Knightian Uncertainty in Economics and Finance

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Outline

How to Invest Your Savings

The Situation

- You have $m > 0$ € left. Savings account or asset market?
- You get the safe return $R > 0$ on the savings account,
- for λ € invested into the asset, you get λX € tomorrow, for an unknown X

Basic Assumptions

- X is a random variable on a probability space $(\Omega, \mathcal{F}, P)$
- the distribution of X is known
- the utility of k is $u(k)$ for an increasing function $u(x)$
- maximize $E^P u((m - \lambda)R + \lambda X)$ over $\lambda \in \mathbb{R}$

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How to Invest: Capital Asset Pricing Model

Further Assumptions

- risk aversion $\Leftrightarrow$ u concave
- the degree of risk aversion at x is $\rho(x) = -\frac{u''(x)}{u'(x)}$
- if $\rho(x) = a > 0$ is constant, then $u(x) = -\exp(-ax)$
- X is normally distributed, mean μ, variance σ^2

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How to Invest: Solution

Optimal Investment

- Expected utility for investment λ

$$E^P u((m-\lambda)R + \lambda X) = -\exp\left(-a(m-\lambda)R - a\lambda\mu + \frac{1}{2}a^2\lambda^2\sigma^2\right)$$

- maximize $(m-\lambda)R + \lambda\mu - \frac{1}{2}a\lambda^2\sigma^2$
- $\lambda^* = \frac{\mu - R}{a\sigma^2}$

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Investment = excess return / (risk aversion · variance)

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Frank Knight: Risk, Uncertainty, and Profit

- Risk = "Roulette" = objective probabilities
- Uncertainty = "Horse Races" = no probabilities
- many entrepreneurial decisions are "horse–races" (start–up)
- financial markets:

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Uncertainty Measures

Probability–Free Ansatz

Approach without fixing a priori a probability measure
a measurable space $(\Omega, \mathscr{F})$
Let $\mathscr{X}$ be the set of all bounded, measurable functions
$X : (\Omega, \mathscr{F}) \to (\mathbb{R}, \mathbb{B})$ = uncertain payoffs, positions

Uncertainty Measure

An uncertainty measure is a mapping $\mathscr{E} : \mathscr{X} \to \mathbb{R}$ that is

- cash invariant: $\mathscr{E}(X + m) = \mathscr{E}(X) + m$ for $m \in \mathbb{R}$
- monotone: $X \geq Y \Rightarrow \mathscr{E}(X) \geq \mathscr{E}(Y)$
- diversification–friendly = concave:
  $\mathscr{E}(\lambda X + (1 - \lambda)Y) \geq \lambda \mathscr{E}(X) + (1 - \lambda)\mathscr{E}(Y)$
- homogenous (maybe): $\mathscr{E}(\lambda X) = \lambda \mathscr{E}(X)$ for $\lambda > 0$

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Uncertainty Measures: Representation

Theorem

Every continuous uncertainty measure has the form

$$\mathscr{E}(X) = \inf_{P \in \mathscr{P}} E^P(X)$$

for a set $\mathscr{P}$ of probability measures on $(\Omega, \mathscr{F})$

Remark

without positive homogeneity:

$$\mathscr{E}(X) = \inf_P E^P(X) + c(P)$$

for a penalty function $c(P)$ c describes the trust in the specification P

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Uncertain Capital Asset Pricing

- keep constant absolute risk aversion, $u(x) = -\exp(-ax)$
- specify a set of distributions for X
- X normal with mean $\mu \in [m, M]$, variance $\sigma^2 \in [s^2, S^2]$

Optimal Investment

- minimize the expected utility over set of priors, or
- $(m - \lambda)R + \lambda\mu - \frac{1}{2}a\lambda^2\sigma^2$
- worst case: maximal variance S^2
- minimal mean m if $\lambda > 0$, maximal mean else
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Prospects of Uncertainty Theory

Mathematics

- Law of Large Numbers go
- Multiple Prior Martingale Theory go
- Modified Hamilton–Jacobi–Bellman Equations

Economics

- Robust Investment
- Optimal Control
- Finance under Volatility Uncertainty (Jörg Vorbrink's Talk)
- Regulation of Financial Markets

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Large Numbers

Probability

Let (X_n) be a sequence of independent and identically distributed random variables with mean $\mu = E^P X_1 \in \mathbb{R}$. Then
$\frac{1}{n}\sum_{k=1}^{n} X_k \to \mu \; a.s.$

Law of Large Numbers under Uncertainty

- Now let $\mathcal{E}$ be an uncertainty measure.
- to clarify: independent, identically distributed

Large Numbers

Probability

Let (X_n) be a sequence of independent and identically distributed random variables with mean $\mu = E^P X_1 \in \mathbb{R}$. Then $\frac{1}{n}\sum_{k=1}^n X_k \to \mu$ *a.s.*

Law of Large Numbers under Uncertainty

- Now let $\mathscr{E}$ be an uncertainty measure.
- to clarify: independent, identically distributed
- suppose $-\infty < m = \inf_{P\in\mathscr{P}} E^P X_1 \leq M = \sup_{P\in\mathscr{P}} E^P X_1 < \infty$
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Value at Risk and the Financial Crisis

"All began with Basel II." Peter Biendarra

Basel II Rules for Capital Requirements

- Value–at–Risk is used to regulate liquidity of banks
- Minimum Capital $\geq 8\% \cdot [\text{Assets} + 12.5 \cdot \text{Value–at–Risk}]$
- What is Value at Risk?
- choose a "small" confidence level $\alpha = 5\%, 1\%, 0.01\%$
- V@R is 10 Mio \$, if the probability to lose more than 10 Mio \$ is α
- $P[-X \geq V@R_\alpha(X)] = \alpha$, i.e. a quantile

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Uncertainty Measures as Risk Measures

Risk Measure

- Monetary risk measures have the form $\rho(X) = \sup_{P \in \mathscr{P}} E^P(-X) = -\mathscr{E}(X)$
- if well constructed: sensitive, manipulation–proof
- convex, i.e. encourage diversification
- Case study (Dana, R.): market breakdown with "stress–testing", perfect markets with uncertainty measures

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Multiple Prior Martingales

Definition

An adapted, bounded process (S_t) is called a multiple prior supermartingale iff

$$S_t \geq \operatorname*{ess\,inf}_{P \in \mathscr{P}} \mathbb{E}^P \left[S_{t+1} \mid \mathscr{F}_t \right]$$

holds true for all $t \geq 0$.
multiple prior martingale: $=$
multiple prior submartingale: $\leq$

Remark

Some assumptions on $\mathscr{P}$ needed:

- *time–consistency*
- *$\mathscr{P}$ weakly compact in $ca(\Omega, \mathscr{F})$*

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- *(S_t) is a multiple prior submartingale iff (S_t) is a $\mathscr{P}$–submartingale.*
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For multiple prior supermartingales: $\Leftarrow$ holds always true.$\Rightarrow$ needs time–consistency. For infinite time horizon: weak compactness in $ca(\Omega, \mathscr{F})$

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