The Meaning of a First-Order Formula, Compositionality and Alphabetic Innocence

The concept of the meaning of a first-order formula is foundational with the reference to the model-theoretic semantics and its applications in natural language semantics. We frame our discussion by asking three basic questions: (1) What are the meanings of first-order formulas (2) What are their modes of composition ? (if there any) (3) Are there any meta-theoretic principles which should govern a semantics ? We select two such principles. Namely, principle of compositionality and principle of alphabetic innocence. The latter principle is not well known. However, it encapsulates a sound pre-theoretic intuition about the meaning of a first-order formula. It has been recently endorsed by a number of authors [2], [11], [12], [18] [19]. Furthermore, it is also recognized on the ground of model-theory [7 p. 26-27]. Alphabetic innocence happens to be highly relevant in the context of translations from a natural language into a first-order language (or its intensional extensions).

In the sequel, we consider signatures without function symbols. Unless stated otherwise we work with ordinary first-order languages L_σ as defined in [14]. Each L_σ is generated by a context-free grammarr (or its algebraic translation) relatively to Var_α; where $Var_\alpha := \{v_i : i<\alpha\}$ for an ordinal α [1]. Due to technical reasons we assume that $\alpha \geq \omega$. By a semantics we mean a function μ in the sense of [5].

We note that the notion of the meaning of a first-order formula is ambiguous. We distinguish between the global and local concept of the meaning of a first-order formula. The global account identifies the meaning of a first-order formula with the class of structures which makes a formula true. It is non-compositional, tied to the expressive power of a language and axiomatizability of classes of structures [1] ,[5]. Due to limited applications outside logic this approach is not in the scope of our interests.

The local accounts provide the meaning of ϕ relatively to a fixed but arbitrary structure $\mathfrak{I}$. Their origins are in the algebraizations of the satisfaction relation. Typically, algebraization of the satisfaction relation for L_σ is given w.r.t the class of cylindric algebras of dimension α. Each first-order structure $\mathfrak{I}$ determines the full cylindric set algebra $\mathfrak{R}(\mathfrak{I})$ of dimension α. $\mathfrak{R}(\mathfrak{I})$ is closed under boolean operations and unary additive cylindrification operators c_i for $i<\alpha$. The meaning assignment $\mu_\mathfrak{I}$ is a homomorphism from a syntactic algebra of L_σ to $\mathfrak{R}(\mathfrak{I})$. Each formula ϕ receives as the value $\phi^\mathfrak{I}$, which is the set of satisfying assignments. Hence, the meaning of ϕ is identified with the satisfaction conditions. Of course, $\mu_\mathfrak{I}$ determines an atomless proper subalgebra $\mathfrak{R}^*(\mathfrak{I})$ of $\mathfrak{R}(\mathfrak{I})$. $\mathfrak{R}^*(\mathfrak{I})$ has two important features: (a) regularity and (b) local finite dimensionality[2].

From the metamathematical perspective the machinery of cylindric algebras is successful. It provides the axiomatization of ordinary first-order logic by a finite set of equation schemas. Moreover, it delivers algebraic proofs of the strong completeness and Downward Skolem Loewenheim Theorem [13],[14]. However, it has been argued that $\mu_\mathfrak{I}$ suffers from the conceptual defects. The first complaint goes under the heading of representionalism [19], [21]. In a nutshell it says that compositionality of $\mu_\mathfrak{I}$ commits us to include sets of sequences as a part of ontology. Indeed, it is typically the case that $|\phi^\mathfrak{I}| := \aleph_1$; even though the denotations of all relation symbols occurring in ϕ have the finite size and $D_\mathfrak{I}$ is countably infinite. Next, the length of $\phi^\mathfrak{I}$ is α-ary. Clearly, such relations are nowhere to be found in the structure $\mathfrak{I}$ at hand.

A closely related worry states that conventional features of syntax (namely variables) are encoded in μ_A. It is then argued that μ_A is not alphabetically innocent. The core intuition of alphabetic innocence is expressed by Albert Visser in [18] as follows " … consider the logical predicate formulas P(x) and P(y). From one point of view, the formulas are different. E.g. there is all the difference in the world between $P(x) \wedge \sim P(x)$ and $P(x) \wedge \sim P(y)$. From another point of view these formulas are the same. The choice of x or y as variables is immaterial. It does not correspond to an 'underlying' meaning".

1This automatically provides canonical ordering on Var_α with the order type of α.

2These two properties are non-trivial. If $|D_\mathfrak{I}| \geq 2$ then $\mathfrak{R}(\mathfrak{I})$ contains elements which are neither regular nor locally finite dimensional. Unfortunately, neither local finite dimensionality nor regularity are not axiomatically definable.

We regard alphabetic innocence as a sound pretheoretic principle. It is especially relevant in the context of a translation with first-order logic as the target language. For example, natural languages expressions containing pronouns such as 'he sees her' are translated into an open formula. Why the choice of variables should determine the model-theoretic meaning of this expressions ? Essentially, the same is valid in the case of mathematical discourse. Why there should be a difference between $v_0 > v_1$ and $v_1 > v_0$ if one just wants to capture the relation of being greater than ? Yet again, this problem emerges when mathematically well-behaved translations between logical languages are concerned. An example is the standard translation of basic modal logic into first-order logic with bounded quantifiers (or its two-variable variant) [17]. For each choice of variable the translation preserves the satisfaction relation. However, it does not preserve the meaning and there is a shift in ontology[3].

Formally, alphabetic innocence is captured by [11] as follows: A semantics μ is alphabetically innocent if and only if for every ϕ and $\phi' \in L_\sigma$, if $\phi' := \tau(\phi)$ then $\mu(\phi) := \mu(\phi')$ where τ stands for bijective simultaneous substitution of free occurrences of variables. With this at hand, alphabetic innocence can be seen as an analogue of meaning preservation of under the relation of α-equivalence. Moreover, it can be interpreted as requiring the invariance of the meaning under a group of permutation of variables (with finite support).

This gives rise to the question: " Is it possible to define an alphabetically innocent semantics μ_A for ordinary first-order languages?". We believe that in order to satisfy alphabetic innocence and get rid off of representationalism (and at least partly retain the intended ontology) it is necessary to disidentify the meaning of ϕ with $\phi^{\mathfrak{I}}$. However, $\phi^{\mathfrak{I}}$ might be a building block of an alphabetically innocent meaning of ϕ. We recall that regularity entails that only values of free variables occurring in ϕ determine the satisfaction condition. Moreover, local finite dimensionality guarantees that there are finitely many of them. These two facts taken together are interpreted as saying that every $\phi^{\mathfrak{I}}$ is a finitary relation in disguise. Hence, it seems that in principle the intended ontology can be recovered from $\mathfrak{R}^*(\mathfrak{I})$. The question is how it can be done ? We stress that Visser's example provides a limitation. It is readily seen that for each μ_A the corresponding synonymy relation can't be a congruence (provided that μ_A 'reasonably' interprets connectives and quantifiers). Consequently, we decide to drop compositionality. This allows to define a desired semantics [4]. Its construction requires two stages.

Firstly, in order to include bound occurrences of variables we strengthen the definition of alphabetically innocent semantics. To this end, we define inductively the operation of bijective variable replacement. It dispenses with capture avoiding side conditions. We prove the substitution lemma for this operation. As a side issue, we show how to define the operation of bijective bound renaming as as a composition of this new operation with the standard simultaneous substitution operation.

Secondly, we define our alphabetically innocent semantics μ_A. The definition μ_A requires (a) an intermediate level of syntactic representation and (b) the auxiliary meaning assignment denoted by μ'_A. We shall represent each $\phi \in L_\sigma$ by the formula in finite variable context $\phi(x_0,...,x_{n-1})$. We stress that strings of the form $\phi(x_0,...,x_{n-1})$ are not elements first-order languages. Strictly speaking, these are pairs whose coordinates are (a) a formula (b) a finite sequence of variables. Each $\phi(x_0,...,x_{n-1})$ determines the relation in $\mathfrak{I}$ as follows: $|\phi(x_0,...,x_{n-1})|^{\mathfrak{I}} := \{<a_0,...,a_{n-1}> \in D_{\mathfrak{I}}^{\ n} : s \in \phi^{\mathfrak{I}}$ and $s(x_j) := a_j$ for each j, $0 \leq j \leq n-1\}$[5]. Hence the meaning of ϕ in $\mathfrak{I}$ is going to parasitic on $\phi(x_0,...,x_{n-1})$. However, there is no unique way to define a sequence of parameters associated with ϕ. In order to

3We remark that problem of alphabetic innocence has a quick solution by eliminating ordinary first-order languages in favour of restricted first-order languages. Since a simultaneous substitution operation is definable by a formula in a restricted form it follows that ordinary and restricted first-order languages have the same expressive power [13]. Nonetheless, working with ordinary first-order languages is clearly desirable. An example is the two variable variant of the standard translation s

4The construction of our semantics relies on the insights provided by Hodges' . We critically emphasize that in [7 p.26-27] it is conjectured that the free alphabetic variants denote the same relation in $\mathfrak{I}$. The semantics defined below can be seen as a proof of Hodges' conjecture.

5 In [20] sets of this form are called extensions of a formula in a structure $\mathfrak{I}$.

avoid ambiguity some systematic method of assembling a formula in finite variable context must be fixed beforehand [8].

Minimally, it is required each $x_i \in Fr(\phi)$ belongs to $(x_0,...,x_{n-1})$. As we have made clear variables outside $Fr(\phi)$ does should not contribute to the meaning of ϕ at all. Therefore, this minimal requirement is also the maximal requirement. Still, in order to avoid ambiguities, we have to determine the intended structure of a list of parameters. We distinguish between (a) an increasing method of ordering parameters and (b) a non-increasing method of ordering parameters. Given the set $Fr(\phi)$, the former method yields the subsequence of canonical order on Var_α [15]. Whereas the latter method yields the sequence which strictly parallels the left to right of occurrence of free variables in ϕ. We opt for the latter method. Clearly, canonical order on Var_α is not necessarily reflected in variable structure of a formula. Furthermore positions in an increasing list of parameters are not necessarily preserved under an arbitrary permutation of variables [6].

With this at hand, we define the representation function $r: L_\sigma \rightarrow L_\sigma \times Var^*$ s.t for each $\phi \in L_\sigma$ $r(\phi)$ is ϕ together with the non-increasing list of parameters in our sense[7]. Next we define the semantics μ'_A s.t (a) the domain of μ'_A is the image of L_σ under r (b) the range of μ'_A is $\wp(D_{\mathfrak{I}}{}^*)$. Finally, by appealing to our substitution lemma, we show that μ'_A is strongly alphabetically innocent . Taking $\mu_A := r \circ \mu'_A$ we obtain a strongly alphabetically innocent semantics for L_σ which retains the intended ontology.

As an application of the semantics μ_A, we refute a claim found in [11]. It states that under any alphabetically innocent semantics it is impossible to define the converse of a relation denoted by a formula of the form $P(x_0,x_1)$. By resorting to relation algebras [8] we show that under μ_A this claim is false. Moreover, we point out a philosophically interesting consequence of our result. There exists formulas which are logically equivalent but they are not μ_A- synonymous in any structure $\mathfrak{I}$. Hence the semantic values of formula under μ_A happen to be fine-grained.

Finally we argue that noncompositionality of μ_A can be explained away by the fact that boolean connectives doesn't have to be necessarily interpreted as corresponding to operations on relations.

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6 Among else this entails that relatively only to the latter method Hodges' conjecture can be shown to be true.

7The symbol * stands for the Kleene star .

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