On Equilibrium Prices in Continuous Time

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Outline

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What is Equilibrium Theory?

In Option Pricing

- Price process (S_t) of an underlying asset given, say geometric Brownian motion
- no arbitrage condition determines prices for a derivative with payoff $D = f(S_T)$ at maturity T
- but where does the price of the underlying come from?
- *supply and demand*
- the competitive market forces of supply and demand are explored in (general) equilibrium theory

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Equilibrium Theory: Model

The framework for equilibrium theory:

- (rational) agents that choose optimal portfolio and consumption plans
- competition: agents take prices as given when maximizing
- prices balance supply and demand

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Equilibrium Theory: Formal Model

- commodity space $\mathscr{X}$: partially ordered topological vector space
- the positive cone $\mathscr{X}_+$ is the *consumption set*
- price : a linear, positive functional $\Psi : \mathscr{X} \to \mathbb{R}$
- prices should be continuous on the consumption set (similar commodities should have similar prices)
- agents have utility functions $U^i : \mathscr{X}_+ \to \mathbb{R}$, continuous, increasing, concave, and an endowment $E^i \in \mathscr{X}_+$
- agents maximize $U^i(C^i)$ subject to the budget constraint $\Psi(C^i - E^i) \leq 0$
- in equilibrium, $\sum_i (C^i - E^i) = 0$.

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What is the use for Mathematical Finance?

Theorem

In equilibrium, there is no arbitrage.

- Usually, the price functional Ψ can be represented by some stochastic process $\psi = (\psi_t)$
- *if* ψ is a semimartingale, it can be used as a state price density
- in other words: Ψ determines the equivalent martingale measure
- Question: is ψ a
- semimartingale with continuous compensator (cumulative interest)?
- semimartingale with absolutely continuous compensator (interest rate)?

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- Arrow, *Le rôle des valeurs boursières pour la repartition la meilleure des risques*
- ⋮
- Duffie, Zame, *The Consumption-Based CAPM*
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  Existence and Uniqueness of Multi-Agent Equilibrium …
  - sufficient conditions for ψ to be a diffusion
  - consumption in rates, $C_t = \int_0^t c_s ds$
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  On Intertemporal Preferences in Continuous Time …
  - consumption as optional random measures
  - economically sensible topology: weak topology for measures (on the time axis)

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The HHK Model

- $\mathscr{X}$ = space of signed optional random measures with L^1–bounded total variation
- $\mathscr{X}$ = rightcontinuous, adapted processes of bounded variation
- order: $X \geq Y$ iff $X - Y$ is nondecreasing
- $\mathscr{X}_+$ = optional random measures C with $\mathbb{E}C_T < \infty$
- topology: small shifts over time should not affect preferences and prices much:

$$||C - C'|| = \mathbb{E}\int_0^T |C_t - C'_t|dt + \mathbb{E}|C_T - C'_T|$$

- on $\mathscr{X}_+$ essentially equivalent to topology of weak convergence in probability

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The Problem

- the HHK topological dual consists of $\Psi(C) = \mathbb{E} \int \psi_t dC_t$ for $\psi = M + A$, where M is a bounded martingale and A an absolutely continuous process
- everything would be great if we had equilibrium with prices in the topological dual
- but: in general, equilibrium prices do not belong to the HHK dual
- reason: the HHK dual is not a lattice (Tanaka!)

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Price Functionals

- Bank, R. 2001: sufficient conditions on utility functions that yield equilibrium with prices $\psi = M + A$, A continuous
- da Rocha, R. 2006: weaken the assumption in BR 01, provide a framework for formulating the most general, and weakest, assumptions that lead to equilibrium

Today's Aim

Characterize all linear, positive functionals $\Psi : \mathscr{X} \to \mathbb{R}$ that are continuous on $\mathscr{X}_+$

Call these functionals *nice*.

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Characterization of Nice Price Functionals

Theorem

A functional $\Psi : \mathscr{X} \to \mathbb{R}$ is nice if and only if it can be written as

$$\Psi(X) = \mathbb{E} \int \psi_t \, dX_t$$

for a process ψ that is

- *bounded, cadlag, and positive,*
- *the optional projection of a raw continuous process ζ with $\mathbb{E} \sup |\zeta_t| < \infty$*

Supermartingale Price Densities

But: didn't we want semimartingale prices?

Theorem

Assume that all individuals are impatient in the sense that

$$U^i(1_{\{t \geq \tau\}}) \geq U^i(1_{\{t \geq \tau'\}})$$

for all stopping times $\tau \leq \tau'$. Then an equilibrium exists with price density ψ that is

- *a bounded, positive supermartingale*
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Proof of the Main Theorem: Sufficiency

Sufficiency part is easy!

- we have $\Psi(C) = \mathbb{E} \int \zeta dC$ as $\psi =^o \zeta$,
- As ζ continuous, weak convergence of (C^n) to C in probability implies $\mathbb{E} \int \zeta dC^n \to \mathbb{E} \int \zeta dC$.

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Necessity

1. Construct a $\mathfrak{T}$–system $(\phi^\tau)_{\tau \text{ stopping time}}$
2. Aggregate the $\mathfrak{T}$–system $(\phi^\tau)_{\tau \text{ stopping time}}$ into an optional process ψ
3. show that ψ is the optional projection of a raw continuous process

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Construction of the $\mathfrak{T}$–system

Fix a stopping time τ. Let $H \in L^1(\Omega, \mathscr{F}_\tau, \mathbb{P})$ and denote by $H 1_{\{t \geq \tau\}}$ the measure with jump of size H at time τ.
Define a linear functional

$$\Psi^\tau : L^1(\Omega, \mathscr{F}_\tau, \mathbb{P}) \to \mathbb{R}$$

via

$$\Psi^\tau(H) = \Psi(H 1_{\{t \geq \tau\}}) .$$

Ψ^τ is continuous on L^1. Thus, there exists a bounded random variable $\phi^\tau \geq 0$ with

$$\Psi^\tau(H) = \mathbb{E} H \phi^\tau .$$

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$$\Psi^\tau(H) = \mathbb{E} H \phi^\tau .$$

Aggregation of the $\mathfrak{T}$–system

Now we have a bounded r.v. ϕ^τ for every stopping time τ. Can one find an optional process ψ such that

$$\psi_\tau = \phi^\tau \textit{ a.s. } ?$$

First condition: the $\mathfrak{T}$–system must be consistent ($\phi^\tau = \phi^\sigma$ on $\{\tau = \sigma\}$, ok here)

Dellacherie/Lenglart, *Sur des Problèmes de Régularisation, de Recollement et d'Interpolation ...*, Sém. de Prob. 1980: (right)continuity in expectation is enough! Let $\tau_n \downarrow \tau$. Then $\lim \mathbb{E}\phi^{\tau_n} = \lim \Psi(1_{\{t \geq \tau_n\}}) \xrightarrow{\Psi \text{ continuous on } \mathscr{X}_+} \Psi(1_{\{t \geq \tau\}}) = \mathbb{E}\phi^\tau$.

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Finding the Raw Continuous Preimage

- The continuity of Ψ on $\mathscr{X}_+$ implies that ψ is continuous in expectation.
- Consequently, it is cadlag and regular (${}^p\psi = \psi_-$).
- Moreover, of class (D)
- Bismut, *Régularité et Continuité des Processus*, 1978 : every such process is the optional projection of a raw continuous process ζ with $\mathbb{E}\sup_{0\le t\le T}|\zeta_t| < \infty$.

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Final Steps

- We have $\Psi(H1_{\{t\geq\tau\}}) = \mathbb{E}\psi_\tau H = \mathbb{E}\int \psi d(H1_{\{t\geq\tau\}})$ for $H \geq 0$.
- Then $\Psi(C) = \mathbb{E}\int \psi dC$ for simple optional random measures.
- By Fatou's lemma, density of simple random optional measures in $\mathscr{X}_+$, and continuity of Ψ on $\mathscr{X}_+$ $\Psi(C) = \mathbb{E}\int \psi dC$ for all $C \in \mathscr{X}_+$ follows.
- One can then even show that ψ is bounded.

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