Optimal Stopping with Multiple Priors

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March 2007

Example from Microeconomics

Market entry

- A firm can invest into a project with profit stream $\delta_0, \delta_1, \delta_2, \ldots$
- Sunk cost $I > 0$, interest rate $r > 0$
- Profit if entry at τ: $G_\tau = \sum_{t=\tau}^{\infty} \delta_t (1+r)^{-(t-\tau)}$
- Assumptions: $\delta_0 = 1, \delta_{t+1} = \delta_t (1 + Z_t)$. (Z_t) iid, $\sim F$
- maximize $\mathbb{E}\,(G_\tau - I)\,(1+r)^{-\tau}$

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Example from Operations Research

Selling a house

- Real estate agent collects bids $p_0, p_1, p_2, \ldots$ for the house
- Running costs $c > 0$, interest rate $r > 0$
- present value of sale at τ: $G_\tau = p_\tau(1+r)^{-\tau} - \sum_{t=0}^{\tau-1} c(1+r)^{-t}$
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American Options

- Buyer has the right to buy the underlying asset for $K > 0$ at some time τ before maturity
- Profit from exercising: $(S_\tau - K)^+$
- Buyer: maximize $\mathbb{E}\,(S_\tau - K)^+\,(1+r)^{-\tau}$
- Seller: ask a price of

$$\max_\tau \mathbb{E}^*\,(S_\tau - K)^+\,(1+r)^{-\tau}$$

where P^* is the pricing measure (equivalent martingale measure)

- Assumptions of the buyer: $S_0 = 1, S_{t+1} = S_t(1 + Z_t)$. (Z_t) iid, $\sim F$
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The Parking Problem

- You drive along a road towars a theatre
- You want to park as close a spossible to the theatre
- Parking spaces are free iid with probability $p > 0$
- When is the right time to stop?

Secretary Problem

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  rejected applicants do not come back
  applicants come in random (uniform) order
  What candidate to take?

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All examples presume that some distribution F is known, and frequently some kind of independence assumption is added

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- Decision Theory
  - Unique prior in the sense of Savage
  - Ellsberg–Paradoxon
  - Weakening of subjective EU (Gilboa–Schmeidler): class of priors
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We choose the following modeling approach

- Let $X_0, X_1, \ldots, X_T$ be a (finite) sequence of random variables
- adapted to a filtration $(\mathscr{F}_t)$
- on a measurable space $(\Omega, \mathscr{F})$,
- let $\mathscr{P}$ be a set of probability measures
- choose a stopping time $\tau \leq T$
- that maximizes

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Assumptions

for the talk: finite state space Ω
more general:

- $\sup_t |X_t| \in \bigcap_{P \in \mathscr{P}} L^1(P)$
- there exists a reference measure P^0: all $P \in \mathscr{P}$ are equivalent to P^0 (wlog, Tutsch, PhD 07)
- agent knows all null sets, Epstein/Marinacci 07
- $\mathscr{P}$ weakly compact in $L^1(\Omega, \mathscr{F}, P^0)$
- inf is always min, Föllmer/Schied 04, Chateauneuf, Maccheroni, Marinacci, Tallon 05

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Snell, Chow/Robbins/Siegmund: Great Expectations

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Solution

- Define the *Snell envelope* U via backward induction:

$$U_T = X_T$$

$$U_t = \max\{X_t, \mathbb{E}[U_{t+1}|\mathscr{F}_t]\} \qquad (t < T)$$

U is the smallest supermartingale $\geq X$.

An optimal stopping time is given by $\tau^* = \inf\{t \geq 0 : X_t = U_t\}$.

Optimal Stopping: Classical Solution

Snell, Chow/Robbins/Siegmund: Great Expectations

- Given a sequence $X_0, X_1, \ldots, X_T$ of random variables
- adapted to a filtration $(\mathscr{F}_t)$
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- choose a stopping time $\tau \leq T$
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Solution

- Define the *Snell envelope* U via backward induction:

$$U_T = X_T$$

$$U_t = \max\left\{X_t, \mathbb{E}\left[U_{t+1}|\mathscr{F}_t\right]\right\} \qquad (t < T)$$

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Extending the General Theory to Multiple Priors

Aims

- Work as close as possible along the classical lines
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- With general $\mathscr{P}$, one runs easily into inconsistencies in dynamic settings
- Time consistency $\iff$ *law of iterated expectations*:

$$\min_{Q\in\mathscr{P}} \mathbb{E}^Q \left[\operatorname{ess\,inf}_{P\in\mathscr{P}} \mathbb{E}^P [X \mid \mathscr{F}_t] \right] = \min_{P\in\mathscr{P}} \mathbb{E}^P X$$

- (Epstein/Schneider,R. , Artzner et al., Detlefsen/Scandolo, Peng, Chen/Epstein) $\iff$ *stability under pasting*:
  - let $P, Q \in \mathscr{P}$ and let $(p_t), (q_t)$ be the density processes
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Minimax Martingales

Definition

An adapted, bounded process (S_t) is called a minimax supermartingale iff

$$S_t \geq \operatorname*{ess\,inf}_{P \in \mathscr{P}} \mathbb{E}^P \left[S_{t+1} \,\middle|\, \mathscr{F}_t \right]$$

holds true for all $t \geq 0$.
Minimax martingale: $=$
Minimax submartingale: $\leq$

Remark

- *Nonlinear notion of martingales.*
- *Different from $\mathscr{P}$-martingale (martingale for all $P \in \mathscr{P}$ simultaneously)*

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- (S_t) *is a minimax submartingale iff* (S_t) *is a* $\mathscr{P}$*–submartingale.*
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Theorem

Let (S_t) be a minimax supermartingale.
Then there exists a minimax martingale M and a predictable, nondecreasing process A with $A_0 = 0$ such that $S = M - A$. Such a decomposition is unique.

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Optional Sampling Theorem

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Let (S_t) be a minimax supermartingale. Let $\sigma \le \tau$ be stopping times. Assume that τ is universally finite in the sense that $P[\tau < \infty] = 1$ for all $P \in \mathscr{P}$. Then

$$\operatorname*{ess\,inf}_{P \in \mathscr{P}} \mathbb{E}^P \left[S_\tau | \mathscr{F}_\sigma \right] \le S_\sigma \,.$$

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Not true without time consistency.

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With the concepts developed, one can proceed as in the classical case!

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Question: what is the relation between the Snell envelopes U^P for fixed $P \in \mathscr{P}$ and the minimax Snell envelope U?

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$$U = \operatorname*{ess\,inf}_{P \in \mathscr{P}} U^P .$$

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Föllmer/Schied, Cvitanić/Karatzas for American Options in incomplete markets

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Under our assumptions, there exists a measure $P^* \in \mathscr{P}$ such that $U = U^{P^*}$. The optimal stopping rule corresponds to the optimal stopping rule under P^*.

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Infinite Time Horizon (and unbounded X)

Backward induction no longer feasible.
Define the value function as

$$V_t = \operatorname*{ess\,sup}_{\tau \geq t} \operatorname*{ess\,inf}_{P \in \mathscr{P}} \mathbb{E}^P \left[X_\tau | \mathscr{F}_t \right]$$

Theorem

- *V is the smallest minimax supermartingale $\geq X$*
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Infinite Time Horizon II

The solution of the finite time horizon problem converge to the infinite time horizon solution.

Theorem

Let U^T be the value function of the optimal stopping problem under ambiguity for time horizon T. Then for all $t \geq 0$

$$\lim_{T \to \infty} U_t^T = V_t$$

Monotonicity and Stochastic Dominance

- Suppose that (Y_t) are iid under $P^* \in \mathscr{P}$ and
- for all $P \in \mathscr{P}$

$$P^*[Y_t \leq x] \geq P[Y_t \leq x] \qquad (x \in \mathbb{R})$$

- and suppose that the payoff $X_t = g(t, Y_t)$ for a function g that is isotone in y,
- then P^* is *for all optimal stopping problems* (X_t) the *worst–case measure*,
- i.e. the robust optimal stopping rule is the optimal stopping rule under P^*.
- Parking problem: choose the smallest p for open lots
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- Parking problem: choose the smallest p for open lots
- House sale: presume the least favorable distribution of bids in the sens of first–order stochastic dominance

Monotonicity and Stochastic Dominance

- Suppose that (Y_t) are iid under $P^* \in \mathscr{P}$ and
- for all $P \in \mathscr{P}$

$$P^*[Y_t \leq x] \geq P[Y_t \leq x] \qquad (x \in \mathbb{R})$$

- and suppose that the payoff $X_t = g(t, Y_t)$ for a function g that is isotone in y,
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More general problems

Knock–out options $\longrightarrow$ ppt
Secretary problems

Ambiguous secretaries (with Tatjana Chudjakow)

- Call applicant j a *candidate* if she is besser than all predecessors
- We are interested in $X_j = Prob[j \text{best} | j \text{candidate}]$
- Here, the payoff X_j is ambiguous — assume that this conditional probability is minimal
- If you compare this probability with the probability that later candidates are best, you presume the *maximal* probability for them!

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- First steps of a theory of minimax martingales
- Extension to Dynamic Variational Preferences (Maccheroni, Marinacci, Rustichini)
- continuous time ...
- secretary problem with learning

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