The Strategic Use of Ambiguity in Games

Frank Riedel Linda Sass

[1]Institute for Mathematical Economics
Bielefeld University

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Outline

Uncertainty versus Risk

- Knightian uncertainty versus (objective) risk
- objective probability versus no probabilities, just uncertain outcomes
- classical approach (Savage, Anscombe-Aumann): even under uncertainty, betting behavior allows to infer *subjective* probability measure P
- since Gilboa–Schmeidler JME 1989, large decision-theoretic literature on ambiguity aversion

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Uncertainty in Games

Classic Game Theory

- Players use pure strategies . . .
- and mixed strategies = roulette wheels
- evaluate payoffs according to expected utility

Our Approach: Ellsberg Urns as Strategies in Uncertainty–Averse Environments

- imagine a player is allowed to use an Ellsberg urn
- . . . credibly, and commit to use it
- for example, through a trustworthy laboratory
- what are the consequences for noncooperative games?

Literature

- Aumann JME 1974, Subjectivity and Correlation in Randomized Strategies
- Bade, GEB 2010, Ambiguous Act Equilibria in two Player Games
- Greenberg, T & D 2000, The Right to Remain Silent
- Dow, Werlang, Eichberger, Kelsey, Klibanoff, Lo, Marinacci . . .

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Ellsberg Urns as Strategies

Let $N = \{1, ..., n\}$ be the set of players. Each player has a finite strategy set $S_i, i = 1, \ldots, N$. Players' payoffs are given by functions

$$u_i : S \to \mathbb{R} \qquad (i \in N)\,.$$

- objective mixed strategies ΔS_i as usual
- Ellsberg urns:
  - player i chooses $(\Omega, \mathscr{F}, \mathscr{P})$
  - $\Omega \neq \emptyset$, $\mathscr{F}$ σ–field, $\mathscr{P}$ set of probability measures on the measurable space $(\Omega, \mathscr{F})$
  - Ellsberg strategy is such a model plus a measurable act $f_i : \Omega \to \Delta S_i$

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Ellsberg Game

- All players are ambiguity–averse in the sense of Gilboa and Schmeidler
- (and this is common knowledge ...)
- payoff of player i for profile $(f_1, \ldots, f_N)$ (and Ellsberg urns $(\Omega_j, \mathscr{F}_j, \mathscr{P}_j)$)

$$U_i(f) = \min_{P_1 \in \mathscr{P}_1, \ldots, P_n \in \mathscr{P}_n} \int_\Omega u_i(f(\omega)) dP_1 \ldots dP_n .$$

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Ellsberg Equilibrium

An *Ellsberg equilibrium* is a profile of Ellsberg urns $\left((\Omega^1, \mathscr{F}^1, \mathscr{P}^1), \ldots, (\Omega^n, \mathscr{F}^n, \mathscr{P}^n)\right)$ and acts $f^* = (f_1^*, \ldots, f_n^*)$ such that no player has an incentive to deviate, i.e. for all players $i \in N$ and all Ellsberg urns $(\Omega_i, \mathscr{F}_i, \mathscr{P}_i)$, and all acts f_i for player i we have

$$U_i(f^*) \geq U_i(f_i, f_{-i}^*) \,.$$

Ellsberg Equilibrium: Reduced Form

As in correlated equilibrium, only the laws of the acts f_i on ΔS_i matter; one can thus go directly to sets of probability measures on ΔS_i

Definition

A *reduced form Ellsberg equilibrium* is a profile of sets of probability measures $\mathscr{P}_i^* \subseteq \Delta S_i$, such that for all players $i \in N$ and all sets of probability measures $\mathscr{P}_i$ on S_i we have

$$\min_{P_1 \in \mathscr{P}_1^*, \ldots, P_n \in \mathscr{P}_n^*} \int_S u_i(s) dP_1 \ldots dP_n \geq \min_{P_i \in \mathscr{P}_i, P_{-i} \in \mathscr{P}_{-i}^*} \int_S u_i(s_i, s_{-i}) dP_1 \ldots dP_n$$

General Remarks

- One does not improve one's own payoff by introducing more ambiguity (as players are ambiguity–averse).
- Consequence: if all other players play classic Nash, the Nash equilibrium strategy is a best reply as well
- Ellsberg equilibrium is a coarsening of the concept of Nash equilibrium
- in particular, Ellsberg equilibria exist
- can one get interesting equilibria outside the set of Nash equilibria?

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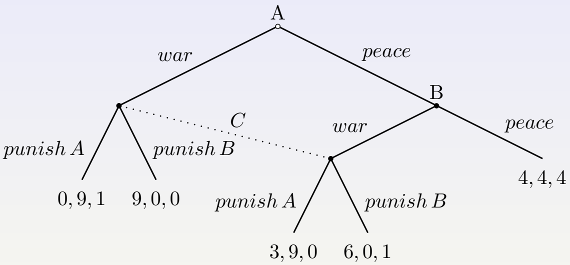
Greenberg's Example



- Unique Nash Equilibrium: A mixes uniformly, B plays War, C mixes uniformly
- war occurs with probability 1

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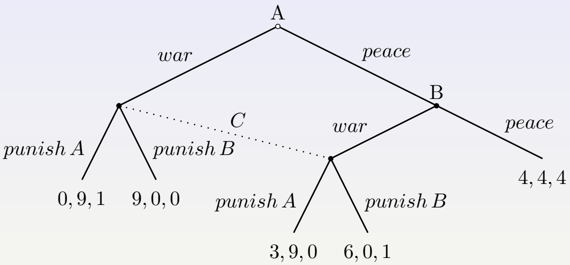

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Greenberg's Example in Normal Form

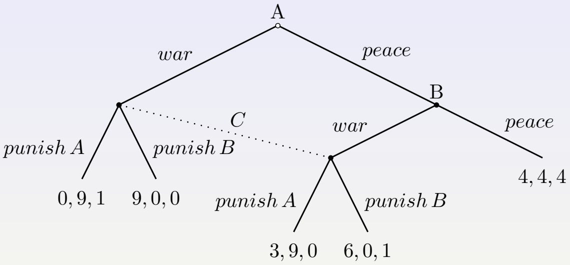
	war	*peace*
war	0, 9, 1	0, 9, 1
peace	3, 9, 0	4, 4, 4

Punish A

	war	*peace*
war	9, 0, 0	9, 0, 0
peace	6, 0, 1	4, 4, 4

Punish B

Greenberg's Example: Ellsberg Equilibrium



- Suppose C plays the Ellsberg strategy $[0, 1]$, i.e. all probabilities between 0 and 1 possible for "Punish A"
- if A playes War, minimal expected payoff 0
- ($peace$, $peace$, $[0, 1]$) is an Ellsberg equilibrium

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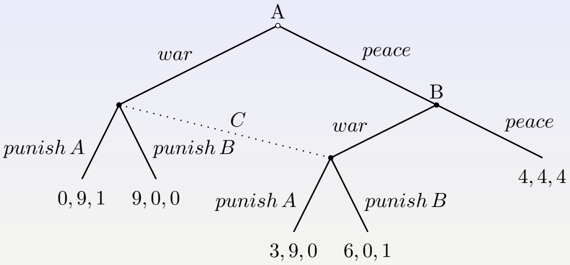

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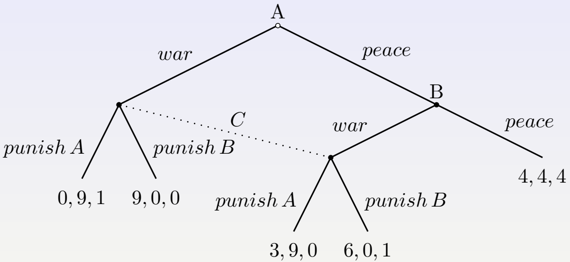

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- there exist Ellsberg equilibria that lie outside the union of the support of all Nash equilibria
- empirically testable conclusion
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Two–Player Games

- Bade 2010: the support of Ellsberg equilibria is contained in the union of supports of Nash equilibria
- so “indistinguishable”? We do not think so. Interesting effects:
- nonlinear payoffs
- immunization against strategic ambiguity
- equilibria “easier” to play than classical mixed strategy equilibria

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Modified Matching Pennies

	HEAD	TAIL
HEAD	3,-1	-1,1
TAIL	-1,1	1,-1

- unique Nash: player 1 plays HEAD with 1/2, layer 2 with 1/3,
- equilibrium payoffs 1/3 and 0

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Ellsberg Equilibria

Full ambiguity is not an equilibrium

- suppose player 2 uses an Ellsberg strategy that allows for any $0 \leq Q \leq 1$ for HEAD
- player 1 has then a unique best reply
- play HEAD with probability $1/3$
- if she does so, payoff always $1/3 \cdot (3Q - (1 - Q)) + 2/3 \cdot (-q + (1 - q)) = 1/3$, independent of Q
- immunization against Q

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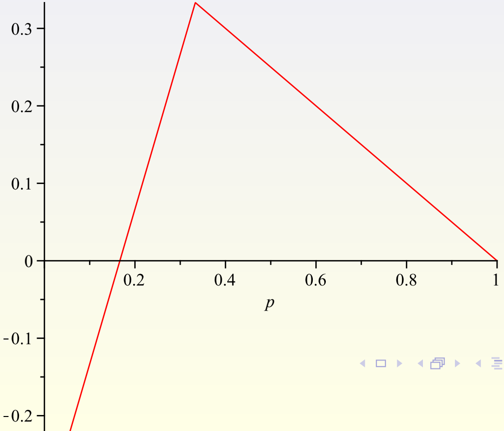
Nonlinear Payoffs

Suppose Player 2 plays the Ellsberg strategy [q0,q1] with q0<1/3<q1, say q0=1/4,q1=2/3

Then the expected payoff from playing p for HEAD is

$$plot\left(\min\left(\frac{3\cdot p\cdot 1}{4}-\frac{p\cdot 3}{4}-\frac{(1-p)\cdot 1}{4}+\frac{(1-p)\cdot 3}{4},\ \frac{3\cdot p\cdot 2}{3}-\frac{p\cdot 1}{3}-\frac{(1-p)\cdot 2}{3}+\frac{(1-p)\cdot 1}{3}\right), p=0..1\right);$$



Ellsberg Equilibria II

How do the Ellsberg equilibria look like?

- player 1 plays HEAD with probability $P \in [1/2, P_1], P_1 \geq 1/2$
- player 2 plays HEAD with probability $Q \in [1/3, Q_1], Q_1 \leq 1/2$
- equilibrium payoffs are the same as in Nash equilibrium
- "easier" to play: player 1 has to play HEAD with probability 50 % or more
- broader prediction than Nash, but still restrictive prediction
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