Backward Stochastic Differential Equations and g-expectations

Introduction to some basic topics

Xue Cheng

Email: chengxue@amss.ac.cn

Institute of Applied Mathematics, Academy of Mathematics and Systems Science, CAS

January, 2008
Background and Motivation

Problem

- How an agent can tell, when facing various risky assets, which one is better?

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- How an agent can evaluate, in a financial market, a contingent claim?
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Theorem (Expected utility Theory, von Neumman and Morgenstein:)

The Existence of Expected utility is equivalent to

- Rational preference
- Continuity Axiom
- Independence Axiom: $L \succeq \tilde{L} \iff \alpha L + (1 - \alpha)\hat{L} \succeq \alpha \tilde{L} + (1 - \alpha)\hat{L}$

- Allais’ Paradox
- Linearity of Expectation
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- **Complete Market:**
  - Exact hedging portfolio
  - Girsanov Transform, Equivalent martingale measure
  - Linear BSDE

- **Incomplete Market:**
  - Not always possible to construct exact hedging portfolio
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Nonlinear Expectations

- Choquet Expectation;
  - Probability $\mapsto$ Capacity; monotone, with $C(\emptyset) = 0$, $C(\Omega) = 1$;
  - Expectation: defined according to this ”nonlinear probability”.

- g-expectation: derived from BSDE

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  - with out classical probability space.
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Example (Black-Scholes type)

Consider a security market that consists of 2 instruments

- \( P_0(t) = e^{rt} \)
- \( P(t) = p \exp[(b - \sigma^2/2)t + \sigma B_t] \)

\[
\begin{align*}
\Delta P_t &= P_t(bdt + \sigma dB_t), \\
P(0) &= p.
\end{align*}
\]

frictionless except that interests for borrowing and lending may not the same...

If an agent plan to receive \( \xi \) at \( T \), then her wealth:

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\Delta V_t &= [rV_t + (b - r)Z_t - (V_t - Z_t)^-(R - r)]dt + Z_t \sigma dB_t, \\
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1. \( R = r \) Black-Scholes Model \( g(y, z) = ry + (b - r)z \)

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V_t = E^g_{t,T}[\xi] = E_Q[\xi], \text{ and } Z_t = \frac{d\langle V, B \rangle_t}{dt}.
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2. \( R \neq r \) Nonlinear, \( V_t = E^g_{t,T}[\xi] \), for \( L^2 \) integrable \( \xi \).
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- Some Basic Properties of BSDE and g-Expectations
- Forward-Backward Stochastic Differential Equations
- Reflected Backward Differential Equations
- Feynman-Kac Formula
- Sublinear Expectations: G-Expectation
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Some Basic Properties of BSDE and g-Expectations

\[ \begin{cases} 
  dy_t = -g(t, y_t, z_t) dt + z_t dB_t, & t \in [0, T] \\
  y_T = \xi 
\end{cases} \]

The Spaces

- \((\Omega, \mathcal{F}, P)\): a complete probability space;
- \(\{B_t, 0 \leq t \leq T\}\): a \(d\)-dimensional standard Brownian motion;
- \(\{\mathcal{F}_t, 0 \leq t \leq T\}\): the natural filtration of \(B_t\);
- \(L^p_\mathcal{F}(\mathbb{R}^m)\): \(\mathbb{R}^m\) valued \(\mathcal{F}\) progressively measurable stochastic process \(\phi\), with \(E[\int_0^T |\phi|^p \, dt] < \infty\);
- \(D^p_\mathcal{F}(\mathbb{R}^m)\): \(\mathbb{R}^m\) valued \(\mathcal{F}\) adapted RCLL stochastic process \(\phi\), with \(E[\sup_{0 \leq t \leq T} |\phi|^p] < \infty\);
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Assumptions:

- the terminal value $\xi \in L^2(\mathcal{F}_T)$;
- for the generator $g$: $[0, T] \times \mathbb{R} \times \mathbb{R}^d \times \Omega \to \mathbb{R}$,
  - (A1), $g(\cdot, x, y)$ is an adapted process with $E[\int_0^T |g(t, 0, 0)|^2 dt] < \infty$;
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  - (A3), $g(\cdot, 0, 0) = 0$, $dt \times dP - a.s.$;
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Theorem (Existence and Uniqueness)

If the parameters of BSDE satisfy the terminal assumption and generator assumption (A1), (A2), the equation has a unique pair of adapted $L^2$ integrable solution $(y_t, z_t) \in S^2_\mathcal{F}(\mathbb{R}) \times L^2_\mathcal{F}$.

We simply give the Stability of solutions in the following sense:

$$E[\sup_{0 \leq t \leq T} |Y^1_t - Y^2_t|^2] + E[\int_0^T |Z^1_s - Z^2_s|^2 ds] \leq cE[|\xi^1 - \xi^2|^2],$$

where $c$ is a constant.
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Definition (g-Evaluation and g-Expectation)

- The so called *g-evaluation* can be defined as
  \[ E^g_{s,t}(y_t) = y_s, \quad 0 \leq s \leq t \leq T; \]
- If furthermore, \( g \) satisfies assumption (A4), then *g-expectation* can be defined as
  \[ E^g(\xi|\mathcal{F}_t) = y_t. \]
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Theorem (Comparison Theorem)

Suppose \((\xi_1, g_1), (\xi_2, g_2)\) satisfy the conditions stated above, let \((Y^1, Z^1), (Y^2, Z^2)\) be the solutions of the corresponding BSDEs, if

\[\xi_1 \geq \xi_2, \quad g_1(t, Y^2_t, Z^2_t) \geq g_2(t, Y^2_t, Z^2_t), \text{ a.s., a.e.}\]

Then we have

\[Y^1_t \geq Y^2_t, \text{ a.s.}\]

And under the above conditions,

\[Y^1_0 = Y^2_0 \iff \xi_1 = \xi_2 \text{ and } g_1(t, Y^2_t, Z^2_t) = g_2(t, Y^2_t, Z^2_t), \text{ a.s., a.e.}\]
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\[ Y^1_0 = Y^2_0 \iff \xi_1 = \xi_2 \text{ and } g_1(t, Y^2_t, Z^2_t) = g_2(t, Y^2_t, Z^2_t), \text{a.s., a.e.} \]

- This is a generalized version.
- It is one of the most important theorems about BSDE, it "plays the same role that the maximum principle in PDE" (El Karoui).
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Some Basic Properties of BSDE and g-Expectations

Theorem (Comparison Theorem)

Suppose \((\xi_1, g_1), (\xi_2, g_2)\) satisfy the conditions stated above, let \((Y^1, Z^1), (Y^2, Z^2)\) be the solutions of the corresponding BSDEs, if

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Some Basic Properties of BSDE and g-Expectations

Theorem (Axiomatic Properties of g-Evaluation and g-Expectation)

(P1) (Monotonicity) \( \mathbb{E}^g_{t,T}[\xi] \leq \mathbb{E}^g_{t,T}[\eta] \), if \( \xi \leq \eta \);

(P2) If \( \xi \in L^2(\mathcal{F}_t) \), then, \( \mathbb{E}^g_{t,t}[\xi] = \xi \), and for g-expectation, \( \mathbb{E}^g_{t,T}[\xi] = \xi \), a.s.

(P3) (Time Consistency) \( \mathbb{E}^g_{s,t}[\mathbb{E}^g_{t,T}[\xi]] = \mathbb{E}^g_{s,T}[\xi] \), \( s \leq t \);

(P4) (”Zero-One Law”) If \( g \) satisfies (A3), \( \mathbb{E}^g_{t,T}[I_A \xi] = I_A \mathbb{E}^g_{t,T}[\xi], A \in \mathcal{F}_t \);

(P5) (Translation Invariance) For g-expectation with \( g \) independent of \( y \),
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\mathbb{E}^g[\xi + \eta | \mathcal{F}_t] = \mathbb{E}^g[\xi | \mathcal{F}_t] + \eta, \text{ a.s. where } \eta \in L^2(\mathcal{F}_t).
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(Bielefeld)
Some Basic Properties of BSDE and g-Expectations

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Theorem (pseudo-linearity)

For $0 \leq s \leq t \leq T$, $X, Y \in L^2(\mathcal{F}_t)$ and $A \in \mathcal{F}_s$, we have

$$E^g_{s,t}[Xl_A + Yl_{A^c}] = E^g_{s,t}[X]l_A + E^g_{s,t}[Y]l_{A^c}$$
Some Basic Properties of BSDE and g-Expectations

Theorem (Monotonic Limit Theorem)

Consider the following sequence of BSDEs:

\[ y_t^i = y_T^i + \int_t^T g(s, y_s^i, z_s^i) ds + (A_T^i - A_t^i) - \int_t^T z_s^i dB_s \]

where \( g \) satisfies (A1) and (A2), \( A_t^i \) is a continuous increasing process with \( A_0^i = 0 \) and \( A_T^i \in L^2(\mathcal{F}_T) \). If \( y_t^i \) converges monotonically up to a process \( y_t \) as \( i \to \infty \), and \( E[\text{esssup}_{0 \leq t \leq T} |y_t|^2] < \infty \). Then there exists stochastic processes \( z \) and \( A \), s.t.

\[ y_t = y_T + \int_t^T g(s, y_s, z_s) ds + (A_T - A_t) - \int_t^T z_s dB_s \]
Some Basic Properties of BSDE and g-Expectations

Determine the generator by the evaluation value......


**Theorem**

- (A1)(A2), $g^1 \geq g^2 \iff \mathbb{E}^{g^1}_{s,t} [\xi] \geq \mathbb{E}^{g^2}_{s,t} [\xi]$, for all $\xi$ and $s, t$;
- (A2)(A4), no $y$, $g^1 \geq g^2 \iff \mathbb{E}^{g^1} [\xi] \geq \mathbb{E}^{g^2} [\xi]$;
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Definition (Martingale under g-Evaluation)

An $\mathcal{F}$-progressively measurable real valued process $y$ with

$$E(\text{ess sup}_{0 \leq t \leq T} |y_t|^2) < \infty,$$

is called a g-martingale (under g-evaluation) if for $\forall 0 \leq s \leq t \leq T$,

$$\mathbb{E}^g_{s,t}(y_t) = y_s, \text{ a.s.}$$

We can also define analogously g-supermartingale and g-submartingale.
Some Basic Properties of BSDE and g-Expectations

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Theorem (Optional Stopping)

Assume the generator $g$ satisfies (A1)(A2), and $y$ is a $g$-supermartingale (resp. submartingale). Then for every two stopping times $\sigma, \tau \leq T$ with $\sigma \leq \tau$, we have

$$E^g_{\sigma,\tau}(y_\tau) \leq y_\sigma, \text{ a.s.}$$

Theorem (Doob-Meyer Type Decomposition Theorem)

Assume that $g$ satisfies (A1)(A2). Let $(y_t)$ be a right continuous $g$-supermartingale on $[0,T]$. Then there exists a unique RCLL increasing process $(A_t)$ with $E[A_T^2] < \infty$ and $A_0 = 0$, such that $(y_t)$ coincides with the unique solution $(y_t)$ of the following BSDE:

$$y_t = y_T + \int_t^T g(s, y_s, z_s)ds + (A_T - A_t) - \int_t^T z_s dB_s, \quad t \in [0, T].$$
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    dX_t &= b(t, X_t, Y_t, Z_t)dt + \sigma(t, X_t, Y_t, Z_t)dB_t, \\
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These equations closely connected with PDEs, thus have lead to interesting results:
- Stochastic representations for PDEs; Generalizing Feynman-Kac formula...
- Monte-Carlo numerical method; Jianfeng Zhang; V.Bally; G.Pages; Nizar Touzi......

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**Theorem (Existence Theorem in Small "Time Durations")**

If the parameters for FBSDE, always denoted $\phi$, satisfy:

1), uniformly Lipschitz in $(x, y, z)$;

2), given $(x, y, z)$, is progressively measurable ($b, h, \sigma$) or measurable in $\mathcal{F}_T$ (for $g$)

3), $\phi(t, 0, \omega) \in L^2_{\mathcal{F}}$

4), the Lipschitz constants of $\sigma$ and $g$, $L_1, L_2: L_1L_2 < 1$

Then there exists a $T_0 > 0$, s.t. for any $T \in (0, T_0]$ and initial value $x$, FBSDE admits a unique adapted solution $(X, Y, Z) \in (S^2_{\mathcal{F}} \times S^2_{\mathcal{F}} \times L^2_{\mathcal{F}})$.

- FBSDE for large time duration might unsolvable.
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Reflected Backward Differential Equations (RBSDE) was first introduced in 1997:


A solution for RBSDE with \((\xi, g, L_t)\), is a triple \((Y, Z, K)\) satisfying

\[
Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s
\]

and \(Y_t \geq L_t\) on \([0, T]\).

\((K_t)\) is a nondecreasing continuous process, s.t.

\[
\int_0^T (Y_s - L_s) dK_s = 0
\]
Reflected Backward Stochastic Differential Equations

- RBSDEs with one continuous barrier;
- RBSDEs with two continuous barriers;
- RBSDEs with $L^2$ barriers;
- RBSDEs with continuous barriers under non-Lipschitz condition.

And the applications:

- The stochastic representation of solutions of PDE with obstacle(s);
- Applications in Finance:
  - Pricing American Options;
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And for the RBSDEs with two continuous barriers;

A solution is a quadruple \((Y, Z, K^+, K^-)\), which satisfies

\[
Y_t = \xi + \int_t^T g(s, Y_s, Z_s) \, ds + K_T^+ - K_t^+ - (K_T^- - K_t^-) - \int_t^T Z_s \, dB_s
\]

\(L_t \leq Y_t \leq U_t\) on \([0, T]\).

\((K_t^+, K_t^-)\) are non decreasing and continuous,

\[
\int_0^T (Y_s - L_s) \, dK_s^+ = 0 \quad \text{and} \quad \int_0^T (Y_s - U_s) \, dK_s^- = 0.
\]
Let \((Y, Z, K)\) be the solution of RBSDE, then

\[
Y_t = \text{ess sup}_{\tau \in \mathcal{T}_t} E \left[ \int_t^\tau g(s, Y_s, Z_s) ds + L_\tau 1_{\{\tau < T\}} + \xi 1_{\{\tau = T\}} | \mathcal{F}_t \right],
\]

where \(\mathcal{T}_t\) is the set of all stopping times valued in \([t, T]\).
Reflected Backward Stochastic Differential Equations

**Theorem**

Let \((Y, Z, K)\) be the solution of RBSDE with 2 continuous barriers, then, for any \(0 \leq t \leq T\) and any stopping times \(\tau, \sigma \in \mathcal{T}_t\), consider

\[
R_t(\sigma, \tau) = \int_t^{\sigma \wedge \tau} g(s, Y_s, Z_s) ds + \xi 1_{\{\sigma \wedge \tau = T\}} + L_\tau 1_{\{\tau < T, \tau \leq \sigma\}} + U_\sigma 1_{\{\sigma < \tau\}}.
\]

\[
\overline{V}_t = \text{ess inf}_{\sigma \in \mathcal{T}_t} \text{ess sup}_{\tau \in \mathcal{T}_t} E[R_t(\sigma, \tau)|\mathcal{F}_t], \quad \underline{V}_t = \text{ess sup}_{\tau \in \mathcal{T}_t} \text{ess inf}_{\sigma \in \mathcal{T}_t} E[R_t(\sigma, \tau)|\mathcal{F}_t]
\]

Then \(Y_t = \overline{V}_t = \underline{V}_t\)
Application in Pricing American Options:


"In some constraint cases, the strategy wealth portfolio \((X_t, \pi)\) ...satisfy the following BSDE

\[-dX_t = b(t, X_t, \pi_t)dt - \pi_t^* \sigma_t dW_t\]

Here \(b\) ... convex with respect to \(x, \pi\). ...suppose that the volatility is invertible and that \((\sigma_t)^{-1}\) is uniformly bounded..."
Consider the payoff process:

\[ \widetilde{S}_s = \xi 1_{\{s=T\}} + S_s 1_{\{s<T\}}. \]

Fix \( t \in [0, T] \), \( \tau \in \mathcal{T}_t \); then exists a unique strategy \((X_s(\tau, \widetilde{S}_\tau), \pi(\tau, \widetilde{S}_\tau))\), which replicate \( \widetilde{S}_\tau \), i.e. for some coefficient \( b \)

\[
-dX^{\tau}_s = b(s, X^{\tau}_s, \pi^{\tau}_s)ds - (\pi^{\tau}_s)^* dB_s, \quad 0 \leq s \leq T,
\]

\[ X^{\tau}_t = \widetilde{S}_\tau. \]

Then the price of the American contingent claim \((\widetilde{S}_s, 0 \leq s \leq T)\) at time \( T \) is given by

\[ X_t = \text{ess} \sup_{\tau \in \mathcal{T}_t} X_t(\tau, \widetilde{S}_\tau). \]

And under the convex assumption, it can be proved that:

\[ X_t = \text{ess} \sup_{\tau \in \mathcal{T}_t} E[ \int_t^\tau b(s, X_s, \pi_s)ds + S_\tau 1_{\{\tau<T\}} + \xi 1_{\{\tau=T\}}|\mathcal{F}_t], \]
Application in Pricing Game Options:

A: can choose cancellation time $\sigma \in \mathcal{T}$

B: choose exercise time $\tau \in \mathcal{T}$

Payoffs: $\infty \geq L_t \geq U_t \geq 0$

Thus A pays B:

$$R(\sigma, \tau) = L_\sigma l_{[\sigma < \tau]} + U_\tau l_{[\tau \leq \sigma]}$$
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And the fair price?

\[ V_{t,T} = \text{ess inf}_{\sigma \in \mathcal{T}_t} \text{ess sup}_{\tau \in \mathcal{T}_t} E[e^{-r\sigma \wedge \tau} R_t(\sigma, \tau) | \mathcal{F}_t] \]

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Feynman-Kac Formula


Consider the following coupled FBSDE:

\[
\begin{align*}
    dX^{x,t}_s &= b(X^{x,t}_s)ds + \sigma(X^{x,t}_s)dW_s, \quad s \in [t, T] \\
    X^{x,t}_t &= x.
\end{align*}
\]

\[
\begin{align*}
    -dY^{x,t}_t &= g(X^{x,t}_s, Y^{x,t}_s, Z^{x,t}_s)ds - Z^{x,t}_s dW_s, \quad s \in [t, T] \\
    Y^{x,t}_T &= \Phi(X^{x,t}_T).
\end{align*}
\]
Assumptions:

(H1) $b, \sigma$ satisfies conditions s.t. the SDE has a unique strong solution;

(H2) $g$ satisfies conditions s.t. the BSDE has a unique solution for every $(x, t) \in \mathbb{R}^n \times [0, T]$;

(H3) $b, \sigma, g$ are deterministic functions.

Set $u(x, t) \triangleq \mathbb{E}^{g}_{t,T}[\Phi(X_{T}^{x,t})]$. 
Theorem

Assume (H1)(H2)(H3), then $u$ is a deterministic function. And if $u \in C^{2,1}$, then it satisfies:

$$\begin{cases}
\frac{\partial u}{\partial t} + L u + f(x, u, \sigma^T \nabla u) = 0, \quad (x, t) \in \mathbb{R}^n \times [0, T], \\
u(x, T) = \Phi(x),
\end{cases}$$

where, $L \phi = \frac{1}{2} \sum_{i,j=0}^{n} a_{i,j}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial \phi}{\partial x_i}, \quad a_{i,j} = [\sigma \sigma^T]_{i,j}$.

Conversely, if the PDE has a $C^{2,1}$-solution, then the solution is unique and is $u$. 

\[\text{(Bielefeld)}\]
Theorem

If (H3) is not satisfied, then $u$ is a $\mathcal{F}$-adapted process. And if $u$ is smooth enough, then it satisfies:

$$
\begin{cases}
- \d u(t, x) = [Lu(t, x) + g(t, x, u, Du\sigma + \phi) + D\phi\sigma]dt - (Du\sigma + \phi)dWt, \\
\end{cases}
$$

$$
u(x, T) = \Phi(x),
$$

and conversely, if $(u, \phi)$ is the solution of BSPDE, then $(u(s, X_{s}^{t,x}), (Du\sigma + \phi)(s, X_{s}^{t,x}))$ is the solution of the related BSDE.
Feynman-Kac Formula

Applications:

- Monte-Carlo methods for PDE;
- Using numerical methods of PDE in solving BSDE.

Example

Consider the Black-Scholes type example. Pricing a contingent claim with a terminal value \((P(T) - q)^+\), we can use the Feynman-Kac formula:

\[
\begin{align*}
\frac{\partial u}{\partial t} + Lu + g(p, u, \sigma^T \nabla u) &= 0, \\
u(p, T) &= (p - q)^+,
\end{align*}
\]

The fair price is just \(y_0 = u(P(0), 0)\).
Feynman-Kac Formula

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Sublinear Expectations: G-Expectation

First let us look at two examples.

Example ("Premium Ambiguity")

Suppose \( dV_t = \frac{dP_t}{P_t} \), and \( r = 0 \),

\[
V_t = V_0 + \int_0^t \sigma_s^T (dB_s + \theta_s ds)
\]

where \( \theta \) denotes the risk premium,

let

\[
\frac{dQ_\theta}{dp} \triangleq \exp\left[ \int_0^T \theta_t dB_t - \frac{1}{2} \int_0^T |\theta_t|^2 dt \right],
\]

then

\[
V_0 = E_{Q_\theta}[V_T],
\]

for the uncertainty of premium, we can give the price to a stochastic asset \( X \) as \( \sup_{Q_\theta} E_{Q_\theta}[X], \theta_t \in \Theta \).
In fact, this can be written into g-expectation framework:

\[ E^g[X] = \sup_{Q_\theta} E_{Q_\theta}[X] = Y_0 \]

\[ E^g : L^2(\Omega, \mathcal{F}_T, P) \to \mathbb{R} \]

where the g-expectation is derived from a BSDE with generator:

\[ g(t, z) \triangleq \sup_{\theta \in \Theta} \sigma_t^{-1} \theta |z| \]
Recall the generalized Feynman-Kac Formula (we might suppose that $\sigma = I$):

$$u(x, t) \triangleq E[\Phi(X_{T}^{x,t})],$$

then $u(x, t)$ is the solution of

$$\begin{cases}
\frac{\partial u}{\partial t} = Ku \\
u(0, x) = \Phi(x).
\end{cases}$$

where

$$K\phi = \frac{1}{2} \sum_{i,j=0}^{n} \frac{\partial^{2} \phi}{\partial x_{i} \partial x_{j}} + \sup_{\theta \in \Theta} \sum_{i=1}^{n} \theta_i \frac{\partial \phi}{\partial x_i}$$
Example

Consider the following nonlinear parabolic PDE:

\[
\begin{cases}
\frac{\partial u}{\partial t} = G\left(\frac{\partial^2 \phi}{\partial x^2}\right) \\
u(0, x) = \Phi(x).
\end{cases}
\]

where

\[
G(x) = \frac{1}{2} \sup_{\sigma \in [\sigma, \bar{\sigma}]} (\sigma^2 x).
\]

And define \( \mathbb{E}[\Phi] \triangleq u(1, 0) \)

Note that this kind of nonlinear expectation cannot be described by g-expectation, and models the so-called "volatility ambiguity".
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Sublinear Expectations: G-Expectation

- Peng, S., G – Brownian Motion and Dynamic Risk Measure under Volatility Uncertainty, Lecture notes in Mini-course of CSFI, Osaka University, 2007.

- $(\Omega, \mathcal{F}, P) \rightarrow (\Omega, \mathcal{H}, \mathbb{E})$, with $\mathcal{H}$ a set of random variables and $\mathbb{E}$ a nonlinear expectation;
- "Probability Language" -- "Expectation Language"
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- Sets v.s. Random Variables
- Measurable?
- Distribution?
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A random variable $\xi$ in a sub-expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called $G_{\underline{\sigma}, \overline{\sigma}}$-normal distributed, denoted by $\xi \sim \mathcal{N}(0; [\underline{\sigma}^2, \overline{\sigma}^2])$, for a given pair $0 \leq \underline{\sigma} \leq \overline{\sigma}$, if for each $\phi \in C_{l,\text{lip}}(\mathbb{R})$, the following function defined by

$$u(t, x) \triangleq \mathbb{E}[\phi(x + \sqrt{t}\xi)], \ (t, x) \in [0, \infty) \times \mathbb{R}$$

is the unique viscosity solution of the PDE in Example2.
Corresponding definition of G-Brownian motion;
Stochastic Calculus of Itô’s type...
Law of Large numbers and Central Limit Theorem...
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Sublinear Expectations: G-Expectation

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Thank you!