Approaches to Knightian Uncertainty in Finance and Economics

Relations of Finance, Insurance, Decision Theory, and Statistics

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University of Oslo, March 15 2023

Outline

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Monetary Risk Measures in Finance

- Let $X(\omega) \in \mathbb{R}$ be a financial position

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- X is a well-defined contract, ω is an observable state of the world ex post
- example: digital option, you get 1 $ if the asset price of Microsoft is above 250
- one likes to write down a probability space, but do we really know P?

Convex Risk Measures

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- for cash m, $\rho(X + m) = \rho(X) - m$
- monotonicity: if $X \geq Y$, $\rho(X) \leq \rho(Y)$
- diversification reduces risk: ρ is convex

Theorem (Föllmer-Schied, Fritelli-Rosazza-Gianin)

Convex risk measures have the form

$$\rho(X) = \sup_{Q} E^{Q}[-X] - \alpha(Q)$$

for some penalty functions $\alpha(Q) \in [0, \infty]$ *for probability measures* Q*.*

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Corollary (Artzner, Delbaen, Eber, Heath)

Positively homogeneous convex risk measures have the form

$$\rho(X) = \sup_{Q \in \mathscr{P}} E^{Q}[-X]$$

for a set of probability measures $\mathscr{P}$.

Monetary Risk Measures in Finance and Model Uncertainty

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- $\alpha = 0$ highest trust, no penalty, $\alpha = \infty$ no trust
- for coherent risk measures, the agent chooses a set of models he trusts and uses a worst-case approach

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Frank Knight, Risk, Uncertainty, and Profit, 1921

Workshop "Uncertainty and Risk" Commemorating the Centenary of Publication of Frank H. Knight's *Risk, Uncertainty, and Profit* and John M. Keynes' *A Treatise on Probability* See https://sites.google.com/view/uncertainty-risk/home

Video of my lecture https://www.youtube.com/watch?v=sf0qDwdGGko

Uncertainty and Risk

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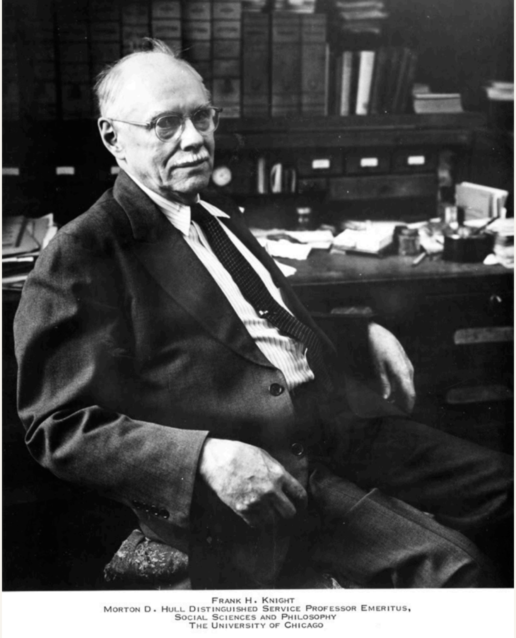
March 17-19, 2021 - Virtual



Frank H. Knight

Photo: University of Chicago Photographic Archive, apf1-03513, Special Collections Research Center.

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- Knight identifies proper uncertainty as a source of profit

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- markets can perfectly price such randomness (insurance)
- *The mathematical type of probability is practically never met with in business. (p.215)*
- In typical business situations, there is no law of large numbers that allows to estimate the probability of success with accuracy.

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- *the income of an entrepreneur is larger ... as there is a scarcity of self-confidence in society combined with the power to make effective guarantees to employees.(p.283)*
- excess profit is the result of confronting uninsurable uncertainty

A Taxonomy of Uncertainty

taken from Lo, Mueller 2010

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4. Imprecise Probabilistic Information
5. Ignorance: data does not help, theories do not help, no quantification is possible

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Insurance Premia under Uncertainty

based on joint work with Max Nendel, Maren Schmeck, Insurance: Mathematics and Economics 2022

- The axiomatization of premium principles is one of the most classic topics in the field of actuarial mathematics.

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- Implicitly, thus, a probabilistic model is assumed
- Model (Knightian) uncertainty is by now widely recognized and crucial for insurance (Solvency II)
- We work in an ex-ante probability-free setting

Astin Bulletin 11 (1980) 52-60

AN ECONOMIC PREMIUM PRINCIPLE

HANS BÜHLMANN *

Eidgenössische Technische Hochschule, Zürich

1. PREMIUM CALCULATION PRINCIPLES VERSUS ECONOMIC PREMIUM PRINCIPLES

(a) The notion of premium calculation principle has become fairly generally accepted in the risk theory literature. For completeness we repeat its definition:

A premium calculation principle is a functional $\mathfrak{H}$ assigning to a random variable X (or its distribution function $F_X(x)$) a real number P. In symbols

$\mathfrak{H}$	:	X	$\rightarrow$	P
premium calculation principle		random variable		real number
	or	$F_X(x)$	$\rightarrow$	P
		distribution function		real number

The interpretation is rather obvious. The random variable X stands for the possible claims of a risk whereas P is the premium charged for assuming this risk.

This is of course formalizing the way *actuaries* think about premiums. In actuarial terms, the premium is a property of the risk (and *nothing else*), e.g.

$$\mathfrak{H}[X] = E[X] + \alpha\sigma[X]$$

$$\mathfrak{H}[X] = (1+\lambda)\, E[X], \text{ etc.}$$

(b) Of course, in *economics* premiums are not only depending on the risk but also on *market conditions*. Let us assume for a moment that we can describe the risk by a random variable X (as under a)), describe the market conditions by a random variable Z.

Then we want to show how an *economic premium principle*

$\mathfrak{E}$:	(X, Z)	$\rightarrow$	P
	pair of random variables		real number

* This paper is greatly influenced by an exchange of ideas with Flavio Pressaco. I am also indebted to Hans Gerber for stimulating discussions on this subject.

Insurance Premia

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- Aim: Provide an axiomatization of premium principles under Knightian uncertainty
- Main result: Insurance premium = risk measure + deviation measure

Setup

Throughout, we consider

- a measurable space $(\Omega, \mathscr{F})$,
- the space $\mathrm{B}_b = \mathrm{B}_b(\Omega, \mathscr{F})$ of all bounded measurable functions $\Omega \to \mathbb{R}$,
- a set $C \subset \mathrm{B}_b$ of insurance claims with $0 \in C$ and $X + m \in C$ for all $X \in C$ and $m \in \mathbb{R}$.

Definition

We say that a map $H\colon C \to \mathbb{R}$ is a *premium principle* if

- $H(X+m) = H(X) + m$ for all $X \in C$ and $m \in \mathbb{R}$.
- $H(X) \geq H(0) = 0$ for all $X \in C$ with $X \geq 0$.

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Some comments:

- The definition of a premium principle implies that $H(m) = m$, for all $m \in \mathbb{R}$, leading to the common assumption of no unjustified risk loading.
- The condition $H(X) \geq 0$, for all $X \in C$ with $X \geq 0$, is a minimal requirement for a sensible notion of a premium principle.

Risk Measures and Deviation Measures

A map $R\colon \mathrm{B}_b \to \mathbb{R}$ is called a risk measure

- $R(0) = 0$ and $R(X + m) = R(X) + m$ for all $X \in \mathrm{B}_b$ and $m \in \mathbb{R}$, note the sign change!
- $R(X) \leq R(Y)$ for all $X, Y \in \mathrm{B}_b$ with $X \geq Y$.

A deviation measure (cf. Rockafellar-Uryasev (2013)) is a map $D\colon C \to \mathbb{R}$ with

- $D(X + m) = D(X)$ for all $X \in C$ and $m \in \mathbb{R}$,
- $D(X) \geq D(0) = 0$ for all $X \in C$.

Premium principles and their decomposition

Theorem

A map $H\colon C \to \mathbb{R}$ *is a premium principle if and only if*

$$H(X) = R(X) + D(X) \quad \textit{for all } X \in C,$$

where $R\colon \mathrm{B}_b \to \mathbb{R}$ *is a risk measure and* $D\colon C \to \mathbb{R}$ *is a deviation measure.*

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- Monetary measure generalizes expected loss
- deviation measure generalizes variance or other measures of fluctuation
- The decomposition needs not be unique

The "maximal" decomposition

Theorem

Let $H\colon C \to \mathbb{R}$ be a premium principle. Define

$$R_{\mathrm{Max}}(X) := \inf\left\{H(X_0)\,\middle|\,X_0 \in C,\ X_0 \geq X\right\},$$
$$D_{\mathrm{Min}}(X) := H(X) - R_{\mathrm{Max}}(X).$$

Then, $R_{\mathrm{Max}}\colon \mathrm{B}_b \to \mathbb{R}$ is a risk measure, $D_{\mathrm{Min}}\colon C \to \mathbb{R}$ is a deviation measure.
For every other decomposition of the form $H(X) = R(X) + D(X)$ with a risk measure R and a deviation measure D, we have $R \leq R_{\mathrm{Max}}$ and $D \geq D_{\mathrm{Min}}$.

Example: variance principle

Consider the variance principle

$$H(X) = \mathbb{E}_{\mathbb{P}}(X) + \frac{\theta}{2}\mathrm{var}_{\mathbb{P}}(X), \quad \text{for } X \in \mathrm{B}_b,$$

with a constant $\theta \geq 0$.

- Here, $R(X) = \mathbb{E}_{\mathbb{P}}(X)$, and $D(X) = \frac{\theta}{2}\mathrm{var}_{\mathbb{P}}(X)$ is *one* possible decomposition of H into risk and deviation.
- However, for $\theta > 0$, this is not the "maximal" decomposition. For $\theta > 0$, the maximal risk measure R_{Max} is given by

$$R_{\mathrm{Max}}(X) = \max_{\mathbb{Q} \in \mathscr{P}} \mathbb{E}_{\mathbb{Q}}(X) - \frac{1}{2\theta} G(\mathbb{Q}|\mathbb{P}),$$

where $\mathscr{P}$ consists of all probability measures $\mathbb{Q}$, which are absolutely continuous w.r.t. $\mathbb{P}$ and satisfy

$$G(\mathbb{Q}|\mathbb{P}) := \mathrm{var}_{\mathbb{P}}\left(\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right) < \infty.$$

The map G is the *Gini concentration index*, see Maccheroni et al. (2006,2009).

Example: Economic Premium Principle

Bühlmann (1980) provides a competitive market foundation for insurance premia. In an expected utility framework for a common prior $\mathbb{P}$, in equilibrium, we have

$$H(X) = \frac{\mathbb{E}_{\mathbb{P}}\big(X\ell'(Z)\big)}{\mathbb{E}_{\mathbb{P}}\big(\ell'(Z)\big)}$$

for the insurer's loss function ℓ and aggregate endowment Z. Deprez and Gerber (1985) consider convex premium principles. Utility indifference leads to

$$\mathbb{E}_{\mathbb{P}}\big(\ell(Z + X - p)\big) = \mathbb{E}_{\mathbb{P}}\big(\ell(Z)\big).$$

For constant risk aversion

$$H(X) = \frac{1}{\alpha} \log \frac{\mathbb{E}_{\mathbb{P}}\big(e^{\alpha(Z+X)}\big)}{\mathbb{E}_{\mathbb{P}}\big(e^{\alpha Z}\big)}, \quad \text{for } X \in \mathrm{B}_b.$$

Choquet Premia

Wang, Young, and Panjer (1997) derive an axiomatic characterization of premium principles in a competitive market setting that results in a representation using Choquet integrals

$$H(X) = \int_{\min X}^{\infty} g\left(\mathbb{P}_X(t)\right) \mathrm{d}t + \min X,$$

where, for $t \geq 0$, $\mathbb{P}_X(t) := \mathbb{P}(X > t)$, and g is a suitable distortion function.

Consistency with Financial Markets

Let H be sublinear.

- Suppose that insurance companies trade in an arbitrage-free financial market given by a linear subspace $M \subset C$ and a nonnegative linear pricing functional $F : M \to \mathbb{R}$ with $F(1) = 1$
- $\mathbb{P}$ is a martingale measure if $\mathbb{E}_{\mathbb{P}}(X) = F(X)$ for $X \in M$
- we need to have $H = F$ on M (competition, arbitrage)
- let R_* be the superhedging functional of the financial market

Consistency with Financial Markets

Theorem

The following statements are equivalent:

1. *The maximal risk measure in the decomposition of H is the superhedging functional, i.e.* $R_{\mathrm{Max}} = R_*$.
2. *A model* $\mathbb{P}$ *is plausible if and only if it is a martingale measure.*

Outline

Monetary Risk Measures in Finance

The Legacy of Frank Knight

Insurance Premia under Uncertainty

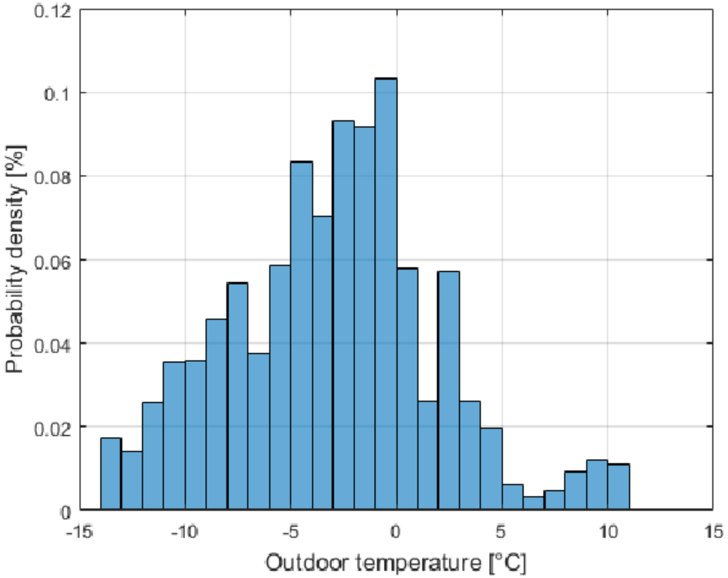
The Smooth Approach to Model Uncertainty and Statistics

Model Uncertainty in Climate Science

Climate Change is Model Uncertainty

- Weather is the current state of temperature, humidity, pressure, rainfall etc at a given location
- Climate is the probability distribution of weather at a given location
- Climate change induces Knightian uncertainty as the change of probabilities is not deterministic

Oslo Temperature Distribution Winter



Oslo

Klimaetaten

Climate Change Vulnerability Analysis for Oslo

Short version

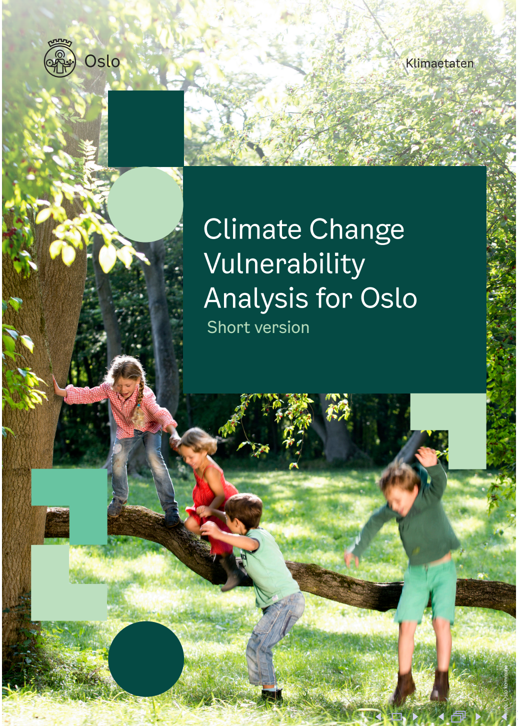


Photo: Oslo kommune

INCREASED PROBABILITY

The number of episodes of heavy precipitation is expected to increase significantly in terms of intensity and frequency. These will also lead to more stormwater and urban floods.

More frequent and heavier floods are expected, and floodwater flows in streams and rivers must be expected to increase.

Increased risk of landslides and flood-related debris flows resulting from increased precipitation.

Higher storm surge levels are expected as a result of sea level rise.

POTENTIALLY INCREASED PROBABILITY

Minor changes are expected in summer precipitation, and higher temperatures and increased evaporation may therefore increase the risk of drought during the summer.

Increased erosion caused by heavy precipitation and increased flooding of rivers and stream may trigger more quick clay slides.

UNCHANGED OR LESS PROBABILITY

Snowmelt floods will occur increasingly earlier in the year and become smaller in scale towards the end of the century.

Shorter ice cover season and reduced ice drift. Coastal rivers will have little ice cover.

UNCERTAIN

It is uncertain whether the incidence and intensity of **strong winds** will change.

•More frequent episodes of heavy precipitation may increase the frequency of rockfalls and rock slides, though mainly smaller **rockfall events.**

Probability for some of the climate related hazards in Oslo towards 2100.
Source: Klimaprofil for Oslo og Akershus (Norsk klimaservicesenter 2017)

These climate changes will create more frequent and more severe climate-related hazards due to more extreme weather and to changes in normal weather. While some of the capital›s future climate-related hazards will be acute, others will emerge more gradually. The most acute hazards will be associated with more extreme precipitation. The increases in precipitation that have occurred and that will continue to occur in Oslo will materialise in the form of heavy and intense rainfall. As a consequence, today›s extreme precipitation may become the new normal. This would increase the likelihood of:

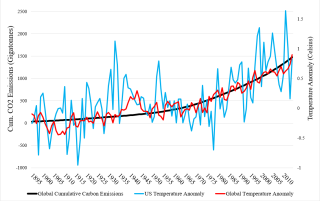
- Stormwater and urban floods. We must reduce the extent of impermeable surfaces in the city, manage stormwater locally, and use it as a resource in the urban landscape.

- River floods. It will be increasingly important to control where water runs when rivers flood, and to secure flood zones along rivers and streams that take a changed climate into account.

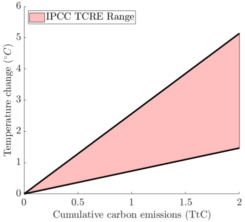
- Landslides and avalanches. Soil deposits and terrain types usually determine where landslides and avalanches occur, and potential slide zones will remain mostly the same, but because they are often triggered by extreme precipitation, future landslides and avalanches in Oslo may become more frequent and cause more damage. This will apply particularly to minor landslides and flood-related debris flows, but also to quick clay slides.

Economics of Climate Change: Uncertainty

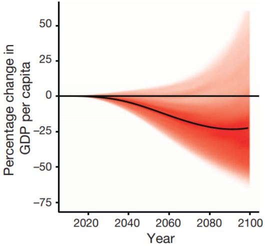
Michael Barnett, Climate Change and Uncertainty: An Asset Pricing Perspective, Management Science, to appear



(a) Source: NASA-GISS, NOAA.



(b) Source: Allen et al. (2018)



(c) Source: Burke et al. (2015)

A Parametric Statistical Model

- Let $\theta \in \Theta = [0, 5]$ (or $\mathbb{R}$) parametrize the change of average temperature in the next 30 years.
- Let $P^0 = N(-2.3, 4)$ be the temperature distribution in Oslo in February. Climate change uncertainty can then be modeled by the family
$$\mathscr{P} = \left(P^\theta\right)_{\theta \in \Theta}.$$
- The different levels of plausibility can be captured by a prior probability μ on Θ.
- How shall we evaluate the outcome of various policies?

Classic Approach: Expected Utility

The Predictive Probability

Classic Approach: Expected Utility

The Predictive Probability

- Let $X(\omega)$ be the outcome of a policy after 30 years (GDP, Employment, Health, ...)

Classic Approach: Expected Utility

The Predictive Probability

- Let $X(\omega)$ be the outcome of a policy after 30 years (GDP, Employment, Health, ...)
- Let u be the Bernoulli utility function of society

Classic Approach: Expected Utility

The Predictive Probability

- Let $X(\omega)$ be the outcome of a policy after 30 years (GDP, Employment, Health, ...)
- Let u be the Bernoulli utility function of society
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- shouldn't we take the uncertainty of θ into account?

New Approach: Smooth Model

Klibanoff, Marinacci, Mukerji, 2005

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- let us model aversion to such second-order uncertainty as we model risk aversion in the first layer

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- Introduce a concave (second-order) utility function v
- Define overall utility as

$$U(X) = \int_\Theta v(c^\theta(X))\,\mu(d\theta) = \mathbb{E}^\mu v\left(c(X)\right)$$

New Approach: Smooth Model

- We also have

$$
\begin{aligned}
U(X) &= \int_{\Theta} v(c^{\theta}(X))\mu(d\theta) \\
&= \mathbb{E}^{\mu}\phi\left(\mathbb{E}^{\theta}u(X)\right)
\end{aligned}
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- if $-\frac{\phi''(y)}{\phi'(y)} \to \infty$, U tends to Gilboa–Schmeidler (maxmin) expected utility $\min_{\theta} \mathbb{E}^{\theta} u(X)$

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- $u(x) = -\exp(-ax), v(y) = -\exp(-by)$
- what is the equivalent insurance premium under risk resp uncertainty?

Application to Insurance

Lemma

The equivalent premium under risk is

$$f^{\theta} = \theta + a\sigma^2/2.$$

The equivalent premium under uncertainty is

$$f = m + a\sigma^2/2 + bv^2/2.$$

Model uncertainty leads to an additional insurance premium.

Identifiability

Definition
A statistical model $(\Omega, \mathscr{F}, cP = \left(P^\theta\right)_{\theta\in\Theta})$ is identifiable if there exist a measurable funktion $k : \Omega \to \Theta$ with $P^\theta[k = \theta] = 1$ for all $\theta \in \Theta$.

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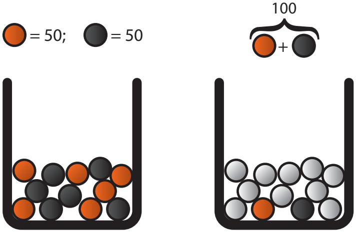
- $P^\theta = \bigotimes_{n \in \mathbb{N}} Q^\theta$
- X_n nth projection
- law of large numbers: $k = \lim \frac{1}{n} \sum_{l=1}^{n} X_l$ identifies θ

Ellsberg Experiment

Ellsberg's Thought Experiment 1



Ellsberg Urn

- An urn contains 100 blue and red balls in unknown proportions
- composition of the urn is verifiable ex post
- $\omega = (c(olor), n(umberofredballs))$
- P_n: the urn contains n red balls
- $k(\omega) = P_n$

Predictive Representation

Theorem (Denti, Pomatto 2022)

If the statistical model $(\Omega, \mathscr{F}, cP = (P^\theta)_{\theta \in \Theta})$ *is identifiable by* k*, then the smooth utility function has the predictive representation*

$$U(X) = \mathbb{E}^{\bar{P}}\left[\phi\left(\mathbb{E}^{\bar{P}}\left[u(X)|k\right]\right)\right].$$

Remark

- *Foundation for decision making under uncertainty in identifiable models*
- $\sigma(k)$ *is the* σ*-field of pure model uncertainty*
- *under identifiability, markets might insure such uncertainty and resolve issued with market incompleteness due to Knightian uncertainty (Friday!!)*