Existence of Financial Equilibria in Continuous Time with Potentially Complete Markets

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Outline

1. Introduction

2. The Analytic Continuous–Time Markov Economy

3. Analytic Transition Densities and Existence of Radner Equilibrium
Existence with Potentially Complete Markets

Existence of Equilibria in Financial Models: Discrete Time

- With (dynamically) complete markets, Arrow–Debreu equilibria can be implemented as financial (Radner) equilibria.
- With nominal assets that span the market, existence of (efficient) equilibria.
- With assets pay dividends in physical goods (real assets), the spanning condition becomes endogenous.
- Magill–Shafer 85: when asset markets are potentially complete, one has generically existence of efficient equilibria.
- Anderson and Raimondo 2008 go a first step to extend Magill–Shafer to continuous time.
Existence with Potentially Complete Markets

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Analytic Economy

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Anderson–Raimondo Approach

Analytic Markov Economy and Non–Standard Analysis

- Information generated by a Brownian motion $W_t$
- All dividends, endowments are real analytic functions of $W_t$
- Bernoulli utilities are real analytic and “nice”
- Financial markets are potentially complete: as many risky assets as dimension of $W_t$
- Asset dividends are linearly independent at maturity $T$
- Main point: asset prices are analytic, and hence, the linear independence carries over from terminal payoffs to prices $\rightarrow$ dynamically complete market
- Method of proof: nonstandard analysis (discrete approximation)
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This Paper

AR leads to two questions

- Many finance applications need more complex dynamics than functions of Brownian motion
  - affine term structure models (Vasiček, Cox–Ingersoll–Ross)
  - stochastic volatility (Heston, Derman–Kani, Dupire)
  - "predictable" returns (Barberis)

Can one extend to general diffusions?

- Can we have a simpler, standard proof of the result?
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Method and Main Results

- State variable general diffusion process $X_t$
- Analytic Markov economy as in Anderson–Raimondo
- Extend Dana 92 to prove existence of an analytic Arrow–Debreu equilibrium
- Give sufficient conditions to show that security prices are analytic functions of $X_t$
- Use analyticity to show dynamic completeness
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Time and Information

- time is continuous $t \in [0, T]$
- the state variable is a diffusion $X_t$ with values in $\mathbb{R}^K$ driven by a $K$-dimensional Brownian motion $W$:
  \[
  X_0 = x, \quad dX_t = b(X_t)dt + \sigma(X_t)dW_t,
  \]
- for Lipschitz continuous functions
  \[
  b : \mathbb{R}^K \rightarrow \mathbb{R}^K
  \]
  and
  \[
  \sigma : \mathbb{R}^K \rightarrow \mathbb{R}^{K \times K}
  \]
  that are called the drift and dispersion function, resp. We let
  \[
  a(x) := \sigma(x)\sigma(x)^T
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  be the diffusion matrix.
- The diffusion matrix satisfies the uniform ellipticity condition
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  \|x \cdot a(x)x\| \geq \varepsilon \|x\|^2
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Introduction

Analytic Economy

Existence

Agents and Commodities

- one physical commodity (no problem to generalize to $D > 1$)
- $I$ agents consuming a flow $(c_t)$ on $[0, T)$ and a terminal consumption $c_T$; write $\nu = dt \otimes \delta_T$
- consumption space $\mathcal{X} = L^p(\Omega \times [0, T], \mathcal{O}, P \otimes \nu)$, $\mathcal{O}$ optional $\sigma$–field, $p \geq 1$
- price space (Arrow–Debreu) $\Psi = \mathcal{X}^* = L^q(\Omega \times [0, T], \mathcal{O}, P \otimes \nu)$
- $U^i(c) = \mathbb{E} \int_0^T u^i(t, c_t) \nu(dt)$
- The period utility functions $u^i$ are nice and analytic on $(0, T) \times \mathbb{R}_{++}$
- agents’ endowment $e^i_t = e^i(t, X_t)$ is an analytic function of time and state
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The period utility functions $u^i$ are continuous on $[0, T] \times \mathbb{R}_+$, differentiably strictly increasing and differentiably strictly concave in consumption on $[0, T] \times \mathbb{R}_{++}$, i.e.

$$\frac{\partial u^i}{\partial c}(t, c) > 0, \quad \frac{\partial^2 u^i}{\partial c^2}(t, c) > 0.$$ 

They satisfy the Inada conditions

$$\lim_{c \downarrow 0} \frac{\partial u^i}{\partial c}(t, c) = \infty$$

and

$$\lim_{c \to \infty} \frac{\partial u^i}{\partial c}(t, c) = 0$$

uniformly in $t \in [0, T]$. 
There are \( K + 1 \) financial assets (Potentially Complete Markets)

- real assets
- dividends

\[ A_t^k = g^k(t, X_t), \quad t \in [0, T] \]

- dividends belong to the consumption set, \( A^k \in \mathcal{X}_+ \).
- \( g^k \) analytic on \((0, T) \times \mathbb{R}^K\).

- Asset 0 is a real zero–coupon bond with maturity \( T \), \( A_T = 1 \),
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Financial Market, ctd.

- Agent \( i \) owns initially \( n^i_k \geq 0 \) shares of asset \( k \)
- A consumption price process is a positive Itô process \( \psi \).
- A (cum–dividend) security price for asset \( k \) is a nonnegative Itô process \( S^k = (S^k_t)_{0 \leq t \leq T} \). We interpret \( S^k \) as the nominal price of the asset \( k \).
- We denote by
  \[
  G^k_t = S^k_t + \int_{[0,t)} A^k_s \psi_s \nu(ds), \quad (0 \leq t \leq T)
  \]
  the (nominal) gain process for asset \( k \).
- A portfolio process is a predictable process \( \theta \) with values in \( \mathbb{R}^{K+1} \) that is \( G \)-integrable.
- A portfolio is admissible for agent \( i \) if its present value plus the present value of the agent's endowment is nonnegative, or
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  V_t + \mathbb{E} \left[ \int_{t+}^T e^i_s \psi_s \nu(ds) \bigg| \mathcal{F}_t \right] \geq 0.
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- A consumption price process is a positive Itô process $\psi$.
- A (cum–dividend) security price for asset $k$ is a nonnegative Itô process $S^k = (S^k_t)_{0 \leq t \leq T}$. We interpret $S^k$ as the nominal price of the asset $k$.
- We denote by
  \[ G^k_t = S^k_t + \int_{[0,t]} A^k_s \psi(s) \nu(ds), \quad (0 \leq t \leq T) \]
  the (nominal) gain process for asset $k$.
- A portfolio process is a predictable process $\theta$ with values in $\mathbb{R}^{K+1}$ that is $G$–integrable.
- A portfolio is admissible for agent $i$ if its present value plus the present value of the agent’s endowment is nonnegative, or
  \[ V_t + \mathbb{E} \left[ \int_{t^+}^T e^i_s \psi(s) \nu(ds) \Bigg| \mathcal{F}_t \right] \geq 0. \]
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Radner Equilibrium

- A portfolio $\theta$ finances a consumption plan $c \in X_+$ for agent $i$ if $\theta$ is admissible for agent $i$ and the intertemporal budget constraint is satisfied for the associated value process $V$:

$$V_t = n^i \cdot S_0 + \int_0^t \theta_u dG_u + \int_0^t (e^i_u - c_u) \psi_u \nu(du).$$

- A Radner equilibrium consists of asset prices $S$, a consumption price $\psi$, portfolios $\theta^i$ and consumption plans $c^i \in X_+$ for each agent $i$ such that $\theta^i$ is admissible for agent $i$ and finances $c^i$, $c^i$ maximizes agent $i$’s utility over all such $i$–feasible portfolio/consumption pairs, and markets clear, i.e. $\sum_{i=1}^I c^i = e$ and $\sum_{i=1}^I \theta^i = N$. 
Radner Equilibrium

A portfolio $\theta$ finances a consumption plan $c \in \mathcal{X}_+$ for agent $i$ if $\theta$ is admissible for agent $i$ and the intertemporal budget constraint is satisfied for the associated value process $V$:

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Main Theorem

There exists a Radner equilibrium \( \left( S, \psi, (\theta^i, c^i)_{i=1,\ldots,l} \right) \) with a dynamically complete market \( (S, A, \psi) \); the prices and dividends are linked by the present value relation

\[
S^k_t = \mathbb{E} \left[ \int_t^T A^k_s \psi_s \nu(ds) \right| \mathcal{F}_t].
\]
**Step 1: Arrow–Debreu Equilibrium**

**Assumption**

*For each agent, the marginal utility of his endowment belongs to the price space $\Psi$:

$$\frac{\partial}{\partial c} u^i(t, \varepsilon^i_t) \in \Psi.$$*

**Theorem**

*There exists an Arrow–Debreu equilibrium $\left( \psi, (c^i)_{i=1,\ldots,I} \right)$ such that

$$\psi_t = \psi(t, X_t), c^i_t = c^i(t, X_t)$$

for continuous functions $\psi, c^i$ that are analytic on $(0, T) \times \mathbb{R}^K$.***
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for continuous functions \( \psi, c^i \) that are analytic on \((0, T) \times \mathbb{R}^K\).
Step 2: Analytic Prices and Completeness

Assumption

The Markov process $X$ has a transition density $P[X_{s+t} \in dy | X_s = x] = p(t, x, y) dy$ for a continuous function $p : (0, T] \times \mathbb{R}^K \times \mathbb{R}^K \to \mathbb{R}_+$ that is analytic on $(0, T) \times \mathbb{R}^K \times \mathbb{R}^K$. Moreover, the transition density $p$ is bounded on $(\eta, T] \times \mathbb{R}^K \times \mathbb{R}^K$ for all $\eta > 0$.

- directly to check for Brownian motion, mean–reverting, Heston etc.
- If $b$ and $\sigma$ as well as its derivatives are bounded, Hölder–continuous, and analytic functions, then the assumption is satisfied.
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Theorem

Define $S_t^k = \mathbb{E} \left[ \int_t^T A_s^k \psi_s \nu(ds) \big| \mathcal{F}_t \right]$. There exist continuous functions $s : [0, T] \times \mathbb{R}^K \rightarrow \mathbb{R}_+$ that are analytic on $(0, T) \times \mathbb{R}^K$ and

$$S_t = s(t, X_t).$$

The first derivatives with respect to $x$, $\frac{\partial s}{\partial x_l}$ are continuous on $[0, T] \times \mathbb{R}^K$ and we have

$$\lim_{t \uparrow T} \frac{\partial s}{\partial x_l}(t, x) = \frac{\partial s}{\partial x_l}(T, x) = \frac{\partial g}{\partial x_l}(T, x).$$
Step 2: Completeness

**Theorem**

*The market \((S, A, \psi)\) is dynamically complete.*

- The market is dynamically complete if the volatility matrix is invertible.
- By Itô’s lemma, the volatility matrix is related to the derivatives of the analytic functions \(s\).
- By continuity, they converge to the linearly independent dividends at maturity.
- By analyticity, the volatility matrix cannot vanish.
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