Existence of Financial Equilibria in Continuous Time with Potentially Complete Markets*

Frank Riedel Frederik Herzberg

Institute of Mathematical Economics
Bielefeld University

Abstract

We prove that in smooth Markovian continuous–time economies with potentially complete asset markets, Radner equilibria with endogenously complete markets exist.

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Introduction

The hallmark of economics is still the general theory of competitive markets as expressed masterfully in the work of Arrow and Debreu. While this theory can be considered as complete, its extension to competitive markets under uncertainty in continuous time remains still imperfect. In discrete models, it is well known that for potentially complete markets of real assets, one generically has a Radner equilibrium with endogenously generated

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complete markets that implement the efficient allocation of the corresponding Arrow–Debreu equilibrium, see Magill and Shafer (1985) or Magill and Quinzii (1998), Theorem 25.7.

Anderson and Raimondo (2008) prove a version of this theorem for specific continuous–time economies where endowments and dividends are smooth functions of Brownian motion and time, and agents have time–separable expected utility functions. They establish their result with the help of nonstandard analysis, an intriguing approach to analysis and stochastics via mathematical logic that allows, e.g., to work with infinitely large and infinitesimally small numbers, and to identify Brownian motion with a random walk of infinite length and infinitesimally small time steps. We believe that such an important theorem deserves a standard proof – we provide it here. [1]

At the same time, we extend the result to more general classes of state variables. Many finance models nowadays rely on more general diffusions; prominent examples include the stochastic volatility models, where the volatility of the risky asset is a mean–reverting process as in Heston (1993), term structure models like Vasicek (1977) or more generally affine term structure models as in Duffie, Pan, and Singleton (2000). It is thus important to have sound equilibrium foundations for such models as well.

The paper is set up as follows. The next section describes a smooth continuous–time Markov economy where all relevant functions are analytic on the open interior of their domain. In this paper, the term "analytic" (=real analytic) refers to infinitely differentiable functions that can be written locally as an infinite power series[2]. Then, we formulate our main theorem on existence of a Radner equilibrium with endogenously dynamically complete markets. The proof is split in several steps. We first recall Dana's 1993 result on existence of an Arrow–Debreu equilibrium and show that in our setup, allocation and prices are analytic functions of time and the state variable. The natural candidates for security prices are the expected present values of future dividends. We show that these can also be expressed as analytic functions of time and the state variable if natural assumptions on

[1]In independent work, Hugonnier, Malamud, and Trubowitz (2012) prove a remarkably similar result to ours. In their first version, these authors *assumed* that equilibrium state prices are analytic. Their last version, which appeared after our paper was available, proves this assumption on endogenous objects, as we do and did in all versions of our paper. We think that our treatment is clearer from an economic point of view, we are more to the point, and we have exact references. We thus hope that our paper provides an interesting reading for our readers. A very elegant generalization of the crucial mathematical part of the analysis can be found in a recent working paper by Kramkov and Predoiu (2011). This paper is motivated by lectures on General Equilibrium Theory that one of the authors (Frank Riedel) gave at Carnegie Mellon University in 2008.

[2]Our reference is Krantz and Parks (2002).

the coefficients of the diffusion are satisfied. On the one hand, if one has a closed–form version of the state variable's transition density, the result holds true. This is straightforward to check in the case of Brownian motion, or mean–reverting diffusions, e.g. From an abstract point of view, it is better to have conditions on the primitive of the model that ensure such a nice transition density. We state sufficient conditions on the drift and dispersion coefficients of our state variable for such a result.

The analyticity of security prices allows us to extend the local independence assumption on terminal dividends to security prices, proving dynamic completeness, as in Anderson and Raimondo (2008). The implementation of the Arrow–Debreu equilibrium as a Radner equilibrium is then standard.

Our paper is closely related to Hugonnier, Malamud, and Trubowitz (2012) which was written and circulated independently and simultaneously. Our approach differs in the sense that we do not assume the analyticity of the transition density, but obtain it from fundamentals by an argument involving the theory of evolution equations on Banach spaces. A very elegant generalization of the crucial mathematical part of the analysis, i.e. the analyticity proof for the transition density, can be found in a recent working paper by Kramkov and Predoiu (2011). The paper by Ehling and Heyerdahl-Larsen (2009) treats the case of multi–good economies.

1 A Diffusion Exchange Economy with Potentially Complete Asset Markets

In this section, we set up an exchange economy in continuous time where the relevant information is generated by a diffusion $X = (X_t)_{t\in[0,T]}$ with values in $\mathbb{R}^K$. It is well known that one needs at least $K + 1$ financial assets to span a dynamically complete market. Assuming that this necessary condition is satisfied, the market is *potentially complete*. Below, we show that in sufficiently smooth economies a Radner equilibrium with dynamically complete markets exists.

1.1 The State Variables

Let W be a K–dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$. Denote by $(\mathcal{F}_t)_{t\geq 0}$ the filtration generated by W augmented by the null sets. We assume that the relevant economic information can be described by the state of a diffusion process X with values in $\mathbb{R}^K$ given by

$$X_0 = x, dX_t = b(X_t)dt + \sigma(X_t)dW_t\,, \tag{1}$$

for an initial state $x \in \mathbb{R}^K$ and measurable functions

$$b : \mathbb{R}^K \to \mathbb{R}^K$$

and

$$\sigma : \mathbb{R}^K \to \mathbb{R}^{K\times K}$$

that are called the drift and dispersion function, resp. We let

$$a(x) := \sigma(x)\sigma(x)^T$$

be the diffusion matrix.

Assumption 1 *The diffusion matrix satisfies the uniform ellipticity condition*

$$|y \cdot a(x)y| \geq \epsilon \left\|y\right\|^2 \tag{2}$$

for some $\epsilon > 0$.

The uniform ellipticity condition (2) ensures that there is enough volatility in every state and the diffusion does not degenerate to a locally deterministic process; in particular, it ensures that the distribution of X has full support, see Stroock and Varadhan (1972).

Assumption 2 *b and σ are analytic functions. b and σ as well as all derivatives up to second order are bounded.*

In particular, this assumption implies that b and σ are Lipschitz–continuous, which ensures that the stochastic differential equation has a unique strong solution and so our state variable is well–defined.

1.2 Commodities and Agents

There is one physical commodity in the economy. Our agents consume a flow $(c_t)_{0\leq t<T}$ and a lump-sum c_T of that commodity at terminal time T.[3] Just like Anderson and Raimondo (2008), we introduce the measure $\nu = dt \oplus \delta_T$, the sum of the Lebesgue measure on $[0,T]$ and the Dirac measure on $\{T\}$. However, the subsequent proofs also work for any other superposition of the Lebesgue measure and finitely many Dirac measures on the time line $[0,T]$, thus allowing for lump-sum dividend payments at exogenously given

[3] As an anonymous referee pointed out, it has become increasingly clear in the literature that the choice of the commodity space and the utility function can play a crucial rôle, see e.g. Hindy and Huang (1992), Bank and Riedel (2001a), Bank and Riedel (2001b), Martins-da Rocha and Riedel (2010).

dates before maturity as well.[4] This allows us to model the consumption plans succinctly as one process $c = (c_t)_{0 \le t \le T}$ in the following way. The commodity space $\mathcal{X}$ consists of p–integrable consumption rate processes and a p–integrable terminal lump sum consumption for some $p \ge 1$,

$$\mathcal{X} = L^p \left(\Omega \times [0,T], \mathcal{O}, P \otimes \nu \right) .$$

The consumption set is the positive cone $\mathcal{X}_+$. We will use occasionally the dual space of $\mathcal{X}$ that we shall call the price space

$$\Psi = L^q \left(\Omega \times [0,T], \mathcal{O}, P \otimes \nu \right)$$

for q with $1/q + 1/p = 1$.

There are $i = 1, \ldots, I$ agents with time–separable expected utility preferences of the form

$$U^i(c) = \mathbb{E} \int_0^T u^i \left(t, c_t \right) \nu(dt)$$

for a period utility function

$$u^i : [0,T] \times \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\} .$$

Assumption 3 *The period utility functions u^i are continuous on $[0,T] \times \mathbb{R}_{++}$ and analytic on $(0,T) \times \mathbb{R}_{++}$. They are differentiably strictly increasing and differentiably strictly concave in consumption on $[0,T] \times \mathbb{R}_{++}$, i.e.*

$$\frac{\partial u^i}{\partial c} \left(t, c \right) > 0, \frac{\partial^2 u^i}{\partial c^2} (t,c) < 0 .$$

They satisfy the Inada conditions

$$\lim_{c \downarrow 0} \frac{\partial u^i}{\partial c} \left(t, c \right) = \infty$$

and

$$\lim_{c \to \infty} \frac{\partial u^i}{\partial c} \left(t, c \right) = 0$$

uniformly in $t \in [0,T]$.

[4] We owe the idea for this generalization to an anonymous referee.

The shape of u^i (strict concavity, differentiability, monotonicity) ensures that U^i is always well-defined and automatically satisfies the linear growth conditions in Dana (1993).[5]

Assumption 4 *Each agent comes with a $P \otimes \nu$–strictly positive entitlement*[6] $e^i \in \mathcal{X}_+$ *that can be written as a function of the state variables:*

$$e^i_t = e^i\left(t, X_t\right)$$

for continuous functions $e^i : [0,T] \times \mathbb{R}^K \to \mathbb{R}$, $i = 1, \ldots, I$. *The functions* e^i *are analytic on* $(0,T) \times \mathbb{R}^K$.

1.3 The Financial Market

Assumption 5 *There are* $K+1$ *financial assets. These are* real *assets in the sense that they pay dividends in terms of the underlying physical commodity. The assets' dividends can be written as*

$$A^k_t = g^k\left(t, X_t\right), t \in [0,T]$$

for continuous functions $g^k : [0,T] \times \mathbb{R}^K \to \mathbb{R}_+$, $k = 0, \ldots, K$. *As for consumption processes, we interpret dividends as a flow on* $[0,T)$ *plus a lump sum payment at time* T.

The dividends belong to the consumption set, $A^k \in \mathcal{X}_+$. *The functions* g^k *are analytic on* $(0,T) \times \mathbb{R}^K$. *Asset* 0 *is a real zero–coupon bond with maturity* T*; it has no intermediate dividends, i.e.* $A^0_t = 0$ *for* $t < T$[7].

[5]To see this, note that a concave utility always grows at most linearly. For example, we have

$$u(t,x) \leq u(t,0) + \frac{\partial}{\partial x} u(t,1)(x-1)\,.$$

By compactness of $[0,T]$, our continuity assumptions, and positivity of marginal utility, this implies

$$u(t,x) \leq Ax + B$$

for

$$A = \max_{0 \leq t \leq T} \frac{\partial}{\partial x} u(t,1)$$

and

$$B = \max_{0 \leq t \leq T} u(t,0)\,.$$

[6]We use the word "entitlement" here to distinguish it from the total initial endowment used below which is the sum of the entitlement and the dividends of assets initially owned by the agent.

[7]We can also work with intermediate dividends. In that case, an additional small detour is necessary in order to construct a suitable numéraire asset. As this part is not at the heart of the present analysis, we do not present this generalization here. The argument is available from the authors.

Agent i owns initially $n^i_k \geq 0$ shares of asset k. Without any trade, the agent is thus endowed with his individual endowment

$$\varepsilon^i_t = e^i_t + n^i \cdot A_t\,.$$

We denote by $N_k = \sum_{i=1}^I n^i_k$ the total number of shares in asset k. The aggregate endowment of agents is then

$$\varepsilon_t = \sum_{i=1}^I e^i_t + \sum_{k=0}^K N_k A^k_t = \sum_{i=1}^I \varepsilon^i_t\,.$$

A consumption price process is a positive Itô process ψ. In the dynamic Radner framework, it is the price of one unit of the consumption good. In the Arrow–Debreu framework, it is interpreted as the initial price for one unit of the consumption good delivered at some time and state of the world. A (cum–dividend) security price for asset k is a nonnegative Itô process $S^k = \left(S^k_t\right)_{0\leq t\leq T}$. We interpret S^k as the nominal price of the asset k. We denote by

$$G^k_t = S^k_t + \int_{[0,t)} A^k_s \psi_s \nu(ds), \qquad (0 \leq t \leq T)$$

the (nominal) gain process for asset k. Note that by no arbitrage we must have $S^k_T = A^k_T \psi_T$ at maturity.

A portfolio process is a predictable[8] process θ with values in $\mathbb{R}^{K+1}$ that is G–integrable, i.e. the stochastic integrals $\int_0^t \theta^k_u dG^k_u$ are well–defined. The value of such a portfolio is $V_t = \theta \cdot S$.

We call a portfolio admissible (without reference to an agent) if its value process is bounded below by a martingale. This admissibility condition rules out doubling strategies[9].

A portfolio is *admissible for agent i* if its present value plus the present value of the agent's endowment is nonnegative, or

$$V_t + \mathbb{E}\left[\int_{t+}^T e^i_s \psi_s \nu(ds) \middle| \mathcal{F}_t\right] \geq 0\,.$$

Note that this implies $V_T \geq 0$ for the terminal value of the portfolio.

[8] A process is *predictable* if and only if it is measurable with respect to the algebra of predictable events, i.e. the smallest σ-algebra with respect to which all $\mathcal{F}$-adapted processes with left-continuous paths are measurable.

[9] Anderson and Raimondo (2008) use a martingale condition to rule out such strategies. This requires to impose a martingale condition on potential security prices. As this martingale property is a consequence of equilibrium, we prefer not to impose this assumption ex ante. Nevertheless, either way works here.

A portfolio θ finances a consumption plan $c \in \mathcal{X}_+$ for agent i if θ is admissible for agent i and the intertemporal budget constraint is satisfied for the associated value process V:

$$V_t = n^i \cdot S_0 + \int_0^t \theta_u dG_u + \int_0^t \left(e_u^i - c_u\right) \psi_u \nu(du)\,.$$

We then call the portfolio/consumption pair (θ, c) i–feasible. More generally, we say that a portfolio θ finances a net consumption plan $z \in \mathcal{X}$ if its value process satisfies

$$V_t = V_0 + \int_0^t \theta_u dG_u + \int_0^t \left(e_u^i - c_u\right) \psi_u \nu(du)\,.$$

A Radner equilibrium consists of asset prices S, a consumption price ψ, portfolios θ^i and consumption plans $c^i \in \mathcal{X}_+$ for each agent i such that θ^i is admissible for agent i and finances c^i, c^i maximizes agent i's utility over all such i–feasible portfolio/consumption pairs, and markets clear, i.e. $\sum_{i=1}^I c^i = \varepsilon$ and $\sum_{i=1}^I \theta^i = N$.

Our way to a Radner equilibrium with dynamically complete markets will lead over the intermediate step of an Arrow–Debreu equilibrium. To this end, some sort of bounds on endowments and marginal utilities are necessary. We impose the following (strong) assumption.

Assumption 6 *Aggregate endowment ε is bounded and bounded away from zero.*

Remark *The previous assumption is quite strong, of course. Its advantage is that it allows to avoid further assumptions on the growth rate of marginal utilities as in Anderson and Raimondo (2008) and Hugonnier, Malamud, and Trubowitz (2012). On the other hand, our proofs would go through with growth assumptions on endowments and bounds on relative risk aversion as in those papers. The crucial point is that the functions m^k in the proof of Theorem 11 below grow at most exponentially.*

If the assets are linearly dependent, there is no hope to span a dynamically complete market. To exclude this, we follow Anderson and Raimondo (2008) and impose a full rank condition on terminal payoffs:

Assumption 7 *On a nonempty open set $V \subset \mathbb{R}^K$, the dividend of the zero–th asset is strictly positive at maturity,*

$$g^0(T, x) > 0, \qquad (x \in V)\,.$$

The functions $h^k : x \mapsto \frac{g^k(T,x)}{g^0(T,x)}$ *are continuously differentiable on* V *for* $k = 1, \dots, K$ *and the Jacobian matrix*

$$Dh(x) = \begin{pmatrix} \frac{\partial h^1(T,x)}{\partial x_1} & \cdots & \frac{\partial h^1(T,x)}{\partial x_K} \\ \vdots & \ddots & \vdots \\ \frac{\partial h^K(T,x)}{\partial x_1} & \cdots & \frac{\partial h^K(T,x)}{\partial x_K} \end{pmatrix}$$

has full rank on V.

The above framework is relatively general and still sufficient for some of our results and even most of the proof of our main result. Nevertheless, for a complete proof of our main theorem, we need the equilibrium state-price density ψ (as a functions of the time and state variables) to be an analytic function from the time line to $L^2(\mathbb{R}^K)$.

Unfortunately, using our present proof methodology, this seems to imply that we must impose rather strong additional assumptions on the utility functions.[10]

Assumption 8 *1. The agents' utility functions are time-independent with a homogeneous discount rate: There exists a real number* r *and for each* i *a function* φ^i *such that for all* i *and* $(t, c) \in [0, T] \times \mathbb{R}_{++}$,

$$u^i(t, c) = e^{-rt} \varphi^i(c).$$

2. The functions e^i *and* g^k *are constant in the first argument for all* i, k.

Part 2 of Assumption 8 ensures that the aggregate endowment ε_t is a time-independent analytic function of the state variable X_t.

2 Existence of Radner Equilibrium with Dynamically Complete Markets

We are now in the position to state our main result. We call the market given by the asset prices S, dividends A, and consumption price ψ dynamically complete if every net consumption plan $z \in \mathcal{X}$ can be financed by an admissible portfolio θ in the sense that its value process satisfies

$$V_t = V_0 + \int_0^t \theta_u dG_u + \int_0^t z_u \psi_u \nu(du)\,.$$

[10] We are grateful to an anonymous referee for raising this point.

Theorem 9 *Under Assumptions 1 to 8, there exists a Radner equilibrium* $\left(S, \psi, (\theta^i, c^i)_{i=1,\dots,I}\right)$ *with a dynamically complete market* (S, A, ψ)*; the prices and dividends are linked by the present value relation*

$$S_t^k = \mathbb{E}\left[\int_t^T A_s^k \psi_s \, \nu(ds) \middle| \mathcal{F}_t\right] . \tag{3}$$

The proof of this theorem runs as follows. In a first step, we establish the existence of an Arrow–Debreu equilibrium. In the current time–additive setup, this is a result by Dana (2002). We extend her result by showing that in our smooth economy the equilibrium consumption price ψ and the allocation $(c^i)_{i=1,\dots,I}$ are analytic functions of time and the state variable. It is well known that one can implement the Arrow–Debreu equilibrium as a Radner equilibrium if one has dynamically complete markets. With *nominal* assets, this is more or less trivial (see Duffie and Huang (1985) and Huang (1987)). Here, our assets pay real dividends, and the completeness depends on the endogenous consumption price ψ and cannot be assumed exogenously.

The natural candidates for our asset prices are, of course, the present values of their future dividends as in (3). We have to show dynamic completeness then. We do this by proving that the (local) linear independence of the dividends at maturity T carries over to the volatility matrix of asset prices. This yields dynamic completeness. This step needs the intermediate mathematical result that our candidate security prices are analytic functions of time and state variable.

The implementation of the Arrow–Debreu equilibrium as a Radner equilibrium is then standard.

2.1 Existence of an Analytic Arrow–Debreu Equilibrium

We quickly recall the notions of classical General Equilibrium Theory. Herein, we assume some familiarity with the theory of continuous-time stochastic processes, as treated, e.g., in Dellacherie (1972); later on, we shall also employ some concepts and results from the theory of real analytic functions (see Krantz and Parks (2002) for an introductory survey).

An allocation is an element $(c^i)_{i=1,\dots,I} \in \mathcal{X}_+^I$. Is is feasible if we have $\sum_{i=1}^I c^i \leq \varepsilon$. A price is a nonnegative, optional process $\psi \in \mathcal{X}_+$.[11] It defines

[11] A process is called *optional* if and only if it is measurable with respect to the *optional σ-algebra*, i.e. the smallest σ-algebra $\mathcal{O}$ with respect to which all $\mathcal{F}$-adapted processes whose paths are a.s. right-continuous and have left limits (*càdlàg* processes) are measurable.

a continuous linear price functional $\Psi(c) = \mathbb{E} \int_0^T c_t \psi_t \, \nu(dt)$ on $\mathcal{X}$.

An Arrow–Debreu equilibrium consists of a feasible allocation $(c^i)_{i=1,\ldots,I}$ and a price ψ such that c^i is budget–feasible and optimal for all agents $i = 1, \ldots, I$, i.e. $\Psi(c^i) \leq \Psi(\varepsilon^i)$, and for all consumption plans $c \in \mathcal{X}_+$ the relation $U^i(c) > U^i(c^i)$ implies $\Psi(c) > \Psi(\varepsilon^i)$.

Existence and uniqueness of Arrow–Debreu equilibria in our separable setting have been clarified by Dana (1993). We recall her existence result and show the additional refinement that equilibrium price and consumption plans are analytic functions of time and the state variable on $(0, T) \times \mathbb{R}^K$.

Theorem 10 *Under Assumptions 3, 4, and 6, there exists an Arrow–Debreu equilibrium* $\left(\psi, (c^i)_{i=1,\ldots,I}\right)$ *such that*

$$\psi_t = \psi(t, X_t)$$
$$c_t^i = c^i(t, X_t)$$

for some continuous functions

$$\psi, c^i : [0, T] \times \mathbb{R}^K \to \mathbb{R}_+$$

that are analytic on $(0, T) \times \mathbb{R}^K$.

If, in addition, Assumption 8 is satisfied, then the function ψ *has the form* $\psi(t, x) = e^{-rt}\chi(x)$ *for some analytic function* $\chi : \mathbb{R}_+ \to \mathbb{R}_+$.

PROOF : By Dana (1993), there exists an equilibrium $(\psi, (c^i))$ with $\psi > 0$ $P \otimes \nu$–a.s. and the allocation (c^i) is the solution of the social planner problem

$$\max_{c \in \mathcal{X}_+^I, \sum c^i \leq \varepsilon} \sum \lambda^i U^i(c^i)$$

for some $\lambda^i > 0$[12].

As we have separable utility functions, the social planner's problem can be solved point– and state–wise; we thus look at the real–valued problem

$$v(t, x) := \max_{\substack{\sum_{i=1}^I x^i = x \\ x^i \geq 0, i=1,\ldots,I}} \sum_{i=1}^I \lambda^i u^i(t, x^i) \,. \tag{4}$$

[12] $\lambda^i = 0$ is not possible. This is already implicit in Dana's proof. Here is another argument based on our Assumption 6. For, if, say, $\lambda^1 = 0$, then $c^1 = 0$ (by Negishi). By the strict monotonicity of utility functions, $c^1 = 0$ is an equilibrium demand only if wealth is zero, i.e. $E \int_0^T \psi_t \varepsilon_t^1 \nu(dt) = 0$. But by Assumption 6 and the Inada assumption, $\varepsilon^1 > 0$ $P \otimes \nu$–a.s. Hence $E \int_0^T \psi_t \varepsilon_t^1 \nu(dt) > 0$, a contradiction.

By Assumption 3, the unique solution of the above real–valued maximization problem is characterized by the equations

$$\lambda^i \frac{\partial u^i}{\partial c}\left(t, x^i\right) = \mu \tag{5}$$

$$\sum_{i=1}^{I} x^i = x \tag{6}$$

for some Lagrange parameter $\mu > 0$. By Dana (1993), Proposition 2.1, the solution of the above equations is given by continuous functions $x^i, \mu : [0, T] \times \mathbb{R}_+ \to \mathbb{R}_+$ of (t, x). By the Analytic Implicit Function Theorem and Assumption 3 (see e.g. Krantz and Parks (2002)), these are even analytic on $(0, T) \times (0, \infty)$ (see also Anderson and Raimondo (2008), page 881). By Dana (1993), we have $c_t^i = x^i\left(t, \varepsilon_t\right)$ and $\psi_t = \mu\left(t, \varepsilon_t\right)$. As aggregate endowment is a function of time and state variable that is continuous on $[0, T] \times \mathbb{R}_+$ and analytic on $(0, T) \times \mathbb{R}_+$ (Assumptions 4 and 5), the general result follows.

If Assumption 8 is satisfied, then the solution argument $(x_i)_{i=1}^I$ of the maximisation problem in Eq. (4) is independent of t (and that $e^{rt}v(t, x)$ is constant in t), whence so is the function x^i. Moreover, Assumption 8 (part 2) ensures in combination with Assumptions 4 and 5 that ε_t is a function of X_t only (since $\varepsilon_t = \sum_{i=1}^I e^i(t, X_t) + \sum_{k=0}^K N_k g^k(t, X_t)$), so that for some analytic function $\bar{e}$, we have

$$\varepsilon_t = \bar{e}(X_t)$$

(at all times t). On the other hand, Eq. (5) and Assumption 8 on the shape of u^i show that

$$\lambda^i e^{-rt} (\varphi^i)' \circ x^i(t, x) = \mu(t, x)$$

for all t, x, so that from the independence of $x^i(t, x)$ from t we may conclude that $e^{rt}\mu(t, x)$ is constant in t, too. Hence, the function $e^{rt}\psi_t = e^{rt}\mu(t, \varepsilon_t) = e^{rt}\mu\left(t, \bar{e}(X_t)\right)$ does not depend on t, but only on X_t, say

$$e^{rt}\psi_t = \chi(X_t).$$

Since ψ_t was shown to be analytic in t and X_t in the general part of the proof, we may conclude that χ is analytic, too.

□

2.2 Analytic Security Prices

The crucial point in the development initiated by Anderson and Raimondo is to establish analyticity of the equilibrium asset price candidates.

Theorem 11 *Define S by (3). Under Assumptions 1 to 8, there exist continuous functions $s : [0,T] \times \mathbb{R}^K \to \mathbb{R}_+$ that are analytic on $(0,T) \times \mathbb{R}^K$ and*

$$S_t = s(t, X_t)\,.$$

The first derivatives with respect to x, $\frac{\partial s}{\partial x_l}$ are continuous on $[0,T] \times \mathbb{R}^K$ and we have

$$\lim_{t \uparrow T} \frac{\partial s}{\partial x_l}(t,x) = \frac{\partial s}{\partial x_l}(T,x) = \frac{\partial g}{\partial x_l}(T,x)$$

PROOF : As X is a Markov process, S_t^k is a function s^k of time t and state X_t, that is

$$s^k(t,x) = \mathbb{E}\left[\int_t^T m^k(s, X_s)\,\nu(ds)|X_t = x\right]$$

for $m^k(t,x) = g^k(t,x)\psi(t,x)$ (the existence of such a function ψ was established in Theorem 10).

Since the equilibrium state-price density ψ must be equal to (a constant multiple of) the marginal felicity at the optimal consumption, Assumption 6 combined with our Assumption 3 entails that m^k is bounded for all k, and hence s^k is bounded as well. By Theorem 5.3 of Chapter 6 on p. 148 of Friedman (1975), or alternatively, by Heath and Schweizer (2000) s^k is a classical $\mathcal{C}^{1,2}$–solution of the Cauchy problem

$$-\frac{\partial}{\partial t}u + \mathcal{L}u = m^k$$

with boundary condition $s^k(T,x) = m^k(T,x)$, $\mathcal{L}$ being the infinitesimal generator of X.

We need to prove that s^k is analytic. We already know that $m^k = g^k\psi$ is jointly analytic (on account of the analyticity of ψ, see Theorem 10, and of the exogenously given g^k).

Now let $d(x) = \exp(-l(x))$ for a smooth function l on $\mathbb{R}^K$ that satisfies $l(x) = \|x\|$ for $\|x\| \geq 1$ and set $u(t,x) = s^k(t,x)d(x)$. Then $u(T,x) = m^k(T,x)d(x) \in L^2(\mathbb{R}^K)$ and u solves the modified Cauchy problem

$$-\frac{\partial}{\partial t}u + \tilde{\mathcal{L}}u = f \tag{7}$$

for some suitable function f and elliptic operator $\tilde{\mathcal{L}}$ with

$$\tilde{\mathcal{L}}(vd) = d\mathcal{L}v \qquad (v \in \mathcal{C}^\infty(\mathbb{R}^K))\,. \tag{8}$$

Moreover, $\tilde{\mathcal{L}}$ is a sectorial operator on $L^2(\mathbb{R}^K)$ (see, for instance, Proposition 3.1.17 of Lunardi (1995) and Theorem 5.2 in Section 2.5 of Pazy (1983)).

Thus, as soon as we know that m^k is even an analytic function from the time line to $L^2(\mathbb{R}^K)$, we may apply standard results on evolution equations in Banach spaces to conclude that u, and then s^k, is analytic in time t. However,

1. in Assumption 8 (part 2), the constancy of g^k in t was explicitly assumed;

2. in Theorem 10 it was established, again on the basis of Assumption 8, that ψ has the form $\psi(t,x) = e^{-rt}\chi(x)$;

3. whence it follows that m^k is of the form $m^k(t,x) = e^{-rt}\tilde{\chi}(x)$ for some analytic function $\tilde{\chi}$;

4. and the boundedness of m^k — and thus of $\tilde{\chi}$ — had been observed earlier on in this proof.

It follows that m^k is an analytic function from the time line to $L^2(\mathbb{R}^K)$.

There are several ways to accomplish the final step — i.e. the application of the theory of evolution equations in Banach spaces to Eq. (8).[13] The most direct route to completion is the application of Theorem 3 of Komatsu (1961), which shows that the solution u of Eq. (7) is in fact an analytic function from the time line to $L^2(\mathbb{R}^K)$. (Alternatively, one may argue as follows: the Corollary on page 209 in Friedman (1969) applies. We can take the Sobolev space $X = \mathbb{W}_2^2$ as the Banach space there. Our operator $\mathcal{L}$ has X as its domain. Hence, condition (E1) there is satisfied. Condition (E2) requires that the resolvent of the Markov process exists in some complex sector around zero. However, this has been proven in Eq. (2.11), Theorem 1 of Yosida (1959), or see the references above. (E3) in that book is automatically satisfied as our operator is independent of time.)

Having established the analyticity of s^k in t, note that s^k is also analytic in x by Theorem 1.2 in Part 3, Chapter 1 of Friedman (1969) and recall that functions which are bounded and separately analytic are jointly analytic (a result of Osgood (1899)).

The continuous differentiability of s^k with respect to the second argument x follows from Theorem 10.3 on p. 143 of Friedman (1969). □

Remark 12 *Perhaps the reader is wondering whether one might obtain analyticity directly by invoking an appropriate general result from the theory of partial differential equations. To be sure, the function s in Theorem 11 solves an inhomogeneous parabolic differential equation (see again Theorem 5.3 in*

[13] We are grateful to an anonymous referee for pointing out a gap in the earlier version of the proof.

Chapter 6 of Friedman (1975), and indeed there does exist a body of literature on analyticity of solutions to linear second-order partial differential equations. For example, De Giorgi and Cattabriga (1971) proved that for the special case of (space-time) dimension 2 (i.e. in our setting $K = 1$) and constant coefficients, any solution to an inhomogeneous linear differential equation with analytic right-hand side (in our case, this is the product of the pricing density, as a function of X and the dividend function g) is again analytic, hence if our process X is just a one-dimensional Brownian motion with drift, then analyticity of s follows. However, as already conjectured by De Giorgi and Cattabriga (1971), this result fails to hold in general for higher dimensions. A counterexample is the heat equation with two-dimensional Laplacian, as was proved by Piccinini (1973), using a right-hand side which grows, while still analytic, at enormous pace. Therefore, one cannot, in general, dispense of growth conditions like those imposed by Anderson and Raimondo (2008) in their Theorem B.4. These negative results have been generalized in numerous papers, in particular by Hörmander (1973) (for the case of partial differential operators with constant coefficients) and Oleĭnik and Radkevič (1973) (for the case of partial differential operators with analytic coefficients).

Oleĭnik and Radkevič (1982) do give sufficient conditions for analyticity of solutions of all solutions — in the distribution sense — of inhomogeneous second-order linear partial differential equations with analytic right-hand side. However, these conditions are limited to the two-dimensional case (which would mean $K = 1$ in our setting) and involve the assumption that the equation can be transformed into another partial differential equations where there are second-order diagonal terms in both variables (in our case this would imply a second-order time-derivative), see Theorem 2 and Theorem 3 of Oleĭnik and Radkevič (1982).

2.3 Dynamically Complete Markets

Theorem 13 *Under Assumptions 1 to 8, the market (S, A, ψ) is dynamically complete.*

PROOF : By Assumption 7, $\psi > 0$ and the fact that X has full support, we have $S_t^0 > 0$ a.s. Hence, we can take asset 0 as a numéraire. Define

$$R_t^k = \frac{S_t^k}{S_t^0}.$$

By Theorem 11, $R_t^k = r^k(t, X_t)$ for continuous functions $r^k : [0,T] \times \mathbb{R}^K \to \mathbb{R}_+$ that are analytic on $(0,T) \times \mathbb{R}^K$, $k = 1, \ldots, K$.

After this change of numéraire, we have a riskless asset (with interest rate 0, of course) and K risky assets, as many as independent Brownian motions.

The asset market is dynamically complete if the volatility matrix is a.s. invertible (see, e.g., Karatzas and Shreve (1998), Theorem 1.6.6)[14]. By Itô's lemma, the volatility matrix is given by $I(t,x)Dr(t,x)\sigma(t,x)$ where Dr is the Jacobian matrix of r and I the triangular matrix

$$I(t,x) = \begin{pmatrix} \frac{1}{r_1(t,x)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{r_K(t,x)} \end{pmatrix}.$$

Now suppose that the volatility matrix has determinant 0 on a set of positive Lebesgue measure. By analyticity and Theorem B.3 in Anderson and Raimondo (2008), we conclude that the determinant vanishes everywhere on $(0,T) \times \mathbb{R}^K$. As Dr, r, and σ are continuous on $[0,T]$, it then follows that

$$\det I(T,x)Dr(T,x)\sigma(T,x) = 0\,.$$

(For Dr and r, this is Theorem 11.) As σ has full rank by Assumption 1 and $I(T,x)$ is triangular, we conclude that

$$\det Dr(T,x) = 0\,.$$

But $r(T,x) = g(T,x)/g^0(T,x) = h(x)$, so

$$\det Dr(T,x) \neq 0$$

on a set of positive measure by Assumption 7. This contradiction shows that the volatility matrix is invertible a.s. We conclude that the market (S, A, ψ) is dynamically complete.

□

With dynamically complete asset markets, it is a standard argument to show that the Arrow–Debreu equilibrium can be implemented as a Radner equilibrium. The basic argument is as in Duffie and Huang (1985), translated to our more complex setting, see also Dana and Jeanblanc (2003), Theorem 7.1.10 (apply this theorem to the asset market with asset 0 as numéraire).

[14]To apply this result, we check quickly that the asset market is also standard in the sense of Karatzas and Shreve (1998): by construction ((3)), the gain processes are martingales; hence, our market is arbitrage–free. As our state–price deflator ψ is in Ψ, also the martingale condition in Karatzas and Kou (1998) is satisfied.

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