The Non–Implementability of Arrow-Debreu Equilibria under Knightian Uncertainty

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Outline

1. Informal Presentation of Results
2. Equivalence under Risk
3. Equilibria under Volatility Uncertainty

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Storyboard

- Financial Market with ambiguity–averse agents
- Individuals face risk and ambiguity at the individual level
- no aggregate ambiguity
- existence of static Arrow–Debreu equilibrium
- Characterization of Equivalence to Radner Equilibrium
- generically, no Radner implementation

Equilibrium in Uncertain Models

Static versus Dynamic Equilibrium

- Arrow–Debreu equilibrium: agents trade efficiently all contingent plans at time 0
- somewhat unrealistic market institution
- perfect insurance in Arrow-Debreu equilibrium
- can these efficient equilibria be implemented by dynamic trading in suitably chosen assets?

Equilibrium in Uncertain Models

Equivalence Results

- The answer is yes in the classic setting at different levels if the asset market is dynamically complete:
  - real assets, endogenous asset prices, "most demanding case"

    Anderson–Zame 2005, Riedel–Herzberg 2014, Malamud et al. 2013
  - "intermediate" case: no dividends, bond

    Duffie–Zame 1989, Dana, Pontier 1992, Karatzas et al. 1990
  - asset prices nominal, can be freely chosen by the market, "easiest case"

    Duffie–Huang
  - Our claim: even in the easiest case, it will be difficult to get equivalence of static and dynamic equilibria under Knightian uncertainty

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Model

Let $(\Omega, \mathcal{F}, P)$ be a probability space endowed with a d–dimensional Brownian motion $W = (W_t)$ and canonical filtration $(\mathcal{F}_t)$

The commodity space is $\mathcal{X} = L^\infty(\Omega, \mathcal{F}_T, P)$ for consumption at terminal time $T > 0$ (L^p–spaces possible if relevant objects are square–integrable)

I Agents with endowments $e^i \in L^\infty_+$ and expected utility functions $U^i(c) = E^P u^i(c)$ for some standard Bernoulli utility function u^i

Arrow–Debreu Model

All trade takes place on a contingent market at time 0

An AD equilibrium consists of a price functional $\Psi : \mathcal{X} \to R$ and an allocation (c^i) such that markets clear, $\sum(c^i - e^i) = 0$ and agents maximize utility subject to their budget constraint: if $U^i(d) > U^i(c^i)$, then $\Psi(d - e^i) > 0$

In general, $\Psi \in ba(\Omega, \mathcal{F}, P)$. With suitable assumptions, $\Psi(x) = E^P \psi x$ for some $\psi \in L^1(\Omega, \mathcal{F}, P)$. With e bounded away from zero, even ψ bounded.

Radner's Dynamic Equilibrium

Agents trade dynamically in financial markets and buy consumption goods on spot markets in the moment of consumption

- A nominal asset market consists of a bond with price $S_t^0 = 1$ (numéraire) and d risky assets with price processes $S_t^j > 0$, given by positive semimartingales
- A feasible trading strategy is a predictable, S–integrable process θ with values in R^d with gains from trade $\int_0^T \theta_u dS_u$
- Agent i finances the consumption plan c^i for a spot consumption price ψ with a feasible trading strategy θ^i such that $(c^i - e^i)\psi = \int_0^T \theta_u^i dS_u$

Radner's Dynamic Equilibrium

A Radner equilibrium consists of trading strategies (θ^i) financing consumption plans (c^i), and a spot consumption price ψ such that markets clear and agents maximize utility subject to their budget constraint

Duffie–Huang Theorem

Let $((c^i), \Psi)$ be an Arrow–Debreu equilibrium.

Ψ can be identified with a positive, suitably bounded random variable ψ

Can we find a Radner equilibrium with the same (efficient) allocation?

Theorem (Duffie, Huang 1985)

Let $M^d, d = 1, \dots, D$ be a martingale generator (e.g., one can take $S_t^0 = 1$ (numéraire) and $S^d = W^d, d = 1, \dots, D$ the Brownian motion itself (Bachelier model of finance))
Then there exist trading strategies θ^i and a spot price ψ such that $((\theta^i, c^i), \psi)$ form a Radner equilibrium.

Intuition: In diffusion models, finitely many assets span the market.

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Uncertain Volatility

Now we assume that the volatility of the Brownian motion W is uncertain. This can be described by a family of probability measures P^σ where σ is an adapted process taking values in some convex, compact subset of R^d
Construction: P^0 Wiener measure on the canonical space with Brownian motion W

$$P^\sigma = \text{law}\left(\int_0^\cdot \sigma_u dW_u\right)$$

Important: the measures are not dominated by one common measure
Quasi–sure Analysis necessary
An event is negligible for agents if it is null simultaneously under all P^σ

The Economy

- I agents with endowment e^i bounded
- Aggregate endowment $e = \sum e^i$ is ambiguity–free:
  for all $P, Q \in \mathcal{P}$ we have $P[e \in \cdot] = Q[e \in \cdot]$
- Utility functions of the Gilboa–Schmeidler expected utility form

$$U^i(c) = \underline{E}u^i(c) = \inf_{P \in \mathcal{P}} E^P u^i(c)$$

  for smooth, strictly increasing, strictly concave Bernoulli utility functions u^i that satisfy an Inada condition
- Ambiguity washes out in the aggregate - option for insurance

Static Equilibrium Notion

Arrow–Debreu Model

- An allocation (c^i) is
  - feasible if we have $\sum_i c^i = e$ quasi–surely
  - efficient if there is no other feasible allocation (d^i) with $U^i(d^i) > U^i(c^i)$ for all agents i
- A price is a positive linear functional $\Psi : \mathcal{X} \to R$
- An equilibrium consists of an allocation (c^i) and a price Ψ such that
  1. $\sum c^i = \sum e^i$
  2. c^i maximizes U^i subject to the budget constraint $\Psi(c) \leq \Psi(e^i)$

Dynamic (Radner) Equilibrium

Agents trade dynamically in a financial market with asset prices $S = \left(S_t^d\right), \quad , d = 0, \dots, D, t \geq 0$; the spot price of consumption at time T is ψ.

1. agents finance net demand $c^i - e^i$, i.e. there are S-integrable portfolio processes θ^i such that

$$\psi(c^i - e^i) = \int_0^T \theta^i dS^d$$

2. asset markets clear : $\sum_{i=1}^I \theta^i = 0$
3. c^i maximizes utility U^i over all c that can be financed with trading dynamically S

Duffie–Huang Theorem (Repetition)

Let $((c^i), \Psi)$ be an Arrow–Debreu equilibrium.

Ψ can be identified with a positive, suitably bounded random variable ψ

Can we find a Radner equilibrium with the same (efficient) allocation? Under risk, in diffusion settings, the answer is yes!

- If the filtration has a martingale generator $M^d, d = 1, \dots, D$, then we can set $S_t^0 = 1$ (numéraire) and $S^d = M^d, d = 1, \dots, D$
- In Brownian settings, one can thus take the Brownian motion itself

  Bachelier model of finance

Our claim: “usually” this result breaks down under Knightian (volatility) uncertainty.

Analysis of the Market: Efficient Allocations

Theorem

Every efficient allocation (c^i) *is ambiguity–free.*
It satisfies the probability–free characterization of identical marginal rates of substitution among agents: for some weights $\alpha^i > 0$ *we have*

$$\alpha^i u^{i\prime}(c^i) = \alpha^j u^{j\prime}(c^j)$$

As a consequence, $c^i = f^i(e)$ *for some monotone, continuous function* f^i*.*

Proof different from Dana, 2002: no comonotonicity, no dominating measure.

Static Equilibrium

We denote $\mathcal{E}^P$ the expected utility economy with homogenous priors P.

Theorem

Let $((c^i), \psi)$ be an AD equilibrium in the expected utility economy $\mathcal{E}^P$. Then $((c^i), \Psi)$ with

$$\Psi(X) = E^P(X\psi)$$

is an AD equilibrium in the economy $\mathcal{E}$.

Remark

The market chooses P and state-price ψ.
Ψ is not unique in general.
Indeterminacy

Implementation under no Aggregate Uncertainty

$e = 1$, no aggregate uncertainty

We use two financial assets, a riskless one with price 1, and the G–Brownian motion W as the "uncertain" asset

Under risk, these assets suffice to span a complete market

Theorem

Implementation of an Arrow–Debreu equilibrium $((c^i), \Psi)$ *is possible if and only if the net trade values* $(c^i - e^i)\psi$ *are mean–ambiguity–free.*
In particular, if some individuals face proper Knightian uncertainty in the mean, implementation will not be possible.

Intuition: It is possible to hedge perfectly under each P^σ, but impossible to do so under all P^σ simultaneously

“Usually” Impementation Fails

Prevalence (Hunt, Sauer, Yorke, Anderson, Zame): a measure–theoretic notion of “large sets” for infinite–dimensional spaces

$A \subset X$ is (finitely) prevalent if there is a finite–dimensional subspace V of X such that for all $x \in X$ the complement of A has Lebesgue measure zero in $x + V$.

Theorem

The set of economies for which no Arrow–Debreu equilibrium can be implemented is (finitely) prevalent.

The Message

Knightian Uncertainty is important

- Asset markets work well when we are faced with risk and diffusions
- risk = well–defined probabilities
- diffusion = no jumps, trembling paths
- asset markets are inefficient when there are jumps (known)
- new: when there is Knightian uncertainty about volatility, even the "nice" asset markets can break down

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