The Fundamental Theorem of Asset Pricing without Probabilistic Prior Assumptions

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Outline

1. Introduction
2. Discrete Models
3. Finance without
   - Topological Model
   - Arbitrage
   - Viability
   - NA Prices
   - Superhedging as a LP
Motivation

- in discrete time and state spaces, finance does not need a probability on the state space
  - probability enters through the no arbitrage axiom
  - as a martingale measure
  - or linear pricing functional
- in complex models, we usually start with \((\Omega, \mathcal{F}, P)\), a probability space
  - \(P^*\) then equivalent martingale measure
- implicit assumption: agents / market knows null sets
- Knightian uncertainty / model uncertainty
- volatility uncertainty (Avellaneda, Peng etc.): singular measures
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- develop the fundamental theorem of asset pricing on a Polish space $(\Omega, d)$
- relation to economic equilibrium (Harrison–Kreps)
- superhedging as a linear program
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Based on Föllmer’s Calcul d’Itô sans probabilités (1981), one can develop the hedging argument of Black and Scholes without prior probabilities (Bick, Willinger 1994).

Föllmer and Schied develop monetary risk measures on $C_b(\Omega, d)$.

The fundamental theorem based on probability spaces goes back to Kreps and Yan (1980); general version due to Dalang-Morton-Willinger. Continuous time by Delbaen and Schachermayer.
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The Fundamental Theorems of Asset Pricing in Discrete Models

- \( \Omega = \{\omega_1, \ldots, \omega_N\} \)
- \( D + 1 \) assets \( d = 0, \ldots, D \)
- price today \( f_d \geq 0 \)
- uncertain payoff tomorrow \( F_d(\omega_k) \)
- summarized by a \( D + 1 \times N \) matrix \( F \)
- safe asset: \( f_0 = 1, F_N(\omega_k) = 1 \)
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A portfolio $\pi \in \mathbb{R}^{D+1}$ is called an arbitrage if

1. no cost: $\pi \cdot f \leq 0$
2. no losses: $F\pi \geq 0$
3. sometimes gain: $F\pi \neq 0$

Theorem

The model $(f, F)$ admits no arbitrage if and only if there exist probabilities $p_k^* > 0$ such that $f = p^* \cdot F$, or $f_d = \sum_{k=1}^{N} p_k^* F_d(\omega_k)$

Remark

No prior probability needed.
The "martingale" measure has full support or is equivalent to the uniform measure on $\Omega$.
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The Topological Model of Finance

- $(\Omega, d)$ Polish space, $\mathcal{F}$ Borel sets
- $D + 1$ assets $d = 0, \ldots, D$, price today $f_d \geq 0$
- uncertain payoff tomorrow $F_d : (\Omega, d) \to \mathbb{R}_+$ continuous

Remark

Continuity not restrictive from modeling perspective, $F_d$ projections, thus continuous.
Model does not work with measurability.
No strictly positive functionals on $\mathcal{B}(\Omega, d)$
Lusin: measurability is close to continuity.
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A portfolio $\pi \in \mathbb{R}^{D+1}$ is called an arbitrage if

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weak definition of arbitrage

with continuity not so weak

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A weak definition of arbitrage with continuity is not considered so weak because an arbitrage generates gains on an open set.
No arbitrage

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Remark

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A probability $P^*$ on $(\Omega, \mathcal{F})$ is called a martingale measure if $f_d = E^{P^*} F_d$ for $d = 1, \ldots, D$. If $P^*$ has full support, it is called a full support martingale measure.
The Fundamental Theorem of Asset Pricing

**Theorem**

The market \((f, F)\) admits no arbitrage if and only if there exists a full support martingale measure.

**Remark**

- Full Support measures exist on Polish spaces.
- Integrability of \(F_t\) comes for free (as in classical case).
- Full support is one natural generalization of the discrete case.
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*The market \((f, F)\) admits no arbitrage if and only if there exists a full support martingale measure.*

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Theorem

The market \((f, F)\) admits no arbitrage if and only if there exists a **full support martingale measure**.

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Economic Equilibrium and No Arbitrage

- $\Omega$ compact

- Economic agent = complete, transitive, continuous relation $\succeq$ on $C(\Omega, d)$. $\succ$ is strictly monotone: if $X \in \mathcal{X}$ satisfies $X \geq 0$ and $X \neq 0$, then for all $Z \in C(\Omega, d)$, we have $Z + X \succ Z$.

- Marketed subspace

  $$M := \langle S_0, S_1, \ldots, S_D \rangle = \left\{ \pi \cdot S; \pi \in \mathbb{R}^{D+1} \right\} .$$

- Under no arbitrage, the price functional

  $$\phi : M \to \mathbb{R}$$

  given by $\phi (\pi \cdot S) = \pi \cdot f$ is well-defined.
Economic Equilibrium and No Arbitrage

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- Economic agent = complete, transitive, continuous relation \( \succeq \) on \( C(\Omega, d) \). \( \succ \) is strictly monotone: if \( X \in \mathcal{X} \) satisfies \( X \geq 0 \) and \( X \neq 0 \), then for all \( Z \in C(\Omega, d) \), we have \( Z + X \succ Z \).
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- Marketed subspace

$$M := \langle S_0, S_1, \ldots, S_D \rangle = \left\{ \pi \cdot S; \pi \in \mathbb{R}^{D+1} \right\}.$$

- Under no arbitrage, the price functional

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**Definition**

We say that the market \((f, S)\) is viable if there exists an agent \(\succeq\) such that no trade is optimal given the budget 0: for all \(\pi \in \mathbb{R}^{D+1}\) with \(\pi \cdot f \leq 0\) we have \(0 \succeq \pi \cdot S\).

**Theorem**

The market \((f, S)\) is viable if and only if there exists a strictly positive linear functional \(\Phi : C(\Omega, d) \rightarrow \mathbb{R}\) such that \(\Phi(X) = \phi(X)\) for \(X \in M\).

**Remark**

\(\Phi\) continuous \((C(\Omega, d)\) Banach lattice).
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The market \((f, S)\) is viable if and only if there exists a strictly positive linear functional \(\Phi : C(\Omega, d) \to \mathbb{R}\) such that \(\Phi(X) = \phi(X)\) for \(X \in M\).

Remark

\(\Phi\) continuous (\(C(\Omega, d)\) Banach lattice).
Finance without Viability

Economic Equilibrium and No Arbitrage

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No Arbitrage Prices

Definition
A derivative or (contingent) claim is a continuous mapping $H : \Omega \to \mathbb{R}_+$. $h \geq 0$ is called a no arbitrage price for $H$ if the extended market with $D + 2$ assets and $f_{D+1} = h$ and $S_{D+1} = H$ admits no arbitrage opportunities.

Corollary
$h$ is a no arbitrage price for a claim $H$ if and only if

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for a full support martingale measure $P$. 
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A portfolio $\pi$ is called a superhedge for the claim $H$ if $\pi \cdot S(\Omega) \geq H(\omega)$ holds true for all $\omega \in \Omega$.

Find the cheapest superhedge for the claim $H$; minimize $\pi \cdot f$ over $\pi \in \mathbb{R}^{D+1}$ subject to $\pi \cdot S(\Omega) \geq H(\omega)$ for all $\omega \in \Omega$. 

Superhedging and Linear Programming

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Embedding into Linear Programming

Linear Programming in Infinite–Dimensional Spaces

- Two dual pairs of vector spaces
- here $X = Y = \mathbb{R}^{D+1}$, $Z = C(\Omega, d)$, $W = ca(\Omega, \mathcal{F})$
- linear constraint $B : \mathbb{R}^{D+1} \to C(\Omega, d)$, $B\pi = \pi \cdot F$
- Adjoint mapping $B^* : ca(\Omega, \mathcal{F}) \to \mathbb{R}^{D+1}$, $B^* \mu = \int Sd\mu$
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no sign constraints on portfolio: equality in the dual program,
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\[ \int S_d d\mu = f_d, \quad d = 0, \ldots, D. \]

\( d = 0 \): probability measure
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**Problem (DSH)**

Minimize the prices \( \int_\Omega H d\mu \) over all martingale measures \( \mu \in \text{ca}(\Omega, \mathcal{F}) \).
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**Theorem**

1. The linear programs $\text{SH}$ and $\text{DSH}$ are dual to each other. Both problems have the same value.

2. Both programs have optimal solutions; in particular, there exists a superhedge $\pi^* \in \mathbb{R}^{D+1}$ and a martingale measure $P^*$ such that

   $$\pi^* \cdot f = \int HdP^*. $$

**Remark**

Full support martingale measures dense in the set of all martingale measures. Hence, superhedging value equal to upper bound of all no arbitrage prices.
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