On Equilibrium Prices in Continuous Time

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Outline

1. General Equilibrium Theory
   - Purpose of Equilibrium Theory
   - Relation with Mathematical Finance
   - The Hindy–Huang–Kreps Approach: Local Substitution
   - Equilibrium Existence Problem

2. Nice Prices
   - Equilibrium with Nice Prices: Previous Work
   - Characterization of Nice Price Functionals
   - Proof of the Main Theorem
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   - Equilibrium with Nice Prices: Previous Work
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What is Equilibrium Theory?

In Option Pricing

- Price process \((S_t)\) of an underlying asset given, say geometric Brownian motion
- no arbitrage condition determines prices for a derivative with payoff \(D = f(S_T)\) at maturity \(T\)
- but where does the price of the underlying come from? supply and demand
- the competitive market forces of supply and demand are explored in (general) equilibrium theory
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The framework for equilibrium theory:

- (rational) agents that choose optimal portfolio and consumption plans
- competition: agents take prices as given when maximizing
- prices balance supply and demand
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Equilibrium Theory: Formal Model

- commodity space \( \mathcal{X} \) is a partially ordered topological vector space
- the positive cone \( \mathcal{X}_+ \) is the consumption set
- price: a linear, positive functional \( \Psi : \mathcal{X} \to \mathbb{R} \)
- prices should be continuous on the consumption set (similar commodities should have similar prices)
- agents have utility functions \( U^i : \mathcal{X}_+ \to \mathbb{R} \), continuous, increasing, concave, and an endowment \( E^i \in \mathcal{X}_+ \)
- agents maximize \( U^i(C^i) \) subject to the budget constraint \( \Psi(C^i - E^i) \leq 0 \)
- in equilibrium, \( \sum_i (C^i - E^i) = 0 \).
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In equilibrium, there is no arbitrage.

- Usually, the price functional $\Psi$ can be represented by some stochastic process $\psi = (\psi_t)$
- If $\psi$ is a semimartingale, it can be used as a state price density
- In other words: $\Psi$ determines the equivalent martingale measure
- Question: is $\psi$ a semimartingale with continuous compensator (cumulative interest)?
- Semimartingale with absolutely continuous compensator (interest rate)?
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- **Debreu**, *Theory of Value*
- **Arrow**, *Le rôle des valeurs boursières pour la repartition la meilleure des risques*
- **Duffie, Zame**, *The Consumption-Based CAPM*
- **Karatzas, Lehoczky, Shreve**, *Existence and Uniqueness of Multi-Agent Equilibrium...*  
  - sufficient conditions for \( \psi \) to be a diffusion  
  - consumption in rates, \( C_t = \int_{0}^{t} c_s ds \)
- **Hindy, Huang, Kreps**, *On Intertemporal Preferences in Continuous Time...*  
  - consumption as optional random measures  
  - economically sensible topology: weak topology for measures (on the time axis)
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The HHK Model

- $\mathcal{X} = \text{space of signed optional random measures with } L^1\text{–bounded total variation}$
- $\mathcal{X} = \text{rightcontinuous, adapted processes of bounded variation}$
- order: $X \geq Y$ iff $X - Y$ is nondecreasing
- $\mathcal{X}_+ = \text{optional random measures } C \text{ with } \mathbb{E}C_T < \infty$
- topology: small shifts over time should not affect preferences and prices much:

$$\|C - C'\| = \mathbb{E} \int_0^T |C_t - C'_t| dt + \mathbb{E}|C_T - C'_T|$$

- on $\mathcal{X}_+$ essentially equivalent to topology of weak convergence in probability
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- $X = \text{rightcontinuous, adapted processes of bounded variation}$
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the HHK topological dual consists of $\Psi(C) = \mathbb{E} \int \psi_t dC_t$ for $\psi = M + A$, where $M$ is a bounded martingale and $A$ an absolutely continuous process.

- everything would be great if we had equilibrium with prices in the topological dual.
- but: in general, equilibrium prices do not belong to the HHK dual.
- reason: the HHK dual is not a lattice (Tanaka!)
The Problem

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Price Functionals

- Bank, R. 2001: sufficient conditions on utility functions that yield equilibrium with prices $\psi = M + A$, $A$ continuous
- da Rocha, R. 2006: weaken the assumption in BR 01, provide a framework for formulating the most general, and weakest, assumptions that lead to equilibrium

Today’s Aim

Characterize all linear, positive functionals $\Psi : \mathcal{X} \rightarrow \mathbb{R}$ that are continuous on $\mathcal{X}_+$

Call these functionals nice.
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Call these functionals *nice*.
A functional \( \Psi : \mathcal{X} \to \mathbb{R} \) is nice if and only if it can be written as
\[
\Psi(X) = \mathbb{E} \int \psi_t \, dX_t
\]
for a process \( \psi \) that is
- bounded, cadlag, and positive,
- the optional projection of a raw continuous process \( \zeta \) with
\[
\mathbb{E} \sup \abs{\zeta_t} < \infty
\]
Supermartingale Price Densities

But: didn’t we want semimartingale prices?

Theorem

Assume that all individuals are impatient in the sense that

\[ U^i(1_{\{t \geq \tau\}}) \geq U^i(1_{\{t \geq \tau'\}}) \]

for all stopping times \( \tau \leq \tau' \). Then an equilibrium exists with price density \( \psi \) that is

- a bounded, positive supermartingale
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Proof of the Main Theorem: Sufficiency

Sufficiency part is easy!

- We have $\psi(C) = \mathbb{E} \int \zeta dC$ as $\psi = \circ \zeta$.
- As $\zeta$ continuous, weak convergence of $(C^n)$ to $C$ in probability implies $\mathbb{E} \int \zeta dC^n \rightarrow \mathbb{E} \int \zeta dC$. 
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Necessity

1. Construct a $\mathcal{F}$–system $(\phi^\tau)_\tau$ stopping time
2. Aggregate the $\mathcal{F}$–system $(\phi^\tau)_\tau$ stopping time into an optional process $\psi$
3. Show that $\psi$ is the optional projection of a raw continuous process
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Fix a stopping time $\tau$. Let $H \in L^1(\Omega, \mathcal{F}_\tau, \mathbb{P})$ and denote by $H1_{\{t \geq \tau\}}$ the measure with jump of size $H$ at time $\tau$. Define a linear functional

$$\psi^\tau : L^1(\Omega, \mathcal{F}_\tau, \mathbb{P}) \to \mathbb{R}$$

via

$$\psi^\tau(H) = \psi(H1_{\{t \geq \tau\}}).$$

$\psi^\tau$ is continuous on $L^1$. Thus, there exists a bounded random variable $\phi^\tau \geq 0$ with

$$\psi^\tau(H) = \mathbb{E}H\phi^\tau.$$
Construction of the $\mathcal{F}$–system

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Construction of the $\mathcal{F}$–system

Fix a stopping time $\tau$. Let $H \in L^1(\Omega, \mathcal{F}_\tau, \mathbb{P})$ and denote by $H1_{\{t \geq \tau\}}$ the measure with jump of size $H$ at time $\tau$.

Define a linear functional

$$\psi^\tau : L^1(\Omega, \mathcal{F}_\tau, \mathbb{P}) \to \mathbb{R}$$

via

$$\psi^\tau(H) = \psi(H1_{\{t \geq \tau\}}).$$

$\psi^\tau$ is continuous on $L^1$. Thus, there exists a bounded random variable $\phi^\tau \geq 0$ with

$$\psi^\tau(H) = \mathbb{E}H\phi^\tau.$$
Aggregation of the $\mathcal{F}$–system

Now we have a bounded r.v. $\phi^\tau$ for every stopping time $\tau$. Can one find an optional process $\psi$ such that

$$\psi_\tau = \phi^\tau \text{ a.s.}$$

First condition: the $\mathcal{F}$–system must be consistent
$(\phi^\tau = \phi^\sigma$ on $\{\tau = \sigma\}$, ok here)

Dellacherie/Lenglart, *Sur des Problèmes de Régularisation, de Recollement et d’Interpolation* . . ., Sém. de Prob. 1980:
(right)continuity in expectation is enough! Let $\tau_n \downarrow \tau$. Then
$$\lim \mathbb{E}\phi^{\tau_n} = \lim \mathbb{E}\psi(1\{t \geq \tau_n\}) \rightarrow \psi \text{ continuous on } \mathcal{B}_+ \mathbb{E}\phi^\tau.$$
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Consequently, it is cadlag and regular ($P\psi = \psi_-$).

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Finding the Raw Continuous Preimage

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Final Steps

- We have $\psi(H1_{\{t \geq \tau\}}) = \mathbb{E}\psi_\tau H = \mathbb{E}\int \psi \, d(H1_{\{t \geq \tau\}})$ for $H \geq 0$.

- Then $\psi(C) = \mathbb{E}\int \psi \, dC$ for simple optional random measures.

- By Fatou’s lemma, density of simple random optional measures in $\mathcal{H}^+$, and continuity of $\psi$ on $\mathcal{H}^+$, $\psi(C) = \mathbb{E}\int \psi \, dC$ for all $C \in \mathcal{H}^+$ follows.

- One can then even show that $\psi$ is bounded.
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- Characterization of Nice Price Functionals
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