A NOTE ON THE SCHOLZ-BRAUER PROBLEM IN ADDITION CHAINS

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1. **Introduction.** An *addition chain* for the positive integer n is a sequence of integers $a_0=1<a_1<a_2<\cdots<a_r=n$ where, for each $i>0$, $a_i=a_j+a_k$, for some $j, k>i$ ($j=k$ is permitted). For example, 1, 2, 4, 8, 10; 1, 2, 3, 6, 9, 10; 1, 2, 4, 6, 10 are three addition chains for $n=10$. By $l(n)$ one means the smallest r for which there is an addition chain for n. One can easily verify that $l(1)=0$, $l(2)=1$, $l(3)=l(4)=2$, $l(5)=l(6)=l(8)=3$, $l(7)=l(9)=l(10)=l(12)=l(16)=4$ and that $l(2^n)=n$.

A. Scholz [2] published the following as problems:

$$m+1 \leqq l(n) \leqq 2m \quad \text{for} \quad 2^m+1 \leqq n \leqq 2^{m+1}, \qquad m \geqq 1, \tag{1}$$

$$l(ab) \leqq l(a)+l(b), \tag{2}$$

$$l(2^q-1) \leqq q+l(q)-1. \tag{3}$$

A. T. Brauer [1] established (1) and (2) and also improved another inequality suggested by Scholz. When considering (3) Brauer showed that (3) holds if the chains admitted in determining $l(q)$ are restricted. So far as the author can discover, the original problem (3) of Scholz has not been solved. In this note we establish (3) for some values of q by a method that may extend to an arbitrary q.

2. **The case** $q=2^s(2^n+1)$.

Lemma 1. $l(2^s+1)=s+1$, $s\geqq 0$.

Proof. Clearly, $l(2^s+1)\leqq s+1$ since

$$1, 2, 2^2, 2^3, \cdots, 2^s, 2^s+1$$

is an addition chain for 2^s+1. Also,

$$2^s-1<2^s+1<2^{s+1}$$

hence, by (1), $l(2^s+1)\geqq s+1$ and the lemma is proved.

Lemma 2. $l(2^q-1)\leqq q+l(q)-1=2^s+s-1$ if $q=2^s$, $s\geqq 0$.

Proof. This is shown by induction. The inequality holds for $s=0$; suppose it holds for $s=r-1$. Then setting $m=2^{r-1}$,

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$$l(2^{2m}-1) = l[(2^m-1)(2^m+1)] \leqq l(2^m-1) + l(2^m+1)$$
$$\leqq 2^r + r - 1$$

by Lemma 1 and the induction hypothesis. This completes the proof.

LEMMA 3. $l(2^{q+1}-1) \leqq q + l(q+1) = 2^s + s + 1$ if $q = 2^s$, $s \geqq 0$.

PROOF. $l(2^{q+1}-1) \leqq l(2^{q+1}-2)+1 \leqq l(2)+l(2^q-1)+1 \leqq 2^s+s+1$ by Lemma 2.

LEMMA 4. $l(2^s(2^n+1)) = n+s+1$, s, $n \geqq 0$.

PROOF. (a) $2^{n+s}-1 < 2^s(2^n+1) < 2^{n+s+1}$ hence $l(2^s(2^n+1)) \geqq n+s+1$ by (1).

(b) $l(2^s(2^n+1)) \leqq l(2^s)+l(2^n+1) = n+s+1$ by Lemma 1 and property (2).

THEOREM 1. $l(2^q-1) \leqq q+l(q)-1 = 2^s(2^n+1)+n+s$ if $q = 2^s(2^n+1)$, s, $n \geqq 0$.

PROOF. This proof will be by induction on s. We have seen in Lemma 3 that the theorem is true if $s=0$. Assume, for the induction proof, that it holds for $s=r-1$ and all $n \geqq 0$. If $m = 2^{r-1}(2^n+1)$,

$$l(2^{2m}-1) \leqq l[(2^m-1)(2^m+1)]$$
$$\leqq 2^{r-1}(2^n+1) + l(2^{r-1}(2^n+1)) + 2^{r-1}(2^n+1)$$

by the inductive hypothesis and Lemma 1. The proof is completed by the use of Lemma 4.

3. **Comments and questions.** Since any positive integer can be written

$$2^{c_1} + 2^{c_2} + 2^{c_3} + \cdots + 2^{c_n}, \qquad \text{where} \quad c_1 > c_2 > \cdots > c_n \geqq 0,$$

one might expect to establish (3) by the ideas of this note (numbers for which $n=2$ having now been disposed of). However, the author has been unable to carry this through.

Other inequalities involving $l(p)$ would be of interest. One can easily show that $l(a+b) \leqq l(a)+l(b)$ if a, $b > 1$. Does the inequality $l(p) < l(2p)$ hold for all $p > 0$? Let $S(n)$ denote the number of solutions of the equation $l(x) = n$. Is it true that $S(n) < S(n+1)$ for all $n > 0$?

BIBLIOGRAPHY

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