Corrigenda: *Stochastic calculus with infinitesimals*

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Page xviii, lines 13–14: characterization of a.s. convergence [60, Theorem <u>7.1</u>]
Page 1, line -6: natural number <u>other than zero</u>
Page 2, line 8: equivalence classes of <u>sequences of</u> real numbers[1]
Page 23–26, Theorem 3.7 and proof:
Theorem 3.7 (Uniqueness of the Itô decomposition). Let $\mu_1, \mu_2, \sigma_1, \sigma_2$ be $\mathcal{F}$-adapted processes. Suppose for all $t \in \mathbb{T} \setminus \{1\}$, we have

$$\mu_1(t)\mathrm{d}t + \sigma_1(t)\mathrm{d}W(t) = \mu_2(t)\mathrm{d}t + \sigma_2(t)\mathrm{d}W(t) + R\left(t + \mathrm{d}t\right)\left(\mathrm{d}t\right)^{1+p}$$

for some constant $p \gg 0$ and a random variable $R\left(t + \mathrm{d}t\right)$ such that $E\left[\int_0^1 R(t+\mathrm{d}t)^2 \mathrm{d}t\right]$ is limited.[2] Then for P-a.e. $\omega \in \Omega$ and ν-a.e. $t \in \mathbb{T} \setminus \{1\}$,

$$\sigma_1(t)(\omega) \simeq \sigma_2(t)(\omega), \qquad \mu_1(t)(\omega) \simeq \mu_2(t)(\omega).$$

Proof. Put $\mu = \mu_1 - \mu_2$ and $\sigma = \sigma_2 - \sigma_1$, so that for all $t \in \mathbb{T} \setminus \{1\}$,

$$\mu(t)\mathrm{d}t - \sigma(t)\mathrm{d}W(t) = R\left(t + \mathrm{d}t\right)\left(\mathrm{d}t\right)^{1+p}.$$

Applying conditional expectations to our assumption $\mu(t)\mathrm{d}t = \sigma(t)\mathrm{d}W(t) + R\left(t + \mathrm{d}t\right)\left(\mathrm{d}t\right)^{1+p}$, we find (using the $\mathcal{F}_t$-linearity of the operator

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[1] I thank Professor Mikhail Katz for spotting this omission.

[2] We denote this random variable by $R\left(t + \mathrm{d}t\right)$ rather than $R\left(t\right)$ because it is $\mathcal{F}_{t+\mathrm{d}t}$-measurable, but in general not $\mathcal{F}_t$-measurable.

$E\left[\cdot\right|\mathcal{F}_t])$

$$\begin{aligned}\mu(t)\mathrm{d}t = E\left[\mu(t)\mathrm{d}t\right|\mathcal{F}_t] &= \sigma(t)\underbrace{E\left[\mathrm{d}W(t)\right|\mathcal{F}_t]}_{=0} \\ &\quad +E\left[R\left(t+\mathrm{d}t\right)\right|\mathcal{F}_t]\left(\mathrm{d}t\right)^{1+p}\end{aligned}$$

so $\mu(t) = E\left[R\left(t+\mathrm{d}t\right)\right|\mathcal{F}_t]\left(\mathrm{d}t\right)^{p}$. Therefore,

$$\mu(t)^2 = E\left[R\left(t+\mathrm{d}t\right)\right|\mathcal{F}_t]^2\ \left(\mathrm{d}t\right)^{2p}$$

and thus, by Jensen's inequality for conditional expectations,

$$\mu(t)^2 \leq E\left[R\left(t+\mathrm{d}t\right)^2\right|\mathcal{F}_t]\ \left(\mathrm{d}t\right)^{2p}.$$

It follows that

$$\begin{aligned}E\left[\int_0^1 \mu(t)^2\mathrm{d}t\right] &\leq E\left[\int_0^1 E\left[R\left(t+\mathrm{d}t\right)^2\right|\mathcal{F}_t]\,\mathrm{d}t\right]\left(\mathrm{d}t\right)^{2p} \\ &= \int_0^1 E\left[E\left[R\left(t+\mathrm{d}t\right)^2\right|\mathcal{F}_t]\right]\left(\mathrm{d}t\right)^{1+2p} \\ &= \int_0^1 E\left[R\left(t+\mathrm{d}t\right)^2\right]\left(\mathrm{d}t\right)^{1+2p} \\ &= E\left[\int_0^1 R\left(t+\mathrm{d}t\right)^2\mathrm{d}t\right]\ \left(\mathrm{d}t\right)^{2p}\end{aligned}$$

Since $E\left[\int_0^1 R\left(t+\mathrm{d}t\right)^2\mathrm{d}t\right]$ was assumed to be limited, we deduce $E\left[\int_0^1 \mu(t)^2\mathrm{d}t\right] = \mathcal{O}\left(\left(\mathrm{d}t\right)^{2p}\right) \simeq 0$, which by the radically elementary Lebesgue Theorem means $\mu(t)(\omega) \simeq 0$ for P-a.e. $\omega \in \Omega$ and ν-a.e. t.

Moreover, a binomial expansion based on $\mu(t)\mathrm{d}t - R\left(t+\mathrm{d}t\right)\left(\mathrm{d}t\right)^{1+p} = \sigma(t)\mathrm{d}W(t)$ yields, when combined with $\left|\mathrm{d}W(t)\right|^2 = \mathrm{d}t$, the equation

$$\sigma(t)^2\mathrm{d}t = \mu(t)^2(\mathrm{d}t)^2 - 2R\left(t+\mathrm{d}t\right)\mu(t)(\mathrm{d}t)^{2+p} + R\left(t+\mathrm{d}t\right)^2\left(\mathrm{d}t\right)^{2+2p},$$

which after taking expectations and applying the Cauchy-Schwarz inequality leads on to

$$\begin{aligned}E\left[\sigma(t)^2\right]\mathrm{d}t &\leq E\left[\mu(t)^2\right](\mathrm{d}t)^2 + E\left[R\left(t+\mathrm{d}t\right)^2\right]\left(\mathrm{d}t\right)^{2+2p} \\ &\quad +2E\left[R\left(t+\mathrm{d}t\right)^2\right]^{1/2}E\left[\mu(t)^2\right]^{1/2}\left(\mathrm{d}t\right)^{2+p}\end{aligned}$$

Therefore,

$$
\begin{aligned}
E\left[\int_0^s \sigma(t)^2 \mathrm{d}t\right] &= \int_0^s E\left[\sigma(t)^2\right] \mathrm{d}t \\
&= \int_0^s E\left[\mu(t)^2\right] (\mathrm{d}t)^2 + \int_0^s E\left[R\left(t+\mathrm{d}t\right)^2\right] (\mathrm{d}t)^{2+2p} \\
&\quad +2\int_0^s E\left[R\left(t+\mathrm{d}t\right)^2\right]^{1/2} E\left[\mu(t)^2\right]^{1/2} (\mathrm{d}t)^{2+p} \\
&\leq E\left[\int_0^1 \mu(t)^2 \mathrm{d}t\right] \mathrm{d}t + E\left[\int_0^1 R(t+\mathrm{d}t)^2 \mathrm{d}t\right] (\mathrm{d}t)^{1+2p} \\
&\quad +2\int_0^1 E\left[R\left(t+\mathrm{d}t\right)^2\right]^{1/2} E\left[\mu(t)^2\right]^{1/2} (\mathrm{d}t)^{2+p}. \qquad (1)
\end{aligned}
$$

The third and last addend can be simplified as follows. First, because $\int_0^1$ is just short hand for $\sum_{t\in\mathbb{T}\setminus\{1\}}$, we have the (rather "wasteful") estimate

$$
\begin{aligned}
&\int_0^1 E\left[R\left(t+\mathrm{d}t\right)^2\right]^{1/2} E\left[\mu(t)^2\right]^{1/2} (\mathrm{d}t)^2 \\
\leq\ & \left(\int_0^1 E\left[R\left(t+\mathrm{d}t\right)^2\right]^{1/2} \mathrm{d}t\right)\left(\int_0^1 E\left[\mu(t)^2\right]^{1/2} \mathrm{d}t\right)
\end{aligned}
$$

Then, using either Jensen's inequality or the Cauchy–Schwarz inequality for the expectation operator $\int_0^1 \cdot \mathrm{d}t$ (which maps each $f : \mathbb{T} \to \mathbb{R}$ to $\frac{1}{N}\sum_{k<N} f(k/N)$), one can further estimate this term:

$$
\begin{aligned}
&\left(\int_0^1 E\left[R\left(t+\mathrm{d}t\right)^2\right]^{1/2} \mathrm{d}t\right)\left(\int_0^1 E\left[\mu(t)^2\right]^{1/2} \mathrm{d}t\right) \\
=\ & \left(\int_0^1 E\left[R\left(t+\mathrm{d}t\right)^2\right]^{1/2} \mathrm{d}t\right)\left(\int_0^1 E\left[\mu(t)^2\right]^{1/2} \mathrm{d}t\right) \\
\leq\ & \left(\int_0^1 E\left[R\left(t+\mathrm{d}t\right)^2\right] \mathrm{d}t\right)^{1/2}\left(\int_0^1 E\left[\mu(t)^2\right] \mathrm{d}t\right)^{1/2} \\
\leq\ & E\left[\int_0^1 R\left(t+\mathrm{d}t\right)^2 \mathrm{d}t\right]^{1/2} E\left[\int_0^1 \mu(t)^2 \mathrm{d}t\right]^{1/2} \qquad (2)
\end{aligned}
$$

Now $E\left[\int_0^1 R\left(t+\mathrm{d}t\right)^2 \mathrm{d}t\right]$ is limited by assumption, and $E\left[\int_0^1 \mu(t)^2 \mathrm{d}t\right]$ is even $\mathcal{O}\left((\mathrm{d}t)^{2p}\right)$, as we have seen before. Hence, the right-hand side of estimate (2) is $\mathcal{O}\left((\mathrm{d}t)^{3p}\right)$, whence the third and last addend in estimate (1) is actually $\mathcal{O}\left((\mathrm{d}t)^{3p}\right)$.

However, the fact that $E\left[\int_0^1 \mu(t)^2 \mathrm{d}t\right]$ is infinitesimal (shown above) and the assumption of $E\left[\int_0^1 R(t+\mathrm{d}t)^2 \mathrm{d}t\right]$ being limited also jointly imply that the first two addends in estimate (1) are infinitesimal (to be more precise, they are $\mathcal{O}\left((\mathrm{d}t)^{2\wedge(1+2p)}\right)$). Thus, estimate (1) actually shows that

$$E\left[\int_0^s \sigma(t)^2 \mathrm{d}t\right] = \mathcal{O}\left((\mathrm{d}t)^{2\wedge(1+2p)\wedge 3p}\right) \simeq 0.$$

(In applications, one will typically have $p \leq 1/2$, so the exponent is just $3p$.) This entails that for P-a.e. $\omega \in \Omega$ and ν-a.e. t, $\sigma(t)(\omega) \simeq 0$ (by the radically elementary Lebesgue Theorem). $\square$