The Fundamental Theorem of Asset Pricing without Probabilistic Prior Assumptions

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July 22, 2011

Outline

Motivation

- in discrete time and state spaces, finance does not need a probability on the state space
- probability enters through the no arbitrage axiom
- as a martingale measure
- or linear pricing functional
- in complex models, we usually start with $(\Omega, \mathscr{F}, P)$, a *probability space*
- P^* then equivalent martingale measure
- implicit assumption: agents / market knows null sets
- Knightian uncertainty / model uncertainty
- volatility uncertainty (Avallaneda, Peng etc.): singular measures

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- develop the fundamental theorem of asset pricing on a Polish space (Ω, d)
- relation to economic equilibrium (Harrison–Kreps)
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Literature

- Based on *Föllmer*'s Calcul d'Itô sans probabilités (1981), one can develop the hedging argument of *Black and Scholes* without prior probabilities (*Bick, Willinger* 1994).
- *Föllmer and Schied* develop monetary risk measures on $C_b(\Omega, d)$.
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- $\Omega = \{\omega_1, \ldots, \omega_N\}$
- $D + 1$ assets $d = 0, \ldots, D$
- price today $f_d \geq 0$
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Theorem

The model (f, F) admits no arbitrage if and only if there exists probabilities $p_k^* > 0$ such that $f = p^* \cdot F$, or $f_d = \sum_{k=1}^{N} p_k^* F_d(\omega_k)$

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Full Support Martingale Measures

Definition

A probability P^* on $(\Omega, \mathscr{F})$ is called a martingale measure if $f_d = E^{P^*} F_d$ for $d = 1, \ldots, D$. If P^* has full support, it is called a full support martingale measure.

The Fundamental Theorem of Asset Pricing

Theorem

The market (f, F) admits no arbitrage if and only if there exists a full support martingale measure.

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- *Full Support measures exist on Polish spaces*
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Economic Equilibrium and No Arbitrage

- Ω compact
- Economic agent = complete, transitive, continuous relation $\succeq$ on $C(\Omega, d)$. $\succ$ is strictly monotone: if $X \in \mathscr{X}$ satisfies $X \geq 0$ and $X \neq 0$, then for all $Z \in C(\Omega, d)$, we have $Z + X \succ Z$.
- Marketed subspace

$$M := \langle S_0, S_1, \ldots, S_D \rangle = \left\{ \pi \cdot S ; \pi \in \mathbb{R}^{D+1} \right\} .$$

- Under no arbitrage, the price functional

$$\phi : M \to \mathbb{R}$$

given by $\phi(\pi \cdot S) = \pi \cdot f$ is well–defined.

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The market (f, S) is viable if and only if there exists a strictly positive linear functional $\Phi : C(\Omega, d) \to \mathbb{R}$ such that $\Phi(X) = \phi(X)$ for $X \in M$.

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A derivative or (contingent) claim is a continuous mapping $H : \Omega \to \mathbb{R}_+$. $h \geq 0$ is called a no arbitrage price for H if the extended market with $D+2$ assets and $f_{D+1} = h$ and $S_{D+1} = H$ admits no arbitrage opportunities.

Corollary

h is a no arbitrage price for a claim H if and only if

$$h = \int H \, dP$$

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Superhedging and Linear Programming

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A portfolio π is called a superhedge for the claim H if $\pi \cdot S(\Omega) \geq H(\omega)$ holds true for all $\omega \in \Omega$.

Problem (Problem SH)

Find the cheapest superhedge for the claim H; minimize $\pi \cdot f$ over $\pi \in \mathbb{R}^{D+1}$ subject to $\pi \cdot S(\Omega) \geq H(\omega)$ for all $\omega \in \Omega$.

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Embedding into Linear Programming

Linear Programming in Infinite–Dimensional Spaces

- Two dual pairs of vector spaces
- here $X = Y = \mathbb{R}^{D+1}$, $Z = C(\Omega, d)$, $W = ca(\Omega, \mathscr{F})$
- linear constraint $B : \mathbb{R}^{D+1} \to C(\Omega, d)$, $B\pi = \pi \cdot F$
- Adjoint mapping $B^* : ca(\Omega, \mathscr{F}) \to \mathbb{R}^{D+1}$, $B^*\mu = \int S d\mu$

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- no sign constraints on portfolio: equality in the dual program, $B^*\mu = f$
- $$\int S_d \, d\mu = f_d, d = 0, \ldots, D\,.$$
- $d = 0$: probability measure
- $d > 0$: martingale measure

Problem (DSH)

Minimize the prices $\int_\Omega H d\mu$ over all martingale measures $\mu \in ca(\Omega, \mathscr{F})$.

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1. *The linear programs SH and DSH are dual to each other. Both problems have the same value.*
2. *Both programs have optimal solutions; in particular, there exists a superhedge $\pi^* \in \mathbb{R}^{D+1}$ and a martingale measure P^* such that*

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