Viability and Arbitrage under Knightian Uncertainty

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Outline

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1. Introduction and Outline
2. Viability under Risk
3. Viability and Arbitrage under Uncertainty
    - Some Issues under Uncertainty
    - Our Model
    - Viability and Arbitrage
    - Sublinear Martingale Expectations and the FTAP
4. The Efficient Market Hypothesis

The Relation of Economics and Finance

- Mathematical Finance:
  - Take a probabilistic model of asset prices as given
  - $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ filtered probability space, (S_t^d) adapted processes
  - impose no arbitrage
  - develop a theory of derivative prices
- Economics:
  - asset prices are endogenous objects
  - derived by demand and supply on a competitive market

Given an arbitrage–free finance model (S_t^d), can one construct a plausible economy such that (S_t^d) are equilibrium prices?

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- or in a “model–free”
- or robust setting
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- Bouchard, Nutz 2015, Burzoni, Fritelli, Maggis 2016 discuss arbitrage in a non–dominated setting
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Harrison, Kreps, Martingales and arbitrage in multiperiod securities markets, Journal of Economic Theory, 1979

Framework: Filtered Probability Space $(\Omega, \mathcal{F}, P, /\mathcal{F}_t))$

- space of contingent claims is $L^2(P)$
- P fixes the notion of "similar commodities", i.e. the topology
- and the notion of "negligible event", here: P-null sets
- and the notion of order, here P-a.s. greater or equal

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Conceivable Agents

- $\mathcal{A}$ is the set of preferences $\preceq$ (complete, transitive orderings) on $\mathcal{X} = L^2(P)$ satisfying convexity, continuity, and strict monotonicity:
- for all $X \in \mathcal{X}$, the upper contour set $\{Z \in \mathcal{X} : X \preceq Z\}$ is convex,
- for all $X \in \mathcal{X}$, the upper contour set $\{Z \in \mathcal{X} : X \preceq Z\}$ is closed under L^2–convergence,
- if $P[R \geq 0] = 1$ and $P[R > 0] > 0$, then $X \prec X + R$ for all $X \in \mathcal{X}$

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Viability and Arbitrage under Risk

Financial Market

- let $(\mathcal{F}_t)_{t=0,\dots,T}$ be a filtration with $\mathcal{F}_0$ trivial, $\mathcal{F}_T \subseteq \mathcal{F}$
- let $S^0_t = 1$ be a numéraire,
- for $d = 1, \dots, D$, let $S^d = (S^d_t)_{t=0,\dots,T}$ be adapted, positive asset prices
- gains from trade for a self–financing portfolio $\theta = (\theta_t)$

$$G^\theta = \sum_{t=1}^{T} \theta_t \cdot \Delta S_t$$

- θ is an arbitrage if $P[G^\theta \geq 0] = 1$ and $P[G^\theta > 0] > 0$

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Viability and Arbitrage under Risk

Representative Agent Equilibrium

- The utility maximization problem for a conceivable agent $\preceq$ is well–posed (at 0) if for every self–financing portfolio θ we have $G^{\theta} \preceq 0$
- The financial market S is viable if for some conceivable agent $\preceq\in\mathcal{A}$, the utility maximization problem is well–posed at 0.
- (Then the market consisting of this “representative agent” is in equilibrium.)

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Viability and Arbitrage under Risk

Theorem (Harrison, Kreps 1979)

The financial market S is viable if and only if there is no arbitrage.

Viability and Arbitrage under Risk

Viability implies no arbitrage

The strict upper contour set at 0 is convex and open in $L^2(P)$ and disjoint from all gains from trade. By the separation theorem, there exists a L^2–continuous, P-strictly positive linear functional that separates the sets. This allows to define an “equivalent martingale measure”, hence no arbitrage.

Viability and Arbitrage under Risk

No arbitrage implies viability

- Modern version: by Dalang, Morton, Willinger 1990, FTAP, there exists an equivalent martingale measure P^*.
- Define a linear preference relation $\preceq$ by

$$X \preceq Y \text{ iff } E^{P^*} X \le E^{P^*} Y$$

- $\preceq$ bis $L^2(P)$–continuous and P–strictly increasing (equivalence!)
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Knightian Uncertainty - Issues

Consider the robust model in which uncertainty is described by a non–dominated class of probability measures $\mathcal{P}$.

- Typical utility function (Gilboa, Schmeidler): $U(X) = \inf_{P \in \mathcal{P}} E^P u(X)$
- θ arbitrage (Vorbrink 2014, Bouchard, Nutz 2015) if
  - $P[G^\theta \geq 0] = 1$ $\mathcal{P}$–quasi surely
  - for some $P \in \mathcal{P}$, $P[G^\theta > 0] > 0$

Knightian Uncertainty - Issues

- Then the utility maximization can be well posed at 0 even if there is arbitrage, $U(G^\theta) = U(0)$
- because $P_0[G^\theta = 0] = 1$ for the worst–case measure P_0
- there do not exist strictly positive linear functionals (compare Beissner, Denis, 2018)
- Even the more general approach of Kreps, 1981 does not apply.

Knightian Uncertainty - Issues

- there is no hope to construct a representative agent equilibrium supporting an arbitrage–free market
- can there be arbitrage in equilibrium?

Knightian Uncertainty - New Approach

The common ordering

- $(\Omega, \mathcal{F})$ measurable space
- $(\mathcal{H}, \tau, \leq)$ pre–ordered topological vector space of measurable functions containing the constants
- $Z \in \mathcal{H}$ is negligible if $Z \leq 0$ and $Z \geq 0$

Marketed Space

The zero cost trades are given by a convex cone $\mathcal{I}$

1. Usually, the set $\mathcal{I}$ consists of (suitably restricted) stochastic integrals
2. of the form $G^\theta = \sum_{t=1}^T \theta_t \cdot \Delta S_t$ in discrete time
3. In Harrison–Kreps, the market is described by a marketed space $M \subset L^2(\Omega, \mathcal{F}, P)$ and a (continuous) linear functional π on M. In this case, $\mathcal{I}$ is the kernel of the price system, i.e.

$$\mathcal{I} = \{X \in M : \pi(X) = 0\} .$$

Relevant Contracts

A non–empty, convex set $\mathcal{R}$ of $\leq$–nonnegative payoffs describes the relevant contracts.
$\mathcal{R}$ contains all strictly positive constant contracts

Relevant Contracts

Examples

- probabilistic model: $\mathcal{R}$ contains the non-zero a.s. nonnegative random variables

$$P[X \geq 0] = 1, P[X > 0] > 0$$

- multiple prior uncertainty: $\mathcal{R}$ contains the non-zero q.s. nonnegative random variables

$$P[X \geq 0] = 1 \text{ for all } P \in \mathcal{P}$$
$$P[X > 0] > 0 \text{ for some } P \in \mathcal{P}$$

- $\mathcal{R} = (0, \infty)$

Agents

The set $\mathcal{A}$ of conceivable agents consists of all preference relations on $\mathcal{H}$ that are

- weakly monotone with respect to the order $\leq$
- convex
- τ–lower semicontinuous: for every sequence $X_n \to X$ with $X_n \preceq Y$ for all $n \in \mathbb{N}$, we have $X \preceq Y$

Viability

Definition

A financial market $(\mathcal{H}, \tau, \leq, \mathcal{I}, \mathcal{R})$ is *viable* if there exists a family of agents $\{\preceq_a\}_{a \in A} \subset \mathcal{A}$ and net trades $(\ell_a^*)_{a \in A} \subset \mathcal{I}$ such that

- l_a^* is optimal for each agent $a \in A$, i.e.

$$\forall a \in A,\ \ell \in \mathcal{I} \quad \ell \preceq_a \ell_a^*, \tag{1}$$

- the market clears, i.e. $\sum_{a \in A} l_a^* = 0$,
- for every relevant contract $R \in \mathcal{R}$ there exists an agent $a \in A$ such that $\ell_a^* \prec_a \ell_a^* + R$

Remarks

The market needs to see relevant contracts

- new property was free in probabilistic setting
- equivalent martingale measures “see” every non-zero positive random variable

Arbitrage

Definition

1. $l^* \in \mathcal{I}$ is an arbitrage if there exists $R \in \mathcal{R}$ with $l \geq R$
2. A sequence $(l_n) \subset \mathcal{I}$ is a free lunch with vanishing risk if there exist a sequence $c_n \downarrow 0$ and $R \in \mathcal{R}$ such that $c_n + l_n \geq R$.

Equivalence

Theorem

A financial market is viable if and only if there is no arbitrage.

The proof is based on strictly positive sublinear instead of linear preferences.

Sublinear Martingale Expectations

Definition

A functional $\mathcal{E} : \mathcal{H} \to \mathbb{R} \cup \{-\infty, \infty\}$ is a sublinear expectation if it is $\leq$-monotone, cash–additive, and sublinear.

$\mathcal{E}$

- is *absolutely continuous*, if $\mathcal{E}(Z) = 0$ for every negligible Z .
- has *full support* if $\mathcal{E}(R) > 0$, for every $R \in \mathcal{R}$.
- has the *(super–)martingale property* if $\mathcal{E}(\ell) \leq 0$ for every $\ell \in \mathcal{I}$.

Fundamental Theorem of Asset Pricing

The viability theorem is closely connected to the fundamental theorem of asset pricing. Let $\mathcal{H}$ be the set of bounded, measurable functions.

Theorem

A financial market is viable if and only if there exists a lower semicontinuous sublinear martingale expectation with full support.

The sublinear martingale expectation is able to "see" all relevant contracts in the case when no strictly positive linear functionals exist.