Backward Stochastic Differential Equations and g-expectations

——Introduction to some basic topics

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Background and Motivation

Problem

- *How an agent can tell, when facing various risky assets, which one is better?*

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- *How an agent can evaluate, in a financial market, a contingent claim?*

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In Economics for Problem 1:

Theorem (Expected utility Theory, von Neumman and Morgenstein:)

The Existence of Expected utility is equivalent to

- *Rational preference*
- *Continuity Axiom*
- *Independence Axiom:* $L \gtrsim \bar{L} \Leftrightarrow \alpha L + (1-\alpha)\hat{L} \gtrsim \alpha \bar{L} + (1-\alpha)\hat{L}$

- Allais' Paradox
- Linearity of Expectation

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- Complete Market:
  - Exact hedging portfolio
  - Girsanov Transform,Equivalent martingale measure
  - Linear BSDE
- Incomplete Market:
  - Not always possible to construct exact hedging portfolio
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Nonlinear Expectations

- Choquet Expectation;
  - Probability $\longmapsto$ Capacity; monotone, with $C(\emptyset) = 0,\ C(\Omega) = 1$;
  - Expectation: defined according to this ”nonlinear probability”.
- g-expectation: derived from BSDE
- Sublinear Expectation; Risk Measure, G-Expectation
  - with out classical probability space.
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Example (Black-Scholes type)

Consider a security market that consists of 2 instruments

- $P_0(t) = e^{rt}$
- $P(t) = p\exp[(b - \sigma^2/2)t + \sigma B_t]$

$$\begin{cases} dP_t = P_t(bdt + \sigma dB_t), \\ P_{(0)} = p. \end{cases}$$

frictionless except that interests for borrowing and lending may not the same...

If an agent plan to receive ξ at T, then her wealth:

$$\begin{cases} dV_t = [rV_t + (b - r)Z_t - (V_t - Z_t)^-(R - r)]dt + Z_t\sigma dB_t, \\ V_T = \xi. \end{cases}$$

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1. $R = r$ Black-Scholes Model $g(y,z) = ry + (b-r)z$

$$V_t = \mathbb{E}^g_{t,T}[\xi] = \mathbb{E}_Q[\xi], \text{ and } Z_t = \frac{d<V,B>_t}{dt}.$$

2. $R \neq r$ Nonlinear, $V_t = \mathbb{E}^g_{t,T}[\xi]$, for L^2 integrable ξ.

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- Some Basic Properties of BSDE and g-Expectations
- Forward-Backward Stochastic Differential Equations
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Some Basic Properties of BSDE and g-Expectations

- $$\begin{cases} dy_t = -g(t, y_t, z_t)dt + z_t dB_t, & t \in [0, T] \\ y_T = \xi \end{cases}$$

◇ Pardoux, E., Peng, S., Adapted solution of a backward stochastic differential equation, Systems Control Letters 14: 55-61, 1990.

◇ Peng, S., BSDE and related g-expectations, Backward Stochastic Differential Equations, El Karoui, N. and Mazliak, L. eds., Paris, 1995-1996, Pitman Research Notes in Mathematics Series, 364, 141-159, Longman, Harlow, 1997.

The Spaces

- $(\Omega, \mathcal{F}, P)$: a complete probability space;
- $\{B_t, 0 \leq t \leq T\}$: a d-dimensional standard Brownian motion;
- $\{\mathcal{F}_t, 0 \leq t \leq T\}$: the natural filtration of B_t;
- $L^p_{\mathcal{F}}(R^m)$: R^m valued $\mathcal{F}$ progressively measurable stochastic process ϕ, with $E[\int_0^T |\phi|^p dt] < \infty$;
- $D^p_{\mathcal{F}}(R^m)$: R^m valued $\mathcal{F}$ adapted RCLL stochastic process ϕ, with $E[\sup_{0 \leq t \leq T} |\phi|^p] < \infty$;
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Assumptions:

- the terminal value $\xi \in L^2(\mathcal{F}_T)$;
- for the generator g: $[0,T] \times R \times R^d \times \Omega \to R$,
  - (A1). $g(\cdot, x, y)$is an adapted process with $E[\int_0^T |g(t,0,0)|^2 dt] < \infty$;
  - (A2). $|g(t, y_1, z_1) - g(t, y_2, z_2)| \le C(|y_1 - y_2| + |z_1 - z_2|)$;
  - (A3). $g(\cdot, 0, 0) = 0, \ dt \times dP - a.s.$;
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Some Basic Properties of BSDE and g-Expectations

Theorem (Existence and Uniqueness)

If the parameters of BSDE satisfy the terminal assumption and generator assumption (A1), (A2), the equation has a unique pair of adapted L^2 integrable solution $(y_t, z_t) \in S^2_{\mathcal{F}}(R) \times L^2_{\mathcal{F}}$.

- We simply give the Stability of solutions in the following sense: $E[\sup_{0\le t\le T}|Y^1_t - Y^2_t|^2] + E[\int_0^T |Z^1_s - Z^2_s|^2 ds] \le cE[|\xi^1 - \xi^2|^2]$,

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Definition (g-Evaluation and g-Expectation)

- *The so called g-evaluation can be defined as* $\mathbb{E}^g{}_{s,t}(y_t) = y_s, \; 0 \le s \le t \le T;$
- *If furthermore, g satisfies assumption (A4), then g-expectation can be defined as* $\mathbb{E}^g(\xi|\mathcal{F}_t) = y_t$.

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Theorem (Comparison Theorem)

Suppose $(\xi_1,\ g_1)$, $(\xi_2,\ g_2)$ *satisfy the conditions stated above, let* (Y^1, Z^1), (Y^2, Z^2) *be the solutions of the corresponding BSDEs, if*

$$\xi_1 \geq \xi_2,\ g_1(t, Y_t^2, Z_t^2) \geq g_2(t, Y_t^2, Z_t^2), a.s., a.e.$$

Then we have

$$Y_t^1 \geq Y_t^2, a.s.$$

And under the above conditions,

$$Y_0^1 = Y_0^2 \Leftrightarrow \xi_1 = \xi_2 \ and \ g_1(t, Y_t^2, Z_t^2) = g_2(t, Y_t^2, Z_t^2), a.s., a.e.$$

- This is a generalized version.
- It is One of the most important theorems about BSDE, it "plays the same role that the maximum principle in PDE" (El Karoui).
- It rules out "Arbitrage Opportunities" when using BSDE in pricing.

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$$\xi_1 \geq \xi_2,\ g_1(t, Y_t^2, Z_t^2) \geq g_2(t, Y_t^2, Z_t^2), a.s., a.e.$$

Then we have

$$Y_t^1 \geq Y_t^2, a.s.$$

And under the above conditions,

$$Y_0^1 = Y_0^2 \Leftrightarrow \xi_1 = \xi_2 \text{ and } g_1(t, Y_t^2, Z_t^2) = g_2(t, Y_t^2, Z_t^2), a.s., a.e.$$

- This is a generalized version.
- It is One of the most important theorems about BSDE, it "plays the same role that the maximum principle in PDE" (El Karoui).
- It rules out "Arbitrage Opportunities" when using BSDE in pricing.

Some Basic Properties of BSDE and g-Expectations

Theorem (Axiomatic Properties of g-Evaluation and g-Expectation)

(P1) *(Monotonicity)* ${\mathbb{E}^g}_{t,T}[\xi] \leq E^g_{t,T}[\eta]$, *if* $\xi \leq \eta$;

(P2) *If* $\xi \in L^2(\mathcal{F}_t)$, *then,* ${\mathbb{E}^g}_{t,t}[\xi] = \xi$, *and for g-expectation,* ${\mathbb{E}^g}_{t,T}[\xi] = \xi, \ a.s.$

(P3) *(Time Consistency)* ${\mathbb{E}^g}_{s,t}[{\mathbb{E}^g}_{t,T}[\xi]] = {\mathbb{E}^g}_{s,T}[\xi], s \leq t$;

(P4) *("Zero-One Law") If g satisfies (A3),* ${\mathbb{E}^g}_{t,T}[I_A\xi] = I_A{\mathbb{E}^g}_{t,T}[\xi], A \in \mathcal{F}_t$;

(P5) *(Translation Invariance) For g-expectation with g independent of y,* $\mathbb{E}^g[\xi + \eta|\mathcal{F}_t] = \mathbb{E}^g[\xi|\mathcal{F}_t] + \eta, \ a.s.$ *where* $\eta \in L^2(\mathcal{F}_t)$.

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Some Basic Properties of BSDE and g-Expectations

Theorem (pseudo-linearity)

For $0 \leq s \leq t \leq T, X, \; Y \in L^2(\mathcal{F}_t)$ *and* $A \in \mathcal{F}_s$, *we have*

$$\mathbb{E}^g{}_{s,t}[X l_A + Y l_{A^c}] = \mathbb{E}^g{}_{s,t}[X] l_A + \mathbb{E}^g{}_{s,t}[Y] l_{A^c}$$

Theorem (Monotonic Limit Theorem)

Consider the following sequence of BSDEs:

$$y_t^i = {y_T}^i + \int_t^T g(s, y_s^i, z_s^i)ds + (A_T^i - A_t^i) - \int_t^T z_s^i dB_s$$

where g satisfies(A1) and(A2), A^i is a continuous increasing process with A_0^i=0 and $A_T^i \in L^2(\mathcal{F}_T)$. If y^i converges monotonically up to a process y as $i \to \infty$, and $E[esssup_{0\le t\le T}|y_t|^2] < \infty$. Then there exists stochastic processes z and A, s.t.

$$y_t = y_T + \int_t^T g(s, y_s, z_s)ds + (A_T - A_t) - \int_t^T z_s dB_s$$

Some Basic Properties of BSDE and g-Expectations

Determine the generator by the evaluation value......

Chen, Z. *C.R.Acad.Sci.Paris, SerieI* 326(4): 483-488 (1998)

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- *(A1)(A2)*, $g^1 \geq g^2 \Leftrightarrow \mathbb{E}^{g^1}_{s,t}[\xi] \geq \mathbb{E}^{g^2}_{s,t}[\xi]$, *for all* ξ *and* s, t;
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Some Basic Properties of BSDE and g-Expectations

Definition (Martingale under g-Evaluation)

An $\mathcal{F}$-progressively measurable real valued process y with

$$E(ess \sup_{0 \leq t \leq T} |y_t|^2) < \infty,$$

is called a g-martingale(under g-evaluation) if for $\forall 0 \leq s \leq t \leq T$,

$$\mathbb{E}^g{}_{s,t}(y_t) = y_s, a.s.$$

- We can also define analogously g-supermartingale and g-submartingale

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Some Basic Properties of BSDE and g-Expectations

Theorem (Optional Stopping)

Assume the generater g satisfies (A1)(A2), and y is a g-supermartingale(resp. submartingale). Then for every tow stopping times $\sigma, \tau \leq T$ with $\sigma \leq \tau$, we have

$$\mathbb{E}^g{}_{\sigma,\tau}(y_\tau) \leq y_\sigma, a.s.$$

Theorem (Doob-Meyer Type Decomposition Theorem)

Assume that g satisfies (A1)(A2).Let (y_t) be a right continuous g-supermartingale on [0,T]. Then there exists a unique RCLL increasing process (A_t) with $E[A_T^2] < \infty$ and $A_0 = 0$, such that (y_t) coincides with the unique solution (y_t) of the following BSDE.

$$y_t = y_T + \int_t^T g(s, y_s, z_s)ds + (A_T - A_t) - \int_t^T z_s dB_s, \quad t \in [0, T].$$

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- BSDE driven by general Levy process
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Forward-Backward Stochastic Differential Equations

$$\begin{cases} dX_t = b(t, X_t, Y_t, Z_t)dt + \sigma(t, X_t, Y_t, Z_t)dB_t, \\ dY_t = h(t, X_t, Y_t, Z_t)dt - Z_t dB_t, \\ X(0) = x, Y(T) = g(X_T). \end{cases}$$

⋄ Antonelli,F, Backward-Forward Stochastic Differential Equations, Ann. App. Prob., 3(1993), 777-793;

⋄ Ma,J and Yong,J, FBSDE and their applications, springer, 1999.

These equations closely connected with PDEs, thus have lead to interesting results:

- Stochastic representations for PDEs; Generalizing Feynman-Kac formula...
- Monte-Carlo numerical method; Jianfeng Zhang; V.Bally; G.Pages; Nizar Touzi......

In a financial market, think of a big investor......

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Forward-Backward Stochastic Differential Equations

Theorem (Existence Theorem in Small "Time Durations")

If the parameters for FBSDE, always denoted ϕ, satisfy:

1), uniformly Lipschitz in (x,y,z);

2), given (x,y,z), is progressively measurable(b,h,σ) or measurable in $\mathcal{F}_T$ (for g)

3), $\phi(t, 0, \omega) \in L^2_{\mathcal{F}}$

4), the Lipschitz constants of σ and g, L_1, L_2: $L_1 L_2 < 1$

Then there exists a $T_0 > 0$, s.t.for any $T \in (0, T_0]$ and initial value x, FBSDE admits a unique adapted solution $(X, Y, Z) \in (S^2_{\mathcal{F}} \times S^2_{\mathcal{F}} \times L^2_{\mathcal{F}})$.

- FBSDE for large time duration might unsolvable.

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Reflected Backward Stochastic Differential Equations

Reflected Backward Differential Equations (RBSDE) was first introduced in 1997:

El Karoui, N., Kapoudjian, C., Pardoux, E., Peng S. and Quenez, M.C., Reflected Solutions of Backward SDE and Related Obstacle Problems for PDEs, Ann. Probab. 25, no2, 702-737, 1997.

A solution for RBSDE with (ξ, g, L_t), is a triple (Y, Z, K) satisfying

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s$$

and $Y_t \geq L_t$ on $[0, T]$.

(K_t) is a nondecreasing continuous process, s.t.

$$\int_0^T (Y_s - L_s)dK_s = 0$$

- RBSDEs with one continuous barrier;
- RBSDEs with two continuous barriers;
- RBSDEs with L^2 barriers;
- RBSDEs with continuous barriers under non-Lipschitz condition.

And the applications:

- The stochastic representation of solutions of PDE with obstacle(s);
- Applications in Finance:
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∗ And for the RBSDEs with two continuous barriers;

A solution is a quadruple (Y, Z, K^+, K^-), which satisfies

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds + K_T^+ - K_t^+ - (K_T^- - K_t^-) - \int_t^T Z_s dB_s$$

$L_t \leq Y_t \leq U_t$ on $[0, T]$.

(K_t^+, K_t^-) are non decreasing and continuous,

$$\int_0^T (Y_s - L_s)dK_s^+ = 0 \text{ and } \int_0^T (Y_s - U_s)dK_s^- = 0.$$

Reflected Backward Stochastic Differential Equations

Theorem

Let (Y, Z, K) be the solution of RBSDE, then

$$Y_t = ess \sup_{\tau \in \mathcal{T}_t} E[\int_t^\tau g(s, Y_s, Z_s)ds + L_\tau 1_{\{\tau < T\}} + \xi 1_{\{\tau = T\}} | \mathcal{F}_t],$$

where $\mathcal{T}_t$ is the set of all stopping times valued in $[t, T]$.

Reflected Backward Stochastic Differential Equations

Theorem

Let (Y, Z, K) be the solution of RBSDE with 2 continuous barriers, then, for any $0 \leq t \leq T$ and any stopping times $\tau,\ \sigma\ \in \mathcal{T}_t$, consider

$$R_t(\sigma,\tau) = \int_t^{\sigma\wedge\tau} g(s, Y_s, Z_s)ds + \xi 1_{\{\sigma\wedge\tau=T\}} + L_\tau 1_{\{\tau<T,\tau\leq\sigma\}} + U_\sigma 1_{\{\sigma<\tau\}}.$$

$$\overline{V}_t = ess \inf_{\sigma\in\mathcal{T}_t} ess \sup_{\tau\in\mathcal{T}_t} E[R_t(\sigma,\tau)|\mathcal{F}_t],\ \underline{V}_t = ess \sup_{\tau\in\mathcal{T}_t} ess \inf_{\sigma\in\mathcal{T}_t} E[R_t(\sigma,\tau)|\mathcal{F}_t]$$

Then $Y_t = \overline{V}_t = \underline{V}_t$

Application in Pricing American Options:

⋄ El Karoui, N., Pardoux, E., and Quenez, M.C., Reflected Backward SDEs and American Options, Numerical methods in finance, Newton Inst. Cambridge Univ. Press, Cambridge, 215-231, 1997.

"In some constraint cases,the strategy wealth portfolio (X_t, π) ...satisfy the following BSDE

$$-dX_t = b(t, X_t, \pi_t)dt - \pi_t^* \sigma_t dW_t$$

Here b... convex with respect to x,π. ...suppose that the volatility is invertible and that $(\sigma_t)^{-1}$ is uniformly bounded..."

Reflected Backward Stochastic Differential Equations

Consider the payoff process:

$$\widetilde{S}_s = \xi 1_{\{s=T\}} + S_s 1_{\{s<T\}}.$$

Fix $t \in [0, T], \tau \in \mathcal{T}_t$; then exists a unique strategy $(X_s(\tau, \widetilde{S}_\tau), \pi(\tau, \widetilde{S}_\tau))$, which replicate $\widetilde{S}_\tau$, i.e. for some coefficient b

$$\begin{aligned} -dX_s^\tau &= b(s, X_s^\tau, \pi_s^\tau)ds - (\pi_s^\tau)^* dB_s, 0 \leq s \leq T, \\ X_\tau^\tau &= \widetilde{S}_\tau. \end{aligned}$$

Then the price of the American contingent claim $(\widetilde{S}_s, 0 \leq s \leq T)$ at time T is given by

$$X_t = ess \sup_{\tau \in \mathcal{T}_t} X_t(\tau, \widetilde{S}_\tau).$$

And under the convex assumption, it can be proved that:

$$X_t = ess \sup_{\tau \in \mathcal{T}_t} E[\int_t^\tau b(s, X_s, \pi_s)ds + S_\tau 1_{\{\tau<T\}} + \xi 1_{\{\tau=T\}} | \mathcal{F}_t],$$

Application in Pricing Game Options:

⋄ Kifer, Y. Game option Finance Stochatic, 4, 442-463, 2000.

⋄ Hamadene S. and Lepeltier, J.-P. Reflected BSDEs and mixed game problem, Stochastics Processes Appl. 85, 177-188. 2000

Reflected Backward Stochastic Differential Equations

- **A**: can choose cancellation time $\sigma \in \mathcal{T}$
- **B**: choose exercise time $\tau \in \mathcal{T}$
- **Payoffs**: $\infty \geq L_t \geq U_t \geq 0$
- Thus A pays B:

$$R(\sigma, \tau) = L_\sigma l_{[\sigma < \tau]} + U_\tau l_{[\tau \leq \sigma]}$$

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And the fair price?

$$V_{t,T} = ess \inf_{\sigma \in \mathcal{T}_t} ess \sup_{\tau \in \mathcal{T}_t} E[e^{-r\sigma \wedge \tau} R_t(\sigma, \tau) | \mathcal{F}_t]$$

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Feynman-Kac Formula

⋄ Pardoux, E., Peng, S., Backward stochastic differential equations and quasilinear parabolic partial differential equations. Lecture Notes in CIS, Vol. 176. Springer-Verlag, 200 – 217, 1992.

⋄ Pardoux, E., Peng, S., Backward doubly stochastic differential equations and systems of quasilinear parabolic SPDEs, Probab. Theory Rel. Fields 98, 209 – 227, 1994.

Consider the following coupled FBSDE:

$$\begin{cases} dX_s^{x,t} = b(X_s^{x,t})ds + \sigma(X_s^{x,t})dW_s, \ s \in [t, T] \\ X_t^{x,t} = x. \end{cases}$$

$$\begin{cases} -dY_t^{x,t} = g(X_s^{x,t}, Y_s^{x,t}, Z_s^{x,t})ds - Z_s^{x,t}dW_s, \ s \in [t, T] \\ Y_T^{x,t} = \Phi(X_T^{x,t}). \end{cases}$$

Assumptions:

(H1) $b,\ \sigma$ satisfies conditions s.t. the SDE has a unique strong solution;

(H2) g satisfies conditions s.t. the BSDE has a unique solution for every $(x,t) \in R^n \times [0,T]$;

(H3) $b,\ \sigma,\ g$ are deterministic functions.

Set $u(x,t) \triangleq \mathbb{E}^g{}_{t,T}[\Phi(X_T^{x,t})]$.

Feynman-Kac Formula

Theorem

Assume (H1)(H2)(H3), then u is a deterministic function. And if $u \in C^{2,1}$, then it satisfies:

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{L}u + f(x, u, \sigma^T \nabla u) = 0, \ (x, t) \in \mathbb{R}^n \times [0, T], \\ u(x, T) = \Phi(x), \end{cases}$$

where, $\mathcal{L}\phi = \frac{1}{2} \sum_{i,j=0}^n a_{i,j}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial \phi}{\partial x_i}, \ \ a_{i,j} = [\sigma \sigma^T]_{i,j}.$

Conversely, if the PDE has a $C^{2,1}$-solution, then the solution is unique and is u.

Feynman-Kac Formula

Theorem

If (H3) is not satisfied, then u is a $\mathcal{F}$-adapted process. And if u is smooth enough, then it satisfies:

$$\begin{cases} -du(t,x) = [\mathcal{L}u(t,x) + g(t,x,u,Du\sigma + \phi) + D\phi\sigma]dt - (Du\sigma + \phi)dWt, \\ u(x,T) = \Phi(x), \end{cases}$$

and conversely, if (u,ϕ) is the solution of BSPDE, then $(u(s,X_s^{t,x}),(Du\sigma + \phi)(s,X_s^{t,x}))$ is the solution of the related BSDE.

Applications:

- Monte-Carlo methods for PDE;
- Using numerical methods of PDE in solving BSDE.

Example

Consider the Black-Scholes type example. Pricing a contingent claim with a terminal value $(P(T) - q)^+$, we can use the Feynman-Kac formula:

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{L}u + g(p, u, \sigma^T \nabla u) = 0, \\ u(p, T) = (p - q)+, \end{cases}$$

The fair price is just $y_0 = u(P(0), 0)$.

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First let us look at two examples.

Example ("Premium Ambiguity")

Suppose $dV_t = \frac{dP_t}{P_t}$*, and* $r = 0$*,*

$V_t = V_0 + \int_0^t \sigma_s^\top (dB_s + \theta_s ds)$

where θ *denotes the risk premium,*

let

$$\frac{dQ_\theta}{dp} \triangleq \exp[\int_0^T \theta_t dB_t - \frac{1}{2}\int_0^T |\theta_t|^2 dt],$$

then

$$V_0 = E_{Q_\theta}[V_T],$$

for the uncertainty of premium, we can give the price to a stochastic asset X *as* $\sup_{Q_\theta} E_{Q_\theta}[X], \theta_t \in \Theta$.

Sublinear Expectations: G-Expectation

In fact, this can be write into g-expectation frame work:

$$\mathbb{E}^g[X] = \sup_{Q_\theta} E_{Q_\theta}[X] = Y_0$$

$\mathbb{E}^g : L^2(\Omega, \mathcal{F}_T, P) \to \mathbb{R}$

where the g-expectation is derived from a BSDE with generator:

$$g(t, z) \triangleq \sup_{\theta \in \Theta} \sigma_t^{-1} \theta |z|$$

Recall the generalized Feynman-Kac Formula (we might suppose that $\sigma = I$):

$$u(x,t) \triangleq E[\Phi(X_T^{x,t})],$$

then $u(x,t)$ is the solution of

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{K}u \\ u(0,x) = \Phi(x). \end{cases}$$

where

$$\mathcal{K}\phi = \frac{1}{2}\sum_{i,j=0}^{n} \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sup_{\theta \in \Theta} \sum_{i=1}^{n} \theta_i \frac{\partial \phi}{\partial x_i}$$

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Example

Consider the following nonlinear parabolic PDE:

$$\begin{cases} \frac{\partial u}{\partial t} = G(\frac{\partial^2 \phi}{\partial x^2}) \\ u(0,x) = \Phi(x). \end{cases}$$

where

$$G(x) = \frac{1}{2} \sup_{\sigma \in [\underline{\sigma},\ \bar{\sigma}]} (\sigma^2 x).$$

And define $\mathbb{E}[\Phi] \triangleq u(1,0)$

- Note that this kind of nonlinear expectation can not be described by g-expectation, and models the so-called ”volatility ambiguity”.

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⋄ Peng, S., G-Expectation, G – Brownian motion and related stochastic calculus of Itô's type, Preprint,(pdf-file available in arXiv:math.PR/0601035v1 3 Jan 2006).

⋄ Peng, S., Multi-dimensional G – Brownian motion and related stochastic calculus under G – expectation, Preprint,(pdf-file available in arXiv:math.PR/0601699 v1 28 Jan 2006).

⋄ Peng, S., G – Brownian Motion and Dynamic Risk Measure under Volatility Uncertainty, Lecture notes in Mini-course of CSFI, Osaka University, 2007.

- $(\Omega, \mathcal{F}, P) \dashrightarrow (\Omega, \mathcal{H}, \mathbb{E})$, with $\mathcal{H}$ a set of randem variables and $\mathbb{E}$ a nonlinear expectation;
- "Probability Language" ⇢ "Expectation Language"

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- Distribution?
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Definition (G-Normal Distribution)

A random variable ξ *in a sub-expectation space* $(\Omega,\ \mathcal{H},\ \mathbb{E})$ *is called* $G_{\underline{\sigma},\bar{\sigma}}$*-normal distributed, denoted by* $\xi \sim \mathcal{N}(0;[\underline{\sigma}^2,\overline{\sigma}^2])$*, for a given pair* $0 \le \underline{\sigma} \le \overline{\sigma}$*, if for each* $\phi \in C_{l,lip}(R)$*, the following function defined by*

$$u(t,x) \triangleq \mathbb{E}[\phi(x+\sqrt{t}\xi)],\ (t,x) \in [0,\infty) \times R$$

is the unique viscosity solution of the PDE in Example2.

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Thank you!