Center for
Mathematical Economics
Working Papers

534

January 2015

Convex vNM–Stable Sets for a Semi Orthogonal Game

Part IV:

Large Economies: The Existence Theorem

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ISSN: 0931-6558

Abstract

Within this paper we establish the existence of a vNM–Stable Set for (cooperative) linear production games with a continuum of players. The coalitional function is generated by $r+1$ "production factors" (non atomic measures). r factors are given by orthogonal probabilities ("cornered" production factors) establishing the core of the game. Factor $r+1$ (the "centralized" production factor) is represented by a nonantomic measure with carrier "across the corners" of the market; i.e., this factor is more abundantly available and the representing measure is not located within the core of the game.

The present paper continues a series of presentations of this topic, for Part I,II,III see [1], [2], [3].

We focus on *convex* vNM–Stable Sets of the game and we present an existence theorem valid for "Large Economies" (the term is not quite orthodox). There are some basic assumptions for the present model which enable us to come up with a comprehensive version of an existence theorem. However, in order to make our presentation tractable (and readable) we wisely restrict ourselves to a simplified model.

As in our previous models there is a (not necessarily unique) imputation outside the core such that the vNM–Stable Set is the convex hull of this imputation and the core. Significantly, this additional imputation can be seen as a truncation of the "centralized" distribution, i.e., the $r+1^{st}$ production factor. Hence there is a remarkable similarity *mutatis mutandis* regarding the Characterization Theorem that holds true for the "purely orthogonal case" ([4],[5]).

1 Notations and Definitions

Within this paper we present a general existence theorem for convex vNM–Stable sets for a Semi Orthogonal Game as introduced in [1] and continued in [2] and [3].

There are some restrictions imposed on the model which are essentially minor. In the present model, the centralized production factor is available in sectors $\boldsymbol{D}^\tau$ of equal size, in other words, the quantities $\lambda_\tau = \boldsymbol{\lambda}(\boldsymbol{D}^\tau) \;\; (\tau \in \mathbf{T})$ are supposed to be equal, i.e., $\lambda_\tau = \frac{1}{t} \;\; (\tau \in \mathbf{T})$.

We use definitions and notations as provided in [1], [2] and previously in [4] and [5]. the reader familiar with this setup may well skip this introductory section . Thus, we consider a (cooperative) ***game*** with a continuum of players, i.e., a triple $(\boldsymbol{I}, \underline{\underline{\mathbf{F}}}, \boldsymbol{v})$ where $\boldsymbol{I}$ is some interval in the reals (the ***players***), $\underline{\underline{\mathbf{F}}}$ is the $\sigma-$field of (Borel) measurable sets (the ***coalitions***) and $\boldsymbol{v}$ (the ***coalitional function***) is a mapping $\boldsymbol{v} : \underline{\underline{\mathbf{F}}} \to \mathbb{R}_+$ which is absolutely continuous w.r.t. the Lebesgue measure $\boldsymbol{\lambda}$. We focus on "linear production games", that is, $\boldsymbol{v}$ is described by finitely many measures $\boldsymbol{\lambda}^\rho, (\rho \in \{0, 1, \ldots, r\})$ via

$$(1.1) \qquad \boldsymbol{v}(S) \; := \; \min \{\boldsymbol{\lambda}^\rho(S) \mid \rho \in \{0, 1, \ldots, r\}\} \quad (S \in \underline{\underline{\mathbf{F}}}).$$

or

$$(1.2) \qquad \boldsymbol{v} = \bigwedge \{\boldsymbol{\lambda}^0, \boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r\} \;\; = \;\; \bigwedge_{\rho \in \mathbf{R}_0} \boldsymbol{\lambda}^\rho \; ,$$

(as previously, we use $\mathbf{R} = \{1, \ldots, r\}$ and $\mathbf{R}_0 = \mathbf{R} \cup \{0\}$). All measures are absolutely continuous w.r.t to Lebesgue measure $\boldsymbol{\lambda}$. The measures $\boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r$ are orthogonal copies of Lebesgue measure on $[0, 1]$. Accordingly, we choose the player set to be $\boldsymbol{I} \; := \; [0, r)$. The carriers $\boldsymbol{C}^\rho \; = \; (\rho - 1, \rho] \;\; (\rho = 0, \ldots, r)$ of the measures $\boldsymbol{\lambda}^\rho$ are the "cartels" commanding commodity ρ. Further details of our notation are exactly those presented in [1].

In particular, the measure $\boldsymbol{\lambda}^0$, $\;(\boldsymbol{\lambda}^0(\boldsymbol{I}) > 1)$ is assumed to have a piecewise constant density $\dot{\boldsymbol{\lambda}}^0$ w.r.t $\boldsymbol{\lambda}$. To this end we consider some family $\{\boldsymbol{D}^\tau\}_{\tau \in \mathbf{T}^\rho}$ that constitutes a partition of the carrier $\boldsymbol{C}^\rho$ of $\boldsymbol{\lambda}^\rho$ such that $\bigcup_{\tau \in \mathbf{T}^\rho} \boldsymbol{D}^\tau = \boldsymbol{C}^\rho$. $\boldsymbol{\lambda}^0$ has constant density h_τ on each $\boldsymbol{D}^\tau$.

For completenes we repeat the basic definitions of our solution concept, the ***vNM–Stable Set*** (von Neumann-Morgenstern [6]). see also the Part I,II,III, i.e., [1],[2],[3].

Definition 1.1. *Let* $(I, \underline{\underline{\mathbf{F}}}, \boldsymbol{v})$ *be a game. An* ***imputation*** *is a measure* $\boldsymbol{\xi}$ *with* $\boldsymbol{\xi}(I) = \boldsymbol{v}(I)$*. An imputation* $\boldsymbol{\xi}$ ***dominates*** *an imputation* $\boldsymbol{\eta}$ *w.r.t* $S \in \underline{\underline{\mathbf{F}}}$ *if* $\boldsymbol{\xi}$ *is* ***effective*** *for* S*, i.e.,*

$$(1.3) \qquad \boldsymbol{\lambda}(S) > 0 \;\; \textit{and} \;\; \boldsymbol{\xi}(S) \leq \boldsymbol{v}(S)$$

and if

$$(1.4) \qquad \boldsymbol{\xi}(T) > \boldsymbol{\eta}(T) \;\; (T \in \underline{\underline{\mathbf{F}}}, \; T \subseteq S, \boldsymbol{\lambda}(T) > 0)$$

holds true. That is, every subcoalition of S (almost every player in S) strictly improves its payoff at $\boldsymbol{\xi}$ versus $\boldsymbol{\eta}$. We write $\boldsymbol{\xi}\,\mathrm{dom}_S\,\boldsymbol{\eta}$ to indicate domination. It is standard to use $\boldsymbol{\xi}\,\mathrm{dom}\,\boldsymbol{\eta}$ whenever $\boldsymbol{\xi}\,\mathrm{dom}_S\,\boldsymbol{\eta}$ holds true for some *coalition $S \in \underline{\underline{\mathbf{F}}}$.*

Definition 1.2. *Let $\boldsymbol{v}$ be a game. A set $\mathcal{S}$ of imputations is called a* ***vNM–Stable Set*** *if*

- *there is no pair $\boldsymbol{\xi}, \boldsymbol{\mu} \in \mathcal{S}$ such that $\boldsymbol{\xi}\,\mathrm{dom}\,\boldsymbol{\mu}$ holds true (*“internal stability”*).*
- *for every imputation $\boldsymbol{\eta} \notin \mathcal{S}$ there exists $\boldsymbol{\xi} \in \mathcal{S}$ such that $\boldsymbol{\xi}\,\mathrm{dom}\,\boldsymbol{\eta}$ is satisfied (*“external stability”*).*

The discrete nature of the density of $\boldsymbol{\lambda}^0$ carries some implications for the establishment of dominance based on discrete analogues of concepts like imputations, coalitions etc. We refer to these analogues as “pre–concepts”. Again see Part 1, i.e., [1] for the details.

2 The Uniform Model

We simplify the shape of the density $\dot{\boldsymbol{\lambda}}^0$ as follows. We assume that the underlying partition is **uniform** in the sense that

$$\lambda_\tau = \boldsymbol{\lambda}(\boldsymbol{D}^\tau) = \frac{1}{t} \quad (\tau = 1, \ldots, rt) \tag{2.1}$$

holds true, in other words, each carrier $\boldsymbol{C}^\rho$ is partitioned into t pieces of equal Lebesgue measure such that

$$\boldsymbol{C}^\rho = \bigcup_{\tau = (\rho-1)t+1}^{\rho t} \boldsymbol{D}^\tau . \tag{2.2}$$

As a consequence, for some vector $\boldsymbol{x} \in \mathbb{R}^{rt}_{+}$ and the generated imputation $\boldsymbol{\vartheta}^{\boldsymbol{x}}$, we have

$$\int \boldsymbol{\vartheta}^{\boldsymbol{x}} d\boldsymbol{\lambda} = \sum_{\tau \in \mathbf{T}} \lambda_\tau x_\tau = \sum_{\tau \in \mathbf{T}} \frac{1}{t} x_\tau ;$$

hence the set of pre–imputations is slightly simplified to be

$$\boldsymbol{J}(v) = \left\{ \boldsymbol{x} \in \mathbb{R}^{rt}_{+} \,\middle|\, \sum_{\tau \in \mathbf{T}} x_\tau = t \right\} . \tag{2.3}$$

In what follows, we shall refer to the sequences $\boldsymbol{\tau}$ as to be the **undercutting** if $\sum_{\rho \in \mathbf{R}} h_{\tau_\rho} < 1$ and **overstepping** if $\sum_{\rho \in \mathbf{R}} h_{\tau_\rho} \geq 1$.

Definition 2.1. *1. Denote by*

$$\overset{\vee}{\boldsymbol{\tau}} = (\overset{\vee}{\tau}_1, \ldots, \overset{\vee}{\tau}_r) \tag{2.4}$$

*the/a **minimizing** sequence, i.e., the sequence with minimal sum*

$$\sum_{\rho \in \mathbf{R}} h_{\overset{\vee}{\tau}_\rho} \leq \sum_{\rho \in \mathbf{R}} h_{\tau_\rho} \quad (\boldsymbol{\tau} \in \mathbf{T}^1 \times \ldots \times \mathbf{T}^r) \ . \tag{2.5}$$

2. Let, for $\sigma \in \mathbf{R}$,

$$\overset{\vee}{\mathbf{T}}{}^\sigma := \left\{ \tau \in \mathbf{T}^\sigma \,\middle|\, \sum_{\rho \in \mathbf{R} \setminus \{\sigma\}} h_{\overset{\vee}{\tau}_\rho} + h_\tau < 1 \right\} \tag{2.6}$$

and put

$$\overset{\vee}{\mathbf{T}} := \bigcup_{\sigma \in \mathbf{R}} \overset{\vee}{\mathbf{T}}{}^\sigma \ . \tag{2.7}$$

That is, $\overset{\vee}{\mathbf{T}}$ is the set of all indices that belong to some undercutting sequence.

3. Furthermore let, for $\sigma \in \mathbf{R}$,

$$\overset{\wedge}{\mathbf{T}}{}^\sigma := \left\{ \tau \in \mathbf{T}^\sigma \,\middle|\, \sum_{\rho \in \mathbf{R} \setminus \{\sigma\}} h_{\overset{\vee}{\tau}_\rho} + h_\tau \geq 1 \right\} \tag{2.8}$$

and put

$$\overset{\wedge}{\mathbf{T}} := \bigcup_{\sigma \in \mathbf{R}} \overset{\wedge}{\mathbf{T}}{}^\sigma \ . \tag{2.9}$$

That is, $\overset{\wedge}{\mathbf{T}}$ is the set of all indices that appear in overshooting sequences only.

Lemma 2.2. *Either $|\overset{\vee}{\mathbf{T}}| \geq r+1$ or $\mathcal{C}(\boldsymbol{v})$ is the unique vNM–Stable Set and not both. In the first case there is some index $\overset{\circ}{\tau}$ such that*

$$\left\{ \overset{\vee}{\tau}_1, \ldots, \overset{\vee}{\tau}_r, \overset{\circ}{\tau} \right\} \subseteq \overset{\vee}{\mathbf{T}} \ . \tag{2.10}$$

$\overset{\circ}{\tau}$ is a "next minimizing" index, i.e.,

$$h_{\overset{\circ}{\tau}} \leq h_\tau \quad \left(\tau \in \mathbf{T} \setminus \left\{ \overset{\vee}{\tau}_1, \ldots, \overset{\vee}{\tau}_r, \right\} \right) \ . \tag{2.11}$$

Proof:

This follows from Theorem 4.9 in Part I ([1])

q.e.d.

We now specify the basic assumptions for the model under consideration within this fourth part of our presentation.

Definition 2.3. *We call*

$$\boldsymbol{v} = \bigwedge \{\boldsymbol{\lambda}^0, \boldsymbol{\lambda}^1, \ldots, \boldsymbol{\lambda}^r\} \quad = \quad \bigwedge_{\rho \in \mathbf{R}_0} \boldsymbol{\lambda}^\rho .$$

a **uniform** *game if the following conditions are satisfied.*

1. $\boldsymbol{\lambda}^0$ *is* **uniform**, *i.e.*

$$\lambda_\tau = \boldsymbol{\lambda}(\boldsymbol{D}^\tau) = \frac{1}{t} \quad (\tau = 1, \ldots, rt) \; . \tag{2.12}$$

2. *There is* $\overset{\circ}{\tau} \in \mathbf{T}$ *as described in Lemma 2.2 such that*

$$\left\{\overset{\vee}{\tau}_1, \ldots, \overset{\vee}{\tau}_r, \overset{\circ}{\tau}\right\} \subseteq \overset{\vee}{\mathbf{T}} \; . \tag{2.13}$$

holds true.

In what follows we will always assume that we are dealing with a uniform game. Thus, in particular the cases treated by Lemma 2.2 and Theorem 4.9 of Part I in which the core turns out to be the unique vNM–Stable Set, are considered to be settled.

Recall the set of preimputations

$$\boldsymbol{H} \;=\; \{\boldsymbol{x} \in \boldsymbol{J}(v) \,|\, \boldsymbol{x}\boldsymbol{a} \geq 1 = v(\boldsymbol{a}) \;\; (\boldsymbol{a} \in \boldsymbol{A}^s)\} \tag{2.14}$$

that serves to provide candidates to generate a vNM–Stable Set. As previously, we will provide a pre–imputation $\overset{\circ}{\bar{\boldsymbol{x}}} \in \boldsymbol{H}$ such that

$$\overset{\circ}{\boldsymbol{H}} \;=\; \mathbf{ConvH}\left\{\overset{\circ}{\bar{\boldsymbol{x}}}, \boldsymbol{e}^{\mathbf{T}^\rho} \;\; (\rho \in \mathbf{R})\right\} \subseteq \boldsymbol{H} \tag{2.15}$$

induces a vNM–Stable Set

$$\overset{\circ}{\mathcal{H}} \;=\; \mathbf{ConvH}\left\{\boldsymbol{\vartheta}^{\overset{\circ}{\bar{\boldsymbol{x}}}}, \boldsymbol{\lambda}^\rho(\rho \in \mathbf{R})\right\} \;=\; \left\{\boldsymbol{\vartheta}^{\boldsymbol{x}} \,\middle|\, \boldsymbol{x} \in \overset{\circ}{\boldsymbol{H}}\right\} \; . \tag{2.16}$$

As a prerequisite we start out by exhibiting a vector $\bar{\boldsymbol{x}}$ that resembles the previous candidates for setting up a vNM–Stable Set in Part I,II,III. However, as it turns out, $\bar{\boldsymbol{x}}$ is in general just a sub pre–imputation and further work is necessary in order to exhibit the pre–imputations $\overset{\circ}{\bar{\boldsymbol{x}}}$ that eventually serve to generate vNM–Stable Sets as above.

Definition 2.4. *1. We define a vector $\bar{\boldsymbol{x}}$ as follows. First of all we put*

$$\overline{x}_\tau \;:=\; h_\tau \quad (\tau \in \overset{\vee}{\mathbf{T}}) \tag{2.17}$$

such that

$$\sum_{\rho\in\mathbf{R}} \overline{x}_{\tau_\rho} < 1 \qquad \textit{whenever} \sum_{\rho\in\mathbf{R}} h_{\tau_\rho} < 1 \;, \tag{2.18}$$

that is, whenever $\boldsymbol{\tau}$ is undercutting.

Note that the minimal sequence is undercutting according to our present convention, i.e.,

$$\sum_{\rho\in\mathbf{R}} h_{\overset{\vee}{\tau}_\rho} \;=\; \sum_{\rho\in\mathbf{R}} \overline{x}_{\overset{\vee}{\tau}_\rho} < 1 \;. \tag{2.19}$$

2. Now, for $\sigma \in \mathbf{R}$ and all *$\tau \in \overset{\wedge}{\mathbf{T}}{}^\sigma$ define*

$$\overline{x}_\tau \;:=\; 1 - \sum_{\rho\in\mathbf{R}\setminus\{\sigma\}} h_{\overset{\vee}{\tau}_\rho} \tag{2.20}$$

such that

$$\sum_{\rho\in\mathbf{R}\setminus\{\sigma\}} h_{\overset{\vee}{\tau}_\rho} + \overline{x}_\tau = \sum_{\rho\in\mathbf{R}\setminus\{\sigma\}} \overline{x}_{\overset{\vee}{\tau}_\rho} + \overline{x}_\tau \;=\; 1 \;. \tag{2.21}$$

Then in particular

$$\sum_{\rho\in\mathbf{R}} \overline{x}_{\tau_\rho} \;=\; 1 \tag{2.22}$$

for any sequence $\boldsymbol{\tau}$ with

$$\tau_\sigma \in \overset{\wedge}{\mathbf{T}}{}^\sigma \;\textit{and}\; \tau_\rho \;=\; \overset{\vee}{\tau}_\rho \quad (\rho \in \mathbf{R}\setminus\{\sigma\}) \;.$$

Remark 2.5. Observe that because of (2.19) and (2.21) we have for $\sigma \in \mathbf{R}$

$$\overline{x}_\tau > h_{\overset{\vee}{\tau}_\sigma} \text{ for all } \tau \in \overset{\wedge}{\mathbf{T}}{}^\sigma \;. \tag{2.23}$$

Hence, for any sequence $\boldsymbol{\tau}$ involving elements $h_{\overset{\vee}{\tau}_\bullet}$ as well as some $\overline{x}_\tau$ for $\overset{\wedge}{\mathbf{T}}{}^\sigma$ the sum of all elements will exceed 1, e.g.,

$$\begin{aligned}
&\overline{x}_{\tau_1} + \overline{x}_{\tau_2} + \overline{x}_{\overset{\vee}{\tau}_3} + \overline{x}_{\overset{\vee}{\tau}_4} + + \ldots + \overline{x}_{\overset{\vee}{\tau}_r} \\
=\;& \overline{x}_{\tau_1} + \overline{x}_{\tau_2} + h_{\overset{\vee}{\tau}_3} + h_{\overset{\vee}{\tau}_4} + \ldots + h_{\overset{\vee}{\tau}_r} \\
\geq\;& \overline{x}_{\tau_1} + h_{\overset{\vee}{\tau}_2} + h_{\overset{\vee}{\tau}_3} + h_{\overset{\vee}{\tau}_4} + \ldots + h_{\overset{\vee}{\tau}_r} \\
=\;& 1
\end{aligned}$$

Moreover, because the sequence $\overset{\vee}{\boldsymbol{\tau}} = \{\overset{\vee}{\tau}_\rho\}_{\rho\in\mathbf{R}}$ has the minimal sum over all elements, it follows that for any sequence $\boldsymbol{\tau}$ involving elemets of $\overset{\wedge}{\mathbf{T}}$ as well as of $\overset{\vee}{\mathbf{T}}$ we have $\sum_{\rho\in\mathbf{R}} \overline{x}_{\tau_\rho} \;\geq\; 1$. Hence, whenever for some sequence $\boldsymbol{\tau}$ we have $\sum_{\rho\in\mathbf{R}} h_{\tau_\rho} \;<\; 1$, then obviously the corresponding relevant vector $\boldsymbol{a}^{\oplus}$ yields $\boldsymbol{a}^{\oplus}\bar{\boldsymbol{x}} = 1$ as $\bar{\boldsymbol{x}}$ coincides with $\boldsymbol{h}$ along the coordinates prescribed by this sequence. We conclude that $\bar{\boldsymbol{x}}$ satisfies all the equations defining $\boldsymbol{H}$ with the possible exception that

$$\bar{\boldsymbol{x}} \in \boldsymbol{J}(v)\ , \quad \text{i.e.,} \quad \bar{\boldsymbol{x}} \geq \mathbf{0}\ , \ \sum_{\tau\in\mathbf{T}} \overline{x}_\tau \;=\; t \tag{2.24}$$

may be violated.

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3 The Extremals of $\boldsymbol{H}$

As previously $\boldsymbol{J} = \boldsymbol{J}(v)$ denotes the pre–imputations of the pre–game v. Using the set $\boldsymbol{A}^s$ of separating pre–coalitions, we we recall the set

$$\text{(3.1)} \qquad \boldsymbol{H} \ = \ \{\boldsymbol{x} \in \boldsymbol{J} \,|\, \boldsymbol{x}\boldsymbol{a} \geq v(\boldsymbol{a}) \ = \ 1 \ \ (\boldsymbol{a} \in \boldsymbol{A}^s)\}$$

of pre–imputations that cannot be dominated via some separating pre–coalition (SECTION 4 of *Part I*). $\boldsymbol{H}$ has been introduced in (4.7.) of Part I (i.e. [1]) and indeed provides a candidate in the special set–up discussed in Parts II and III. Within the framework establishend in that context, it turned out that $\boldsymbol{H}$ had just one extremal point apart from the vectors $\boldsymbol{e}^{\mathbf{T}^\rho}$ $(\rho \in \mathbf{R})$.

Within this section we will illuminate the general situation in the context of uniform games. We will exhibit all the extremals of $\boldsymbol{H}$ which, in general are finitely many. Of course, all the extremals of the core, i.e., the vectors $\boldsymbol{e}^{\mathbf{T}^\rho}$ $(\rho \in \mathbf{R})$, are extremals of $\boldsymbol{H}$ as well, we mean to specify the remaining ones. To this end, define

$$\text{(3.2)} \qquad \Delta \ := \ t - \sum_{\tau \in \mathbf{T}} \overline{x}_\tau \ = \ t - \left\{ \sum_{\tau \in \overset{\vee}{\mathbf{T}}} h_\tau + \sum_{\tau \in \overset{\wedge}{\mathbf{T}}} \overline{x}_\tau \ . \right\}$$

If $\Delta \ < \ 0$, then we know that

$$\text{(3.3)} \qquad \boldsymbol{H} \ = \ \mathbf{ConvH} \left\{ \boldsymbol{e}^{T^\rho} \ \ (\rho \in \mathbf{R}) \right\} \ .$$

That is, $\boldsymbol{H} \ = \ \boldsymbol{C}(v)$ equals the pre–core, this is the alternative case mentioned in Lemma 2.2 and excluded by our assumtion about the uniform model. In the uniform case under consideration we have $\Delta \geq 0$.

Definition 3.1. *For $\Delta \geq 0$ we define*

$$\text{(3.4)} \qquad \bar{\boldsymbol{x}}^\sigma \ := \ \bar{\boldsymbol{x}} + \Delta \boldsymbol{e}^\sigma \ \ (\sigma \in \overset{\wedge}{\mathbf{T}}) \ .$$

We are going to prove that

$$\text{(3.5)} \qquad \boldsymbol{H} \ = \ \mathbf{ConvH} \left\{ \boldsymbol{e}^{T^\rho} \ \ (\rho \in \mathbf{R}) \ , \ \ \bar{\boldsymbol{x}} + \Delta \boldsymbol{e}^\sigma \ \ (\sigma \in \overset{\wedge}{\mathbf{T}}) \right\} \ .$$

holds true.

Theorem 3.2. *Within the uniform model, i.e., for $\Delta \geq 0$, the pre–imputations $\bar{\boldsymbol{x}}^\sigma$ are extremals of $\boldsymbol{H}$.*

Proof:

1$^{\text{st}}$STEP : According to Remark 2.5 we know that $\bar{\boldsymbol{x}}$ satisfies all the inequalities determining $\boldsymbol{H}$ with the exception of the imputation equation $\sum_{\tau \in \mathbf{T}} x_\tau \ = \ 1$ and possibly non–negativity. As we assume $\Delta \ \geq \ 0$, we know that $\bar{\boldsymbol{x}} \geq \mathbf{0}$ and hence all the $\bar{\boldsymbol{x}}^\sigma \in \boldsymbol{H}$ $(\sigma \in \mathbf{R})$ are imputations as $\sum_{\tau\in\mathbf{T}} x_\tau = 1$ results from the construction provided in Definition 3.1.

2$^{\text{nd}}$STEP :

Now we show that every $\bar{\boldsymbol{x}}^\sigma$ is uniquely defined by a set of equations chosen from the inequalities determining $\boldsymbol{H}$.

Indeed, pick any relevant vector $\boldsymbol{a}^\oplus$ listed in Theorem 3.5 of Part I (i.e. [1]) and let $\overline{\tau}_1, \ldots, \overline{\tau}_r$ denote the non–zero coordinates. Now, to any such relevant vector there appear also the permuted versions, say

$$
\begin{aligned}
&\bar{\boldsymbol{a}}^{\oplus\sigma} := \\
&(0,\ldots,0,1,0\ldots,0,1,0,\frac{1-\sum_{\rho\in\mathbf{R}\setminus\{\sigma\}} h_{\overline{\tau}_\rho}}{h_{\overline{\tau}_\sigma}},0,\ldots,0,1,0,\ldots,0)\ .
\end{aligned}
\tag{3.6}
$$

with non–zero coordinates at the same positions. Hence there are r equations

$$
\bar{\boldsymbol{x}}\bar{\boldsymbol{a}}^{\oplus\sigma} = 1 \quad (\sigma \in \mathbf{R})\ . \tag{3.7}
$$

satisfied by $\overline{x}_{\overline{\tau}_1}, \ldots, \overline{x}_{\overline{\tau}_r}$. The r coordinates involved are not elements of $\overset{\vee}{\mathbf{T}}$. Hence the coordinates along $\overline{\tau}_1, \ldots, \overline{\tau}_r$ of $\bar{\boldsymbol{x}}$ and the ones of every $\bar{\boldsymbol{x}}^\sigma$ coincide, actually they equal the coordinates of $\boldsymbol{h}$. Thus we have also

$$
\bar{\boldsymbol{x}}^\sigma\bar{\boldsymbol{a}}^{\oplus\sigma} = 1 \quad (\sigma \in \mathbf{R})\ . \tag{3.8}
$$

Now consider the linear system of equations suggested for the r coordinates under consideration. The coefficient matrix of this system is given by the vectors $\bar{\boldsymbol{a}}^{\oplus\sigma}$ hence it is

$$
G := \begin{pmatrix} g_1, 1, \ldots, 1 \\ 1, g_2, \ldots, 1 \\ \ldots \\ 1, 1, \ldots, g_r \end{pmatrix} \tag{3.9}
$$

using $g_\sigma = \frac{1-\sum_{\rho\in\mathbf{R}\setminus\{\sigma\}} h_{\overline{\tau}_\rho}}{h_{\overline{\tau}_\sigma}} > 1$. The determinant of this matrix is

$$
\begin{vmatrix} g_1, 1, \ldots, 1 \\ 1, g_2, \ldots, 1 \\ \ldots \\ 1, 1, \ldots, g_r \end{vmatrix} = \begin{vmatrix} g_1-1, 0, \ldots, 0 \\ 0, g_2-1, \ldots, 0 \\ \ldots \\ 0, 0, \ldots, g_r-1 \end{vmatrix} = \prod_{\rho\in\mathbf{R}} (g_\rho - 1) \ > \ 0\ .
$$

Hence the linear system of equations (3.8) which involves variable $x_{\overline{\tau}_1}, \ldots, x_{\overline{\tau}_r}$ has exactly the solution $x_{\overline{\tau}_1} = h_{\overline{\tau}_1}, \ldots, x_{\overline{\tau}_r} = h_{\overline{\tau}_r}$. These are the coordinates of $\bar{\boldsymbol{x}}$ as well as the ones of $\bar{\boldsymbol{x}}^\sigma$ for all $\sigma \in \mathbf{R}$.

Consequently, all coordinates of any $\bar{\boldsymbol{x}}^\sigma$ for indices $\tau \in \overset{\wedge}{\mathbf{T}}$ are uniquely defined by equations resulting from the inequalitites of $\boldsymbol{H}$.

3$^{\text{rd}}$STEP :

However, the coordinates in $\overset{\vee}{\mathbf{T}}$ of $\bar{\boldsymbol{x}}$ are obviously defined by their very definition which involves equations resulting from inequalities of $\boldsymbol{H}$ as described in Definition 3.1. But then the coordinates in $\overset{\vee}{\mathbf{T}}$ of every $\bar{\boldsymbol{x}}^\sigma$ apart from σ are uniquely defined by (2.20). Finally coordinate σ is defined by the imputation equation which is equivalent to (2.21), that is, an equation from the inequalities defining $\boldsymbol{H}$.

q.e.d.

Theorem 3.3. *The extremals of the pre–core* $\left\{\boldsymbol{e}^{T^\rho}\right\}_{\rho\in\mathbf{R}}$ *and the pre–imputations* $\{\bar{\boldsymbol{x}}^\sigma\}_{\sigma\in\mathbf{R}}$ *are exactly the extremals of* $\boldsymbol{H}$*.*

Proof:

We know that the pre–core extremals and the $\{\bar{\boldsymbol{x}}^\sigma\}_{\sigma\in\mathbf{R}}$ are extremals of $\boldsymbol{H}$. We have to show that these are the *only* extremals of $\boldsymbol{H}$.

To this end, fix some extremal $\widehat{\boldsymbol{x}}$ of $\boldsymbol{H}$.

1st STEP :

Let $\boldsymbol{\tau} = (\tau_1, \ldots, , \tau_r)$ be a sequence such that

$$\mathbf{T}^{\boldsymbol{\tau}} \ := \ \{\tau_1, \ldots, , \tau_r\} \subseteq \overset{\vee}{\mathbf{T}} \ .$$

Let $\boldsymbol{a}^{\oplus\rho}$ be the corresponding separating vectors. The inequalities defining $\boldsymbol{H}$ in context with the sequence $\boldsymbol{\tau}$ and the family $\boldsymbol{a}^{\oplus\rho}$ are given by $\boldsymbol{a}^{\oplus\rho}\widehat{\boldsymbol{x}} \geq 1$. We write $\boldsymbol{x}^{\boldsymbol{\tau}} \ := \ \widehat{\boldsymbol{x}}_{|T^{\boldsymbol{\tau}}}$ for the coordinates of $\widehat{\boldsymbol{x}}$ restricted to the sequence $\boldsymbol{\tau}$. Then the above inequalities can be described by using the matrix G given by (3.9) in the $2^{nd}STEP$ of the previous proof via

$$G\boldsymbol{x}^{\boldsymbol{\tau}} \geq 1 \ .$$

Now, inspect the set

$$\boldsymbol{H}^{\boldsymbol{\tau}} \ = \ \{\boldsymbol{x} \in \mathbb{R}_{\mathbf{T}^{\boldsymbol{\tau}}} \,|\, \boldsymbol{x} \geq \boldsymbol{0} \ , \ \ G\boldsymbol{x} \geq \boldsymbol{e} = (1, , \ldots, 1)\} \tag{3.10}$$

The extremals of this set are given by the projection $\boldsymbol{h}^{\boldsymbol{\tau}} \ = \ \boldsymbol{h}_{|\mathbf{T}^{\boldsymbol{\tau}}}$ and the unit vectors $\boldsymbol{e}^{\tau_\rho}$. These unit vectors in turn are the projections of the $\boldsymbol{e}^{\mathbf{T}^\rho}$ on $\boldsymbol{H}^{\boldsymbol{\tau}}$. Figure 3.1 indicates the situation.

2nd STEP : Suppose now, that there are at least two indices $\sigma, \sigma' \in \mathbf{R}$ such that there is no equation in the corresponding rows of G, i.e., we have

$$\boldsymbol{a}^{\oplus\sigma}\widehat{\boldsymbol{x}} > 1 \ , \ \boldsymbol{a}^{\oplus\sigma'}\widehat{\boldsymbol{x}} > 1 \ . \tag{3.11}$$

First assume that $\widehat{\boldsymbol{x}}$ has positive coordinates $\tau_\sigma, \tau_{\sigma'}$

For $\varepsilon > 0$ let

$$\widehat{\boldsymbol{x}}^{\pm\varepsilon} \ := \ \widehat{\boldsymbol{x}} \pm \varepsilon\boldsymbol{e}^{\tau_\sigma} \mp \varepsilon\boldsymbol{e}^{\tau_{\sigma'}} \ . \tag{3.12}$$

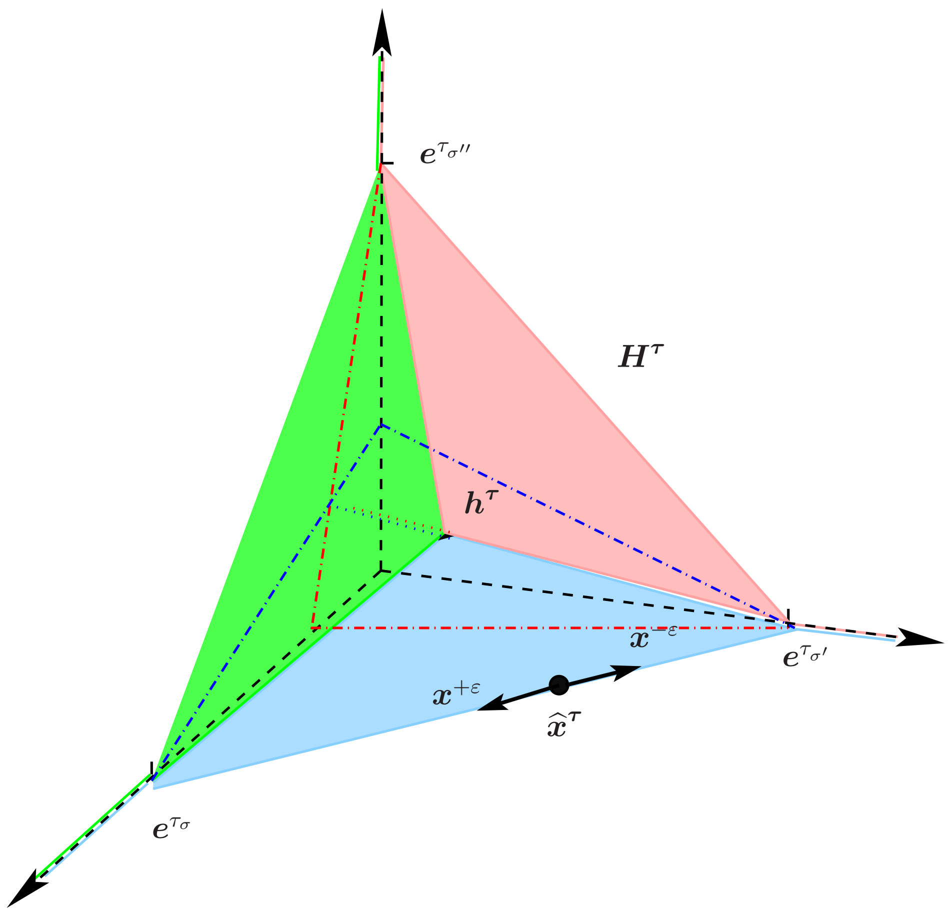

Figure 3.1: The shape of $\boldsymbol{H}^{\boldsymbol{\tau}}$

Then, if ε is sufficiently small, the strict inequalities (3.11) are being preserved. The other inequalities or equations are being preserved as the vectors $\boldsymbol{a}^{\oplus\rho}$ have a unit at both coordinates $\tau^\sigma, \tau^{\sigma'}$. See Figure 3.1. Obviously, the total coordinate sum $\sum_{\tau\in\mathbf{T}} x_\tau = 1$ is preserved as well. Hence $\widehat{\boldsymbol{x}}^\varepsilon$ and $\widehat{\boldsymbol{x}}^{-\varepsilon}$ are imputations and $\widehat{\boldsymbol{x}}^\varepsilon, \widehat{\boldsymbol{x}}^{-\varepsilon} \in \boldsymbol{H}$. Now $\frac{\widehat{\boldsymbol{x}}^\varepsilon+\widehat{\boldsymbol{x}}^{-\varepsilon}}{2} = \widehat{\boldsymbol{x}}$, contradicting our assumption that $\widehat{\boldsymbol{x}}$ is extremal in $\boldsymbol{H}$.

Next, it could happen that, say $\widehat{x}_{\tau_1} = 0$. Then (inspect Figure 3.1) essentially the case that $\widehat{x}_{|\boldsymbol{\tau}} = t\boldsymbol{e}^{\tau_1}$ for some $\tau > 1$ could pose a problem. Replace $\tau_1 \in \mathbf{T}^1$ by some $\tau_1' \in \mathbf{T}^1$ and repeat the argument. Now, not all the $\tau_1'' \in \mathbf{T}^1$ can yield $\widehat{x}_{|\boldsymbol{\tau}} = t''\boldsymbol{e}^{\tau_1''}$ for some $t'' > 1$ as it would follow that the total $\sum_{\tau\in\mathbf{T}^1} x_\tau > 1$ exceeds 1 and $\widehat{\boldsymbol{x}}$ would not be an imputation. Hence we are either back at the beginning of this step or there is at most *one* coordinate σ that yields a strict inequality like in (3.11).

3rdSTEP : So now there is at most *one* coordinate σ that yields a strict inequality like in (3.11), let this be coordinate 1. That is we have

$$\boldsymbol{a}^{\oplus 1}\widehat{\boldsymbol{x}} > 1\ , \quad \boldsymbol{a}^{\oplus\rho}\widehat{\boldsymbol{x}} = 1 \ \ (\rho \in \mathbf{R} \setminus \{1\}) \tag{3.13}$$

(the coordinates correspond to $\boldsymbol{\tau}$, so $\boldsymbol{a}^{\oplus\rho}$ has the coordinate $\neq 1$ at τ_ρ).

Now, again inspecting

$$\boldsymbol{H}^{\boldsymbol{\tau}} \;=\; \{\boldsymbol{x} \in \mathbb{R}_{\mathbf{T}^\tau} \,|\, \boldsymbol{x} \geq \boldsymbol{0}\ , \quad G\boldsymbol{x} \geq \boldsymbol{e} = (1,,\ldots,1)\} \tag{3.14}$$

one observes that $\widehat{\boldsymbol{x}}$ must be located on an edge of $\boldsymbol{H}^{\boldsymbol{\tau}}$ connecting $\boldsymbol{h}_{|\mathbf{T}^\tau}$ and a unit vector $\boldsymbol{e}^{\tau_1}$; see again Figure 3.1 .

4thSTEP : Now, by the same argument as used in the second step of the proof of Theorem 3.2, but reduced to the coefficient matrix G with row σ deleted, we find that actually $\widehat{x}_{\tau_\rho} = h_{\tau_\rho} =$ for $\rho \in \mathbf{R} \setminus \{1\}$.

Combining we see that $\widehat{\boldsymbol{x}}$ projected to the coordinates of $\boldsymbol{\tau}$ is a convex combination of the projections of $\boldsymbol{h}$ and $\boldsymbol{e}^{\tau_1}$, i.e., for some α, $0 \leq \alpha \leq 1$, we have

$$\widehat{\boldsymbol{x}}^{\boldsymbol{\tau}} \;=\; \alpha\boldsymbol{h}^{\boldsymbol{\tau}} + (1-\alpha)\boldsymbol{e}^{\tau_1}\ . \tag{3.15}$$

Now, replace one $\tau_\rho \in \overset{\vee}{\mathbf{T}}{}^\rho$ $(\rho > 1)$ by some $\tau_{\rho'} \in \overset{\vee}{\mathbf{T}}{}^\rho$. Repeat the argument provided in the $2^{nd}STEP$. Now again, if there is a ("second") inequality $\boldsymbol{a}^{\oplus\rho'}\widehat{\boldsymbol{x}} > 1$, then we see at once that $\widehat{\boldsymbol{x}}$ is not extremal in $\boldsymbol{H}$. Otherwise we have as previously $\widehat{x}_{\tau_{\rho'}} = h_{\tau_{\rho'}}$. Continuing this way, we find that

$$\widehat{x}_\tau \;=\; \alpha h_\tau \quad (\tau \in \overset{\vee}{\mathbf{T}}{}^\rho) \quad (\rho > 1)\ . \tag{3.16}$$

5thSTEP : Now exchange $\tau_1 \in \overset{\vee}{\mathbf{T}}{}^1$ by some $\tau_1' \in \overset{\vee}{\mathbf{T}}{}^1$. Then exactly as above we have, for some $\beta > 0$,

$$\widehat{\boldsymbol{x}}^{\boldsymbol{\tau}'} \;=\; \beta\boldsymbol{h}^{\boldsymbol{\tau}'} + (1-\beta)\boldsymbol{e}^{\tau_1'}\ . \tag{3.17}$$

But the coordinates of $\widehat{\boldsymbol{x}}$ in $\overset{\vee}{\mathbf{T}}{}^2 \cup \overset{\vee}{\mathbf{T}}{}^3 \cup \ldots \cup \overset{\vee}{\mathbf{T}}{}^r$ have already been established to be αh_τ, from which we conclude that $\alpha = \beta$.

This can be done for all $\tau_1 \in \overset{\vee}{\mathbf{T}}{}^1$, so that we come up at this stage with

(3.18) $$\widehat{\boldsymbol{x}}_{|\overset{\vee}{\mathbf{T}}} \;=\; \alpha \boldsymbol{h}_{|\overset{\vee}{\mathbf{T}}} + (1-a)\boldsymbol{e}^{\mathbf{T}^1}{}_{|\overset{\vee}{\mathbf{T}}}$$

for the coordinates of $\widehat{\boldsymbol{x}}$ at $\overset{\vee}{\mathbf{T}}$.

6^th STEP : Within this step we will show that, similarly to (3.18), for the coordinates in $\overset{\wedge}{\mathbf{T}}$ we have

(3.19) $$\widehat{\boldsymbol{x}}_{|\overset{\wedge}{\mathbf{T}}} \;\geq\; \alpha \boldsymbol{h}_{|\overset{\wedge}{\mathbf{T}}} + (1-a)\boldsymbol{e}^{\mathbf{T}^1}{}_{|\,\overset{\wedge}{\mathbf{T}}} .$$

Return to a sequence $\boldsymbol{\tau} = (\tau_1, \ldots, , \tau_r)$ such that

(3.20) $$\mathbf{T}^{\boldsymbol{\tau}} \;:=\; \{\tau_1, \ldots, , \tau_r\} \subseteq \overset{\vee}{\mathbf{T}} .$$

as in the $1^{st}STEP$. We know that

(3.21) $$\widehat{\boldsymbol{x}}_{|\overset{\vee}{\mathbf{T}}} \;=\; \alpha \boldsymbol{h}_{|\overset{\vee}{\mathbf{T}}} + (1-a)\boldsymbol{e}^{\mathbf{T}^1}{}_{|\,\overset{\wedge}{\mathbf{T}}} .$$

Now we replace $\tau_1 \in \overset{\vee}{\mathbf{T}}{}^1$ by $\tau_1' \in \overset{\wedge}{\mathbf{T}}{}^1$. Write $\boldsymbol{\tau}' : \;=\; (\tau_1', \ldots, , \tau_r)$, then by definition of $\overset{\vee}{\mathbf{T}}$ it follows that

$$\widehat{\boldsymbol{x}}^{\boldsymbol{\tau}'} \;=\; \widehat{x}_{\tau_1'} \boldsymbol{e}^{\tau_1'} + \alpha\,(h_{\tau_2}, \ldots, h_{\tau_r})$$

satisfies

(3.22) $$1 \leq \widehat{x}_{\tau_1'} + \sum_{\tau \in \mathbf{T}^{\boldsymbol{\tau}'}} \widehat{x}_\tau \;=\; \widehat{x}_{\tau_1'} + \alpha \sum_{\rho \in \mathbf{R}\setminus\{1\}} h_{\tau_\rho} .$$

Hence

(3.23) $$\begin{aligned} \widehat{x}_{\tau_1'} \;&\geq\; 1 - \alpha \sum_{\rho \in \mathbf{R}\setminus\{1\}} h_{\tau_\rho} \\ &=\; (1-\alpha) + \alpha \left(1 - \sum_{\rho \in \mathbf{R}\setminus\{1\}} h_{\tau_\rho} \right) . \end{aligned}$$

Now let us choose for $\boldsymbol{\tau}$ in particular the *minimizing* sequence $\overset{\vee}{\boldsymbol{\tau}}$ as introduced in Definition 2.1. Then by Definition 2.4, (2.20) we have

(3.24) $$\overline{x}_{\tau_1'} = 1 - \sum_{\rho \in \mathbf{R}\setminus\{1\}} h_{\overset{\vee}{\tau}_\rho} .$$

Combining (3.23) and (3.24) we obtain

$$\widehat{x}_{\tau_1'} \geq (1-\alpha) + \alpha \overline{x}_{\tau_1'} \tag{3.25}$$

for all $\tau_1' \in \overset{\wedge}{\mathbf{T}}{}^1 = \overset{\wedge}{\mathbf{T}} \cap \mathbf{T}^1$.

Next, the same argument can be applied if instead of τ_1 we replace, say $\tau_2 \in \overset{\vee}{\mathbf{T}}{}^2$, by some $\tau_2'' \in \overset{\wedge}{\mathbf{T}}{}^2$. Writing $\boldsymbol{\tau}'' : = (\tau_1, \tau_2'', \ldots, , \tau_r)$ and referring to (3.21), we have this time

$$\widehat{\boldsymbol{x}}^{\boldsymbol{\tau}''} = \left((1-\alpha), \widehat{x}_{\tau_2''}, h_{\tau_3}, \ldots, h_{\tau_r}\right) .$$

Again, as $\tau_2'' \in \overset{\wedge}{\mathbf{T}}{}^2$ we have

$$\begin{aligned} 1 &\leq \sum_{\rho \in \mathbf{R}} \widehat{x}_{\tau_\rho''} \\ &= (1-\alpha) + \widehat{x}_{\tau_2''} + \alpha \sum_{\rho \in \mathbf{R} \setminus \{1,2\}} h_{\tau_\rho} . \end{aligned} \tag{3.26}$$

Hence

$$\begin{aligned} \widehat{x}_{\tau_2''} &\geq \alpha - \alpha \sum_{\rho \in \mathbf{R} \setminus \{1,2\}} h_{\tau_\rho} \\ &= \alpha \left(1 - \sum_{\rho \in \mathbf{R} \setminus \{1,2\}} h_{\tau_\rho} \right) \\ &\geq \alpha \left(1 - \sum_{\rho \in \mathbf{R} \setminus \{2\}} h_{\tau_\rho} \right) . \end{aligned} \tag{3.27}$$

Specifying τ to $\overset{\vee}{\tau}$ once again we now obtain - again consulting (2.20) -

$$\widehat{x}_{\tau_2''} \geq \alpha \overline{x}_{\tau_2''} \tag{3.28}$$

for all $\tau_2'' \in \overset{\wedge}{\mathbf{T}}{}^2 = \overset{\wedge}{\mathbf{T}} \cap \mathbf{T}^2$. Of course a similar argument holds true for $\rho \in \mathbf{R} \setminus \{1, 2\}$, thus actually

$$\widehat{x}_{\tau_\rho'''} \geq \alpha \overline{x}_{\tau_\rho'''} \tag{3.29}$$

for all $\tau_\rho''' \in \overset{\wedge}{\mathbf{T}}{}^\rho = \overset{\wedge}{\mathbf{T}} \cap \mathbf{T}^\rho$, $\rho \in \mathbf{R} \setminus \{1\}$.

Combining (3.25) and (3.29) we observe that indeed for the coordinates $\tau \in \overset{\wedge}{\mathbf{T}}$ we have

$$\widehat{\boldsymbol{x}}_{\,|\,\overset{\wedge}{\mathbf{T}}} \geq (1-\alpha)\boldsymbol{e}^{\mathbf{T}^1}{}_{|\,\overset{\wedge}{\mathbf{T}}} + \alpha \overline{\boldsymbol{x}}_{\,|\,\overset{\wedge}{\mathbf{T}}} , \tag{3.30}$$

i.e., (3.19). This concludes the present step.

7[th]**STEP** : In view of (3.30) we can define a nonnegative set of coefficients

$$\boldsymbol{\delta}_{\bullet} \;=\; \{\delta_\tau\}_{\tau\in\hat{\mathbf{T}}} \tag{3.31}$$

via

$$\widehat{\boldsymbol{x}}_{\,|\,\hat{\mathbf{T}}} \;=:\; (1-\alpha)\boldsymbol{e}^{\mathbf{T}^1}{}_{|\,\hat{\mathbf{T}}} + \alpha\bar{\boldsymbol{x}}_{\,|\,\hat{\mathbf{T}}} \;+\; \alpha\Delta\delta_{\bullet}\;, \tag{3.32}$$

using the constand Δ that has been specified in (3.2). Then (3.18) and (3.32) imply

$$\widehat{\boldsymbol{x}} \;=:\; (1-\alpha)\boldsymbol{e}^{\mathbf{T}^1} + \alpha\bar{\boldsymbol{x}} \;+\; \alpha\Delta\delta_{\bullet}\;. \tag{3.33}$$

As $\widehat{\boldsymbol{x}}$ is an imputation, we have

$$\begin{aligned} t \;&=\; \sum_{\tau\in\mathbf{T}} \widehat{x}_\tau \\ &=\; \alpha\sum_{\tau\in\mathbf{T}} h_\tau + t(1-\alpha) + \sum_{\sigma\in\hat{\mathbf{T}}} \alpha\Delta\delta_\sigma \end{aligned} \tag{3.34}$$

That is

$$\begin{aligned} \sum_{\sigma\in\hat{\mathbf{T}}} \alpha\Delta\delta_\sigma \;&=\; t - t(1-\alpha) - \alpha\sum_{\tau\in\mathbf{T}} h_\tau \\ &=\; \alpha\left(t - \sum_{\tau\in\mathbf{T}} h_\tau\right) \;=\; \alpha\Delta \end{aligned} \tag{3.35}$$

in view of the definition of Δ, see (3.2). Thus $\sum_{\tau\in\hat{\mathbf{T}}}\delta_\tau \;=\; 1$, i.e., $\delta_{\bullet}$ is a set of "convex coefficients".

Concluding we come up with

$$\begin{aligned} \widehat{\boldsymbol{x}} \;&=\; (1-\alpha)\boldsymbol{e}^{\mathbf{T}^1} + \alpha\bar{\boldsymbol{x}} + \alpha\Delta\boldsymbol{\delta}_{\bullet} \\ &=\; (1-\alpha)\boldsymbol{e}^{\mathbf{T}^1} + \alpha\left(\bar{\boldsymbol{x}} + \sum_{\sigma\in\{\hat{\mathbf{T}}\}} \delta_\sigma\Delta\boldsymbol{e}^\sigma\;.\right) \\ &=\; (1-\alpha)\boldsymbol{e}^{\mathbf{T}^1} + \alpha\left[\sum_{\sigma\in\hat{\mathbf{T}}} \delta_\sigma\left(\bar{\boldsymbol{x}} + \Delta\boldsymbol{e}^\sigma\right)\right] \\ &=\; (1-\alpha)\boldsymbol{e}^{\mathbf{T}^1} + \alpha\left(\sum_{\sigma\in\hat{\mathbf{T}}} \delta_\sigma\bar{\boldsymbol{x}}^\sigma\right)\;, \end{aligned} \tag{3.36}$$

that is, $\widehat{\boldsymbol{x}}$ is a convex combination of the extremal vectors exhibited in Theorem 3.2. As $\widehat{\boldsymbol{x}}$ is assumed to be extremal, this shows that this convex combination must be a trivial one, i.e., $\widehat{\boldsymbol{x}}$ is one of the extremals already known.

q.e.d.

4 The Effective Pre–Imputations

By definition the elements of $\boldsymbol{H}$ are effective for the separating relevant vectors, i.e., the vectors that are of "first type" $\boldsymbol{a}^{\odot}$ and of "second type" $\boldsymbol{a}^{\ominus}$ as introduced in *Theorem 3.5* of *Part I*. Now obviously the question arises whether effectiveness can be established with respect to the third type of relevant vectors, i.e., the non-separating vectors $\boldsymbol{a}^{\ominus}$. Necessarily we must have a clue to this situation as we want to create a vNM–Stable Set that calls for using all types of relevant vectors in order to establish internal and external domination.

As we have seen, the extremals of $\boldsymbol{H}$ apart from those of the core are obtained by constructing $\bar{\boldsymbol{x}}$ and - as this vector is not a pre–imputation – then distributing the remaining mass Δ in a natural way over $\overset{\wedge}{\boldsymbol{T}}$. That is, we have formula (5.7) which we repeat here:

$$\text{(4.1)} \qquad \boldsymbol{H} \;=\; \textbf{ConvH}\left\{ \boldsymbol{e}^{T^{\rho}} \quad (\rho \in \mathbf{R})\;,\;\; \bar{\boldsymbol{x}} + \Delta \boldsymbol{e}^{\sigma} \quad (\sigma \in \overset{\wedge}{\mathbf{T}}) \right\} \;.$$

Now, within this section we exhibit those pre–imputations in $\boldsymbol{H}$ that in additiona are also effective for the relevant vectors of the second type $\boldsymbol{a}^{\ominus}$. This amounts to restricting the distribution of the free mass Δ over the basis vectors $\{\boldsymbol{e}^{\sigma}\}_{\sigma \in \overset{\wedge}{\mathbf{T}}}$ in a suitable way.

We start out by discussing a several examples in detail as this clears the path to the comprehensiv treatment.

Example 4.1. Let r $=$ t $=$ 2 and consider $\boldsymbol{h} \;=\; (\varepsilon, \varepsilon; \varepsilon, h_4)$; necessarily assuming $\boldsymbol{\lambda}^0(\boldsymbol{I}) = \frac{1}{2}\{3\varepsilon + h_4\} > 1$, i.e.

$$\text{(4.2)} \qquad h_4 > 2 - 3\varepsilon \;; \quad \varepsilon < \frac{1}{3} \;.$$

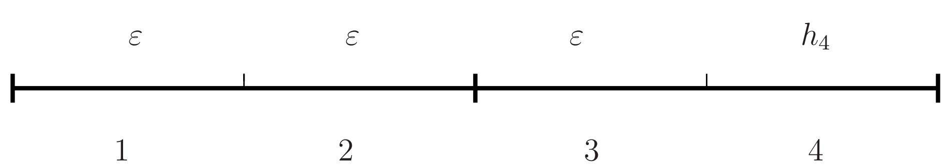

Figure 4.1: Discussing $\boldsymbol{H}$ in a 2×2 case

For completeness we list relevant vectors

$$\boldsymbol{a}^{\oplus} = (0, 1; \frac{1-\varepsilon}{e}, 0) \;;\; \text{normalized:} \quad \bar{\boldsymbol{a}}^{\oplus} = (0, \varepsilon; 1-\varepsilon, 0)$$

satisfying

$$\boldsymbol{e}^{12}\bar{\boldsymbol{a}}^{\oplus} = \varepsilon = \boldsymbol{c}^0\bar{\boldsymbol{a}}^{\oplus} \;=\; v(\bar{\boldsymbol{a}}^{\oplus}) \;<\; (1-\varepsilon) \;=\; \boldsymbol{e}^{34}\bar{\boldsymbol{a}}^{\oplus}$$

and its twin $(0, \varepsilon; 1-\varepsilon, 0)$ as well as $(1-\varepsilon, 0; \varepsilon, 0)$ and $(0, 1-\varepsilon; \varepsilon, 0)$. There are two relevant vectors of the first type, namely

$$\boldsymbol{a}^{\odot} \;=\; (1,0;,0,1) \text{ and } \boldsymbol{a}^{\bigcirc} \;=\; (0,1;,0,1)\ .$$

The inequalities resulting, i.e.,

$$\begin{aligned} x_1 + x_4 &\geq 1 \\ x_2 + x_4 &\geq 1 \end{aligned} \tag{4.3}$$

do not in general determine $\boldsymbol{H}$ nor do they imply $\boldsymbol{H} \;=\; \boldsymbol{C}(v)$.

However, we have at aonce

$$\overset{\vee}{\mathbf{T}} \;=\; \{1,2;3\}\ ;\quad \overset{\wedge}{\mathbf{T}} \;=\; \{4\}\ ,$$

hence we find for $\bar{\boldsymbol{x}}$ the coordinates

$$\overline{x}_\tau \;=\; h_\tau = \varepsilon \quad (\tau \;=\; 1,2,3).$$

As $\overline{x}_4 = 1 - e$ we observe that this does not yield an imputation, rather the only extremal is obtained from the imputation equation $\sum_{\tau \in \mathbf{T}} x_\tau \;=\; 2$; that is we obtain

$$\bar{\boldsymbol{x}}^4 \;=\; (\varepsilon, \varepsilon; \varepsilon, 2 - 3\varepsilon)\ . \tag{4.4}$$

note that this extremal satisfies none of the inequalities provided by (4.3) with an equation . We have two minimal sequences and clearly

$$\boldsymbol{H} \;=\; \left\{\boldsymbol{e}^{12}, \boldsymbol{e}^{34}, \bar{\boldsymbol{x}}\right\} \tag{4.5}$$

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Example 4.2. Let $r = 2$ and $t = 3$. Without specifying $\boldsymbol{h}$ in advance let

$$\overset{\vee}{\mathbf{T}} \;=\; \{1,2;4\} \quad \overset{\wedge}{\mathbf{T}} \;=\; \{3;5,6\} \tag{4.6}$$

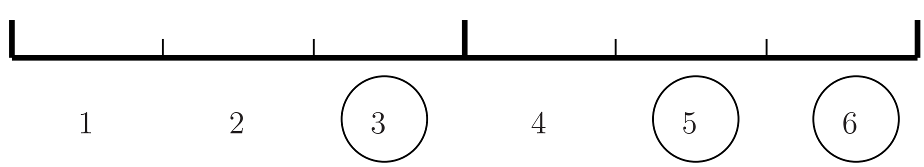

Figure 4.2: Discussing $\boldsymbol{H}$ in a 2×3 case

Considering the relevant vectors $\boldsymbol{a}^{\odot}$ of the first type we obtain the resulting inequalities

$$\begin{array}{lllllll} x_1 & & & & +x_5 & & \geq 1 \\ x_1 & & & & & +x_6 & \geq 1 \\ & x_2 & & & +x_5 & & \geq 1 \\ & x_2 & & & & +x_6 & \geq 1 \\ & & x_3+ & x_4 & & & \geq 1 \\ & & x_3+ & x_4 & & & \geq 1\ . \end{array} \tag{4.7}$$

summing up yields

$$2\sum_{\tau\in\mathbf{T}} x_\tau \geq 6,$$

that is, the coordinates of $\bar{\boldsymbol{x}}$ have to satisfy

$$t = 3 \geq \sum_{\tau\in\mathbf{T}} x_\tau \geq 3\ .$$

Consequently *all* inequalitites involved *must* be equations. then it follows at once that

$$x_1 = x_2 \text{ and } x_5 = x_6\ .$$

therefore, unless $h_1 = h_2$, the vectors $\boldsymbol{e}^{123}$ and $\boldsymbol{e}^{456}$ are the only solutions of $\boldsymbol{J}(v)$ to the inequalitiy system above. On the other hand, if we put $h_1 = h_2 := \varepsilon$, then it follows that $x_5 = x_6 = 1-\varepsilon$; hence $\bar{\boldsymbol{x}}$ has the shape

$$\bar{\boldsymbol{x}} = (\varepsilon, \varepsilon, x_3; x_4, 1-\varepsilon, 1-\varepsilon) \tag{4.8}$$

with $x_3 + x_4 = 1$. Now, according to Theorem 3.3 we have

$$\bar{\boldsymbol{x}} = (\varepsilon, \varepsilon, 1-h_4; h_4, 1-\varepsilon, 1-\varepsilon) \tag{4.9}$$

with $h_4 < 1-\varepsilon$ so that again we have two minimal sequences $\overset{\vee}{\boldsymbol{\tau}}$ namely $(1,4)$ and $(2,4)$. We have in this case

$$\boldsymbol{H} = \left\{\boldsymbol{e}^{123}, \boldsymbol{e}^{456}, \bar{\boldsymbol{x}}\right\} \tag{4.10}$$

and $\bar{\boldsymbol{x}}$ is not only the extremal but also satisfies all inequalities (4.7) with an equation as well as it satisfies the imputation equation $\sum_{\tau\in\mathbf{T}} x_\tau = 3$.

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Example 4.3. A similar occurrence is observed in the following example with $\rho = 3$ and $t = 2$. We assume

$$\overset{\vee}{\mathbf{T}} = \{1,2;3;5\} \quad \overset{\wedge}{\mathbf{T}} = \{4;6\} \tag{4.11}$$

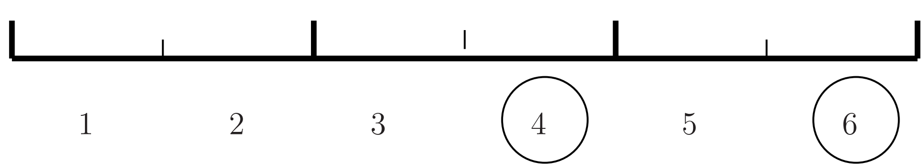

Figure 4.3: Discussing $\boldsymbol{H}$ in a 3×2 case

The relevant vectors $\boldsymbol{a}^{\odot}$ of the first type result in inequalities

$$\begin{array}{cccccccl} x_1 & & +x_3 & & & & +x_6 & \geq 1 \\ & x_2 & +x_3 & & & & +x_6 & \geq 1 \\ x_1 & & & +x_4 & +x_5 & & & \geq 1 \\ & x_2 & & +x_4 & +x_5 & & & \geq 1 \end{array} \tag{4.12}$$

which, again by summing up yields

$$2\sum_{\tau\in\mathbf{T}} x_\tau \geq 4.$$

Again the coordinates of $\bar{\boldsymbol{x}}$ have to satisfy

$$t = 2 \geq \sum_{\tau\in\mathbf{T}} x_\tau \geq 2\ .$$

Consequently *all* inequalitites involved *must* be equations. Then (unless $\boldsymbol{H}$ equals the core) it follows at once that

$$x_1 = x_2 \ =:\ \varepsilon$$

and

$$x_3 + x_6 = x_4 + x_5 = 1 - \varepsilon\ .$$

Now again the Extremal Characterization Theorem 3.3 tells us that $x_3 = h_3$ and $x_5 = h_5$ for the coordinates of $\bar{\boldsymbol{x}}$; hence we come up with

$$\bar{\boldsymbol{x}} \ = \ (\varepsilon, \varepsilon; h_3, 1 - \varepsilon - h_5; h_5, 1 - \varepsilon - h_3)\ . \tag{4.13}$$

Again $\bar{\boldsymbol{x}}$ is the extremal of

$$\boldsymbol{H} = \left\{\boldsymbol{e}^{12}, \boldsymbol{e}^{34}, \boldsymbol{e}^{56}, \bar{\boldsymbol{x}}\right\} \tag{4.14}$$

and it satisfies all the equations resulting from relevant vectors $\boldsymbol{a}^{\odot}$ as well as the imputation equation regarding total $\sum_{\tau\in\mathbf{T}} x_\tau = 2$.

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The above examples show that $\Delta \ = \ 0$ may occur in abundance, in which case we have no problem with effectivenes regarding the third type of relevant vectors. The following example shows a different picture.

Example 4.4. The example is significant: it turns out that $\Delta > 0$ holds true. We choose $r = 2$ and $t = 4$ and assume

$$\overset{\vee}{\mathbf{T}} \ = \ \{1, 2, 3; 5, 6\} \quad \overset{\wedge}{\mathbf{T}} \ = \ \{4; 7, 8\} \tag{4.15}$$

For $\varepsilon < \frac{1}{2}$ and $h_4, h_7, h_8 \geq 1 - \varepsilon > \frac{1}{2}$ we represent $\boldsymbol{\lambda}^0$ by

$$\boldsymbol{h} \ = \ (\varepsilon, \varepsilon, \varepsilon, h_4; \varepsilon, \varepsilon, h_7, h_8)\ .$$

Then that $\boldsymbol{\lambda}^0(\boldsymbol{I}) > 1$ is guaranteed by

$$5\varepsilon + h_4 + h_7 + h_8 > 4 \ \text{ i.e., by } \ h_4 + h_7 + h_8 > 4 - 5\varepsilon\ .$$

In particular, if we choose

$$\boldsymbol{h} \ = \ (\varepsilon, \varepsilon, \varepsilon, 1; \varepsilon, \varepsilon, 1, 1)\ , \tag{4.16}$$

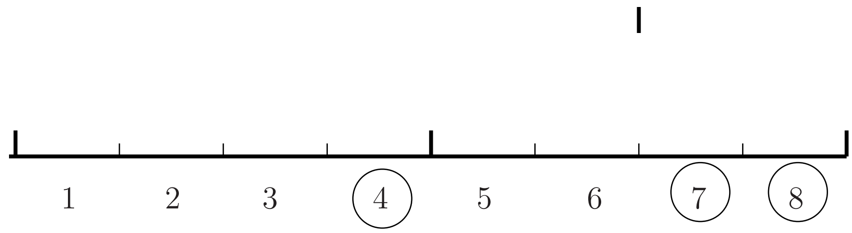

Figure 4.4: $\boldsymbol{H}$ in a 2×4 case with $\Delta > 0$

then

$$\frac{1}{5} < \varepsilon < \frac{1}{2} \text{ is equivalent to } 1 - \varepsilon > \varepsilon \ , \quad \boldsymbol{\lambda}^0(\boldsymbol{I}) > 1 \ . \tag{4.17}$$

There are several minimal sequences all of them calling for

$$x_4 = x_7 = x_8 = 1 - \varepsilon \ ,$$

that is

$$\bar{\boldsymbol{x}} \;=\; (\varepsilon, \varepsilon, \varepsilon, 1 - \varepsilon; \varepsilon, \varepsilon, 1 - \varepsilon, 1 - \varepsilon) \tag{4.18}$$

with a total sum

$$\sum_{\tau \in \mathbf{T}} \overline{x}_\tau = 5\varepsilon + 3(1 - \varepsilon) = 3 + 2\varepsilon < 4 = t \ .$$

Thus, $\bar{\boldsymbol{x}}$ is not an imputation. We find $\Delta = 4 - (3 + 2\varepsilon) = 1 - 2\varepsilon > 0$ and hence the three extremals

$$\begin{aligned} \bar{\boldsymbol{x}}^4 &= (\varepsilon, \varepsilon, \varepsilon, 2 - 3\varepsilon; \varepsilon, \varepsilon, 1 - \varepsilon, 1 - \varepsilon) \\ \bar{\boldsymbol{x}}^7 &= (\varepsilon, \varepsilon, \varepsilon, 1 - \varepsilon; \varepsilon, \varepsilon, 2 - 3\varepsilon, 1 - \varepsilon) \\ \bar{\boldsymbol{x}}^8 &= (\varepsilon, \varepsilon, \varepsilon, 1 - \varepsilon; \varepsilon, \varepsilon, 1 - \varepsilon, 2 - 3\varepsilon) \ . \end{aligned} \tag{4.19}$$

Therefore

$$\boldsymbol{H} \;=\; \left\{ \boldsymbol{e}^{1234}, \boldsymbol{e}^{5678}, \bar{\boldsymbol{x}}^4, \bar{\boldsymbol{x}}^7, \bar{\boldsymbol{x}}^8 \right\} \ . \tag{4.20}$$

Now the decisive relevant vectors are those of the type $\boldsymbol{a}^\ominus$, e.g.

$$\boldsymbol{a}^\ominus \;=\; \boldsymbol{a}^{\ominus 158} \;=\; \left(1, 0, 0, 0; \frac{\varepsilon}{1 - \varepsilon}, 0, 0, \frac{1 - 2\varepsilon}{1 - \varepsilon} \right) \ .$$

The extremal $\bar{\boldsymbol{x}}^8$ yields

$$\bar{\boldsymbol{x}}^8 \boldsymbol{a}^\ominus \;=\; \frac{2 - 6\varepsilon(1 - \varepsilon)}{1 - \varepsilon} \;=\; \frac{2}{1 - \varepsilon} - 6\varepsilon \ .$$

computing the zeros of the quadratic functin shows that

$$\bar{\boldsymbol{x}}^8 \boldsymbol{a}^\ominus \;=\; \left\{ \begin{array}{ll} > \ 1 & 0 < \varepsilon < \frac{1}{3} \\ < \ 1 & \frac{1}{3} < \varepsilon < \frac{1}{2} \end{array} \right\} \ . \tag{4.21}$$

That is, the extremals of $\boldsymbol{H}$ cannot serve for external domination via $\boldsymbol{a}^{\ominus}$ for the values $\frac{1}{5} < \varepsilon < \frac{1}{3}$. However, we are successfull when turning to the barycenter of $\boldsymbol{H}$. Indeed, let

$$(4.22)\quad \mathring{\bar{\boldsymbol{x}}} := \frac{1}{3}\left(\bar{\boldsymbol{x}}^4 + \bar{\boldsymbol{x}}^7 + \bar{\boldsymbol{x}}^8\right) = \left(\varepsilon, \varepsilon, \varepsilon, \frac{4-5\varepsilon}{3}; \varepsilon, \varepsilon, \frac{4-5\varepsilon}{3}, \frac{4-5\varepsilon}{3}\right) ,$$

then we obtain

$$(4.23)\qquad \mathring{\bar{\boldsymbol{x}}}\boldsymbol{a}^{\ominus} = \frac{4-10\varepsilon(1-\varepsilon)}{3(1-\varepsilon)} = \frac{4}{3(1-\varepsilon)} - \frac{10}{3}\varepsilon$$

which yields

$$(4.24)\qquad \mathring{\bar{\boldsymbol{x}}}\boldsymbol{a}^{\ominus} < 1 \quad \text{for} \quad \frac{1}{5} < \varepsilon < \frac{1}{2} .$$

In view of the specification (4.17) this is exactly the condition we need for to make sure that $\mathring{\bar{\boldsymbol{x}}}$ can be employed for external domination via the relevant vector $\boldsymbol{a}^{\ominus}$.

Now within the context of this example, we turn to the general case, i.e., instead of (4.16) we choose

$$(4.25)\qquad \boldsymbol{h} = (h_1, h_2, h_3, 1; h_5, h_6, 1, 1) ,$$

with

$$(4.26)\qquad h_1 + h_2 + h_3 + h_5 + h_6 > 1$$

in order to ensure $\boldsymbol{\lambda}^0(\boldsymbol{I}) > 1$ and

$$(4.27)\qquad h_{\tau_1} + h_{\tau_2} < 1 \quad (\tau_1 \in \mathbf{T}^1,\ \tau_2 \in \mathbf{T}^2).$$

in order to ensure

$$(4.28)\qquad \overset{\vee}{\mathbf{T}} = \{1, 2, 3; 5, 6\} \quad \overset{\wedge}{\mathbf{T}} = \{4; 7, 8\}$$

as previously. We assume that the minimizing sequence is represented by (h_1, h_5), i.e.,

$$(4.29)\qquad \overset{\vee}{\boldsymbol{\tau}} = (1, 5) .$$

Then we obtain

$$(4.30)\qquad \bar{\boldsymbol{x}} = (h_1, h_2, h_3; 1 - h_5; h_5, h; 6, 1 - h_1, 1 - h_1)$$

which implies

$$(4.31)\qquad \begin{aligned} \Delta &= 4 - [(h_1 + h_2 + h_3 + h_5 + h_6) + 2(1 - h_1) + (1 - h_5)] \\ &= 1 + h_1 - (h_2 + h_3 + h_6) . \end{aligned}$$

Now we attempt to distribute the mass Δ on coordinates $\tau = 4, 7, 8 \in \overset{\wedge}{\mathbf{T}}$ obtaining a vector

$$\overset{\circ}{\bar{\boldsymbol{x}}} = \bar{\boldsymbol{x}} + \Delta_4 \boldsymbol{e}^4 + \Delta_7 \boldsymbol{e}^7 + \Delta_8 \boldsymbol{e}^8 \tag{4.32}$$

with $\Delta_4 + \Delta_7 + \Delta_8 = 1$. Suitably we choose $\Delta_7 = \Delta_8 =: \Delta_2$ and $\Delta_4 =: \Delta_1$ this way enumerating the terms by $\rho = 1, 2 \in \mathbf{R}$. Then we consider

$$\overset{\circ}{\bar{\boldsymbol{x}}} = \bar{\boldsymbol{x}} + \Delta_1 \boldsymbol{e}^4 + \Delta_2 \boldsymbol{e}^7 + \Delta_2 \boldsymbol{e}^8 \quad ; \quad \Delta_1 + 2\Delta_2 = 1 \ . \tag{4.33}$$

Recall that $\bar{\boldsymbol{x}}$ and $\boldsymbol{c}^0$ coincide on coordinates $\tau \in \overset{\vee}{\mathbf{T}}$, they equal h_τ. Hence, if we consider a relevant vector $\boldsymbol{a}^{\ominus}$ and its correponding sequence $\boldsymbol{\tau}$, then along this sequence the vectors $\bar{\boldsymbol{x}}$ and $\boldsymbol{c}^0$ differ exactly on the last coordinate, that is, $\overline{\tau}_2 \in \overset{\wedge}{\mathbf{T}}{}^2$. The same is obviously true for $\overset{\circ}{\bar{\boldsymbol{x}}}$. E.g, we have along coordinates 158 (i.e., inspecting $\boldsymbol{a}^{\ominus 158}$)

$$\begin{aligned} \overset{\circ}{\bar{\boldsymbol{x}}}_{158} &= \boldsymbol{c}^0_{158} + \left(\overset{\circ}{\overline{x}}_8 - c^0_8\right) \boldsymbol{e}^8 \\ &= \boldsymbol{c}^0_{158} + \left((1 - h_1) + \Delta_2 - 1\right) \boldsymbol{e}^8 \\ &= \boldsymbol{c}^0_{158} + \left(\Delta_2 - h_1\right) \boldsymbol{e}^8 \end{aligned} \tag{4.34}$$

.

Similarly, i.e., inspecting the sequence 157 that is attached to $\boldsymbol{a}^{\ominus 157}$,

$$\begin{aligned} \overset{\circ}{\bar{\boldsymbol{x}}}_{157} &= \boldsymbol{c}^0_{157} + \left(\overset{\circ}{\overline{x}}_7 - c^0_7\right) \boldsymbol{e}^7 \\ &= \boldsymbol{c}^0_{157} + \left((1 - h_1) + \Delta_2 - 1\right) \boldsymbol{e}^7 \\ &= \boldsymbol{c}^0_{157} + \left(\Delta_2 - h_1\right) \boldsymbol{e}^7 \end{aligned} \tag{4.35}$$

,

while for 541 that is attached to $\boldsymbol{a}^{\ominus 514}$,

$$\begin{aligned} \overset{\circ}{\bar{\boldsymbol{x}}}_{514} &= \boldsymbol{c}^0_{514} + \left(\overset{\circ}{\overline{x}}_4 - c^0_4\right) \boldsymbol{e}^4 \\ &= \boldsymbol{c}^0_{514} + \left((1 - h_5) + \Delta_1 - 1\right) \boldsymbol{e}^4 \\ &= \boldsymbol{c}^0_{514} + \left(\Delta_1 - h_5\right) \boldsymbol{e}^4 \end{aligned} \tag{4.36}$$

.

Now, scalar multiplication with the relevant $\boldsymbol{a}^{\ominus}$ yields

$$\begin{aligned} \overset{\circ}{\bar{\boldsymbol{x}}} \boldsymbol{a}^{\ominus 158} = \overset{\circ}{\bar{\boldsymbol{x}}}_{158} \boldsymbol{a}^{\ominus 158} &= \boldsymbol{c}^0_{158} \boldsymbol{a}^{\ominus 158} + (\Delta_2 - h_1) a_8^{\ominus 158} \\ &= \boldsymbol{c}^0 \boldsymbol{a}^{\ominus 158} + (\Delta_2 - h_1) a_8^{\ominus 158} \\ &= 1 + (\Delta_2 - h_1) a_8^{\ominus 158} \ . \end{aligned} \tag{4.37}$$

Analogously

$$\begin{aligned} \overset{\circ}{\bar{\boldsymbol{x}}} \boldsymbol{a}^{\ominus 157} = \overset{\circ}{\bar{\boldsymbol{x}}}_{157} \boldsymbol{a}^{\ominus 157} &= \boldsymbol{c}^0_{157} \boldsymbol{a}^{\ominus 157} + (\Delta_2 - h_1) a_7^{\ominus 157} \\ &= 1 + (\Delta_2 - h_1) a_7^{\ominus 157} \ , \end{aligned} \tag{4.38}$$

and finally

$$
\begin{aligned}
\text{(4.39)} \qquad \mathring{\bar{\boldsymbol{x}}}\boldsymbol{a}^{\ominus 514} \;=\; \mathring{\bar{\boldsymbol{x}}}_{514}\boldsymbol{a}^{\ominus 514} &= \boldsymbol{c}^{0}_{514}\boldsymbol{a}^{\ominus 514} + (\Delta_1 - h_5)a_4^{\ominus 514} \\
&= 1 + (\Delta_1 - h_5)a_4^{\ominus 514} \;.
\end{aligned}
$$

Therefore, if we can find Δ_1, Δ_2 such that

$$
\text{(4.40)} \qquad \Delta_1 + 2\Delta_2 = \Delta \;, \quad \Delta_2 < h_1 \;, \quad \Delta_1 < h_5 \;,
$$

then

$$
\text{(4.41)} \qquad
\begin{aligned}
\mathring{\bar{\boldsymbol{x}}}\boldsymbol{a}^{\ominus 158} &= \mathring{\bar{\boldsymbol{x}}}_{158}\boldsymbol{a}^{\ominus 158} < 0 \;, \\
\mathring{\bar{\boldsymbol{x}}}\boldsymbol{a}^{\ominus 157} &= \mathring{\bar{\boldsymbol{x}}}_{157}\boldsymbol{a}^{\ominus 157} < 0 \;, \\
\mathring{\bar{\boldsymbol{x}}}\boldsymbol{a}^{\ominus 514} &= \mathring{\bar{\boldsymbol{x}}}_{514}\boldsymbol{a}^{\ominus 514} < 0 \;.
\end{aligned}
$$

That is, we see, that

$$
\text{(4.42)} \qquad \mathring{\bar{\boldsymbol{x}}}\boldsymbol{a}^{\ominus} < 0 \;,
$$

for the relevant vectors of the third type listet in (4.41). This inequality follows for all other relevant vectors of the third type. E.g., for $\boldsymbol{a}^{\ominus 268}$ we come up immediately with

$$
\text{(4.43)} \qquad \mathring{\bar{\boldsymbol{x}}}_{268}\boldsymbol{a}^{\ominus 268} \;=\; \boldsymbol{c}^{0}_{268}\boldsymbol{a}^{\ominus 258} + (\Delta_2 - h_1)a_8^{\ominus 268} \;<\; 1
$$

as coordinate 8 is again the only one for $\boldsymbol{c}^0$ and $\mathring{\bar{\boldsymbol{x}}}$ to differ. Hence, (4.42) holds true for *any* relevant vector of the third type.

$$
\text{(4.44)} \qquad \mathring{\bar{\boldsymbol{x}}}\boldsymbol{a}^{\ominus} < 0 \;,
$$

But condition (4.40) can be

satisfied as in view of (4.26) we have

$$
\text{(4.45)} \qquad
\begin{aligned}
h_1 + h_2 + h_3 + h_5 + h_6 &> 1 \\
h_5 + 2h_1 &> 1 + h_1 - h_2 - h_3 - h_6 \\
h_5 + 2h_1 &> \Delta \\
h_5 + 2h_1 &> \Delta = \Delta_1 + 2\Delta_2
\end{aligned}
$$

allows for a choice of Δ_1, Δ_2 satisfying (4.40). This way we have found a candidate $\mathring{\bar{\boldsymbol{x}}}$ for the third member of a vNM–Stable Set.

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Based in these considerations we are now in the position to formulate the general theorem. For simplicity we assume $h_\tau = 1$ for $\tau \in \hat{\mathbf{T}}$.

Definition 4.5. *1. Given the minimizing sequence* $\overset{\vee}{\boldsymbol{\tau}}$ *let*

$$h^{\star}_{\sigma} \;:=\; \sum_{\rho\in\mathbf{R}\setminus\{\sigma\}} h_{\overset{\vee}{\tau}_{\rho}} \;. \tag{4.46}$$

2. We say that a vector $\boldsymbol{\Delta} = (\Delta_1,\ldots,\Delta_r)$ *is an* ***admissible distribution of mass*** Δ *if*

$$\Delta \;=\; \sum_{\rho\in\mathbf{R}} \overset{\wedge}{\mathbf{t}}_{\rho}\Delta_{\rho} \quad \textit{and} \quad \Delta_{\rho} < h^{\star}_{\rho} \;\; (\rho\in\mathbf{R}). \tag{4.47}$$

Theorem 4.6. *There exists* $\overset{\circ}{\bar{\boldsymbol{x}}} \in \boldsymbol{H}$ *such that*

$$\overset{\circ}{\bar{\boldsymbol{x}}}\boldsymbol{a}^{\ominus} < 1 \tag{4.48}$$

holds true for any relevant vector $\boldsymbol{a}^{\ominus}$ *of the second type. this vector is induced by an admissible distribution of mass* Δ *over the vectors* $\{\boldsymbol{e}^{\sigma}\}_{\sigma\in\overset{\wedge}{\mathbf{T}}}$ *.*

Proof: Denote

$$\overset{\wedge}{\mathbf{t}}_{\rho} \;:=\; |\overset{\wedge}{\mathbf{T}}^{\rho}| \quad (\rho\in\mathbf{R}).$$

Then, because of $\boldsymbol{\lambda}^0(\boldsymbol{I}) > 1$ we have

$$\boldsymbol{\lambda}^0(\boldsymbol{I}) \;=\; \sum_{\tau\in\overset{\vee}{\mathbf{T}}} h_{\tau} + \sum_{\rho\in\mathbf{R}} \overset{\wedge}{\mathbf{t}}_{\rho} > t \;. \tag{4.49}$$

Next, using the minimizing sequence $\overset{\vee}{\boldsymbol{\tau}}$ and the definition

$$h^{\star}_{\sigma} \;:=\; \sum_{\rho\in\mathbf{R}\setminus\{\sigma\}} h_{\overset{\vee}{\tau}_{\rho}} \;,$$

we have (using some self explaining notation)

$$\bar{\boldsymbol{x}} \;=\; (h_1,\ldots,h_{\rho_1}, 1-h^{\star}_1,\ldots 1-h^{\star}_1;\ldots; h_{\rho_{r-1}},\ldots h_{\rho_r}, 1-h^{\star}_r,\ldots,1-h^{\star}_r) \;. \tag{4.50}$$

The total mass is

$$\sum_{\tau\in\mathbf{T}} \overline{x}_{\tau} = \sum_{\tau\in\overset{\vee}{\mathbf{T}}} h_{\tau} + \sum_{\rho\in\mathbf{R}} (1-h^{\star}_{\rho})\overset{\wedge}{\mathbf{t}}_{\rho} \tag{4.51}$$

and hence Δ computes to

$$\Delta \;=\; t - \left(\sum_{\tau\in\overset{\vee}{\mathbf{T}}} h_{\tau} + \sum_{\rho\in\mathbf{R}} (1-h^{\star}_{\rho})\overset{\wedge}{\mathbf{t}}_{\rho} \right) \;. \tag{4.52}$$

In view of (4.49) we have

$$t - \sum_{\tau\in\overset{\vee}{\mathbf{T}}} h_{\tau} < \sum_{\rho\in\mathbf{R}} \overset{\wedge}{\mathbf{t}}_{\rho} \;,$$

an inserting this in (4.52) we obtain

$$\Delta < \sum_{\rho\in\mathbf{R}} \overset{\wedge}{\mathbf{t}}_\rho - \sum_{\rho\in\mathbf{R}} (1-h^\star_\rho)\overset{\wedge}{\mathbf{t}}_\rho \;=\; \sum_{\rho\in\mathbf{R}} h^\star_\rho \overset{\wedge}{\mathbf{t}}_\rho \tag{4.53}$$

Therefore we can choose a set of reals $\boldsymbol{\Delta} = (\Delta_\rho)_{\rho\in\mathbf{R}}$ such that

$$\Delta \;=\; \sum_{\rho\in\mathbf{R}} \overset{\wedge}{\mathbf{t}}_\rho \Delta_\rho \quad \text{and} \quad \Delta_\rho < h^\star_\rho \tag{4.54}$$

holds true, that is, we can choose an admissible distribution of mass Δ. Using this distribution we define

$$\begin{aligned}
\overset{\circ}{\bar{\boldsymbol{x}}} \;:=\; & \bar{\boldsymbol{x}} + \sum_{\rho\in\mathbf{R}} \Delta_\rho \sum_{\tau\in\overset{\wedge}{\mathbf{T}}^\rho} \boldsymbol{e}^\tau \\
=\; & (h_1,\dots,h_{\rho_1}, 1-h^\star_1+\Delta_1,\dots 1-h^\star_1+\Delta_1;\dots \\
& \dots; h_{\rho_{r-1}},\dots h_{\rho_r}, 1-h^\star_r+\Delta_r,\dots,1-h^\star_r+\Delta_r)
\end{aligned} \tag{4.55}$$

which is an imputation in view of (4.47). Now consider a relevant vector $\boldsymbol{a}^\ominus$ with corresponding sequence $\boldsymbol{\tau} = (\tau_1,\dots,\tau_r,\overline{\tau}_r)$ with some $\overline{\tau}_r \in \overset{\wedge}{\mathbf{T}}^r$. Then, as $\overset{\circ}{\bar{\boldsymbol{x}}}$ and $\boldsymbol{c}^0$ coincide on coordinates $\tau \in \overset{\wedge}{\mathbf{T}}$, we have

$$\begin{aligned}
\overset{\circ}{\bar{\boldsymbol{x}}}\boldsymbol{a}^\ominus \;&=\; \boldsymbol{c}^0\boldsymbol{a}^\ominus + (\overset{\circ}{\bar{x}}_{\overline{\tau}_r} - c^0_{\overline{\tau}_r})a^\ominus_{\overline{\tau}_r} \\
&=\; 1 + ((1-h^\star_r)+\Delta_r - 1)a^\ominus_{\overline{\tau}_r} \\
&=\; 1 + (\Delta_r - h^\star_r)a^\ominus_{\overline{\tau}_r} \\
&<\; 1
\end{aligned} \tag{4.56}$$

the strict inequality in the last line resulting from (4.47).

We may have to consider relevant vectors with corresponding sequences $\boldsymbol{\tau}$ that are obtained by permuting the ordering, so that the element say $\overline{\tau}_r$ appears in $\overset{\wedge}{\mathbf{T}}^\rho$ instead of $\overset{\wedge}{\mathbf{T}}^r$. This problem is obviously solved by replacing r by ρ in (4.56).

q.e.d.

Corollary 4.7. *Let the convex (relatively open) set*

$$\begin{aligned}
\overset{\circ}{\boldsymbol{H}} \;:=\; \Bigg\{ \overset{\circ}{\bar{\boldsymbol{x}}} \in \boldsymbol{H} \;\Bigg|\; & \overset{\circ}{\bar{\boldsymbol{x}}} \;:=\; \bar{\boldsymbol{x}} + \sum_{\rho\in\mathbf{R}} \Delta_\rho \sum_{\tau\in\overset{\wedge}{\mathbf{T}}^\rho} \boldsymbol{e}^\tau \;, \\
& \sum_{\rho\in\mathbf{R}} \overset{\wedge}{\mathbf{t}}_\rho \Delta_\rho \;=\; \Delta,\; \Delta_\rho < h^\star_\rho \;\; (\rho\in\mathbf{R}) \Bigg\} \;\neq \emptyset
\end{aligned} \tag{4.57}$$

denote the elements of $\boldsymbol{H}$ *that are obtained by an admissible distribution of mass* Δ. *Then* $\overset{\circ}{\boldsymbol{H}}$ *consists exactly of the* strictly *effective pre–imputations of* $\boldsymbol{H}$.

Remark 4.8. Any $\mathring{\bar{\boldsymbol{x}}} \in \mathring{\boldsymbol{H}}$ together with the pre–core can dominate any other element $\widehat{\boldsymbol{x}} \in \boldsymbol{H}$ that is located within the convex hull of the $\boldsymbol{x}^{\sigma}$. Consequently, the convex hull of such an $\mathring{\bar{\boldsymbol{x}}} \in \mathring{\boldsymbol{H}}$ and the core can dominate any element of $\boldsymbol{H}$ that is not located within this convex hull.

For, recall that all preimputations in $\mathring{\boldsymbol{H}}$ coincide on coordinates in $\overset{\vee}{\mathbf{T}}$, i.e.,

$$\widehat{x}_{\tau} \;=\; \mathring{\bar{x}}_{\tau} \;=\; h_{\tau} \quad (\tau \in \overset{\vee}{\mathbf{T}}) \;,$$

hence the only coordinate for these vectors to differ along a sequence $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_r, \overline{\tau}_r)$ defining a vector $\boldsymbol{a}^{\ominus}$ is $\overline{\tau}_r \in \overset{\wedge}{\mathbf{T}}$.

Therefore, given $\widehat{\boldsymbol{x}}$, $\mathring{\bar{\boldsymbol{x}}} \in \mathring{\boldsymbol{H}}$, choose some $\overline{\tau}_r \in \overset{\wedge}{\mathbf{T}}^r$ (we assume r for convenience) such that

$$\widehat{x}_{\overline{\tau}_r} > \mathring{\bar{x}}_{\overline{\tau}_r} \;,$$

then choose $\tau_1, \ldots, \tau_r \in \overset{\vee}{\mathbf{T}}$ arbitrary such that

$$\widehat{x}_{\tau_\rho} \;=\; \mathring{\bar{x}}_{\tau_\rho} \;=\; h_{\tau_\rho} \quad (\rho \in \mathbf{R}) \;.$$

then, for sufficiently small $\varepsilon_1 > 0$ the imputation

$$\mathring{\boldsymbol{x}}^1 \;:=\; \varepsilon_1 \boldsymbol{e}^{\mathbf{T}^1} + (1-\varepsilon_1)\mathring{\bar{\boldsymbol{x}}} \in \mathring{\boldsymbol{H}}$$

exceeds $\widehat{\boldsymbol{x}}$ at coordinates τ_1 and $\overline{\tau}_r$. For, clearly, ε_1 can be choosen such that (“strict”) effectiveness is preserved, i.e., such that $\boldsymbol{a}^{\ominus}\mathring{\boldsymbol{x}}^1 < 1$ holds true. Continuing this way we see that for sufficiently small $\varepsilon_r > 0$

$$\mathring{\boldsymbol{x}}^r \;:=\; \varepsilon_r \boldsymbol{e}^{\mathbf{T}^r} + (1-\varepsilon_r)\mathring{x}^{r-1} \in \mathring{\boldsymbol{H}}$$

exceeds $\widehat{\boldsymbol{x}}$ at coordinates $\tau_1, \ldots, \tau_r, \overline{\tau}_r$ and still effectiveness is preserved, i.e., $\boldsymbol{a}^{\ominus}\mathring{\boldsymbol{x}}^r < 1$ holds true. Thus we have

$$\mathring{\boldsymbol{x}}^r \;\operatorname{dom}_{\boldsymbol{a}^{\ominus}} \widehat{\boldsymbol{x}} \;. \tag{4.58}$$

To prove the somewhat more general claim at the beginning of this remark, if $\widehat{\widehat{\boldsymbol{x}}}$ is an imputation in $\boldsymbol{H}$ then $\widehat{\widehat{\boldsymbol{x}}} \;=\; \alpha\boldsymbol{e} + (1-\alpha)\widehat{\boldsymbol{x}}$ with a suitable core element $\boldsymbol{e}$ and some $\widehat{\boldsymbol{x}}$ as above. Obviously $\alpha\boldsymbol{e} + (1-\alpha)\mathring{\bar{\boldsymbol{x}}}$ serves to dominate $\widehat{\widehat{\boldsymbol{x}}}$ via the same $\boldsymbol{a}^{\ominus}$.

Thus we observe that any $\mathring{\bar{\boldsymbol{x}}} \in \mathring{\boldsymbol{H}}$ suggests a vNM–solution $\mathcal{G}$ to be constructed via

$$\boldsymbol{G} \;:=\; \mathbf{ConvH}\left\{\boldsymbol{e}^{\mathbf{T}^1}, \ldots, \boldsymbol{e}^{\mathbf{T}^r}, \mathring{\bar{\boldsymbol{x}}}\right\} \;.$$

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5 The vNM–Stable Set

The results of the previous sections suggest obvious candidates for the construction of a vNM–Stable Set. One has to compute the vector $\bar{\boldsymbol{x}}$ and then distribute the remaining mass Δ in an admissible way, that is, take an element of $\overset{\circ}{\boldsymbol{H}}$.Tthe convex hull of this element and the core extremals will yield the desired vNM–Stable Set.

Theorem 5.1. *1. Let $\widehat{\boldsymbol{\tau}}$ be an undercutting sequence, i.e.,*

$$\sum_{\rho \in \mathbf{R}} h_{\widehat{\tau}_\rho} < 1$$

and let $\boldsymbol{a}^{\oplus}$ be the corresponding relevant vector of the second type. Then $\boldsymbol{a}^{\oplus}$ is efficient for any $\widehat{\boldsymbol{x}} \in \boldsymbol{H}$, more precisely,

$$\widehat{\boldsymbol{x}}\boldsymbol{a}^{\oplus} = 1 = v(\boldsymbol{a}^{\oplus}) \ . \tag{5.1}$$

2. Next, let $\widehat{\boldsymbol{\tau}}$ be an undercutting sequence and let $\overline{\tau} \in \overset{\vee}{\mathbf{T}}$ such that $\overline{\boldsymbol{\tau}} = (\widehat{\boldsymbol{\tau}}, \overline{\tau})$ is overshooting. Let $\boldsymbol{a}^{\ominus}$ denote the relevant vector of the third type corresponding to $\overline{\boldsymbol{\tau}}$. Then

$$\bar{\boldsymbol{x}}\boldsymbol{a}^{\ominus} < 1 \ , \tag{5.2}$$

that is, $\boldsymbol{a}^{\ominus}$ is ("strictly") effective for $\bar{\boldsymbol{x}}$ (but $\bar{\boldsymbol{x}}$ is not necessarily a n imputation).

3. Finally, let $\overline{\boldsymbol{\tau}} = (\widehat{\boldsymbol{\tau}}, \overline{\tau})$ and $\boldsymbol{a}^{\ominus}$ be chosen as in the second item *above. Then, for $\overset{\circ}{\bar{\boldsymbol{x}}} \in \overset{\circ}{\boldsymbol{H}}$*

$$\overset{\circ}{\bar{\boldsymbol{x}}}\boldsymbol{a}^{\ominus} < 1 \ , \tag{5.3}$$

that is, $\boldsymbol{a}^{\ominus}$ is ("strictly") effective for (the imputation) $\overset{\circ}{\bar{\boldsymbol{x}}}$.

Proof:

1$^{\text{st}}$STEP : Obviously by our construction we have for the extremal points of $\boldsymbol{H}$

$$\bar{\boldsymbol{x}}^{\sigma}\bar{\boldsymbol{a}}^{\oplus} \;=\; \bar{\boldsymbol{x}}\bar{\boldsymbol{a}}^{\oplus} = 1 \quad (\sigma \in \mathbf{R}) \ ; \tag{5.4}$$

thus *item* 1 is an immediate consequence.

2$^{\text{nd}}$STEP :

Next, regarding $\bar{\boldsymbol{x}}$ as constructed in Definition 2.4 we have

(5.5)

$$\begin{aligned}\bar{\boldsymbol{x}}\boldsymbol{a}^{\ominus} &= (h_{\widehat{\tau}_1}, \ldots, h_{\widehat{\tau}_r}, \overline{x}_{\overline{\overline{\tau}}_r}) \left(1, \ldots, 1, \frac{(h_{\widehat{\tau}_1} + \ldots + h_{\widehat{\tau}_{r-1}} + h_{\overline{\overline{\tau}}_r}) - 1}{h_{\overline{\overline{\tau}}_r} - h_{\widehat{\tau}_r}} \, , \frac{1 - (h_{\widehat{\tau}_1} + \ldots + h_{\widehat{\tau}_r})}{h_{\overline{\overline{\tau}}_r} - h_{\widehat{\tau}_r}} , \right) \\ &= h_{\widehat{\tau}_1} + \ldots + h_{\widehat{\tau}_{r-1}} + \alpha h_{\widehat{\tau}_r} + \beta \overline{x}_{\overline{\overline{\tau}}_r} \ ,\end{aligned}$$

where α, β are the last two coordinates of $\boldsymbol{a}^{\ominus}$ which are positive and sum up to 1. Hence, if $h_{\widehat{\tau}_r} \geq \overline{x}_{\tau_r}$, then

$$\bar{\boldsymbol{x}}\boldsymbol{a}^{\ominus} \leq \sum_{\rho \in \mathbf{R}} h_{\widehat{\tau}_\rho} < 1 \; .$$

On the other hand, if $h_{\widehat{\tau}_r} < \overline{x}_{\tau_r}$, then

$$\bar{\boldsymbol{x}}\boldsymbol{a}^{\ominus} \leq \sum_{\rho \in \mathbf{R} \setminus \{r\}} h_{\widehat{\tau}_\rho} + \overline{x}_{\tau_r} < h_{\tau_1} + \sum_{\rho \in \mathbf{R} \setminus \{1,r\}} h_{\widehat{\tau}_\rho} + \overline{x}_{\tau_r} = \; 1 \; ,$$

in view of equations (2.20) (2.21), or (2.22).

3rdSTEP : Follows from 4.8.

q.e.d.

Naturally we define

$$\overset{\circ}{\bar{\boldsymbol{\vartheta}}} \; := \; \boldsymbol{\vartheta}^{\overset{\circ}{\bar{\boldsymbol{x}}}} \quad (\overset{\circ}{\bar{\boldsymbol{x}}} \in \overset{\circ}{\boldsymbol{H}}) \; . \tag{5.6}$$

We fix some $\overset{\circ}{\bar{\boldsymbol{x}}} \in \overset{\circ}{\boldsymbol{H}}$. Then a candidate for a vNM–Stable set is provided by

$$\overset{\circ}{H} \; := \; \mathbf{ConvH}\left\{ \overset{\circ}{\bar{\boldsymbol{x}}} \; , \quad \boldsymbol{e}^{T^\rho} \quad (\rho \in \mathbf{R}) \right\} \; . \tag{5.7}$$

and

$$\overset{\circ}{\mathcal{H}} \; = \; \mathbf{ConvH}\left\{ \overset{\circ}{\bar{\boldsymbol{\vartheta}}} \; , \; \boldsymbol{\lambda}^{\rho} \;\; (\rho \in \mathbf{R}) \right\} \; . \tag{5.8}$$

Now we have

Theorem 5.2. *The set $\overset{\circ}{\mathcal{H}}$, i.e., the set of imputations induced by $\overset{\circ}{\boldsymbol{H}}$, is internally stable.*

Proof:

We can more or less directly appeal to Theorem 3.11 of Part II as $\overset{\circ}{\boldsymbol{H}}$ has just one extremal apart from the $\boldsymbol{e}^{\mathbf{T}^\rho}$. For completenes we repeat the argument.

1stSTEP : Whenever $\boldsymbol{a}^{\oplus}$ is a relevant vector of the first or second kind (i.e. a separating pre–coalitions), then we know that $\boldsymbol{x}\boldsymbol{a}^{\oplus} \geq 1 = v(\boldsymbol{a})$ holds true. Hence, no separating relevant vector induces a coalition that yields a domination. Therefore, we can restrict ourselves to domination via the non–separating relevant vectors of the third type $\boldsymbol{a}^{\ominus}$ described by items $2, 3$ of Theorem 5.1.

These vectors $\bar{\boldsymbol{a}}^{\ominus}$ are given by a sequence $(\widehat{\tau}_1, \ldots, \widehat{\tau}_r, \overline{\overline{\tau}}_r)$ by

$$\begin{aligned}
\overline{a}_{\widehat{\tau}_\rho} &= 1 \quad (\rho \in \mathbf{R} \setminus \{r\}) \\
\overline{a}_{\widehat{\tau}_r} &= \frac{(h_{\widehat{\tau}_1} + \ldots + h_{\widehat{\tau}_{r-1}} + h_{\overline{\overline{\tau}}_r}) - 1}{h_{\overline{\overline{\tau}}_r} - h_{\widehat{\tau}_r}} \\
\overline{a}_{\overline{\overline{\tau}}_r} &= \frac{1 - (h_{\widehat{\tau}_1} + \ldots + h_{\widehat{\tau}_r})}{h_{\overline{\overline{\tau}}_r} - h_{\widehat{\tau}_r}}, \\
\overline{a}_\tau &= 0 \quad \text{otherwise}
\end{aligned} \tag{5.9}$$

with

$$h_{\widehat{\tau}_1} + \ldots + h_{\widehat{\tau}_r} < \; 1 \; < h_{\widehat{\tau}_1} + \ldots + h_{\widehat{\tau}_{r-1}} + h_{\overline{\overline{\tau}}_r} \; .$$

There also the permuted versions $\boldsymbol{a}^{\odot\sigma}$, but for simplicity we assume that domination takes place via some vector given by (5.9).

Introduce vectors $\boldsymbol{a}^{\odot}$ of first type with value 1 at coordinates corresponding to

$$h_{\widehat{\tau}_1}, \ldots, h_{\widehat{\tau}_{r-1}}, h_{\overline{\overline{\tau}}_r}$$

and vector $\boldsymbol{a}^{\oplus}$ with non–vanishing coordinates at coordinates corresponding to

$$h_{\widehat{\tau}_1}, \ldots, h_{\widehat{\tau}_{r-1}}, h_{\widehat{\tau}_r}$$

Now according to Remark 2.5 we have for the vector $\bar{\boldsymbol{x}}$

$$\sum_{\rho \in \mathbf{R} \setminus \{r\}} \overline{x}_{\widehat{\tau}_\rho} + \overline{x}_{\overline{\overline{\tau}}_r} \geq 1 \quad \text{that is} \quad \bar{\boldsymbol{x}}\boldsymbol{a}^{\odot} \geq 1 \; . \tag{5.10}$$

The vector $\mathring{\bar{\boldsymbol{x}}}$ exceeds $\bar{\boldsymbol{x}}$ exactly at coordinate $\overline{\overline{\tau}}_r$. Hence,

$$\mathring{\bar{\boldsymbol{x}}}\boldsymbol{a}^{\odot} \; \geq \; \bar{\boldsymbol{x}}\boldsymbol{a}^{\odot} \; \geq \; 1 \; . \tag{5.11}$$

The vectors of $\mathring{\boldsymbol{H}}$ are of the form

$$\boldsymbol{x} \; = \; \sum_{\rho \in \mathbf{R}} \alpha_\rho \boldsymbol{e}^{T^\rho} + \overline{\alpha}\mathring{\bar{\boldsymbol{x}}} \tag{5.12}$$

with a "convex" coefficients $(\alpha_1, \ldots, \alpha_r, \overline{\alpha})$ (i.e., nonnegative and summing up to 1). Suppose now that $\boldsymbol{x} \operatorname{dom}_{\boldsymbol{a}^{\ominus}} \boldsymbol{y}$ holds true for some $\boldsymbol{x},\ \boldsymbol{y} \in \mathring{\boldsymbol{H}}$. Then $\boldsymbol{y}$ is of a similar form, say,

$$\boldsymbol{y} \; = \; \sum_{\rho \in \mathbf{R}} \beta_\rho \boldsymbol{e}^{T^\rho} + \overline{\beta}\mathring{\bar{\boldsymbol{x}}} \; , \tag{5.13}$$

again with a "convex" coefficients $(\beta_1, \ldots, \beta_r, \overline{\beta})$.

2$^{\text{nd}}$STEP : Recall that $\mathring{\bar{\boldsymbol{x}}}$ looks like $\bar{\boldsymbol{x}}$ along the positive coordinates of $\boldsymbol{a}^{\odot}$. Now we write

$$\boldsymbol{x} \; = \; \left(\sum_{\sigma \in \mathbf{R}} \alpha_\sigma \right) \sum_{\rho \in \mathbf{R}} \frac{\alpha_\rho}{\sum_{\sigma \in \mathbf{R}} \alpha_\sigma} \boldsymbol{e}^{T^\rho} + \overline{\alpha}\mathring{\bar{\boldsymbol{x}}} \; =: \; (1 - \overline{\alpha})\boldsymbol{e} + \overline{\alpha}\mathring{\bar{\boldsymbol{x}}} \; ; \tag{5.14}$$

in other words, any $\boldsymbol{x} \in \boldsymbol{H}$ is a convex combination of a pre–core element and $\mathring{\bar{\boldsymbol{x}}}$. Note that any $\boldsymbol{e}^{T^\rho}$ $(\rho \in \mathbf{R})$ and hence any vector $\boldsymbol{e}$ of the pre–core satisfies

(5.15)
$$\sum_{\rho \in \mathbf{R}} e_{\widehat{\tau}_\rho} = 1 \quad ,$$

no matter whether the separating sequence ends up with or without $\overline{\overline{\tau}}_r$.

Similarly

(5.16)
$$\boldsymbol{y} \; = \; (1-\overline{\beta})\boldsymbol{e}' + \overline{\beta}\mathring{\bar{\boldsymbol{x}}} \; .$$

Now, if domination takes place between $\boldsymbol{x}$ and $\boldsymbol{y}$ via $\boldsymbol{a}^{\ominus}$, then

(5.17)
$$(1-\overline{\alpha})\boldsymbol{e} + \overline{\alpha}\mathring{\bar{\boldsymbol{x}}} \; > \; (1-\overline{\beta})\boldsymbol{e}' + \overline{\beta}\mathring{\bar{\boldsymbol{x}}}$$

for coordinates $(\widehat{\tau}_1, \ldots, \widehat{\tau}_r, \overline{\overline{\tau}}_r)$.

First of all, consider the separating sequence obtained by omitting $\widehat{\tau}_r$, i.e., $(\widehat{\tau}_1, \ldots, \overline{\overline{\tau}}_r)$. Then, according to (5.11) and (5.15) we find by taking the sum $\sum_{\rho \in \mathbf{R}\setminus\{r\}} x_{\widehat{\tau}_\rho} + x_{\overline{\overline{\tau}}_r} \; \geq \; 1$ on both sides and writing $\xi \; := \; \sum_{\rho \in \mathbf{R}\setminus\{r\}} x_{\widehat{\tau}_\rho} + x_{\overline{\overline{\tau}}_r}$

$$(1-\overline{\alpha}) + \overline{\alpha}\xi > (1-\overline{\beta}) + \overline{\beta}\xi$$

i.e.

$$\overline{\alpha}(\xi - 1) > \overline{\beta}(\xi - 1)$$

hence necessarily

$$\overline{\alpha} > \overline{\beta} \; .$$

Now we perform the same operation along the sequence $(\widehat{\tau}_1, \ldots, \widehat{\tau}_r)$ *not* including $\overline{\overline{\tau}}_\rho$. Now coordinate $\overline{\overline{\tau}}_r$ is not involved and we have $\eta \; := \; \sum_{\rho \in \mathbf{R}} x_{\widehat{\tau}_\rho} < 1$ can be employed so that summation along the sequence now produces

$$(1-\overline{\alpha}) + \overline{\alpha}\eta > (1-\overline{\beta}) + \overline{\beta}\eta$$

i.e.

$$\overline{\alpha}(\eta - 1) > \overline{\beta}(\eta - 1)$$

$$\overline{\alpha}(1 - \eta) < \overline{\beta}(1 - \eta)$$

hence

$$\overline{\alpha} < \overline{\beta} \; .$$

This contradiction proves that domination cannot take place inside $\boldsymbol{H}$ via a non–separating sequence resulting from a relevant vector described by (5.9).

q.e.d.

Theorem 5.3. *1. Let $\bar{\boldsymbol{a}}^{\oplus}$ be a separating vector of the second kind with a sequence $\widehat{\boldsymbol{\tau}}$ of positive coordinates (all elements of $\overset{\vee}{\mathbf{T}}$). Let $\mathring{\boldsymbol{x}}$ be an imputation such that*

(5.18)
$$\sum_{\rho \in \mathbf{R}} \mathring{x}_{\widehat{\tau}_\rho} < \sum_{\rho \in \mathbf{R}} h_{\widehat{\tau}_\rho} \; .$$

Then there exists $\overset{\circ}{\widehat{\boldsymbol{x}}} \in \overset{\circ}{\boldsymbol{H}}$ *such that*

$$\overset{\circ}{\widehat{\boldsymbol{x}}} \operatorname{dom}_{\bar{\boldsymbol{a}}^{\oplus}} \overset{\circ}{\boldsymbol{x}} \ . \tag{5.19}$$

2. *Let* $\boldsymbol{\vartheta}$ *be an imputation with minima vector* $\boldsymbol{m}$*. If*

$$\sum_{\rho \in \mathbf{R}} m_{\widehat{\tau}_\rho} < \sum_{\rho \in \mathbf{R}} h_{\widehat{\tau}_\rho} \ , \tag{5.20}$$

then, for sufficiently small $\varepsilon > 0$*, there exists an* ε–$\bar{\boldsymbol{a}}^{\oplus}$ *relevant coalition* $T^\varepsilon = T^{\varepsilon \bar{\boldsymbol{a}}^{\oplus}}$ *and* $\overset{\circ}{\widehat{\boldsymbol{x}}} \in \boldsymbol{H}$ *such that* $\overset{\circ}{\widehat{\boldsymbol{\vartheta}}} := \boldsymbol{\vartheta}^{\overset{\circ}{\widehat{x}}}$ *yields*

$$\overset{\circ}{\widehat{\boldsymbol{\vartheta}}} \operatorname{dom}_{T^\varepsilon} \boldsymbol{\vartheta} \ . \tag{5.21}$$

Proof:

1st STEP : Assume w.l.g. that r minimizes the quotients $\frac{\overset{\circ}{x}_{\widehat{\tau}_\rho}}{h_{\widehat{\tau}_\rho}}$ $(\rho \in \mathbf{R})$, i.e, $\frac{\overset{\circ}{x}_{\widehat{\tau}_r}}{h_{\widehat{\tau}_r}} \leq \frac{\overset{\circ}{x}_{\widehat{\tau}_\rho}}{h_{\widehat{\tau}_\rho}}$, or

$$\frac{\overset{\circ}{x}_{\widehat{\tau}_r}}{h_{\widehat{\tau}_r}} h_{\widehat{\tau}_\rho} \leq \overset{\circ}{x}_{\widehat{\tau}_\rho} \quad (\rho \in \mathbf{R}) \ . \tag{5.22}$$

Define $\overline{\alpha} := \frac{\overset{\circ}{x}_{\widehat{\tau}_r}}{h_{\widehat{\tau}_r}} < 1$. Now because of

$$1 - \sum_{\rho \in \mathbf{R}} \overset{\circ}{x}_{\widehat{\tau}_\rho} > 1 - \sum_{\rho \in \mathbf{R}} h_{\widehat{\tau}_\rho}$$

it follows that

$$\frac{\left(1 - \sum_{\rho \in \mathbf{R}} \overset{\circ}{x}_{\widehat{\tau}_\rho}\right) + \overset{\circ}{x}_{\widehat{\tau}_r}}{\left(1 - \sum_{\rho \in \mathbf{R}} h_{\widehat{\tau}_\rho}\right) + h_{\widehat{\tau}_r}} > \frac{\overset{\circ}{x}_{\widehat{\tau}_r}}{h_{\widehat{\tau}_r}} = \overline{\alpha} \ ,$$

or, equivalently

$$\frac{1 - \sum_{\rho \in \mathbf{R} \setminus \{r\}} \overset{\circ}{x}_{\widehat{\tau}_\rho}}{1 - \sum_{\rho \in \mathbf{R} \setminus \{r\}} h_{\widehat{\tau}_\rho}} > \overline{\alpha} \ ,$$

$$1 - \sum_{\rho \in \mathbf{R} \setminus \{r\}} \overset{\circ}{x}_{\widehat{\tau}_\rho} > \overline{\alpha} \left(1 - \sum_{\rho \in \mathbf{R} \setminus \{r\}} h_{\widehat{\tau}_\rho}\right)$$

which is

$$1 - \overline{\alpha} > \sum_{\rho \in \mathbf{R} \setminus \{r\}} \left(\overset{\circ}{x}_{\widehat{\tau}_\rho} - \overline{\alpha} h_{\widehat{\tau}_\rho} \ .\right) \tag{5.23}$$

Because of (5.22) the terms under sum in (5.23) are all non negative. Therefore, (5.23) permits to choose positive reals $\alpha_1, \ldots, \alpha_r$ such that

$$1 - \overline{\alpha} \; > \; 1 - \alpha_r \; > \sum_{\rho \in \mathbf{R} \setminus \{r\}} \left(\overset{\circ}{x}_{\widehat{\tau}_\rho} - \overline{\alpha} h_{\widehat{\tau}_\rho} \; . \right) \tag{5.24}$$

$$\alpha_\rho > \overset{\circ}{x}_{\widehat{\tau}_\rho} - \alpha_r h_{\widehat{\tau}_\rho} \quad (\rho \in \mathbf{R} \setminus \{r\}) \; , \tag{5.25}$$

and

$$1 - \alpha_r \; = \sum_{\rho \in \mathbf{R} \setminus \{r\}} \alpha_\rho \tag{5.26}$$

holds true. In other words, the α_ρ are positive convex coefficients,

$$\sum_{\rho \in \mathbf{R}} \alpha_\rho \; = \; 1 \; . \tag{5.27}$$

Also, we have

$$\alpha_r > \overline{\alpha} = \frac{\overset{\circ}{x}_{\widehat{\tau}_r}}{h_{\widehat{\tau}_r}} \; . \tag{5.28}$$

Now, as $\widehat{\tau}_1, \ldots, \widehat{\tau}_r \in \overset{\vee}{\mathbf{T}}$ the vector

$$\widehat{\boldsymbol{x}} \; := \sum_{\rho \in \mathbf{R} \setminus \{r\}} \alpha_\rho \boldsymbol{e}^{\boldsymbol{T}^\rho} + \alpha_r \bar{\boldsymbol{x}} \tag{5.29}$$

and the vector

$$\overset{\circ}{\widehat{\boldsymbol{x}}} \; = \sum_{\rho \in \mathbf{R} \setminus \{r\}} \alpha_\rho \boldsymbol{e}^{\boldsymbol{T}^\rho} + \alpha_r \overset{\circ}{\bar{\boldsymbol{x}}} \tag{5.30}$$

coincide along the coordinates of $\widehat{\boldsymbol{\tau}}$. Then clearly for $\rho \in \mathbf{R} \setminus \{r\}$ we have

$$\widehat{x}_{\widehat{\tau}_\rho} \; = \; \alpha_\rho + \alpha_r h_{\widehat{\tau}_\rho} \; > \; \overset{\circ}{x}_{\widehat{\tau}_\rho} \tag{5.31}$$

(in view of (5.25)), and for $\rho = r$

$$\widehat{x}_{\widehat{\tau}_r} \; = \; \alpha_r h_{\widehat{\tau}_r} > \overline{\alpha} h_{\widehat{\tau}_r} \; = \; \overset{\circ}{x}_{\widehat{\tau}_r} \tag{5.32}$$

(in view of (5.28)). Moreover

$$\begin{aligned} \overset{\circ}{\widehat{\boldsymbol{x}}} \boldsymbol{a}^{\oplus} \; &= \; \widehat{\boldsymbol{x}} \boldsymbol{a}^{\oplus} \\ &= \sum_{\rho \in \mathbf{R} \setminus \{r\}} \alpha_\rho \boldsymbol{e}^{\boldsymbol{T}^\rho} \boldsymbol{a}^{\oplus} + \alpha_r \overset{\circ}{\bar{\boldsymbol{x}}} \boldsymbol{a}^{\oplus} \\ &= \sum_{\rho \in \mathbf{R} \setminus \{r\}} \alpha_\rho \boldsymbol{e}^{\boldsymbol{T}^\rho} \boldsymbol{a}^{\oplus} + \alpha_r \bar{\boldsymbol{x}} \boldsymbol{a}^{\oplus} \; = \; \sum_{\rho \in \mathbf{R}} \alpha_r = 1 \; . \end{aligned} \tag{5.33}$$

Now (5.31),(5.32) and (5.33) imply

$$\hat{\mathring{\boldsymbol{x}}} \,\mathrm{dom}_{\boldsymbol{a}^{\oplus}}\, \mathring{\boldsymbol{x}} \ .$$

2^ndSTEP : If $\boldsymbol{\vartheta}$ is an imputation satisfying the condition specified for $\boldsymbol{m}$, then $\boldsymbol{m}$ can play the role of $\mathring{\boldsymbol{x}}$. Hence by Theorem 4.5. of [1] we find, for $\varepsilon > 0$ sufficiently small, an ε–relevant coalition $T^{\varepsilon} = T^{\varepsilon \boldsymbol{a}^{\oplus}}$ such that

$$\boldsymbol{\vartheta}^{\hat{\mathring{x}}} \,\mathrm{dom}_{T^{\varepsilon}}\, \boldsymbol{\vartheta} \ ,$$

q.e.d.

Theorem 5.4. *Let* $\boldsymbol{a}^{\ominus}$ *be a pre–coalition of the third kind with a corresponding sequence* $\widehat{\boldsymbol{\tau}} = (\widehat{\tau}_1, \ldots, \widehat{\tau}_r, \overline{\overline{\tau}})$ *of positive coordinates. Let* $\mathring{\boldsymbol{x}} \in \mathbb{R}^{rt}$ *satisfy the following conditions.*

1.

$$1 > \sum_{\rho \in \mathbf{R}} \mathring{x}_{\widehat{\tau}_\rho} \geq \sum_{\rho \in \mathbf{R}} h_{\widehat{\tau}_\rho} \ . \tag{5.34}$$

2.

$$\mathring{x}_{\widehat{\tau}_1} + \mathring{x}_{\widehat{\tau}_2} +, \ldots, + \mathring{x}_{\widehat{\tau}_{r-1}} + \mathring{x}_{\overline{\overline{\tau}}} \ \geq \ 1 \ . \tag{5.35}$$

3.

$$\sum_{\tau \in \mathbf{T}} \lambda_\tau \mathring{x}_\tau \ \leq \ 1 \ \textit{i.e.} \ \sum_{\tau \in \mathbf{T}} \mathring{x}_\tau \leq t \tag{5.36}$$

that is, $\mathring{\boldsymbol{x}}$ *is a "pre–subimputation".*

Then there exists $\mathring{\widehat{\boldsymbol{x}}} \in \mathring{\boldsymbol{H}}$ *such that*

$$\mathring{\widehat{\boldsymbol{x}}} \, \mathrm{dom}_{\boldsymbol{a}^{\ominus}} \, \mathring{\boldsymbol{x}} \tag{5.37}$$

holds true.

4. *Let* $\boldsymbol{\vartheta}$ *be an imputation with minima vector* $\boldsymbol{m}$*. If* $\boldsymbol{m}$ *satisfies the above conditions for* $\mathring{\boldsymbol{x}}$*, then, for sufficiently small* $\varepsilon > 0$*, there exists an* ε–$\bar{\boldsymbol{a}}^{\oplus}$ *relevant coalition* $T^\varepsilon = T^{\varepsilon \bar{\boldsymbol{a}}^{\oplus}}$ *and* $\widehat{\boldsymbol{x}} \in \boldsymbol{H}$ *such that*

$$\boldsymbol{\vartheta}^{\widehat{x}} \, \mathrm{dom}_{T^\varepsilon} \, \boldsymbol{\vartheta} \ . \tag{5.38}$$

Proof:

1$^{\text{st}}$STEP :

Define

$$\overline{\alpha} \ := \ \frac{1 - \sum_{\rho \in \mathbf{R}} \mathring{x}_{\widehat{\tau}_\rho}}{1 - \sum_{\rho \in \mathbf{R}} h_{\widehat{\tau}_\rho}} \ , \quad 0 < \overline{\alpha} < 1 \ , \tag{5.39}$$

and

$$\overline{\alpha}_\rho \ := \ \mathring{x}_{\widehat{\tau}_\rho} - \overline{\alpha} h_{\widehat{\tau}_\rho} \quad \rho \in \mathbf{R} \ . \tag{5.40}$$

First of all we *assume* that

$$\overline{\alpha}_\rho \geq 0 \quad \text{for} \quad \rho \in \mathbf{R}, \tag{5.41}$$

for then $\overline{\alpha}_1, \ldots, \overline{\alpha}_r, \overline{\alpha}$ constitute a set of "convex coefficients", that is, nonnegative and summing up to 1.

We will have to get rid of this assumption by means of some additional argument, this will be presented in the $5^{th}STEP$.

2nd STEP :

Now let

$$\overset{*}{\boldsymbol{x}} := \sum_{\rho \in \mathbf{R}} \overline{\alpha}_\rho \boldsymbol{e}^{T^\rho} + \bar{\alpha}\bar{\boldsymbol{x}} \ .$$

(5.42) and

$$\overset{\circ}{\widehat{\boldsymbol{x}}} := \sum_{\rho \in \mathbf{R}} \overline{\alpha}_\rho \boldsymbol{e}^{T^\rho} + \bar{\alpha}\overset{\circ}{\bar{\boldsymbol{x}}} \in \boldsymbol{H} \ .$$

Then, along coordinates $\widehat{\tau}_\rho \in \overset{\vee}{\mathbf{T}}$ $(\rho \in \mathbf{R})$ we observe that $\overset{\circ}{\widehat{x}}_{\widehat{\tau}_\rho} = \overset{*}{x}_{\widehat{\tau}_\rho}$ while for $\overline{\overline{\tau}}_r$ we have $\overset{\circ}{\widehat{x}}_{\overline{\overline{\tau}}_r} \geq \overset{*}{x}_{\overline{\overline{\tau}}_r}$.

Now clearly

$$\overset{\circ}{\overline{x}}_{\widehat{\tau}_\rho} = \overset{*}{x}_{\widehat{\tau}_\rho} = \overline{\alpha}_\rho + \overline{\alpha} h_{\widehat{\tau}_\rho} \geq \overset{\circ}{x}_{\widehat{\tau}_\rho} \quad (\rho \in \mathbf{R}) \tag{5.43}$$

by just rewriting (5.40). Moreover, by Theorem 5.1, *items* 2,3 i.e., by (5.2) and (5.3) we know that $\bar{\boldsymbol{x}}\boldsymbol{a}^\ominus \leq \overset{\circ}{\bar{\boldsymbol{x}}}\boldsymbol{a}^\ominus < 1$ and as $\overline{\alpha} > 0$ it follows that

$$\overset{*}{\boldsymbol{x}}\boldsymbol{a}^\ominus < 1 \ \ , \overset{\circ}{\widehat{\boldsymbol{x}}}\boldsymbol{a}^\ominus < 1 \ \ . \tag{5.44}$$

3rd STEP :

Essentially it remains to show that

$$\overset{\circ}{\widehat{x}}_{\overline{\overline{\tau}}} \geq \overset{*}{x}_{\overline{\overline{\tau}}} \ = \ \overline{\alpha}_r + \overline{\alpha x_{\overline{\overline{\tau}}}} > \overset{\circ}{x}_{\overline{\overline{\tau}}} \tag{5.45}$$

holds true. To this end we insert $\overline{\alpha}_r \ = \ \overset{\circ}{x}_{\widehat{\tau}_r} - \overline{\alpha} h_{\widehat{\tau}_r}$ so that equivalently we need to show $\overset{\circ}{x}_{\widehat{\tau}_r} - \overline{\alpha} h_{\widehat{\tau}_r} + \overline{\alpha x_{\overline{\overline{\tau}}}} > \overset{\circ}{x}_{\overline{\overline{\tau}}}$, i.e.

$$\overset{\circ}{x}_{\overline{\overline{\tau}}} - \overset{\circ}{x}_{\widehat{\tau}_r} \ < \ \overline{\alpha}(\overline{x}_{\overline{\overline{\tau}}} - \overline{x}_{\widehat{\tau}_r}) \tag{5.46}$$

Now recall that for the *minimal* sequence $\overset{\vee}{\boldsymbol{\tau}}$ we have

$$\overline{x}_{\overline{\overline{\tau}}} = 1 - \sum_{\rho \in \mathbf{R} \setminus \{r\}} h_{\overset{\vee}{\tau}_\rho} = 1 - \sum_{\rho \in \mathbf{R} \setminus \{r\}} \overline{x}_{\overset{\vee}{\tau}_\rho} \tag{5.47}$$

by Definition 2.20. For all other undercutting sequences $\widehat{\boldsymbol{\tau}}$ we have clearly

$$\overline{x}_{\overline{\overline{\tau}}} \geq 1 - \sum_{\rho \in \mathbf{R} \setminus \{r\}} h_{\widehat{\tau}_\rho} = 1 - \sum_{\rho \in \mathbf{R} \setminus \{r\}} \overline{x}_{\widehat{\tau}_\rho} \ . \tag{5.48}$$

Hence

$$\overline{x}_{\overline{\overline{\tau}}} - \overline{x}_{\widehat{\tau}_r} \geq 1 - \sum_{\rho \in \mathbf{R}} \overline{x}_{\widehat{\tau}_\rho} \tag{5.49}$$

On the other hand, in view of our assumption (5.34), we have

$$\overset{\circ}{x}_{\overline{\overline{\tau}}} < 1 - \sum_{\rho \in \mathbf{R} \setminus \{r\}} \overset{\circ}{x}_{\widehat{\tau}_\rho} \; . \tag{5.50}$$

Hence

$$\overset{\circ}{x}_{\overline{\overline{\tau}}} - \overset{\circ}{x}_{\widehat{\tau}_r} < 1 - \sum_{\rho \in \mathbf{R}} \overset{\circ}{x}_{\widehat{\tau}_\rho} \; . \tag{5.51}$$

Consequently

$$\begin{aligned} \overset{\circ}{x}_{\overline{\overline{\tau}}} - \overset{\circ}{x}_{\widehat{\tau}_r} &< 1 - \sum_{\rho \in \mathbf{R}} \overset{\circ}{x}_{\widehat{\tau}_\rho} \; . \\ &= \left(1 - \sum_{\rho \in \mathbf{R}} \overset{\circ}{x}_{\widehat{\tau}_\rho}\right) \frac{1 - \sum_{\rho \in \mathbf{R}} \overline{x}_{\widehat{\tau}_\rho}}{1 - \sum_{\rho \in \mathbf{R}} \overline{x}_{\widehat{\tau}_\rho}} \\ &\leq \left(1 - \sum_{\rho \in \mathbf{R}} \overset{\circ}{x}_{\widehat{\tau}_\rho}\right) \frac{(\overline{x}_{\overline{\overline{\tau}}} - \overline{x}_{\widehat{\tau}_r})}{1 - \sum_{\rho \in \mathbf{R}} \overline{x}_{\widehat{\tau}_\rho}} \\ &= \overline{\alpha}(\overline{x}_{\overline{\overline{\tau}}} - \overline{x}_{\widehat{\tau}_r}) \; , \end{aligned} \tag{5.52}$$

which is (5.46). Hence (5.45) is verified.

4$^{\text{th}}$STEP :

Now, inequalitites (5.43) have to be rendered to be strict in order to yield dominance, i.e.,

$$\overset{\circ}{\widehat{\boldsymbol{x}}} \operatorname{dom}_{\boldsymbol{a}^{\ominus}} \overset{\circ}{\boldsymbol{x}} \; . \tag{5.53}$$

Now, as $\sum_{\rho \in \mathbf{R}} \overset{\circ}{x}_{\widehat{\tau}_\rho} < 1$ by (5.34), there exists some $\delta > 0$ such that

$$\sum_{\rho \in \mathbf{R}} (\overset{\circ}{x}_{\widehat{\tau}_\rho} + \delta) = 1$$

and hence the vector

$$\boldsymbol{e} \;:=\; \sum_{\rho \in \mathbf{R}} (\overset{\circ}{x}_{\widehat{\tau}_\rho} + \delta) \boldsymbol{e}^{\mathbf{T}^\rho} \in \boldsymbol{C}(v)$$

has coordinates $e_{\widehat{\tau}_\rho} > \overset{\circ}{x}_{\widehat{\tau}_\rho}$ exceeding the coordinates of $\overset{\circ}{\boldsymbol{x}}$ $(\rho \in \mathbf{R})$. Thus, for sufficiently small but positive $\varepsilon > 0$ the vector

$$\overset{\circ}{\widetilde{\boldsymbol{x}}} \;:=\; (1 - \varepsilon)\overset{\circ}{\widehat{\boldsymbol{x}}} + \varepsilon \boldsymbol{e} \in \overset{\circ}{\boldsymbol{H}}$$

yields

$$\overset{\circ}{\widetilde{x}}_{\widehat{\tau}_\rho} > \overset{\circ}{x}_{\widehat{\tau}_\rho} \quad (\rho \in \mathbf{R})$$

without disturbing inequalities (5.44) and (5.45), i.e., preserving

$$\tilde{\mathring{x}}_{\overline{\overline{\tau}}_r} > \mathring{x}_{\overline{\overline{\tau}}_r} \quad \text{and} \quad \tilde{\mathring{\boldsymbol{x}}}\boldsymbol{a}^{\ominus} < 1,$$

i.e., we have

$$\tilde{\mathring{\boldsymbol{x}}}\,\mathrm{dom}_{\boldsymbol{a}^{\ominus}}\,\mathring{\boldsymbol{x}}\ .$$

5thSTEP :

We still have to deal with assumption (5.41) in the $1^{st}STEP$. This is done as in the proof of Theorem 3.14 in Part II; for completeness we copy the procedure.

Rewrite the terms in the first step such that we have

$$\widehat{\alpha} := \frac{1 - \sum_{\rho \in \mathbf{R}} \mathring{x}_{\widehat{\tau}_\rho}}{1 - \sum_{\rho \in \mathbf{R}} h_{\widehat{\tau}_\rho}}\ , \quad 0 < \widehat{\alpha} < 1\ , \tag{5.54}$$

and

$$\widehat{\alpha}_\rho := \mathring{x}_{\widehat{\tau}_\rho} - \widehat{\alpha} h_{\widehat{\tau}_\rho} \quad \rho \in \mathbf{R}\ . \tag{5.55}$$

If $\widehat{\alpha}_\rho \geq 0$ for $\rho \in \mathbf{R}$, then $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_r, \widehat{\alpha}$ constitutes a set of "convex coefficients", our assumption within the $1^{st}STEP$.

If this is not so, then we adjust the coefficients as follows. First of all observe

$$\widehat{\alpha}(1 - \sum_{\rho \in \mathbf{R}} h_{\widehat{\tau}_\rho}) = (1 - \sum_{\rho \in \mathbf{R}} \mathring{x}_{\widehat{\tau}_\rho})$$

i.e.

$$\sum_{\rho \in \mathbf{R}} \mathring{x}_{\widehat{\tau}_\rho} - \widehat{\alpha} \sum_{\rho \in \mathbf{R}} h_{\widehat{\tau}_\rho} = 1 - \widehat{\alpha} \quad \text{or} \quad \sum_{\rho \in \mathbf{R}} (\mathring{x}_{\widehat{\tau}_\rho} - \widehat{\alpha} h_{\widehat{\tau}_\rho}) = 1 - \widehat{\alpha}$$

i.e.

$$\sum_{\rho \in \mathbf{R}} \widehat{\alpha}_\rho = 1 - \widehat{\alpha}\ .$$

Tentatively we write $\alpha^+ := \max\{0, \alpha\}$ for real α. Now consider the function

$$L(\bullet) : [0,1] \to [0,1]$$

given by

$$L(\alpha) := \sum_{\rho \in \mathbf{R}} (\mathring{x}_{\widehat{\tau}_\rho} - \alpha h_{\widehat{\tau}_\rho})^+ \quad (\alpha \in [0,1])$$

which is continuous and decreasing in α. We have

$$L(0) = \sum_{\rho \in \mathbf{R}} \mathring{x}_{\widehat{\tau}_\rho} < 1$$

$$L(1) \geq \sum_{\rho \in \mathbf{R}} \mathring{x}_{\widehat{\tau}_\rho} - \sum_{\rho \in \mathbf{R}} h_{\widehat{\tau}_\rho} > 0$$

Compare this with the decreasing function $\alpha \to 1-\alpha$ on $[0,1]$ which has values 1 and 0 at arguments 0 and 1. Clearly we can find some $\overline{\alpha} \in [0,1]$, $\overline{\alpha} \leq \widehat{\alpha}$, such that both functions are equal, that is

$$(5.56) \qquad 1-\overline{\alpha} \;=\; \sum_{\rho \in \mathbf{R}} (\mathring{x}_{\widehat{\tau}_\rho} - h_{\widehat{\tau}_\rho})^+ \geq \sum_{\rho \in \mathbf{R}} (\mathring{x}_{\widehat{\tau}_\rho} - h_{\widehat{\tau}_\rho}) \;> 0 \;.$$

Define $\overline{\alpha}_1, \ldots, \overline{\alpha}_r, \geq 0$ by

$$(5.57) \qquad \overline{\alpha}_\rho \;:=\; (\mathring{x}_{\widehat{\tau}_\rho} - \overline{\alpha} h_{\widehat{\tau}_\rho})^+ \geq (\mathring{x}_{\widehat{\tau}_\rho} - \overline{\alpha} h_{\widehat{\tau}_\rho})$$

then

$$\sum_{\rho \in \mathbf{R}} \overline{\alpha}_\rho = 1-\overline{\alpha}\;, \qquad \overline{\alpha} < \widehat{\alpha}\;.$$

Now, the set of coefficients $\overline{\alpha}_1, \ldots, \overline{\alpha}_r, \overline{\alpha}$ can replace the initial set $\widehat{\alpha}_1, \ldots, \widehat{\alpha}_r, \widehat{\alpha}$ in a way that (5.45) is satisfied and we may proceed with ourt proof as in the $2^{nd}STEP$.

We can then proceed as described in the above proof beginning in the $2^{nd}STEP$.

q.e.d.

Remark 5.5. If $\mathring{\boldsymbol{x}}$ satisfies

$$\mathring{x}_{\tau_1} + \mathring{x}_{\tau_2} +, \ldots, + \mathring{x}_{\tau_{r-1}} + \mathring{x}_{\tau_r} \;\geq\; 1\;,$$

for *all* undercutting sequences $\boldsymbol{\tau} \;=\; (\tau_1, \tau_2, \ldots, \tau_{r-1}, \tau_r)$, then $\mathring{\boldsymbol{x}}$ equals some $\boldsymbol{e}^{\mathsf{T}^\rho}$. This follows from Lemma 4.8 and Theorem 4.9 of [1]. Therefore, if $\boldsymbol{\vartheta}$ is an imputation such that the minima vector $\boldsymbol{m}$ satisfies

$$\mathring{m}_{\tau_1} + \mathring{m}_{\tau_2} +, \ldots, + \mathring{m}_{\tau_{r-1}} + \mathring{m}_{\tau_r} \;\geq\; 1\;,$$

for *all* sequences $\boldsymbol{\tau} \;=\; (\tau_1, \tau_2, \ldots, \tau_{r-1}, \tau_r)$, then $\boldsymbol{m} \;=\; \boldsymbol{e}^{\mathsf{T}^\rho}$ for some $\rho \in \mathbf{R}$. Hence the minima vector is a pre–imputation from which it follows at once that $\boldsymbol{\vartheta} = \boldsymbol{\vartheta}^{\boldsymbol{m}} = \boldsymbol{\vartheta}^{\boldsymbol{e}^{\mathsf{T}^\rho}} = \boldsymbol{\lambda}^\rho$.

◦ ∼∼∼∼∼∼ ◦

Theorem 5.6. *Let $\boldsymbol{v}$ be a uniform game. Let $\mathring{\overline{\boldsymbol{x}}} \in \boldsymbol{H}^0$, $\mathring{\overline{\boldsymbol{\vartheta}}} \;:=\; \boldsymbol{\vartheta}^{\mathring{\overline{\boldsymbol{x}}}}$, and let*

$$(5.58) \qquad \mathring{\boldsymbol{H}} \;=\; \boldsymbol{ConvH}\left\{\mathring{\overline{\boldsymbol{x}}}, \boldsymbol{e}^{\mathsf{T}^\rho} \quad (\rho \in \mathbf{R})\right\} \subseteq \boldsymbol{H}^0.$$

Then

$$(5.59) \qquad \mathring{\mathcal{H}} \;=\; \boldsymbol{ConvH}\left\{\mathring{\overline{\boldsymbol{\vartheta}}}, \boldsymbol{\lambda}^\rho (\rho \in \mathbf{R})\right\} \;=\; \left\{\boldsymbol{\vartheta}^{\boldsymbol{x}} \,\middle|\, \boldsymbol{x} \in \mathring{\boldsymbol{H}}\right\}$$

is externally stable, hence a vNM–Stable Set.

Proof: Extern stability follows from Theorems 5.3, 5.4 and the above Remark 5.5. Intern stability has been verified by Theorem 5.2 ,

q.e.d.

Remark 5.7. The existence theorem provides a generalization of our previous results studied in Part I,II,III. We exhibit a set of pre–imputations $\boldsymbol{H}^0$ outside the core every element of which, together with the core establishes a vNM–Stable Set. This set is based on the vector (sub pre–imputation) $\bar{\boldsymbol{x}}$ which is obtained by a truncation of the density of $\boldsymbol{\lambda}^0$ and a suitable adjustment.

The elements of $\boldsymbol{H}^0$ are then obtained by suitably adding mass on coordinates $\tau \in \hat{\mathbf{T}}$ such that an imputation results. One should compare this to the Characterization Theorem in [4] and [5].

The resemblance is striking. The above existence theorem again points out an imputation $\mathring{\bar{\boldsymbol{\vartheta}}}$ that is absolutely continuous w.r.t. $\boldsymbol{\lambda}^0$ and has a density bounded by 1; exactly as in the previous Characterization Theorem. Other than previously however, one cannot choose all densities with these requirements but has to observe further restrictions.

At this stage we do not have a charcterization Theorem. Also, our result is limited to a piecewise constant density $\dot{\boldsymbol{\lambda}}^0$. These questions will have to be delt with in due time.

◦ ∼∼∼∼∼∼∼ ◦

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