On Irreversible Investment

Frank Riedel[1] and Xia Su[2]

[1]Institute of Mathematical Economics
Bielefeld University [2]Bonn Graduate School of Economics

First Version: June 2006, This Version: October 2007

Irreversibility

Many investment decisions involve

- Sunk Costs (investment is irreversible)
  - firm specific capital
  - lemons problem
  - workers: dismissal protection
  - advertisement, marketing etc.
- the option to wait for better times
  - demand or macroeconomic shocks
  - opportunity to gather information

Irreversibility

Many investment decisions involve

- Sunk Costs (investment is irreversible)
  - firm specific capital
  - lemons problem
  - workers: dismissal protection
  - advertisement, marketing etc.
- the option to wait for better times
  - demand or macroeconomic shocks
  - opportunity to gather information

Irreversibility

Many investment decisions involve

- Sunk Costs (investment is irreversible)
  - firm specific capital
  - lemons problem
  - workers: dismissal protection
  - advertisement, marketing etc.
- the option to wait for better times
  - demand or macroeconomic shocks
  - opportunity to gather information

Irreversibility

Many investment decisions involve

- Sunk Costs (investment is irreversible)
  - firm specific capital
  - lemons problem
  - workers: dismissal protection
  - advertisement, marketing etc.
- the option to wait for better times
  - demand or macroeconomic shocks
  - opportunity to gather information

Dynamic Capacity Choice

- We consider a firm that chooses a dynamic capacity expansion plan in a stochastic environment
- investment into capacity is irreversible
- capacity depreciates at some rate $\delta \geq 0$
- period profits are affected by an exogenous stochastic factor X
- firm maximizes net present value of cash flows at interest rate $r > 0$

Dynamic Capacity Choice

- We consider a firm that chooses a dynamic capacity expansion plan in a stochastic environment
- investment into capacity is irreversible
- capacity depreciates at some rate $\delta \geq 0$
- period profits are affected by an exogenous stochastic factor X
- firm maximizes net present value of cash flows at interest rate $r > 0$

Dynamic Capacity Choice

- We consider a firm that chooses a dynamic capacity expansion plan in a stochastic environment
- investment into capacity is irreversible
- capacity depreciates at some rate $\delta \geq 0$
- period profits are affected by an exogenous stochastic factor X
- firm maximizes net present value of cash flows at interest rate $r > 0$

Dynamic Capacity Choice

- We consider a firm that chooses a dynamic capacity expansion plan in a stochastic environment
- investment into capacity is irreversible
- capacity depreciates at some rate $\delta \geq 0$
- period profits are affected by an exogenous stochastic factor X
- firm maximizes net present value of cash flows at interest rate $r > 0$

Dynamic Capacity Choice

- We consider a firm that chooses a dynamic capacity expansion plan in a stochastic environment
- investment into capacity is irreversible
- capacity depreciates at some rate $\delta \geq 0$
- period profits are affected by an exogenous stochastic factor X
- firm maximizes net present value of cash flows at interest rate $r > 0$

Literature

- Arrow 1968, Deterministic Model, π general
- Pindyck 1988, Bertola 1988, Dixit 92, X geometric Brownian motion, $\pi(x, c) = x^{\alpha} c^{\beta}$, also Abel/Eberly 97
- Kobila, 1993, X geometric Brownian motion, π general
- Boyarchenko, 2004, X exponential Lévy process, π Cobb–Douglas, also Oksendal 2000
- Guo, Miao, Morellec, 2004, X geometric Brownian motion with regime shifts in drift and volatility, π Cobb–Douglas
- for the relation to durable goods and consumption with memory: Bank, R. 2001, Bank, El Karoui 2004

Literature

- Arrow 1968, Deterministic Model, π general
- Pindyck 1988, Bertola 1988, Dixit 92, X geometric Brownian motion, $\pi(x,c) = x^{\alpha}c^{\beta}$, also Abel/Eberly 97
- Kobila, 1993, X geometric Brownian motion, π general
- Boyarchenko, 2004, X exponential Lévy process, π Cobb–Douglas, also Oksendal 2000
- Guo, Miao, Morellec, 2004, X geometric Brownian motion with regime shifts in drift and volatility, π Cobb–Douglas
- for the relation to durable goods and consumption with memory: Bank, R. 2001, Bank, El Karoui 2004

Literature

- Arrow 1968, Deterministic Model, π general
- Pindyck 1988, Bertola 1988, Dixit 92, X geometric Brownian motion, $\pi(x, c) = x^{\alpha} c^{\beta}$, also Abel/Eberly 97
- Kobila, 1993, X geometric Brownian motion, π general
- Boyarchenko, 2004, X exponential Lévy process, π Cobb–Douglas, also Oksendal 2000
- Guo, Miao, Morellec, 2004, X geometric Brownian motion with regime shifts in drift and volatility, π Cobb–Douglas
- for the relation to durable goods and consumption with memory: Bank, R. 2001, Bank, El Karoui 2004

Literature

- Arrow 1968, Deterministic Model, π general
- Pindyck 1988, Bertola 1988, Dixit 92, X geometric Brownian motion, $\pi(x, c) = x^{\alpha} c^{\beta}$, also Abel/Eberly 97
- Kobila, 1993, X geometric Brownian motion, π general
- Boyarchenko, 2004, X exponential Lévy process, π Cobb–Douglas, also Oksendal 2000
- Guo, Miao, Morellec, 2004, X geometric Brownian motion with regime shifts in drift and volatility, π Cobb–Douglas
- for the relation to durable goods and consumption with memory: Bank, R. 2001, Bank, El Karoui 2004

Literature

- Arrow 1968, Deterministic Model, π general
- Pindyck 1988, Bertola 1988, Dixit 92, X geometric Brownian motion, $\pi(x, c) = x^\alpha c^\beta$, also Abel/Eberly 97
- Kobila, 1993, X geometric Brownian motion, π general
- Boyarchenko, 2004, X exponential Lévy process, π Cobb–Douglas, also Oksendal 2000
- Guo, Miao, Morellec, 2004, X geometric Brownian motion with regime shifts in drift and volatility, π Cobb–Douglas
- for the relation to durable goods and consumption with memory: Bank, R. 2001, Bank, El Karoui 2004

Literature

- Arrow 1968, Deterministic Model, π general
- Pindyck 1988, Bertola 1988, Dixit 92, X geometric Brownian motion, $\pi(x, c) = x^{\alpha} c^{\beta}$, also Abel/Eberly 97
- Kobila, 1993, X geometric Brownian motion, π general
- Boyarchenko, 2004, X exponential Lévy process, π Cobb–Douglas, also Oksendal 2000
- Guo, Miao, Morellec, 2004, X geometric Brownian motion with regime shifts in drift and volatility, π Cobb–Douglas
- for the relation to durable goods and consumption with memory: Bank, R. 2001, Bank, El Karoui 2004

Results

- Existence and Uniqueness
- new solution method
  - first–order approach (martingale approach)
  - stochastic backward equation
  - complete solution for all semimartingale models
- qualitative theory

Results

- Existence and Uniqueness
- new solution method
  - first–order approach (martingale approach)
  - stochastic backward equation
  - complete solution for all semimartingale models
- qualitative theory

Results

- Existence and Uniqueness
- new solution method
  - first–order approach (martingale approach)
  - stochastic backward equation
  - complete solution for all semimartingale models
- qualitative theory

Results

- Existence and Uniqueness
- new solution method
  - first–order approach (martingale approach)
  - stochastic backward equation
  - complete solution for all semimartingale models
- qualitative theory
  - underinvestment in general
  - free intervals: marginal profit flow equals user cost of capital
  - blocked intervals: marginal profit flow equals user cost of capital on average
  - comparative statics

Results

- Existence and Uniqueness
- new solution method
  - first–order approach (martingale approach)
  - stochastic backward equation
  - complete solution for all semimartingale models
- qualitative theory
  - underinvestment in general
  - free intervals: marginal profit flow equals user cost of capital
  - blocked intervals: marginal profit flow equals user cost of capital on average
  - comparative statics

Results

- Existence and Uniqueness
- new solution method
  - first–order approach (martingale approach)
  - stochastic backward equation
  - complete solution for all semimartingale models
- qualitative theory
  - underinvestment in general
  - free intervals: marginal profit flow equals user cost of capital
  - blocked intervals: marginal profit flow equals user cost of capital *on average*
  - comparative statics

Results

- Existence and Uniqueness
- new solution method
  - first–order approach (martingale approach)
  - stochastic backward equation
  - complete solution for all semimartingale models
- qualitative theory
  - underinvestment in general
  - free intervals: marginal profit flow equals user cost of capital
  - blocked intervals: marginal profit flow equals user cost of capital *on average*
  - comparative statics

Results

- Existence and Uniqueness
- new solution method
  - first–order approach (martingale approach)
  - stochastic backward equation
  - complete solution for all semimartingale models
- qualitative theory
  - underinvestment in general
  - free intervals: marginal profit flow equals user cost of capital
  - blocked intervals: marginal profit flow equals user cost of capital *on average*
  - comparative statics
    - supermodular operating profit: investment increasing in X
    - investment decreasing in user cost of capital $r + \delta$

Results

- Existence and Uniqueness
- new solution method
    - first–order approach (martingale approach)
    - stochastic backward equation
    - complete solution for all semimartingale models
- qualitative theory
    - underinvestment in general
    - free intervals: marginal profit flow equals user cost of capital
    - blocked intervals: marginal profit flow equals user cost of capital *on average*
    - comparative statics
        - supermodular operating profit: investment increasing in X
        - investment decreasing in user cost of capital $r + \delta$

Results

- Existence and Uniqueness
- new solution method
    - first–order approach (martingale approach)
    - stochastic backward equation
    - complete solution for all semimartingale models
- qualitative theory
    - underinvestment in general
    - free intervals: marginal profit flow equals user cost of capital
    - blocked intervals: marginal profit flow equals user cost of capital *on average*
    - comparative statics
        - supermodular operating profit: investment increasing in X
        - investment decreasing in user cost of capital $r+\delta$

Results

- Existence and Uniqueness
- new solution method
  - first–order approach (martingale approach)
  - stochastic backward equation
  - complete solution for all semimartingale models
- qualitative theory
  - underinvestment in general
  - free intervals: marginal profit flow equals user cost of capital
  - blocked intervals: marginal profit flow equals user cost of capital *on average*
  - comparative statics
    - supermodular operating profit: investment increasing in X
    - investment decreasing in user cost of capital $r + \delta$

Results

- Existence and Uniqueness
- new solution method
  - first–order approach (martingale approach)
  - stochastic backward equation
  - complete solution for all semimartingale models
- qualitative theory
  - underinvestment in general
  - free intervals: marginal profit flow equals user cost of capital
  - blocked intervals: marginal profit flow equals user cost of capital *on average*
  - comparative statics
    - supermodular operating profit: investment increasing in X
    - investment decreasing in user cost of capital $r + \delta$

The Model

- $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t))$ filtered probability space
- firm chooses $I(t)$, cumulative investment up to time t, an adapted process
- irreversibility: I nondecreasing
- I leads to a capacity $C = C^I$ given by
  $C(0-) = 0, dC(t) = dI(t) - \delta C(t)dt$
  where δ is depreciation rate
- profit flow in t: $\pi(X(t), C(t))$
  where X is an exogenous stochastic process (semimartingale, Lévy process, Brownian motion), π strictly concave, $\uparrow$, $\mathscr{C}^1$ in capacity,

The Model

- $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t))$ filtered probability space
- firm chooses $I(t)$, cumulative investment up to time t, an adapted process
- irreversibility: I nondecreasing
- I leads to a capacity $C = C^I$ given by
  $C(0-) = 0, dC(t) = dI(t) - \delta C(t)dt$
  where δ is depreciation rate
- profit flow in t: $\pi(X(t), C(t))$
  where X is an exogenous stochastic process (semimartingale, Lévy process, Brownian motion), π strictly concave, $\uparrow$, $\mathscr{C}^1$ in capacity,

The Model

- $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t))$ filtered probability space
- firm chooses $I(t)$, cumulative investment up to time t, an adapted process
- irreversibility: I nondecreasing
- I leads to a capacity $C = C^I$ given by
  $C(0-) = 0, dC(t) = dI(t) - \delta C(t)dt$
  where δ is depreciation rate
- profit flow in t: $\pi(X(t), C(t))$
  where X is an exogenous stochastic process (semimartingale, Lévy process, Brownian motion), π strictly concave, $\uparrow$, $\mathscr{C}^1$ in capacity,

The Model

- $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t))$ filtered probability space
- firm chooses $I(t)$, cumulative investment up to time t, an adapted process
- irreversibility: I nondecreasing
- I leads to a capacity $C = C^I$ given by
  $C(0-) = 0, dC(t) = dI(t) - \delta C(t)dt$
  where δ is depreciation rate
- profit flow in t: $\pi(X(t), C(t))$
  where X is an exogenous stochastic process (semimartingale, Lévy process, Brownian motion), π strictly concave, $\uparrow$, $\mathscr{C}^1$ in capacity,

The Model

- $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t))$ filtered probability space
- firm chooses $I(t)$, cumulative investment up to time t, an adapted process
- irreversibility: I nondecreasing
- I leads to a capacity $C = C^I$ given by
  $C(0-) = 0, dC(t) = dI(t) - \delta C(t)dt$
  where δ is depreciation rate
- profit flow in t: $\pi(X(t), C(t))$
  where X is an exogenous stochastic process (semimartingale, Lévy process, Brownian motion), π strictly concave, $\uparrow$, $\mathscr{C}^1$ in capacity,

The Control Problem

- total profit is

$$\Pi(I) = \mathbb{E} \int_0^T e^{-rt} \left(\pi\left(X(t), C^I(t)\right) dt - dI(t)\right)$$

for an interest rate $r > 0$

- Problem: choose the optimal investment plan I!
- singular stochastic control problem

The Control Problem

- total profit is

$$\Pi(I) = \mathbb{E} \int_0^T e^{-rt} \left(\pi \left(X(t), C^I(t) \right) dt - dI(t) \right)$$

for an interest rate $r > 0$

- Problem: choose the optimal investment plan I!
- singular stochastic control problem

The Control Problem

- total profit is

$$\Pi(I) = \mathbb{E} \int_0^T e^{-rt} \left(\pi \left(X(t), C^I(t) \right) dt - dI(t) \right)$$

  for an interest rate $r > 0$
- Problem: choose the optimal investment plan I!
- singular stochastic control problem

Benchmark: The Reversible Case

Under perfect reversibility of investment, the firm equates at every point of time marginal profit and user cost of capital:

$$\pi_c(X(t), C(t)) = r + \delta$$

- Capacity is stock, $r + \delta$ rental cost per period
- "myopic" behavior because of reversibility

Benchmark: The Reversible Case

Under perfect reversibility of investment, the firm equates at every point of time marginal profit and user cost of capital:

$$\pi_c\left(X(t), C(t)\right) = r + \delta$$

- Capacity is stock, $r + \delta$ rental cost per period
- "myopic" behavior because of reversibility

Existence: Sketch

- maximization of a concave functional
- usual idea: continuity and appropriate kind of compactness (subsequence) principle
- continuity through dominated convergence (integrability constraint)
- subsequence principle: Komlos' Theorem

Existence: Sketch

- maximization of a concave functional
- usual idea: continuity and appropriate kind of compactness (subsequence) principle
- continuity through dominated convergence (integrability constraint)
- subsequence principle: Komlos' Theorem

Existence: Sketch

- maximization of a concave functional
- usual idea: continuity and appropriate kind of compactness (subsequence) principle
- continuity through dominated convergence (integrability constraint)
- subsequence principle: Komlos' Theorem

Existence: Sketch

- maximization of a concave functional
- usual idea: continuity and appropriate kind of compactness (subsequence) principle
- continuity through dominated convergence (integrability constraint)
- subsequence principle: Komlos' Theorem

Existence: Details

- Let $\pi^*(t) = \max_i \pi(X_t, i) - ri$, and $i^*(t)$ be the maximizer (optimal reversible policy).
- Assumption: The reversible problem is well posed: $\mathbb{E} \int_0^T e^{-rt} (\pi^*(t))\, dt < \infty$
- Assumption: $\mathbb{E} \sup_{0 \leq s \leq t} i^*(s) < \infty$ for $t < T$.
- Then the problem has a finite value. Moreover, one can restrict to plans I with $I(t) \leq \sup_{0 \leq s \leq t} i^*(s) \quad a.s.$

Existence: Details

- Let $\pi^*(t) = \max_i \pi(X_t, i) - ri$, and $i^*(t)$ be the maximizer (optimal reversible policy).
- Assumption: The reversible problem is well posed: $\mathbb{E} \int_0^T e^{-rt} (\pi^*(t))\, dt < \infty$
- Assumption: $\mathbb{E} \sup_{0 \le s \le t} i^*(s) < \infty$ for $t < T$.
- Then the problem has a finite value. Moreover, one can restrict to plans I with $I(t) \le \sup_{0 \le s \le t} i^*(s) \quad a.s.$

Existence: Details

- Let $\pi^*(t) = \max_i \pi(X_t, i) - ri$, and $i^*(t)$ be the maximizer (optimal reversible policy).
- Assumption: The reversible problem is well posed: $\mathbb{E} \int_0^T e^{-rt} (\pi^*(t))\, dt < \infty$
- Assumption: $\mathbb{E} \sup_{0 \le s \le t} i^*(s) < \infty$ for $t < T$.
- Then the problem has a finite value. Moreover, one can restrict to plans I with $I(t) \le \sup_{0 \le s \le t} i^*(s) \quad a.s.$

Existence: Details

- Let $\pi^*(t) = \max_i \pi(X_t, i) - ri$, and $i^*(t)$ be the maximizer (optimal reversible policy).
- Assumption: The reversible problem is well posed: $\mathbb{E} \int_0^T e^{-rt} (\pi^*(t))\, dt < \infty$
- Assumption: $\mathbb{E} \sup_{0 \leq s \leq t} i^*(s) < \infty$ for $t < T$.
- Then the problem has a finite value. Moreover, one can restrict to plans I with $I(t) \leq \sup_{0 \leq s \leq t} i^*(s) \quad a.s.$

Existence: Details II

Now one can apply Komlos Theorem:

Theorem (Komlos, Kabanov)

Let (I^n) be a sequence of optional random measures with $\sup_n \mathbb{E} I^n(T) < \infty$. Then there exists a subsequence (J^n) and an optional random measure J^* such that

$$\frac{1}{n}\sum_{k=1}^{n} J^n(t) \to J^*(t) \qquad a.s.$$

for every point of continuity t of J^*.

J^* is the optimal solution for our problem!

First–Order Condition

Under irreversibility, marginal profit from investment at time t

$$\nabla\Pi(I)(t) = \mathbb{E}\left[\int_t^T e^{-rs}\pi_C\left(X(s), C(s)\right)e^{-\delta(s-t)}\,ds \,\middle|\, \mathscr{F}_t\right] - e^{-rt}$$

Optimality: $\nabla\Pi(I)(t) \leq 0 \quad a.s.$ and equality, whenever $dI(t) > 0$!

Problem: the FOC does not bind frequently!

First–Order Condition

Under irreversibility, marginal profit from investment at time t

$$\nabla\Pi(I)(t) = \mathbb{E}\left[\int_t^T e^{-rs}\pi_C\left(X(s), C(s)\right) e^{-\delta(s-t)}\, ds \,\middle|\, \mathscr{F}_t \right] - e^{-rt}$$

Optimality: $\nabla\Pi(I)(t) \le 0$ $\quad a.s.$ and equality, whenever $dI(t) > 0$!

Problem: the FOC does not bind frequently!

First–Order Condition

Under irreversibility, marginal profit from investment at time t

$$\nabla\Pi(I)(t) = \mathbb{E}\left[\int_t^T e^{-rs}\pi_C\left(X(s), C(s)\right) e^{-\delta(s-t)}\, ds \,\middle|\, \mathscr{F}_t\right] - e^{-rt}$$

Optimality: $\nabla\Pi(I)(t) \leq 0 \quad a.s.$ and equality, whenever $dI(t) > 0$!

Problem: the FOC does not bind frequently!

First–Order Condition

Under irreversibility, marginal profit from investment at time t

$$\nabla\Pi(I)(t) = \mathbb{E}\left[\int_t^T e^{-rs}\pi_C\left(X(s), C(s)\right)e^{-\delta(s-t)}\,ds \,\middle|\, \mathscr{F}_t\right] - e^{-rt}$$

Optimality: $\nabla\Pi(I)(t) \leq 0 \quad a.s.$ and equality, whenever $dI(t) > 0$!
Problem: the FOC does not bind frequently!

Solution

Ansatz: there is a *base capacity level* $L(t)$

the optimal strategy is to keep capacity C just above L

Aim: characterize base capacity L

- via optimal stopping problems
- via a backward equation

Result: the backward equation can be solved

Solution

Ansatz: there is a *base capacity level* $L(t)$
the optimal strategy is to keep capacity C just above L
Aim: characterize base capacity L

- via optimal stopping problems
- via a backward equation

Result: the backward equation can be solved

Solution

Ansatz: there is a *base capacity level* $L(t)$

the optimal strategy is to keep capacity C just above L

Aim: characterize base capacity L

- via optimal stopping problems
- via a backward equation

Result: the backward equation can be solved

- explicitly for Cobb–Douglas, infinite horizon, and

  numerically for all semimartingales X and all profit functions

Solution

Ansatz: there is a *base capacity level* $L(t)$
the optimal strategy is to keep capacity C just above L
Aim: characterize base capacity L

- via optimal stopping problems
- via a backward equation

Result: the backward equation can be solved

- explicitly, for Cobb–Douglas, infinite horizon, and
- numerically, for all semimartingales X and all profit functions

Solution

Ansatz: there is a *base capacity level* $L(t)$
the optimal strategy is to keep capacity C just above L
Aim: characterize base capacity L

- via optimal stopping problems
- via a backward equation

Result: the backward equation can be solved

- explicitly, for Cobb–Douglas, infinite horizon, and
  - exponential Lévy processes
  - homogeneous diffusions
  - regime shifts
- numerically, for all semimartingales X and all profit functions

Solution

Ansatz: there is a *base capacity level* $L(t)$

the optimal strategy is to keep capacity C just above L

Aim: characterize base capacity L

- via optimal stopping problems
- via a backward equation

Result: the backward equation can be solved

- explicitly, for Cobb–Douglas, infinite horizon, and
    - exponential Lévy processes
    - homogeneous diffusions
    - regime shifts
- numerically, for all semimartingales X and all profit functions

Solution

Ansatz: there is a *base capacity level* $L(t)$
the optimal strategy is to keep capacity C just above L
Aim: characterize base capacity L

- via optimal stopping problems
- via a backward equation

Result: the backward equation can be solved

- explicitly, for Cobb–Douglas, infinite horizon, and
  - exponential Lévy processes
  - homogeneous diffusions
  - regime shifts
- numerically, for all semimartingales X and all profit functions

Solution

Ansatz: there is a *base capacity level* $L(t)$
the optimal strategy is to keep capacity C just above L
Aim: characterize base capacity L

- via optimal stopping problems
- via a backward equation

Result: the backward equation can be solved

- explicitly, for Cobb–Douglas, infinite horizon, and
  - exponential Lévy processes
  - homogeneous diffusions
  - regime shifts
- numerically, for all semimartingales X and all profit functions

Solution

Ansatz: there is a *base capacity level* $L(t)$
the optimal strategy is to keep capacity C just above L
Aim: characterize base capacity L

- via optimal stopping problems
- via a backward equation

Result: the backward equation can be solved

- explicitly, for Cobb–Douglas, infinite horizon, and
    - exponential Lévy processes
    - homogeneous diffusions
    - regime shifts
- numerically, for all semimartingales X and all profit functions

Solution

Ansatz: there is a *base capacity level* $L(t)$

the optimal strategy is to keep capacity C just above L

Aim: characterize base capacity L

- via optimal stopping problems
- via a backward equation

Result: the backward equation can be solved

- explicitly, for Cobb–Douglas, infinite horizon, and
  - exponential Lévy processes
  - homogeneous diffusions
  - regime shifts
- numerically, for all semimartingales X and all profit functions

Base Capacity Level: Heuristics

- Fix a point in time t and assume $\delta = 0$
- Assume that it is optimal to invest at t, to wait until τ, and to invest again
- Call the corresponding level L_t^τ
- From the (two!) FOC, we have

$$\mathbb{E}\left[\int_t^\tau e^{-rs}\pi_c(X(s), L_t^\tau)\,ds \,\middle|\, \mathscr{F}_t\right] = \mathbb{E}\left[e^{-rt} - e^{-r\tau} \,\middle|\, \mathscr{F}_t\right]$$

- or

$$\mathbb{E}\left[\int_t^\tau e^{-rs}\left[\pi_c(X(s), L_t^\tau) - r\right] ds \,\middle|\, \mathscr{F}_t\right] = 0$$

- stochastic equation in L_t^τ $\to$ solvable

Base Capacity Level: Heuristics

- Fix a point in time t and assume $\delta = 0$
- Assume that it is optimal to invest at t, to wait until τ, and to invest again
- Call the corresponding level L_t^τ
- From the (two!) FOC, we have

$$\mathbb{E}\left[\int_t^\tau e^{-rs}\pi_c(X(s), L_t^\tau)\,ds \,\middle|\, \mathscr{F}_t\right] = \mathbb{E}\left[e^{-rt} - e^{-r\tau} \,\middle|\, \mathscr{F}_t\right]$$

- or

$$\mathbb{E}\left[\int_t^\tau e^{-rs}\left[\pi_c(X(s), L_t^\tau) - r\right] ds \,\middle|\, \mathscr{F}_t\right] = 0$$

- stochastic equation in L_t^τ → solvable

Base Capacity Level: Heuristics

- Fix a point in time t and assume $\delta = 0$
- Assume that it is optimal to invest at t, to wait until τ, and to invest again
- Call the corresponding level L_t^τ
- From the (two!) FOC, we have

$$\mathbb{E}\left[\int_t^\tau e^{-rs}\pi_c\left(X(s), L_t^\tau\right) ds \,\middle|\, \mathscr{F}_t\right] = \mathbb{E}\left[e^{-rt} - e^{-r\tau} \,\middle|\, \mathscr{F}_t\right]$$

- or

$$\mathbb{E}\left[\int_t^\tau e^{-rs}\left[\pi_c\left(X(s), L_t^\tau\right) - r\right] ds \,\middle|\, \mathscr{F}_t\right] = 0$$

- stochastic equation in L_t^τ $\rightarrow$ solvable

Base Capacity Level: Heuristics

- Fix a point in time t and assume $\delta = 0$
- Assume that it is optimal to invest at t, to wait until τ, and to invest again
- Call the corresponding level L_t^τ
- From the (two!) FOC, we have

$$\mathbb{E}\left[\int_t^\tau e^{-rs}\pi_c\left(X(s), L_t^\tau\right) ds \,\middle|\, \mathscr{F}_t\right] = \mathbb{E}\left[e^{-rt} - e^{-r\tau} \,\middle|\, \mathscr{F}_t\right]$$

- or

$$\mathbb{E}\left[\int_t^\tau e^{-rs}\left[\pi_c\left(X(s), L_t^\tau\right) - r\right] ds \,\middle|\, \mathscr{F}_t\right] = 0$$

- stochastic equation in L_t^τ $\rightarrow$ solvable

Base Capacity Level: Heuristics

- Fix a point in time t and assume $\delta = 0$
- Assume that it is optimal to invest at t, to wait until τ, and to invest again
- Call the corresponding level L_t^τ
- From the (two!) FOC, we have

$$\mathbb{E}\left[\int_t^\tau e^{-rs}\pi_c\left(X(s), L_t^\tau\right) ds \,\middle|\, \mathscr{F}_t\right] = \mathbb{E}\left[e^{-rt} - e^{-r\tau} \,\middle|\, \mathscr{F}_t\right]$$

- or

$$\mathbb{E}\left[\int_t^\tau e^{-rs}\left[\pi_c\left(X(s), L_t^\tau\right) - r\right] ds \,\middle|\, \mathscr{F}_t\right] = 0$$

- stochastic equation in $L_t^\tau \longrightarrow$ solvable

Base Capacity Level: Heuristics

- Fix a point in time t and assume $\delta = 0$
- Assume that it is optimal to invest at t, to wait until τ, and to invest again
- Call the corresponding level L_t^τ
- From the (two!) FOC, we have

$$\mathbb{E}\left[\int_t^\tau e^{-rs}\pi_c\left(X(s), L_t^\tau\right) ds \,\middle|\, \mathscr{F}_t\right] = \mathbb{E}\left[e^{-rt} - e^{-r\tau} \,\middle|\, \mathscr{F}_t\right]$$

- or

$$\mathbb{E}\left[\int_t^\tau e^{-rs}\left[\pi_c\left(X(s), L_t^\tau\right) - r\right] ds \,\middle|\, \mathscr{F}_t\right] = 0$$

- stochastic equation in L_t^τ $\rightarrow$ solvable

Characterizing L via Optimal Stopping

- claim: base capacity level is

$$L(t) = \operatorname{essinf}_{\tau > t} L_t^{\tau}$$

- optimal stopping problem
- find the smallest level that makes FOC binding
- "cautious" approach

Characterizing L via Optimal Stopping

- claim: base capacity level is

$$L(t) = \text{essinf}_{\tau > t} L_t^\tau$$

- optimal stopping problem
- find the smallest level that makes FOC binding
- "cautious" approach

Characterizing L via Optimal Stopping

- claim: base capacity level is

$$L(t) = \operatorname{essinf}_{\tau > t} L_t^\tau$$

- optimal stopping problem
- find the smallest level that makes FOC binding
- "cautious" approach

Characterizing L via Optimal Stopping

- claim: base capacity level is

$$L(t) = \operatorname{essinf}_{\tau > t} L_t^{\tau}$$

- optimal stopping problem
- find the smallest level that makes FOC binding
- "cautious" approach

Characterizing L via Backward Equation

- Keep capacity just above base capacity L:

$$C(t) = \sup_{s \le t} L(s)$$

- From the FOC

$$\mathbb{E}\left[\int_t^T e^{-rs}\pi_C\left(X(s), \sup_{0\le u\le s} L(u)\right) ds \,\middle|\, \mathscr{F}_t\right] \le e^{-rt}$$

- Stochastic Backward Equation for the Base Capacity:

$$\mathbb{E}\left[\int_t^T e^{-rs}\pi_C\left(X(s), \sup_{t\le u\le s} L(u)\right) ds \,\middle|\, \mathscr{F}_t\right] = e^{-rt}$$

Characterizing L via Backward Equation

- Keep capacity just above base capacity L:

$$C(t) = \sup_{s \le t} L(s)$$

- From the FOC

$$\mathbb{E}\left[\int_t^T e^{-rs}\pi_C\left(X(s), \sup_{0 \le u \le s} L(u)\right) ds \,\middle|\, \mathscr{F}_t\right] \le e^{-rt}$$

- Stochastic Backward Equation for the Base Capacity:

$$\mathbb{E}\left[\int_t^T e^{-rs}\pi_C\left(X(s), \sup_{t \le u \le s} L(u)\right) ds \,\middle|\, \mathscr{F}_t\right] = e^{-rt}$$

Characterizing L via Backward Equation

- Keep capacity just above base capacity L:

$$C(t) = \sup_{s \le t} L(s)$$

- From the FOC

$$\mathbb{E}\left[\int_t^T e^{-rs}\pi_C\left(X(s), \sup_{0 \le u \le s} L(u)\right) ds \,\middle|\, \mathscr{F}_t\right] \le e^{-rt}$$

- Stochastic Backward Equation for the Base Capacity:

$$\mathbb{E}\left[\int_t^T e^{-rs}\pi_C\left(X(s), \sup_{t \le u \le s} L(u)\right) ds \,\middle|\, \mathscr{F}_t\right] = e^{-rt}$$

Main Theorem

It is enough to solve the backward equation!

Theorem

If L solves the stochastic backward equation

$$\mathbb{E}\left[\int_t^T e^{-rs}\pi_C\left(X(s), \sup_{t\le u\le s} L(u)\right) ds \,\middle|\, \mathscr{F}_t\right] = e^{-rt}$$

then the corresponding capacity

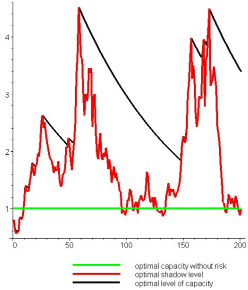
$$C(t) = \sup_{s\le t} L(s)$$

with investment plan $I(t) = C(t)$ is optimal.

Explicit Solutions

Lévy Processes

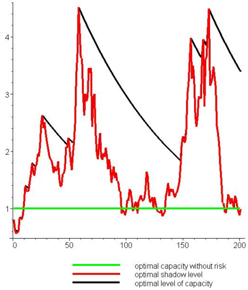
- Cobb–Douglas profit: $\pi(x, c) = xc^{1-\alpha}$
- $X(t) = \exp(\xi(t))$, ξ Markov, i.i.d. increments
- infinite horizon $T = \infty$
- here: X martingale
- "keep marginal profit just below user cost of capital times a markup factor"
- $\pi_c(X(t), C(t)) \leq (r + \delta)\frac{\alpha+\gamma}{\gamma}$ where γ depends on ξ



Explicit Solutions

Lévy Processes

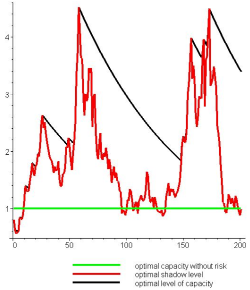
- Cobb–Douglas profit: $\pi(x, c) = xc^{1-\alpha}$
- $X(t) = \exp(\xi(t))$, ξ Markov, i.i.d. increments
- infinite horizon $T = \infty$
- here: X martingale
- "keep marginal profit just below user cost of capital times a markup factor"
- $\pi_c(X(t), C(t)) \leq (r + \delta)\frac{\alpha+\gamma}{\gamma}$ where γ depends on ξ



Explicit Solutions

Lévy Processes

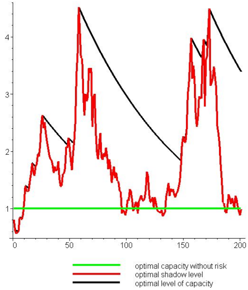
- Cobb–Douglas profit: $\pi(x, c) = xc^{1-\alpha}$
- $X(t) = \exp(\xi(t))$, ξ Markov, i.i.d. increments
- infinite horizon $T = \infty$
- here: X martingale
- "keep marginal profit just below user cost of capital times a markup factor"
- $\pi_c(X(t), C(t)) \leq (r + \delta)\frac{\alpha+\gamma}{\gamma}$ where γ depends on ξ



Explicit Solutions

Lévy Processes

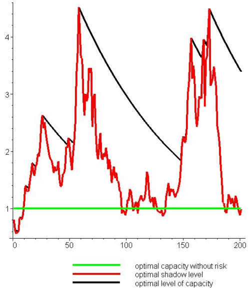
- Cobb–Douglas profit: $\pi(x, c) = xc^{1-\alpha}$
- $X(t) = \exp(\xi(t))$, ξ Markov, i.i.d. increments
- infinite horizon $T = \infty$
- here: X martingale
- "keep marginal profit just below user cost of capital times a markup factor"
- $\pi_c(X(t), C(t)) \leq (r + \delta) \frac{\alpha+\gamma}{\gamma}$ where γ depends on ξ



Explicit Solutions

Lévy Processes

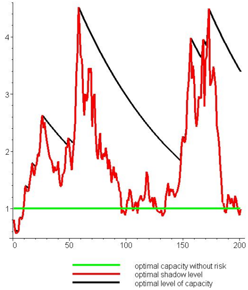
- Cobb–Douglas profit: $\pi(x, c) = xc^{1-\alpha}$
- $X(t) = \exp(\xi(t))$, ξ Markov, i.i.d. increments
- infinite horizon $T = \infty$
- here: X martingale
- "keep marginal profit just below user cost of capital times a markup factor"
- $\pi_c(X(t), C(t)) \leq (r + \delta)\frac{\alpha+\gamma}{\gamma}$ where γ depends on ξ



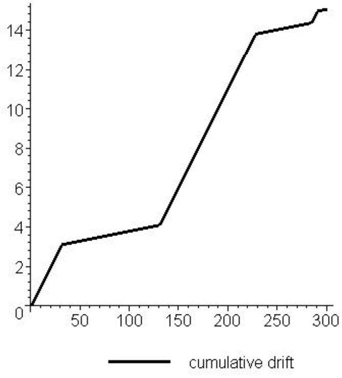
Explicit Solutions

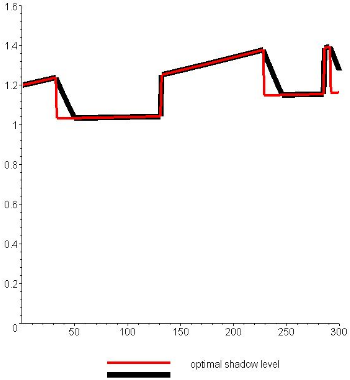
Lévy Processes

- Cobb–Douglas profit: $\pi(x, c) = xc^{1-\alpha}$
- $X(t) = \exp(\xi(t))$, ξ Markov, i.i.d. increments
- infinite horizon $T = \infty$
- here: X martingale
- "keep marginal profit just below user cost of capital times a markup factor"
- $\pi_c(X(t), C(t)) \leq (r + \delta)\frac{\alpha+\gamma}{\gamma}$ where γ depends on ξ



Markov Chain with Poisson Clock





Qualitative Theory

- lump sum investment at $t = 0$
- investment plan I consists of three components:
  - smooth: $dI^a(t) = i(t)dt$
  - jump part: $I^j(t)$ sum of jumps
  - singular: $I^\perp$, singular with respect to dt

    only when a diffusion is present

Qualitative Theory

- lump sum investment at $t = 0$
- investment plan I consists of three components:
  - smooth: $dI^a(t) = i(t)dt$
  - jump part: $I^j(t)$ sum of jumps
  - singular: $I^\perp$, singular with respect to dt
    only when a diffusion is present

Qualitative Theory

- lump sum investment at $t = 0$
- investment plan I consists of three components:
  - smooth: $dI^a(t) = i(t)dt$
  - jump part: $I^j(t)$ sum of jumps
  - singular: $I^\perp$, singular with respect to dt
    only when a diffusion is present

Qualitative Theory

- lump sum investment at $t = 0$
- investment plan I consists of three components:
  - smooth: $dI^a(t) = i(t)dt$
  - jump part: $I^j(t)$ sum of jumps
  - singular: $I^\perp$, singular with respect to dt

    only when a diffusion is present

Qualitative Theory, ctd.

Free Interval: only $dI^a(t) > 0$

Theorem

On a free interval, marginal profit equals user cost of capital:

$$\pi_c(X(t), C(t)) = r + \delta$$

Blocked Interval $[t, \tau]$: $dI(s) = 0$ for all $s \in [t, \tau]$

Theorem

On a blocked interval, marginal profit equals user cost of capital on average:

$$\mathbb{E}\left[\int_t^\tau e^{-(r+\delta)s}\left[\pi_c(X(s), C(t)) - (r+\delta)\right] ds \,\middle|\, \mathscr{F}_t\right] = 0$$

Qualitative Theory, ctd.

Free Interval: only $dI^a(t) > 0$

Theorem

On a free interval, marginal profit equals user cost of capital:

$$\pi_c(X(t), C(t)) = r + \delta$$

Blocked Interval $[t, \tau]$: $dI(s) = 0$ for all $s \in [t, \tau]$

Theorem

On a blocked interval, marginal profit equals user cost of capital on average:

$$\mathbb{E}\left[\int_t^\tau e^{-(r+\delta)s} \left[\pi_c(X(s), C(t)) - (r+\delta)\right] ds \,\middle|\, \mathscr{F}_t \right] = 0$$

Qualitative Theory, ctd.

Free Interval: only $dI^a(t) > 0$

Theorem

On a free interval, marginal profit equals user cost of capital:

$$\pi_c(X(t), C(t)) = r + \delta$$

Blocked Interval $[t, \tau]$: $dI(s) = 0$ for all $s \in [t, \tau]$

Theorem

On a blocked interval, marginal profit equals user cost of capital on average:

$$\mathbb{E}\left[\int_t^\tau e^{-(r+\delta)s}\left[\pi_c(X(s), C(t)) - (r+\delta)\right] ds \,\middle|\, \mathscr{F}_t\right] = 0$$

Qualitative Theory, ctd.

Free Interval: only $dI^a(t) > 0$

Theorem

On a free interval, marginal profit equals user cost of capital:

$$\pi_c(X(t), C(t)) = r + \delta$$

Blocked Interval $[t, \tau]$: $dI(s) = 0$ for all $s \in [t, \tau]$

Theorem

On a blocked interval, marginal profit equals user cost of capital on average:

$$\mathbb{E}\left[\int_t^\tau e^{-(r+\delta)s}\left[\pi_c(X(s), C(t)) - (r+\delta)\right] ds \,\middle|\, \mathscr{F}_t\right] = 0$$

Jumps and Surprises

Arrow: no jumps after time 0

Theorem

After time 0, jumps occur only at information surprises.

Information surprise (Hindy, Huang): unpredictable stopping time

Theorem

Suppose that the optimal investment plan has a jump at the stopping time τ. Then we have

$$\pi_c\left(X_\tau, C^l_\tau\right) \geq r + \delta\,.$$

One never jumps to excess capacity.

Jumps and Surprises

Arrow: no jumps after time 0

Theorem

After time 0, jumps occur only at information surprises.

Information surprise (Hindy, Huang): unpredictable stopping time

Theorem

Suppose that the optimal investment plan has a jump at the stopping time τ. Then we have

$$\pi_c\left(X_\tau, C^I_\tau\right) \geq r + \delta\,.$$

One never jumps to excess capacity.

Jumps and Surprises

Arrow: no jumps after time 0

Theorem

After time 0*, jumps occur only at information surprises.*

Information surprise (Hindy, Huang): unpredictable stopping time

Theorem

Suppose that the optimal investment plan has a jump at the stopping time τ*. Then we have*

$$\pi_c\left(X_\tau, C^I_\tau\right) \geq r + \delta\,.$$

One never jumps to excess capacity.

Jumps and Surprises

Arrow: no jumps after time 0

Theorem

After time 0, jumps occur only at information surprises.

Information surprise (Hindy, Huang): unpredictable stopping time

Theorem

Suppose that the optimal investment plan has a jump at the stopping time τ. Then we have

$$\pi_c\left(X_\tau, C^I_\tau\right) \geq r + \delta\,.$$

One never jumps to excess capacity.

Comparative Statics

Theorem

Let $X \geq Y$ a.s. Assume that operating profit function is supermodular: $\frac{\partial^2 \pi}{\partial x \partial c} \geq 0$. Then the optimal policy under X is greater or equal the optimal policy under Y.

Theorem

The optimal policy is nondecreasing in user cost of capital $r + \delta$.

Comparative Statics

Theorem

Let $X \geq Y$ a.s. Assume that operating profit function is supermodular: $\frac{\partial^2 \pi}{\partial x \partial c} \geq 0$. Then the optimal policy under X is greater or equal the optimal policy under Y.

Theorem

The optimal policy is nondecreasing in user cost of capital $r + \delta$.

Conclusion and Outlook

- new method for singular control problems
- general solution for all semimartingale models
- qualitative theory

- Robust Irreversible Investment

- General Equilibrium Theory, Game Theory
- Finance implications

Conclusion and Outlook

- new method for singular control problems
- general solution for all semimartingale models
- qualitative theory
- Robust Irreversible Investment
  - Unknown distribution of profits: minimax-approach
    Gilboa–Schmeidler minimax EU
    dynamic coherent risk measures
- General Equilibrium Theory, Game Theory
- Finance implications

Conclusion and Outlook

- new method for singular control problems
- general solution for all semimartingale models
- qualitative theory
- Robust Irreversible Investment
  - Unknown distribution of profits, minimax–approach
  - Gilboa–Schmeidler minimax EU
  - dynamic coherent risk measures
- General Equilibrium Theory, Game Theory
- Finance implications

Conclusion and Outlook

- new method for singular control problems
- general solution for all semimartingale models
- qualitative theory
- Robust Irreversible Investment
  - Unknown distribution of profits, minimax–approach
  - Gilboa–Schmeidler minimax EU
  - dynamic coherent risk measures
- General Equilibrium Theory, Game Theory
- Finance implications

Conclusion and Outlook

- new method for singular control problems
- general solution for all semimartingale models
- qualitative theory

- Robust Irreversible Investment
  - Unknown distribution of profits, minimax–approach
  - Gilboa–Schmeidler minimax EU
  - dynamic coherent risk measures
- General Equilibrium Theory, Game Theory
- Finance implications

Conclusion and Outlook

- new method for singular control problems
- general solution for all semimartingale models
- qualitative theory

- Robust Irreversible Investment
  - Unknown distribution of profits, minimax–approach
  - Gilboa–Schmeidler minimax EU
  - dynamic coherent risk measures
- General Equilibrium Theory, Game Theory
- Finance implications

Conclusion and Outlook

- new method for singular control problems
- general solution for all semimartingale models
- qualitative theory

- Robust Irreversible Investment
    - Unknown distribution of profits, minimax–approach
    - Gilboa–Schmeidler minimax EU
    - dynamic coherent risk measures
- General Equilibrium Theory, Game Theory
- Finance implications

Conclusion and Outlook

- new method for singular control problems
- general solution for all semimartingale models
- qualitative theory

- Robust Irreversible Investment
  - Unknown distribution of profits, minimax–approach
  - Gilboa–Schmeidler minimax EU
  - dynamic coherent risk measures
- General Equilibrium Theory, Game Theory
- Finance implications

Conclusion and Outlook

- new method for singular control problems
- general solution for all semimartingale models
- qualitative theory

- Robust Irreversible Investment
  - Unknown distribution of profits, minimax–approach
  - Gilboa–Schmeidler minimax EU
  - dynamic coherent risk measures
- General Equilibrium Theory, Game Theory
- Finance implications