Existence of Financial Equilibria in Continuous Time with Potentially Complete Markets

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Existence with Potentially Complete Markets

Existence of Equilibria in Financial Models: Discrete Time

- With (dynamically) complete markets, Arrow–Debreu equilibria can be implemented as financial (Radner) equilibria
- With nominal assets that span the market, existence of (efficient) equilibria
- With assets pay dividends in physical goods (real assets), the spanning condition becomes endogenous
- Magill–Shafer 85: when asset markets are *potentially complete*, one has *generically* existence of efficient equilibria
- Anderson and Raimondo 2008 go a first step to extend Magill–Shafer to continuous time

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Anderson–Raimondo Approach

Analytic Markov Economy and Non–Standard Analysis

- Information generated by a Brownian motion W_t
- All dividends, endowments are real analytic functions of W_t
- Bernoulli utilities are real analytic and "nice"
- Financial markets are potentially complete: as many risky assets as dimension of W_t
- Asset dividends are linearly independent at maturity T
- main point: asset prices are analytic, and hence, the linear independence carries over from terminal payoffs to prices $\longrightarrow$ dynamically complete market
- method of proof: nonstandard analysis (discrete approximation)

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AR leads to two questions

- Many finance applications need more complex dynamics than functions of Brownian motion
  - affine term structure models (Vasiĉek, Cox–Ingersoll–Ross)
  - stochastic volatility (Heston, Derman–Kani, Dupire)
  - "predictable" returns (Barberis)

  Can one extend to general diffusions?
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- State variable general diffusion process X_t
- Analytic Markov economy as in Anderson–Raimondo
- extend Dana 92 to prove existence of an analytic Arrow–Debreu equilibrium
- give sufficient conditions to show that security prices are analytic functions of X_t
- uses analyticity to show dynamic completeness
- implement Arrow–Debreu as a Radner equilibrium

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Time and Information

- time is continuous $t \in [0, T]$
- the state variable is a diffusion X_t with values in $\mathbb{R}^K$ driven by a K–dimensional Brownian motion W:

$$X_0 = x, dX_t = b(X_t)dt + \sigma(X_t)dW_t \,,$$

- for Lipschitz continuous functions

$$b : \mathbb{R}^K \to \mathbb{R}^K$$

  and

$$\sigma : \mathbb{R}^K \to \mathbb{R}^{K \times K}$$

  that are called the drift and dispersion function, resp. We let

$$a(x) := \sigma(x)\sigma(x)^T$$

  be the diffusion matrix.
- The diffusion matrix satisfies the uniform ellipticity condition

$$\|x \cdot a(x)x\| \geq \varepsilon \|x\|^2$$

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Agents and Commodities

- one physical commodity (no problem to generalize to $D > 1$)
- I agents consuming a flow (c_t) on $[0, T)$ and a terminal consumption c_T; write $\nu = dt \otimes \delta_T$
- consumption space $\mathscr{X} = L^p(\Omega \times [0, T], \mathscr{O}, P \otimes \nu)$, $\mathscr{O}$ optional σ–field, $p \geq 1$
- price space (Arrow–Debreu) $\Psi = \mathscr{X}^* = L^q(\Omega \times [0, T], \mathscr{O}, P \otimes \nu)$
- $U^i(c) = \mathbb{E} \int_0^T u^i(t, c_t) \nu(dt)$
- The period utility functions u^i are *nice* and analytic on $(0, T) \times \mathbb{R}_{++}$.
- agents' endowment $e^i_t = e^i(t, X_t)$ is an analytic function of time and state

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Nice

The period utility functions u^i are continuous on $[0, T] \times \mathbb{R}_+$, differentiably strictly increasing and differentiably strictly concave in consumption on $[0, T] \times \mathbb{R}_{++}$, i.e.

$$\frac{\partial u^i}{\partial c}(t, c) > 0, \frac{\partial^2 u^i}{\partial c^2}(t, c) > 0 \,.$$

They satisfy the Inada conditions

$$\lim_{c \downarrow 0} \frac{\partial u^i}{\partial c}(t, c) = \infty$$

and

$$\lim_{c \to \infty} \frac{\partial u^i}{\partial c}(t, c) = 0$$

uniformly in $t \in [0, T]$.

Financial Market

- There are $K+1$ financial assets (Potentially Complete Markets)
- real assets
- dividends

$$A_t^k = g^k(t, X_t), t \in [0, T]$$

  - dividends belong to the consumption set, $A^k \in \mathscr{X}_+$.
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- Asset 0 is a real zero–coupon bond with maturity T, $A_T = 1$,

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Financial Market, ctd.

- Agent i owns initially $n^i_k \geq 0$ shares of asset k
- A consumption price process is a positive Itô process ψ.
- A (cum–dividend) security price for asset k is a nonnegative Itô process $S^k = \left(S^k_t\right)_{0 \leq t \leq T}$. We interpret S^k as the nominal price of the asset k.
- We denote by

$$G^k_t = S^k_t + \int_{[0,t)} A^k_s \psi_s \nu(ds), \qquad (0 \leq t \leq T)$$

  the (nominal) gain process for asset k.
- A portfolio process is a predictable process θ with values in $\mathbb{R}^{K+1}$ that is G–integrable
- A portfolio is admissible for agent i if its present value plus the present value of the agent's endowment is nonnegative, or

$$V_t + \mathbb{E}\left[\int_{t+}^{T} e^i_s \psi_s \nu(ds) \,\middle|\, \mathscr{F}_t\right] \geq 0 .$$

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- A (cum–dividend) security price for asset k is a nonnegative Itô process $S^k = \left(S_t^k\right)_{0 \leq t \leq T}$. We interpret S^k as the nominal price of the asset k.
- We denote by

$$G_t^k = S_t^k + \int_{[0,t)} A_s^k \psi_s \nu(ds), \qquad (0 \leq t \leq T)$$

  the (nominal) gain process for asset k.
- A portfolio process is a predictable process θ with values in $\mathbb{R}^{K+1}$ that is G–integrable
- A portfolio is admissible for agent i if its present value plus the present value of the agent's endowment is nonnegative, or

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Financial Market, ctd.

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Radner Equilibrium

- A portfolio θ finances a consumption plan $c \in \mathscr{X}_+$ for agent i if θ is admissible for agent i and the intertemporal budget constraint is satisfied for the associated value process V:

$$V_t = n^i \cdot S_0 + \int_0^t \theta_u dG_u + \int_0^t \left(e_u^i - c_u\right) \psi_u \nu(du) .$$

- A Radner equilibrium consists of asset prices S, a consumption price ψ, portfolios θ^i and consumption plans $c^i \in \mathscr{X}_+$ for each agent i such that θ^i is admissible for agent i and finances c^i, c^i maximizes agent i's utility over all such i–feasible portfolio/consumption pairs, and markets clear, i.e. $\sum_{i=1}^I c^i = e$ and $\sum_{i=1}^I \theta^i = N$.

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Main Theorem

Theorem

There exists a Radner equilibrium $\left(S, \psi, (\theta^i, c^i)_{i=1,\dots,I}\right)$ *with a dynamically complete market* (S, A, ψ)*; the prices and dividends are linked by the present value relation*

$$S_t^k = \mathbb{E}\left[\int_t^T A_s^k \psi_s \, \nu(ds) \,\middle|\, \mathscr{F}_t\right]. \tag{1}$$

Step 1: Arrow–Debreu Equilibrium

Assumption

For each agent, the marginal utility of his endowment belongs to the price space Ψ:

$$\frac{\partial}{\partial c} u^i(t, \varepsilon_t^i) \in \Psi \, .$$

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There exists an Arrow–Debreu equilibrium $\left(\psi, (c^i)_{i=1,\dots,I}\right)$ such that

$$\psi_t = \psi(t, X_t), c_t^i = c^i(t, X_t)$$

for continuous functions ψ, c^i that are analytic on $(0, T) \times \mathbb{R}^K$.

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Step 2: Analytic Prices and Completeness

Assumption

The Markov process X has a transition density $P[X_{s+t} \in dy | X_s = x] = p\,(t, x, y)\, dy$ for a continuous function

$$p : (0, T] \times \mathbb{R}^K \times \mathbb{R}^K \to \mathbb{R}_+$$

that is analytic on $(0, T) \times \mathbb{R}^K \times \mathbb{R}^K$. Moreover, the transition density p is bounded on $(\eta, T] \times \mathbb{R}^K \times \mathbb{R}^K$ for all $\eta > 0$.

- directly to check for Brownian motion, mean–reverting, Heston etc.
- If b and σ as well as its derivatives are bounded, Hölder–continuous, and analytic functions, then the assumption is satisfied.

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Theorem

Define $S_t^k = \mathbb{E}\left[\int_t^T A_s^k \psi_s \, \nu(ds) \,\middle|\, \mathscr{F}_t\right]$. *There exist continuous functions* $s : [0, T] \times \mathbb{R}^K \to \mathbb{R}_+$ *that are analytic on* $(0, T) \times \mathbb{R}^K$ *and*

$$S_t = s(t, X_t) \, .$$

The first derivatives with respect to x, $\frac{\partial s}{\partial x_l}$ *are continuous on* $[0, T] \times \mathbb{R}^K$ *and we have*

$$\lim_{t \uparrow T} \frac{\partial s}{\partial x_l}(t, x) = \frac{\partial s}{\partial x_l}(T, x) = \frac{\partial g}{\partial x_l}(T, x)$$

Step 2: Completeness

Theorem

The market (S, A, ψ) is dynamically complete.

- the market is dynamically complete if the volatility matrix is invertible
- By Itô's lemma, the volatility matrix is related to the derivatives of the analytic functions s
- by continuity, they converge to the linearly independent dividends at maturity
- by analyticity, the volatility matrix cannot vanish

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