bar{\boldsymbol{\xi}} + (1 - \beta_0) \boldsymbol{e}^\beta \ . \tag{4.74}$$

Now we can apply Theorem 4.4 and distinguish the following alternatives.

2ndSTEP :

First consider an e–relevant coalition S as described in *item* 1 or *item* 3 . In both cases we have

$$\boldsymbol{\lambda}^1(S) \;=\; \ldots \;=\; \boldsymbol{\lambda}^r(S) \leq \boldsymbol{\lambda}^0(S) \ ,$$

and hence

$$\varepsilon = \boldsymbol{v}(S) = \boldsymbol{e}^\alpha(S) = \boldsymbol{e}^\beta(S) \ . \tag{4.75}$$

As $\widehat{\boldsymbol{\xi}}$ is effective for S we have

$$\varepsilon \geq \boldsymbol{v}(S) \geq \widehat{\boldsymbol{\xi}}(S) = \alpha_0 \bar{\boldsymbol{\xi}}(S) + (1 - \alpha_0) \boldsymbol{e}^\alpha(S) = \alpha_0 \bar{\boldsymbol{\xi}}(S) + (1 - \alpha_0) \varepsilon \ ,$$

that is

$$0 \geq \alpha_0 \left(\bar{\boldsymbol{\xi}}(S) - \varepsilon\right) \ .$$

Consequently

$$\bar{\boldsymbol{\xi}}(S) \leq \varepsilon \quad \text{or} \quad \alpha_0 = 0 \ . \tag{4.76}$$

On the other hand we deduce from $\widehat{\boldsymbol{\xi}} > \widehat{\boldsymbol{\eta}}$ "on S" that $\widehat{\boldsymbol{\xi}}(S) > \widehat{\boldsymbol{\eta}}(S)$, hence

$$\alpha_0 \bar{\boldsymbol{\xi}}(S) + (1-\alpha_0)\boldsymbol{e}^{\alpha}(S) > \beta_0 \bar{\boldsymbol{\xi}}(S) + (1-\beta_0)\boldsymbol{e}^{\beta}(S) \ ,$$

i.e.,

$$\alpha_0 \bar{\boldsymbol{\xi}}(S) + (1-\alpha_0)\varepsilon > \beta_0 \bar{\boldsymbol{\xi}}(S) + (1-\beta_0)\varepsilon \ ,$$

that is

$$(\alpha_0 - \beta_0)\bar{\boldsymbol{\xi}}(S) > (\alpha_0 - \beta_0)\varepsilon \ .$$

Consequently

$$\bar{\boldsymbol{\xi}}(S) > \varepsilon \ . \tag{4.77}$$

Combining (4.76) and (4.77) we obtain $\alpha_0 = 0$ and hence $\widehat{\boldsymbol{\xi}} = \boldsymbol{e}^{\alpha}$. Now, if $\beta_0 > 0$ then

$$\widehat{\boldsymbol{\eta}} = \beta_0 \bar{\boldsymbol{\xi}} + (1-\beta_0)\boldsymbol{e}^{\beta} > \beta_0 \varepsilon + (1-\beta_0)\varepsilon = \varepsilon = \boldsymbol{v}(S) \geq \widehat{\boldsymbol{\xi}}(S)$$

which is impossible, hence $\beta_0 = 0$ and $\widehat{\boldsymbol{\eta}} = \boldsymbol{e}^{\beta}$. Therefore (4.72) results in

$$\boldsymbol{e}^{\alpha} \operatorname{dom}_S \boldsymbol{e}^{\beta} \ ,$$

but dominance between two core elements is impossible as well.

3rdSTEP :

Now consider the case presented in *item* 2. This time we have

$$\boldsymbol{\lambda}^1(S) \ = \ \ldots \ = \ \boldsymbol{\lambda}^{r-1}(S) \ = \ \boldsymbol{\lambda}^0(S) \ \leq \boldsymbol{\lambda}^r(S) \ .$$

Note that $S \subseteq \overset{\vee}{\boldsymbol{E}}$ implies

$$\bar{\boldsymbol{\xi}} = \boldsymbol{\lambda}^0 \quad \text{on} \quad S$$

and therefore

$$\widehat{\boldsymbol{\xi}} \ = \ \alpha_0 \boldsymbol{\lambda}^0 + \sum_{\rho \in \mathbf{R}} \boldsymbol{\lambda}^{\rho}$$

on S. Hence we write now

$$\widehat{\boldsymbol{\xi}} = \alpha_r \boldsymbol{\lambda}^r + (1-\alpha_r)\boldsymbol{e}^0 \ , \tag{4.78}$$

where

$$\boldsymbol{e}^0 \ := \ \sum_{\sigma \in \mathbf{R}\setminus\{r\}} \left(\frac{\alpha_\sigma}{\sum_{\rho \in \mathbf{R}\setminus\{r\}\cup\{0\}} \alpha_\rho} \right) \boldsymbol{\lambda}^{\sigma} + \frac{\alpha_0}{\sum_{\rho \in \mathbf{R}\setminus\{r\}\cup\{0\}} \alpha_\rho} \boldsymbol{\lambda}^0$$

satisfies $\boldsymbol{e}^0(S) = \boldsymbol{v}(S) = \delta$. Similarly

$$\widehat{\boldsymbol{\eta}} = \beta_r \boldsymbol{\lambda}^r + (1 - \beta_r)\boldsymbol{e}^{00} \ , \tag{4.79}$$

with $\boldsymbol{e}^{00}(S) = \boldsymbol{v}(S) = \varepsilon$. Now, if $\boldsymbol{\lambda}^r(S) = \varepsilon$, then we can apply the procedure as in the $2^{nd} STEP$. If, on the other hand, $\boldsymbol{\lambda}^r(S) > \varepsilon$, then $\widehat{\boldsymbol{\xi}}(S)\varepsilon$ or $\widehat{\boldsymbol{\eta}}(S) > \varepsilon$ would result; none of which is compatible with $\varepsilon = \boldsymbol{v}(S) \geq \widehat{\boldsymbol{\xi}}(S) > \widehat{\boldsymbol{\eta}}(S)$. Hence necessarily $\alpha_r = 0$. Thus we obtain

$$\boldsymbol{e}^0 \operatorname{dom}_S \boldsymbol{e}^{00}$$

but $\boldsymbol{v}(S) = \boldsymbol{e}^0(S) > \boldsymbol{e}^{00}(S) = \varepsilon$ is by no means possible.

q.e.d.

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